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Review of *The Riemann Hypothesis* and *Prime Obsession*

What is the greatest unsolved problem in mathematics? Ask the next person you meet that, and hopefully you'll get nothing worse than a quizzical stare. But even your typical mathematician might not immediately have an answer for you, as they are likely either engrossed in their own subdiscipline, a very heavy teaching load, or (more likely) both, and even an answer will more likely reflect the biases of the responder, not a detailed understanding of all the top problems out there. We can only hope that Fermat's Last Theorem is no longer on your friend's list, but otherwise you might not find immediate consensus. Yet in about the last five years no fewer than *seven* books, all aimed squarely at the layman, treat precisely this question!

Overall there is indeed consensus as to what the most important problems (plural) are, even if there is disagreement over which one receives top billing. In fact, the Clay Institute for Mathematics (www.claymath.org) has recently offered million dollar prizes for the solution of seven of these. Many of these are well-known to a general mathematical audience, such as the P=NP question in computer science (interestingly, the only 'Millenium Prize' problem for which either a yes *or* no answer is acceptable).

On the other hand, some are rather difficult to explain to non-experts. The Hodge Conjecture, for instance, essentially asserts that any element of a certain natural subgroup of the cohomology of a complex projective variety X comes from cocycles associated to some subvariety of X. Got that? Nonetheless, in a recent book, veteran math popularizer Keith Devlin, attempts to describe all seven of the 'Millenium Prize' problems, though with as little success on the two most thorny[i] as in the official publication from the AMS; indeed, he acknowledges the difficulty of his task in the book.

Another one of the problems, the Poincaré Conjecture, has inspired two more[ii] similar books, describing the intense human drama of its recent solution. Given the compelling math (this conjecture has led to most of algebraic topology in one way or another, and is a beautiful characterization of the 3-sphere by its vanishing fundamental group), and the story's culmination in the explicit rejection of the Fields Medal[iii] by Grigory Perelman, the choice of this topic for a book is no surprise. This implies there is a market for books which introduce great problems with a human face.

And despite some lack of consensus, there seems to be growing recognition that yet another of the Millenium Prize problems is the most important remaining open question in our field – the Riemann Hypothesis. True to form, *all four* of the other recent popular books tackle this conjecture (the RH, for short). We here review the two of these of greatest interest to JACMS readers.

With the history of post-Napoleonic Europe as a lush backdrop, John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* attempts to unpack the problem itself; on the other hand, Karl Sabbagh's *The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics* has a far lighter treatment of the math, focusing on "the humanity of mathematicians", in particular those mathematicians working on the RH on the cusp of the 21st century. Both achieve their major goals[iv], though each book has certain deficiencies I'll outline later.

The Clay Math Institute's description for non-mathematicians of the RH is as follows:

Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g., 2, 3, 5, 7, etc. Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826 - 1866) observed that the frequency of prime numbers is very closely related to the behavior of an elaborate function "$\zeta(s)$" called the Riemann Zeta function. The Riemann hypothesis asserts that all interesting solutions of the equation $\zeta(s) = 0$ lie on a straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of primes.

Hopefully the reader can add that $\zeta(s) = 1+1/2^s+1/3^s+1/4^s+\ldots$, the function behind the Basel problem, which Euler rather famously solved by proving that $\zeta(2) = \pi^2/6$; some readers may also know that since $\zeta(s)$ is defined on the complex plane via analytic continuation, 'interesting' means 'not for s a negative integer'. But what that 'close relation' is, the gentle reader may be forgiven if they don't know, for this is the heart of this most difficult question.

Now, given the spate of popular math books of the last two decades, and the relative inaccessibility of the problem, why would anyone write or read about it? In Sabbagh's book, the interest is to some extent in TV-style voyeurism. Lest anyone think I mean a reality-show race to wealth, keep in mind that the problem is 145 years old, has been seriously worked on for over a century, and will almost certainly not be solved anytime soon[v].

Instead of *Survivor*, the researchers in *The Riemann Hypothesis* are on *All My Children*. We hear shouts of controversy over two claims on the first elementary[vi] proof of the Prime Number Theorem (that primes are, roughly speaking, distributed logarithmically among all positive integers) by Erdös and Selberg; we witness a totally chance conversation at Princeton that yields one of the main means of attacking the RH, via randomly distributed Hermitian matrices; we see the late entrance of the 'lovely' Fields medalist Alain Connes as a knight to (possibly) save the day. There is even a black sheep in the family, Louis de Branges[vii]; his story begins like the rest, but becomes a tragic portrait of a man 'for whom there is nothing in the world but mathematics and persecution,' for even if his latest proof is correct, he has cried wolf too often for anyone to listen any longer.

Sabbagh attempts to portray what drives mathematicians (as readers of this journal will know, not the money), to describe the flash of the moment of insight, and give a feel for the RH itself at a basic level, and he succeeds – but at a cost. Despite his attempt to portray his subjects as human, his attitude is captured well in the prologue; "It is given to them to see truths…" Sabbagh's mathematicians are people, yes, but people who operate on a totally different plane when it comes to math, and often don't operate so well elsewhere. That isn't to say he is all wrong; his description of a typical math conference rings so true that, while reading the book at such a gathering, I thought maybe he was hiding behind the chalkboard, taking notes. It is good reading, but he makes his case too strongly; his claim that mathematicians find thinking abstractly "a more satisfying activity than any of the other pleasures the world has to offer" needs a correction of 'any' to 'many'.

Derbyshire's goals are totally different, and so are his mathematicians. *Prime Obsession* is loosely written as a two-track book, one which goes into some detail about the math of the RH, the other following the history behind each advance along the way. As both tracks are following paper trails to some degree, the book is more grounded in reality. That's not to say he shies from hyperbole; in the chapter 'Turning the Golden Key', he refers to Euler's product formula for the zeta function ($\sum_n \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}$, with n ranging over positive integers and p over all primes), which his prose has built up to for half the book, by saying, "I simply cannot tell you how wonderful this result is."

But the historical view places such comments in context. We really meet Bernhard Riemann, who proposed the now-famous hypothesis in a then little-noticed 1859 paper, in his milieu of the still-fragmented pre-Bismarck Germany. For those who know about Riemann from teaching integration or geometry, there is a lot to learn and bring back to the classroom. Like Sabbagh's de Branges (who appears in a single endnote here), Riemann seems "a rather sad and slightly pathetic character" and brilliant mathematician, but we also discover his "very pious" Lutheranism and his devotion to his family. Each player in the history is thus portrayed, sympathetically and within the overall saga of the RH. If the first book is a soap opera, this one is at least on the History Channel, maybe even PBS[viii]. On the other hand, the story never grows boring, and it starts to prepare the reader for the in-depth mathematical treatment.

The mathematics does not totally drive the book – Derbyshire even suggests that the reader skip every other chapter once it gets too hairy! Still, it clearly excites him, and rightly so. I have never seen a treatment of the Riemann Hypothesis that builds up so clearly and with a writer's flair for suspense. That is the great strength of this 'remarkable book', as Nobel laureate John Nash's jacket blurb relates, and it is also the great weakness of an otherwise exceptional math popularization. Another blurb[ix] puts the dilemma best; the book 'explains the hypothesis in ways understandable by ordinary mathematicians and even by laymen.'

Indeed! The early going is manageable, and some of the graphics are truly spectacular. On the other hand, advanced undergraduates have difficulty understanding big-Oh notation and finite fields on the first try (and later)! To be fair, I think Derbyshire really does an excellent job of making things like complex-valued functions accessible, and especially the last few math chapters are quite appropriate for a classroom setting in a senior seminar or number theory class, as is perhaps appropriate for a book published indirectly via the National Academy of Science. But these chapters are not simply an idle beach read.

The Riemann Hypothesis also tries to gives some idea of what the RH is all about, but Sabbagh's tendency to over-elevate the (math-related) thoughts of mathematicians also infects his treatment of the math itself. He describes the mechanics of the statement at a level many readers will be able to understand; however, this leaves mysterious why Riemann should have made such a statement in the first place. This would be fine – that's not his goal, after all - if he didn't at the same time leave behind some other readers completely. The 'Toolkit' appendices will help some readers who have forgotten the concepts involved, but they are probably not enough to teach a neophyte what a logarithm or an eigenvalue is.

The nice thing is that these mathematical caveats don't obscure why someone would want to read the books. Every math book ever written has a specific level of sophistication in mind – no less so here. But these books are different from the usual crop of math popularization, in that the stories about people are not the window dressing to entice the reader into exploring a number of interesting concepts like infinity or the golden mean. Here the focus really is on people. Especially in *Prime Obsession,* the amazing problem is also of key significance, but we delve into the lives of those who either are developing it now, or who brought it to its current prominence. Though I consider *Prime Obsession* a quite strong book, *The Riemann Hypothesis* is also worth a look for different reasons; neither is right for everyone.

One final aspect of these books will be of particular interest to readers of this journal. The past century has seen some great debates about whether mathematics is created or invented, and how much of what is out there will ever really be discovered; both authors take rather strong positions for non-experts.

Sabbagh makes it clear he believes math exists to be discovered, but includes some interesting anecdotes by various mathematicians working on the RH regarding what they think, from the strong realism of Connes to one mathematician's comparing his Serbian Orthodoxy to believing the RH is true (in a very weak sense); even those who disagree seem so involved in the pursuit of the answer that they become arguments for the independent existence of mathematical ideas. Unfortunately, he doesn't dig deeper, and talks of such abstract truth-seeking as a curious property of his researchers, "apart from, say, meditation or religion."

Derbyshire spends less time on such issues, but his statement is even stronger – he ends the book by quoting the eternal progress theme of the great German mathematician David Hilbert, that "We must know, we will know[x]," whether the RH is true. It is not clear if this is warranted[xi]; however, thanks to Derbyshire writing his people in their *Sitz im Leben,* we still find humility in the mathematicians themselves.

He attributes mathematician Leonhard Euler's 'serenity and inner strength' to his 'rock-solid religious faith'; even more so, Riemann's 'daily self-examination before the face of God' comes up several times, including in the epilogue, where Riemann dies, his wife reciting the Lord's Prayer at his side. His epitaph? "All things work together for good to them that love God," Romans 8:28. Neither the solution of the greatest problem in mathematics – nor its absence – can change that.

[An earlier version of this review appeared in *Books and Culture*.]

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[i] The Hodge Conjecture and the Birch-Swinnerton-Dyer Conjecture.

[ii] These are George Szpiro's *Poincaré's Prize* and Donal O'Shea's *The Poincaré Conjecture*.

[iii] Apparently for reasons even more enigmatic than for any Nobel rejection, certainly more so than Sartre's or Pasternak's.

[iv] Neither of the two books actually written by mathematicians (highly respected ones, in fact) treats the contemporary human element as well as Sabbagh, or the mathematics and history as deeply as Derbyshire; nonetheless, Marcus du Sautoy's *The Music of the Primes* and Dan Rockmore's *Stalking the Riemann Hypothesis*) also generated reasonable accolades upon publication.

[v] For example, see the article "The Riemann Hypothesis" in the March 2003 issue of *Notices of the American Mathematical Society*. Aimed at experts and written by someone who directs an institute with solving the RH as a primary objective, the view taken is long-term.

[vi] I.e. not using complex analysis – however, still very *hard*.

[vii] De Branges is most famous for solving the Bieberbach Conjecture in the early 1980s – also after a lot of controversy over whether it really was a solution.

[viii] For instance, the treatment of the Dreyfus affair is quite detailed – and even adds to the overall exposition, because Hadamard (one of the original two provers of the Prime Number Theorem) was Dreyfus' brother-in-law; still, this is not your typical math popularization.

[ix] By the great mathematics-as-recreation advocate Martin Gardner.

[x] Wir müssen wissen, wir werden wissen.

[xi] In the article mentioned in the first footnote, the author gives as an argument for the validity of the RH, "It seems unlikely that nature is that perverse!" which, though possibly true, does not exactly exude confidence.