Proceedings of the Twentieth Conference of the Association of Christians in the Mathematical Sciences

Redeemer University College
May 27 – 30, 2015

Contents

Preface

The 20th Biennial Conference of the Association of Christians in the Mathematical Sciences was hosted by the Mathematics and Computer Science Departments at Redeemer University College, May 27–30, 2015. This was the first time the association has held its meeting outside of the USA. The conference included over 100 participants from many different colleges and universities across North America. Derek Schuurman, Kyle Spyksma, and Kevin Vander Meulen served on the local organizing committee. The conference was sponsored by Redeemer University College and Edifide.

The ACMS conference was preceded by a Pre-Conference Faculty Development Workshop, May 25–27. The workshop was organized by Maria Zack and with various sessions led by Greg Crow, Matt DeLong, Dave Klanderman, Derek Schuurman, Anthony Tongen, and Maria Zack. Also, starting the afternoon before the conference, Amanda Harsy-Ramsay led several graduate student workshops. Bob Brabenec distributed a historical overview of the ACMS 1977 –2015.

The ACMS conference included an afternoon of outings on Thursday May 28. One group went to visit Niagara Falls, another visited the Bertrand Russell archives at McMaster University and also enjoyed a walking tour of Princess Point led by Shari-Ann Kuiper from the Christian environmental organization A Rocha. During the banquet, Bob Brabenec, Lori Carter, Jeremy Case, and Mike Stob described the contributions of recently deceased ACMS members Peter Rothmaler, Kim Kihlstrom, David Neuhouser, and Paul Zwier respectively.

The ACMS conference included two dynamic keynote speakers: Dr. Annalisa Crannell, professor of mathematics from Franklin & Marshall College in Pennsylvania and Dr. Matthew Dickerson, professor of computer science at Middlebury College in Vermont. An author of the book *Viewpoints: Mathematical Perspective and Fractal Geometry*, Dr. Crannell spoke about mathematics and art, reflecting on the projective geometry in the context of perspective artists. Her keynote lectures were entitled "Math and Art: the Good, the Bad and the Pretty" and "In the Shadow of Desargues." Dr. Dickerson, author of *The Mind and the Machine: What it Means to be Human and Why it Matters*, gave the two keynote lectures "Can Computers Reason?" and "On Mind, Body and Abstractions: the Ecologies of Physicalism vs. Christianity."

The main speakers presented on topics that have been used elsewhere, so their presentations were not submitted to these proceedings. The conference also included over 50 different presentations from participants covering the history of mathematics, philosophy and theology of mathematics, as well as pedagogical issues in computer science and mathematics. These proceedings include non-refereed contributed papers based on presentations given at the conference.

Thomas Price, Derek Schuurman, and Kevin Vander Meulen
Co-Editors of the Proceedings.

The 21st Biennial Conference of the ACMS
will be held at
Charleston Southern University
May 31 – June 3, 2017.

Abstracts for Invited Talks

In the Shadow of Desargues

Annalisa Crannell, Franklin & Marshall College, PA

Those of us who teach projective geometry often nod to perspective art as the spark from which projective geometry caught fire and grew. This talk looks directly at projective geometry as a tool to illuminate the workings of perspective artists. We will particularly shine the light on at Desargues' triangle theorem (which says that any pair of triangles that is perspective from a point is perspective from a line), together with an even simpler theorem (you have to see it to believe it!). Given any convoluted, complicated polygonal object, these theorems allow us to draw that object together with something that is related to it— its shadow, reflection, or other rigid symmetries—and we'll show how this works. (If you enjoy doodling or sketching, bring your pencil, a good eraser, and a straightedge.)

Math and Art: The Good, the Bad, and the Pretty

Annalisa Crannell, Franklin & Marshall College, PA

How do we fit a three-dimensional world onto a two-dimensional canvas? Answering this question will change the way you look at the world, literally: we'll learn where to stand as we view a painting so it pops off that two-dimensional canvas seemingly out into our three-dimensional space. In this talk, we'll explore the mathematics behind perspective paintings, which starts with simple rules and will lead us into really lovely, really tricky puzzles. Why do artists use vanishing points? What's the difference between 1-point and 3-point perspective? Why don't your vacation pictures don't look as good as the mountains you photographed? Dust off those old similar triangles, and get ready to put them to new use in looking at art!

Can Computers Reason?

Matthew Dickerson, Middlebury College, VT

Modern computers can execute complex programs that quickly process vast quantities of data. The philosophy of physicalism as espoused by well-known modern figures such as biologist Richard Dawkins, philosopher Daniel Dennett, or engineer-futurist Raymond Kurzweil tells us that humans are simply complex biochemical computers. But can computational devices (however fast or powerful) reason? That is, can they determine what is true in any normative or valid way? At the core, this is a philosophical question, and not an engineering or technical one. This talk will argue that the answer is “no” - ironically by drawing upon and critiquing the very words of Dawkins, Dennett, and other well-known physicalists. But if computers are not capable of reason, then either humans are equally incapable of reason or else physicalism must be false.

On Mind, Body, and Abstractions: The Ecologies of Physicalism vs. Christianity

Matthew Dickerson, Middlebury College, VT

The philosophy of physicalism, in denying a spiritual reality, reduces the mind to a biological brain. Ironically, Raymond Kurzweils physicalist vision of the future, while affirming only a body, actually devalues both the human body and the created world. A Christian worldview offers a stronger bases for a healthy care of creation. An exploration of why suggests as a side warning to mathematicians and computer scientists that there might be a danger in loving abstractions too much.

Abstracts of Contributed Talks

On the use of applets when teaching about the Enigma machine

Brandon Bate, Houghton College, NY

The Enigma machine, along with the heroic efforts required to break it, constitutes one of the most interesting topics in the history of cryptography. In this talk, I will share my approach to presenting this material, which makes use of applets which mimic both the Enigma machine and the Polish bombes (the electro-mechanical devices constructed by the Polish Cipher Bureau to break Enigma). By utilizing such software, students not only encrypt messages using Enigma, but also break encrypted messages using the same techniques employed by the Polish Cipher Bureau. In doing so, students gain a deeper understanding of the group theory underlying the cryptanalysis of the Enigma machine and find an appreciation for the practical power of mathematics.

The Mysterious Mathematical Madame X

Brian Beasley, Presbyterian College, SC

Agnes Meyer Driscoll, a 20th century pioneer of cryptanalysis, served first in the United States Navy and later in the Armed Forces Security Agency for a total of forty-one years. Yet few details are available about her cryptographic work and her personal life. Driscoll's expertise in both mathematics and foreign languages enabled her to break several complex Japanese code systems in the 1920s and 1930s. However, she could not duplicate that success later with the German Enigma machine. Why did Driscoll resist the offer of British assistance with Enigma? What effect if any did a serious automobile accident have upon her? More questions than answers concerning "Madame X" remain; this talk will provide a brief summary of both the known and the unknown in the life of this fascinating individual.

Using Free Web-based Google Software to Enhance Your Course

Nick Boros, Olivet Nazarene University, IL

We will discuss how one can use Google Drive (Docs, Sheets, Slides, Forms, etc.), Google forums, and Google hangouts in place of a learning management system and also in a way that improves upon such existing approaches. We will also discuss how they can be used to improve the classroom experience in general: creating "Clicker"-type questions and surveys, all the way to easy ways of forming groups and giving students convenient ways of best contributing in group activities.

Random Numbers and God's Nature

James Bradley, Calvin College, MI

In this talk, I start with mathematical Platonism, an ancient stream of thought that views numbers as transcending physical reality. I join this to recent insights into mathematical randomness from theoretical computer science. Joining these streams—one ancient, one recent—yields the surprising conclusion that randomness, defined in a particular way, is part of the nature of God. I then explore some of the implications of this conclusion for our understanding of the doctrine of God's infinitude.

Understanding Everyday Use of the Computer: How Philosophy can be Practical

Nick Breems, Dordt College, IA

The issues which must be considered when attempting to understand the totality of behavior when a human uses a computer are both multifarious and diverse. We thus have difficulties in gaining insight into the meaning that use of the computer creates in our lives, particularly in everyday use. Such understanding is important because insight based on holistic understanding is our best hope for exploiting the God-created potentials for human flourishing that is part of the promise of computing technology. As computer and internet technologies continue to embed themselves in the fabric of our everyday living, the number and complexity of use situations grows, and all facets of these situations must be apprehended and appreciated in order to respond wisely and act normatively.

In this presentation, I will summarize the Human Use of Computers Framework (HUCF) developed by Basden (2008), a tool for producing insight into complex everyday computer use situations, and show several examples of this framework in use. This will demonstrate how the HUCF can help prevent overlooking areas that are crucial for understanding the human experience of using the computer, and will exhibit the practical implications of a philosophically-based tool for understanding.

Biology Inspired Computing Exercises

Lori Carter, Point Loma Nazarene University, CA

Research has shown that computing courses taught within a context aid in retention. Biology in general and genetics in particular is a context full of interesting problems to solve computationally. Additionally, the field of biology has vast collections of data available to be manipulated and analyzed. This presentation will give examples of biology-related programming problems that can motivate students at a variety of levels

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Mathematics without Apologies by Michael Harris–A Review

Jeremy Case, Taylor University, IN

The subtitle of *Mathematics without Apologies* by Michael Harris is a "Portrait of a Problematic Vocation," and that problematic vocation involves pure mathematics. What do pure mathematicians do, and why should they do it? Harris critiques the usual answers of truth, beauty, and potential applications as he gives a contemporary revision of G.H. Hardys *A Mathematicians Apology.* Described as a post-post-modern book, *Mathematics without Apologies* provides a cultural, sociological, philosophical and psychological landscape of the profession and the life of a mathematician. We will explore whether a romantic view of the profession satisfactorily answers the questions and how it might compare to a Christian perspective.

Math Teachers' Circle: A Bridge from the College to the Community

Thomas Clark, Dordt College, IA

Although much of the training pre-service mathematics teachers receive is either in mathematical content or educational theory and pedagogy, research indicates that mathematical knowledge for teaching (MKT), which lies in a sense at the intersection of the two, is a factor in teacher quality. Unfortunately, none of the recent improvements in teacher preparation necessarily affect in-service mathematics teachers. One successful program affecting in-service teachers nationwide is the Math Teachers' Circle Network.

In this presentation I will discuss some of the ways in which a math teachers' circle can be a benefit to the college and the community. The mission of my institution involves "equipping the broader community" as well as our own students. Starting a circle in my area is one way I have been living out this mission. Math teachers' circle brings faculty, teachers, and students together to solve interesting mathematical problems. I will discuss the ways in which a circle can help local teachers to improve their teaching through deepening their mathematical knowledge, students to learn firsthand from teachers what it is like in a classroom, and faculty to have an impact beyond their own classrooms into the community as well as learn a few things themselves. Finally I'll provide some resources for those interested in learning more about math teachers' circles.

The God of Mathematics

Annalisa Crannell, Franklin & Marshall College, PA

In the summer of 2009, Crannell taught a seven-week adult Sunday School class called "The God of Mathematics." She designed this course both for the congregants of her own church (Wheatland Presbyterian Church) and also as an outreach to her local community (Lancaster, Pennsylvania). The class explored ways that mathematical understanding can provide metaphors that might help to illuminate faith; it included gentle introductions to infinity, paradox, symmetry, and non-orientable manifolds, among other topics. In this contributed talk, Crannell will discuss both the design of the course and some reactions to it from congregants and the community.

Quantitative literacy - What is it and what can we do with it?

Catherine Crockett, Point Loma Nazerene University, CA

Simply stated quantitative literacy, also known as quantitative reasoning, can be described as the ability to understand quantitative data and then use it to create sound arguments supported by that data. Until the later part of the twentieth century, there wasnt much focus in university-level education on competency in quantitative literacy (or QL). However, that changed after the findings of several national and international surveys exposed deficiencies in QL in the general population. During spring 2015, I spent my sabbatical exploring quantitative literacy. The objective for my sabbatical was to research current best practices for the teaching and assessing of QL, plan the implementation of those practices into existing GE math courses at PLNU and learn how QL fundamentals can be integrated university wide in the GE program. The goal of this presentation is to give an introduction to QL and examples of how to incorporate QL into your existing courses.

A Successful Implementation of the Flipped Classroom in Mathematics

Bryan Dawson, Union University, TN

The structure of a successful implementation of the flipped classroom for Calculus I and II and for Introduction to Analysis will be described. Examples of recorded lectures and the technology that produces them, lecture exercises, and in-class activities will be given. Student results and reactions will also be discussed.

Lessons Learned Modeling Killer Whales

Matthew Dickerson, Middlebury College, VT

A look at many interesting questions that have arisen in a collaborate research project on a spatially explicit individual-based (or agent-based) model of transient killer whales in southeast Alaska. We will survey some of the research questions that motivated the work, at some of the questions that the initial model has addressed (and how we approached them), and at some of the new questions raised by the work and the obstacles to overcome in order for the work to advance.

Designing for Mistrust

Eric Gossett, Bethel University, MN

The 2014 ACM North Central Region programming contest contained a problem about a group of v bandits who want to use multiple locks to seal their treasure and distribute keys in such a way that no group of less than m bandits can open all the locks. The problem asks for an algorithm that will determine the number of locks needed for any set of parameters (v, m).

I will present an analytic solution that produces a minimum number of locks, a recurrence relation solution, and a constructive algorithm that can print out a table showing the locks and which subset of bandits hold keys for each lock. Each table forms a balanced incomplete block design (BIBD). The parameters of the BIBD can be uniquely determined from v and m.

Software Engineering: Teaching Challenges

Paul C. Grabow, Baylor University, TX

The term software engineering can be traced to the late 1960s in response to large-scale, software development problems. Since then it has evolved as a discipline, both within industry and the academy. There have been distinct educational successes: "Standard practice" has matured (and found its way into more textbooks), the ACM and IEEE Computer Society have published curriculum guidelines, computer science programs commonly offer at least one software engineering course, and software engineering degrees (undergraduate or graduate) are more common. However, software engineering still presents a challenge. The term itself has become contorted by companies (and society in general); software has become far more diverse (along with the environments in which software engineers work); industrial software processes are not easily replicated in the classroom; what students are expected to know (once they are employed) has expanded significantly; software tools change rapidly (affecting student expectations); and the discipline involves far more than "good programming" (or a large programming project). This talk describes these challenges – and suggestions for dealing with them – in light of my 30 years teaching software engineering in a university.

Inviting the Nations In: Aiding International Graduate Students at Clemson

Rachel Grotheer, Clemson University, SC

The teaching of undergraduates by graduate students is an essential component of the mathematics graduate program at many universities. As a result, many universities require international students to demonstrate proficiency in spoken English in order to maintain their assistantships and status in the program. At Clemson, we have recognized that this requirement existed without any support from the university to enable the students to meet the requirement. In collaboration with the departments graduate chair and a few other faculty members, we have developed a class to aid international graduate students in developing their spoken English skills, as well as their awareness of cultural norms and differences, to help them succeed not only as graduate teaching assistants, but in all aspects of their professional and personal life in the United States. In this talk we will give an overview of the program, its early success, and the importance of and challenges in encouraging, supporting and welcoming students from all nations into a graduate mathematics program.

Computer Science Doxology

Joe Hoffert, Indiana Wesleyan University, IN

In both the secular and Christian academic communities computer science has generally not been associated with Christianity. One approach to the interaction of faith and computer science is to look at the various areas of computer science that reflect God's character. This presentation proposal presents examples of computer science subdisciplines and how they give glory to God.

Software System Development and Aesthetics: Software patterns "beautify" software code via repetition and abstraction. Patterns can be combined as "sentences" of a "language" to communicate higher levels of abstraction. The beauty and communication of patterns point to God's beauty and relationship.

Artificial Intelligence and Imago Dei: The complexity of humans points to our Creator's depth and complexity. AI illuminates the complexity of human thought and points to what it means to be human, to be separate from computational machines. AI also points to the creativity of humans as endowed by God.

Chaos and Order with Information: People are inundated with information. Database management systems (DBMSs) were created to bring order out of the chaos of data. This endeavor highlights the goodness of order which reflects God's bringing order out of chaos in creation. DBMSs also show human finitude since we need tools to manage the data we have created.

Website Development and Relationships: Website development underscores the importance of human relationships as a means of communication. Humans are relational because they are made in the image of a relational Triune God. Website development doxology values people's time and perspectives.

Symbolic Powers of Ideals: Problems and Progress

Mike Janssen, Dordt College, IA

Symbolic powers of ideals have been the focus of much recent study in commutative algebra and algebraic geometry. Problems in algebraic geometry (e.g., Warings problem) and commutative algebra (e.g., the question of containment in ordinary powers of ideals) have motivated much of this work, but symbolic powers also have applications to other fields, such as computer science, combinatorics, and graph theory. We will explore the algebraic and geometric questions that have motivated the study of symbolic powers of ideals and share recent results in this direction.

Can You Make a Christian Video Game?

Michael Janzen and Josh Noble*, The King's University, AB

We explore the possibility of creating a video game that evidences Christian content and principles. Our focus is on the playing experience and inherent meaning of a game, as opposed to the social or time costs of playing video games. We decompose aspects of video games to components: story, code execution, rules, and appearance. We further consider two distinct types of appearance: sensory and action. We apply this decomposition to four types of games: single player puzzles, multiplayer conflict, single player sandbox games, and multiplayer sandbox games. We conclude that story is subject to the same critique as other media, code execution is not amenable to morally good or bad values, while rules and appearance can carry Christian content and principles. We include a sketch showing equivalence of code execution between different games, where some are objectionable while others are not. This sketch follows an approach where we reduce the code execution of one game to the other.

Pressure and Impulse in Student Learning: What I Learned From Teaching Physics

Kim Jongerius, Northwestern College, IA

In the fall of 2014, a one-semester gap between the departure of one physics professor and the arrival of the next afforded me the opportunity(?) to teach a first-semester, calculus-based physics class. The thirty-year gap between the last (of three) physics courses I had taken myself and this course I was to teach, combined with a two-week notice prior to the start of the semester, placed me in the interesting position of learning alongside my students. Wading through an unfamiliar text, trying to understand publisher-produced lecture slides, learning from and getting frustrated with online homework, entering review sessions fearful of what students might askall these things gave me a much clearer understanding of and empathy with the student experience and prepared me to make effective changes in other courses I teach.

God: One and Infinite

Daniel Kiteck, Indiana Wesleyan University, IN

I see the most mathematically significant verse as Deut. 6:4 where God says He is ONE. (And I don't believe that it is an accident that the greatest commandment to love God with all we are immediately follows.) What is the concept of "one" in relationship to God? Is God dependent on the concept of "one?" What if "one" is ultimately always a comparison going back to God? God is also commonly viewed as infinite. How is this connected to our understanding of the mathematical continuum? Could this help us see how God is foundational both to discrete and continuous mathematics? These and related topics and questions will be explored.

Experiencing a Paradigm Shift: Teaching Statistics through Simulation-based Inference

Dave Klanderman, Mandi Maxwell, Trinity Christian College, IL
and *Nathan Tintle*, Dordt College, IA

For decades, statistics has been taught as an application of formulas, making use of normal and other distributions, and relying heavily on algebraic skills of students, in short, emphasizing mathematical thinking. More recently, several textbook author teams have published statistics books that place an increased emphasis on simulation and randomization methods, and a corresponding decreased emphasis on the algebraic manipulation in formulas (e.g., Lock et al., 2012; Tintle et al., 2015) as a way to encourage better statistical thinking. This session describes simulation-based inference curricula more fully, reports on the necessary steps towards implementation of such an approach, and provides both qualitative and quantitative comparisons of this new pedagogical approach with a more traditional approach. Appropriate justification of this approach to teaching and learning statistics is also provided, along with providing an overview of recent trends to shift to this approach in statistics courses taught at the high school, junior college, and university levels across North America, including a number of Christian colleges and universities affiliated with ACMS.

The Mathematics of Evolution

Steven R. Lay, Lee University, TN

Like many universities, Lee University has a non-major's course for liberal arts students. The course typically includes a potpourri of topics: logical thinking, scientific notation, linear functions, estimation, and probability. At Lee, we have found a way to conclude the course that applies these varied topics to an issue designed to engage student interest and promote critical thinking. We have developed a series of three lessons on "The Mathematics of Evolution." The first lesson is on radiometric dating. The second lesson is on the origin and progression of life. And the third lesson deals with the nature of the DNA genetic code. In this presentation we will provide examples from each lesson, as time permits. In so doing, we will address the following questions:

1. What are the dangers in sampling data over a short period of time (say 100 years) and extrapolating this over a much longer period (say a million years)?

2. What is the probability that a "simple" life form was produced by a series of random actions and how long might this be expected to take?

3. What is the probability that a "lower" life form evolved into a "higher" life form through a sequence of random mutations?

4. What does the poly-functional nature of the DNA code say about the long-term viability of species? Are we evolving upward or devolving downward?

Optimizing the *Introduction to Proofs* course: semantics, syntax, and style

Bryant Mathews, Azusa Pacific University, CA

For professional mathematicians, theorem-proving is a process that combines intuition, deduction, and communication. Put another way, mathematicians rely on their facility with both semantic and syntactic reasoning and on their familiarity with standard proof-writing style. In teaching our Discrete Mathematics and Proof course, I have found that my students often struggle with each of these aspects of the proving process. When they lose track of the meaning of the statements involved in a proof, they have trouble developing an effective strategy. When they ignore the logical structure of the statement to be proved, they fail to identify and apply relevant proof techniques. When they lack fluency with mathematical language and writing style, they struggle to clearly communicate their reasoning to others.

Traditionally, math students have been expected to develop these three theorem-proving skills by "immersion:" by watching "native speakers" prove theorems and then trying it out themselves. This approach has much to commend it, but it seems not to work very well (or, at least, not quickly enough) for students who are less prepared to "pick up" the new "language." These students need a "coach" to separate out the three skills from one another for demonstration, practice, and feedback, and then to put them back together again. In this talk, I will describe the methods and materials I use to help improve students' intuition, deduction, and communication, with a particular focus on how I train students to integrate these three modes of thought.

Ten Mathematicians Who Recognized Gods Hand in their Work

Dale McIntyre, Grove City College, PA

Scottish philosopher David Hume (1711-1776) once observed that

> "Whoever is moved by *faith* to assent to [the Christian religion], is conscious of a continued miracle in his own person, which subverts all the principles of his understanding, and gives him a determination to believe what is most contrary to custom and experience." [Hume]

Evidently Humes cynical pronouncement did not apply to Euler, Cauchy, Cantor, and other profound thinkers who believed God had commissioned and equipped them to glorify Him in their pursuit of truth through mathematics And based on their extraordinary achievements the principles of their *understanding* do not appear to have been *subverted* too badly!

Leading mathematicians of the past commonly affirmed that God created and sovereignly rules the universe and that He providentially sustains and nurtures His creatures. Despite Hume's assertion, history teaches us that faith often informs rational inquiry and vice versa. In many cases Christian commitment stimulated intellectual activity; sometimes mathematical understanding led to spiritual insight. In this paper, ten of history's most influential mathematicians express the role faith in God and religious conviction played in their work *in their own words*.

[Hume, David, *An Enquiry Concerning Human Understanding*, LaSalle, IL: Open Court, 1966 (First Published 1748), p. 145.]

Mystery of the Infinite: Developing a Mathematical Summer Scholars Program

Christopher Micklewright, Eastern University, PA

For the past two years, the Templeton Honors College, at Eastern University, has been conducting a Summer Scholars Program for high school students. The program, which has offered courses in a variety of topics, brings students to campus for an intensive residential program, coupled with pre-program and post-program work, for which students can earn college credit. In summer 2014, I helped to develop a mathematically based course, to include rigorous study of discrete mathematics, as well as a variety of lectures and extracurricular activities integrating faith and philosophy with mathematics. In this presentation, I will give an overview of the program, highlighting some of the successes as well as noting some of the challenges. It is my hope that this presentation will facilitate productive conversation and collaboration with others who might be involved in developing similar programs.

Overview of the Consumer Price Index Sampling

Moon Jung Cho, U.S. Bureau of Labour Statistics, NE

The U.S. Consumer Price Index (CPI) is a complex product that combines economic theory with sampling and statistical techniques. It uses data from several surveys to produce a measure of average price change for the consumer sector of the American economy. Production of the CPI requires the skills of many professionals, including economists, statisticians, computer scientists, data collectors, and other supporting professionals. In my presentation, I will overview the CPI sampling and highlight its intricacy.

Preparing Students to Read a Calculus Textbook

Doug Phillippy, Messiah College, PA

Consider the exercise of reading the textbook before class. While most educators agree that this practice leads to better learning, too often students enrolled in a calculus class do not find pre-class reading a valuable use of their time, and their commitment to doing so fades. Why is this? As instructors, we hope that these students will be well-versed in the fundamental concepts of the subject by the time they prepare for their final exam, but as they progress through the course and encounter new concepts, they may not be ready for the technical language of the standard calculus textbook. Further, their conceptual understanding of the subject matter *why is it important?, how is it relevant?, does this connect to something I already understand?* is probably not well developed. As a result they may not be ready for an explanation that includes precise terminology, presupposes a students interest in the end application, and fails to make explicit ties to prior knowledge. This talk will describe an alternate approach to reading a calculus text that places its reading after the lecture. The main focus of this talk will be the pre-lecture reading assignment and activities that are not intended to replace the reading of the calculus text but simply displace it to after the lecture.

Math, God and Politics - A Fight over Geometry in 19th Century Italy

Donna Pierce, Whitworth University, WA

In 1839 a polemic, reminiscent of the Renaissance public challenges over mathematical problems, was issued by the leader of the synthetic school of geometry, Vincent Flauti, to the analytical school, headed by Fortunato Padula. Three geometric problems were proposed, all carefully chosen to guarantee a victory for the synthetic school. The judges were from the Royal Academy of Sciences, men also favorable to the synthetic method. Why then did the analytics take up this challenge, and who were the real victors? This was not just a fight over the 'correct' way to do geometry, it was a fight over politics, a changing society, and most importantly, the Godly way to do mathematics. In this talk we will learn the goals and motivations of the synthetic school and the analytical school, the means they used to achieve those goals, and the final outcomes for mathematics and Italy.

Forming a consulting group

Thomas E. Price, University of Akron, OH

When I began consulting, I unwittingly limited the contracts available to me by underestimating and under promoting my abilities. Over the years, my consulting experiences have taught me that my mathematical training and related skills prepared me for a broader variety of contract work than I had originally thought. I also learned how to better market myself and the skill sets of my subcontractors. The goal of this presentation is to encourage and assist those interested in contract work by summarizing the strategies I used to develop a gratifying and productive consultancy.

David and the Census (II Sam 24, I Chr 21): Lessons for Mathematicians

John Roe, Penn State, PA

This is something I've been thinking about for a while, the idea being that this story shows us counting (the quintessence of mathematics) as an example of "technique" in the sense of Ellul: not simply a neutral analysis, but something which imports its own value system which may or may not be appropriate.

Parables to a Mathematician

Melvin Royer, Indiana Wesleyan University, IN

Jesus frequently used parables in His ministry, usually short narratives illustrating the outcomes of people's choices. In John 3:12 and Matthew 13:10-15, He explained that one reason was to be sure that people who genuinely wanted to understand His message would be able to do so. Since most of His audience was familiar with an agrarian economy, Jesus spoke extensively of wheat, fish, trees, wine, debt, tenants, lamps, etc. Many people have speculated on parables Jesus might have used had He lived in a different society. This non-scholarly (but hopefully thought-provoking) talk will propose parables targeted toward groups of mathematicians with various levels of Christian background. One parable for Christian "beginners" is as follows:

To introduce the ratio test for the convergence of infinite series, a calculus professor asked the class to discuss in groups the convergence of $\sum_{n=1}^{\infty} \frac{n^{400}}{2^n}$. One group reported divergence by the n-th term test. When questioned, they displayed a calculator graph of the sequence $a_n = \frac{n^{400}}{2^n}$ for $1 \leq n \leq 100$, showing monotonic increase. The professor suggested trying even larger values of n; the group then reported that (a_n) is monotonic for n as large as 300, at which point their calculator had a numerical overflow. Overhearing this discussion, several other groups switched their answers to divergent. The ratio test was then discussed. Anyone who hears this should pay attention.

Exploring appropriate computing for education in developing nations

Derek Schuurman, Redeemer University College, ON

This presentation will describe a pilot project involving the use of the Raspberry Pi for use in Christian elementary schools in Nicaragua. Some of the existing computers and software currently used in computing labs will be described along with some of the challenges that are faced. The features of the Raspberry Pi will be described along with how it might be a well-suited technology for use in schools in developing nations. Some feedback from a pilot project involving 30 Raspberry Pi's will be shared along with ideas for prospective future projects.

Mathematical Lessons from Finland and Sweden

Rebecca Seaberg, Bethel University, MN

I will describe lessons learned from visiting mathematics classrooms in Finland and Sweden for a month during the spring of 2013. Finland has become well-known for educational success in international comparisons; Sweden is similar to Finland in many ways but more like the U.S. in educational results (both are around average compared to other OECD (Organization for Economic Cooperation and Development) countries). Two major differences that set Finland apart is the high level of respect given the teaching profession within society and the greater educational requirements for becoming a teacher. All Finnish secondary mathematics teachers must have a Masters Degree in mathematics. Swedish teachers complain of lack of respect for the profession and as in the U.S., a bachelors degree is all that is necessary. Some other differences in Finland are fewer teaching hours required so that more time is available for lesson planning and focus on students, teachers often work with their students for three years, enabling closer relationships, a strong emphasis on problem solving in the math curriculum and de-emphasis on memorization, and a focus on equity and cooperation rather than competition. Both Finland and Sweden have more opportunities for vocational education at the upper secondary level and more students choose to focus there as well, in contrast to the prevailing thought in the U.S. that almost everyone should go to college. I will conclude with some changes both Sweden and the United States are making that resemble Finnish methods.

The *Best* Religious Calendar

Andrew Simoson, King University, TN

Many religions have deep roots in the rhythms of the moon. And ever since at least the fifth century BC man has known that the moon repeats itself every $n = 19$ years. Is this integer value n the best of all choices? Easter follows such a calendar. We briefly show that 19 is *second* best. And then we run time backwards, and give a rationale as to why a certain species of cicada has a life cycle of 17 years. The answer involves the moon, the Farey series, and Kepler's laws of motion.

The Math Olympian, by Richard Hoshino – A Review

Moriah Magcalas, University of Waterloo, ON, and
Kyle Spyksma, Redeemer University College, ON

This presentation will be a review of the novel *The Math Olympian* written by ACMS member Richard Hoshino.

The Remarkable Mrs. Somerville

Richard Stout, Gordon College, MA

As a woman growing up in the late eighteenth century, Mary Somerville (1780-1872) was denied access to most formal education and getting a university education was completely out of the question. Yet her interests in nature, science, and mathematics, coupled with an intense curiosity and tenacious desire to learn led her to eventually be known and respected by scientists, mathematicians, and intellectuals in both Britain and France. She is one of the important woman in the history of mathematics, even though she did not publish original work. However, she was a talented writer, producing several significant works, including *Mechanism of the Heavens*, a translation and amplification of Laplaces great work, and, at the end of her life, a series of *Personal Reflections*. Reading through her reflections gives an interesting glimpse into her personality, her opportunities for social networking, and some of what motivated her work. In this talk I hope to use these reflections to summarize aspects of her life and to introduce her work. We will see how, as a self-taught woman, she was able to gain access to the upper echelons of scientific society and how, as a committed Christian, her faith was evident in both her life and her work.

Data Mining as an undergraduate course

Deborah Thomas, Bethel University, MN

Data mining is an important area of research today. Our students are therefore better prepared if they are exposed to it while pursuing their undergraduate degree. However, the typical level of such a course is too rigorous for the average undergraduate student. In my talk, I will present the curriculum we developed to teach data mining to Juniors and Seniors as a seminar course. In the course, we discussed a variety of algorithms and their purposes. We focused on the algorithms described in Top 10 algorithms in data mining by Wu, et. al. We used a variety of software packages to solve the problems, including the Weka library, Java and Excel. Some students also used Mathematica and Python. A big focus of the course was the final project, where the students were given one of three different datasets from the Bethel University library archives. There were no restrictions on how to solve the problems and what they could use to answer the questions posed. I will describe the datasets in detail, the questions we were trying to answer and the strategies the students used to answer the questions. I will conclude with some of the results of their research and how their work might be expanded to be submitted to a publication in the future.

Reading Scripture Logically

Derek Thompson, Taylor University, IN

This is a talk for undergraduates currently taking (or who have taken) discrete mathematics, and/or for professors who teach mathematical logic. After covering biconditionals and logical arguments, I take some class time give this talk about the importance of applying the logic they've learned to how they read Scripture, so that they can always be prepared to give an answer (1 Peter 3:15), demolish arguments against God (2 Corinthians 10:5), and to correct, rebuke, and encourage (2 Timothy 4:2).

Physical activity in a Theory of Computing class

Nancy Tinkham, Rowan University, NJ

Physical activity breaks, sometimes called brain breaks, are beginning to gain attention among K-12 teachers as a way to keep their students alert and engaged in the classroom. In the Fall 2014 semester, faced with the task of teaching an introductory course in Theory of Computing in a once-a-week, 2 1/2-hour format, I decided to try incorporating physical activity into my own classroom. Time is precious in the college classroom, so any physical activities have to be directly related to the course material. I will describe some physically active exercises that I used in the classroom to teach students about regular expressions, finite automata, and other theoretical concepts. During the semester, I found that these exercises helped students to have fun and to stay connected to the material, even at the end of this long, late-night class. I also found that the exam averages and the overall course average were higher in Fall 2014 than they had been during the previous four years of teaching this night class. This invites further experimentation with the technique in future semesters.

Mathematics of wooden pliers

Anthony Tongen, James Madison University, VA

Functional wooden pliers can be constructed from a rectangular block of wood using ten cuts, with negligible loss of volume. These cuts form a hexagonal joint, with two reflectional symmetries, around which the pliers can open. A two-dimensional model describing the mechanics of the three-dimensional pliers was constructed based on the lengths of the cuts and the angles at which the cuts are placed. This model fully predicts whether or not pliers constructed with an arbitrary set of cuts can open and, if so, how far those pliers will open, based solely on the parametrization of the hexagonal joint by a characteristic length, λ, and an angle, θ. Additionally, techniques from linear algebra and analysis are utilized to determine the set of possible pivot points and to derive a closed form solution for the maximum angle of opening, given an arbitrary pivot point.

Mathematics as Culture-Making Activity

Kevin Vander Meulen, Redeemer University College, ON

Drawing on Andy Crouch's book *Culture Making: Recovering Our Creative Calling*, as well as the neo-Calvinist position in the second half of the chapter 'Ontology' in *Mathematics Through the Eyes of Faith*, I reflect on the nature of doing mathematics and its implications. I plan to explore the non-neutral character of mathematics in the context of the cultural mandate and the nature of naming.

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10 Statisticians, 20 Slides, 30 Centuries

Clifford H. Wagner, Pennsylvania State University at Harrisburg, PA

A slide presentation for introducing an elementary statistics course, this survey covers three thousand years in the history of statistics, with attention to personalities, techniques, and ethical issues.

Learning Catalytics - Students Engaging in the Classroom

Aaron Warnock, Highline College, WA

Come try out Learning Catalytics - a new classroom response system that allows students to respond to questions using any web-enabled device such as smartphones, tablets, or laptops. Learning Catalytics accepts free response, mathematical expressions, and even drawn or graphical submissions, which goes well beyond the multiple-choice limitations of traditional classroom response systems. Teachers will love how the system can automatically group students based on their dissimilar answers for discussion and resubmission. Aggregate student responses can be anonymously displayed on the board for even further discussion. This is a great way to engage every student in answering classroom questions (rather than calling on them one at a time). In addition to its mathematical strengths, Learning Catalytics is powerful beyond the classroom and can be used to facilitate group submission of assignments in a hybrid or online setting. Bring your electronic devices to participate or look on with a friend!

Cultivating Mathematical Affections: The Influence of Christian Faith on Mathematics Pedagogy

Joshua B. Wilkerson, Texas State University, TX

The goal of this paper is to make the case that Christian faith has an opportunity to impact the discussion on best practices in mathematics not primarily through the cognitive discussion on objectives and standards, but through the affective discussion on the formation of values, the cultivation of mathematical affections not merely knowing, but also loving, and practicing the truth, beauty, and goodness inherent in mathematics. First I will outline the work being done on affect in mathematics education, examining what values are actually endorsed by the community of mathematics educators. After summarizing this work on affect it will be clear that, even in the words of leading researchers, the field is lacking any cohesive, formal approach to analyzing and assessing the affective domain of learning. In part two of this paper I will argue the thesis that Christian faith offers solutions to the frustrations and shortcomings admitted by researchers on affect in mathematics education. Christian faith offers insight into how mathematical affections might actually be shaped. Here I will draw heavily on the work of philosopher James K.A. Smith and make explicit connection between his work and the mathematics classroom. Finally, I will conclude with a call to action discussing how we as Christian educators might begin to have fruitful contributions to and dialogue with the current research being done in mathematics education.

Supercomputing for Everyone - Easy Distributed Grid Architecture for Research

Brent Wilson, George Fox University, OR

Current distributed systems present several challenges to both scientists and students who may not be very skilled at programming parallel applications for use on such systems. Grid computing is a cost effective means of providing supercomputing computation for both scientists and students of computing. Easy Distributed Grid Architecture for Research (EDGAR) is a grid computing solution that meets two critical constraints, namely ease of application programming for users and also platform independence in its implementation. Satisfying these two constraints makes EDGAR one of the only time and cost effective grid computing solutions. EDGAR creates an easily used, high performance solution for scientists and students in solving computationally intense problems and provides easy access to supercomputing. This presentation introduces EDGAR and demonstrates its ease of use.

A Triune Philosophy of Mathematics

Dusty Wilson, Highline College, WA

What is mathematics and is it discovered or invented? The Humanist, Platonist, and Foundationalist each provide answers. But are the options within the philosophy of mathematics so limited? Rather than viewing and describing mathematics in a mutually exclusive manner, each of these approaches includes components of truth from a greater triune philosophy of mathematics. This talk will introduce this inclusive triune paradigm through which to explore fundamental questions about mathematics.

Home Primes and Foreign Primes

Nicholas Zoller, Southern Nazarene University, OK

Home primes and foreign primes are produced by a simple recipe that blends prime factorizations with recursion. The *home prime* of a positive integer n is formed by concatenating the prime factors of n in nondecreasing order. If the resulting integer is prime, then we have found the home prime of n. If not, then we repeat the process as many times as needed to obtain a prime. For instance, $35 = 5 \cdot 7$. After concatenation, we have $57 = 3 \cdot 19$, which is followed by $319 = 11 \cdot 29$, which is followed by 1129, which is prime. Thus, the home prime of 35 is 1129. To obtain the *foreign prime* of a positive integer n, we form the next integer by concatenating the prime factors of n in nonincreasing order. For example, starting with $35 = 7 \cdot 5$, we next consider $75 = 5 \cdot 5 \cdot 3$, followed by $553 = 79 \cdot 7$, followed by 797, which is prime. Thus, 797 is the foreign prime of 35. In this talk we give some results about home primes and foreign primes for integers $n < 100$. As one might expect from the arbitrary nature of the concatenation process, there are few easily discernible patterns.

A Subject-Centered Approach to Integrate Faith and Learning

Valorie L. Zonnefeld, Dordt College, IA

K-12 mathematics educators, with the National Council of Teachers of Mathematics at the forefront, have moved from teacher-centered to student-centered classroom pedagogies. This move was made in an effort to engage students more deeply in their learning. As a high school teacher at the beginning of the 21st century, I made this change as well, but have since become increasingly uncomfortable with aspects of student-centered teaching. I now believe that the choice between student-centered and teacher-centered approaches is a false dichotomy. In this session we will explore Parker Palmer's conception of a subject-centered classroom, what a subject-centered classroom looks like in mathematics, and how a subject-centered classroom can be a more faithful way to integrate faith and learning.

Abstracts for Panels and Workshops

Hybrid Classes in Mathematics and Computer Science

PLNU Group, Point Loma Nazarene University, CA

Panel Members
Ryan Botts: Associate Professor of Mathematics Point Loma Nazarene University
Lori Carter: Professor of Computer Science PLNU
Catherine Crockett: Assistant Professor of Mathematics PLNU
Greg Crow: Professor of Mathematics PLNU
Jesus Jimenez: Professor of Mathematics PLNU
Maria Zack: Professor of Mathematics and Chair of the Mathematical, Information, and Computer Science Department, PLNU

Offering hybrid (also called blended) courses is an approach being considered by many universities. A hybrid course combines both online and in-class learning activities, generally reducing the total in-class time required. The assumed benefits include increased time-flexibility for students, the opportunity for students to learn at their own pace, less pressure on precious classroom space, and the opportunity to introduce additional pedagogical strategies that address different styles of learning.

Three years ago the Mathematical, Information, and Computer Science department at Point Loma Nazarene University embarked on the adventure of designing and offering courses in the hybrid format. To date, four courses have been offered in the hybrid format: Introduction to Computer Programming, Introduction to Statistics, Math for Elementary Teachers, and Problem Solving (our GE course for upper division students). This panel will discuss the process of building the courses, the experience of teaching them, the feedback received from students, and the real benefits and drawbacks observed.

Following up with *The Mind and the Machine*

Joe Hoffert, Indiana Wesleyan University, IN

An informal conversation, with Matthew Dickerson, chaired by Joe Hoffert, reflecting on the Matthew Dickerson's book *The Mind and the Machine: What It Means to Be Human and Why It Matters*. How should Christians respond to transhumanism (e.g., what venues would this take, how to engage with respect, whether or not to respond publicly, reacting/responding vs. proactive ongoing research/pedagogy)?

Graduate Student Workshop 1: "The Application"

Amanda Harsy-Ramsay, Lewis University, IL

- The timeline of applying for jobs and where to find job postings
  - How to stay organized and keep on top of deadlines
- Advice for writing the following:
  - CV, Cover Letter
  - Teaching Philosophy Statement
  - Research Statement
  - Faith Statement and other supplemental material for some Christian universities

- Advice about obtaining letters of recommendation
  - Who to ask? When to ask?
  - Advice for assisting letter writers for non-mathjobs.org
- End with a breakout writing session for them to work on any of these documents

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Graduate Student Workshop 2: "Pre-Campus Interviews"

Amanda Harsy-Ramsay, Lewis University, IL

- Typical questions and general advice for pre-campus interviews
  - Phone Interview
  - Skype Interview
  - The JMM Interview
- Advice about what questions to ask during a pre-campus interview
- Advice about what to wear to a pre-campus interview
- End session with practicing interviews

Graduate Student Workshop 3: "The On-Campus Interview"

Amanda Harsy-Ramsay, Lewis University, IL

- General Advice
  - What to wear? What to bring?
  - What does a typical interview day(s) look like?
- Preparing to meet with faculty members, students, Deans, Provosts, and Presidents
  - What questions to ask
- Preparing your Research Talk
  - How to make it accessible, undergraduate student friendly, and cool.
- Preparing your Teaching Demonstration
- Break out into groups to practice describing your research to a non-expert

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Graduate Student Workshop 4: "Questions and Answers"

Amanda Harsy-Ramsay, Lewis University, IL

The organizers and possibly other invited ACMS participants will hold a panel where they will take questions and describe their personal experiences with the job search. In particular we will discuss how we recognized Gods provision during this stressful season of life.

Final Goal: Depending on how many students and organizers we have, I would like to put together a mentoring network where we pair up graduate students with faculty members from the ACMS.

Contributed Papers

On Random Numbers and God's Nature *

James Bradley, Department of Mathematics,
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Abstract

I start with mathematical Platonism, an ancient stream of thought that views numbers as transcending physical reality. I join this to recent insights into mathematical randomness from theoretical computer science. Joining these streams – one ancient, one recent – yields the surprising conclusion that randomness, defined in a particular way, is part of the nature of God. I then explore some of the implications of this conclusion for our understanding of the doctrine of God's infinitude.

Computer scientists approach randomness via random numbers rather than random events. But the term "random number" is not the same as that used by experimenters for whom a random number is one generated by a process that makes any of a collection of numbers equally likely, as occurs when tossing a single die. Rather, computer scientists take a single number and ask if it is random. In the binary, "base 2," language used in computers, where "1" and "0" are the only options, all numbers are strings of 1's and 0's. Consider, for example, the numbers $.1010101010101010\ldots$ and $.0111111001111011011 1$. Intuitively, the latter seems more random since predicting the next digit appears impossible. (The latter number was, in fact, generated by flipping a coin, which confirms that the next digit would be unpredictable.) Computer scientists ask what "random" means for such numbers.

The first attempt to formulate a concept of randomness for sequences of numbers was by Richard Von Mises in 1919. This approach starts with the idea that for a random number written in binary, each consecutive bit should be equally likely to be a 0 or a 1. This means that as we look at increasing numbers of bits, the bits will come closer to being half 0's and half 1's – this is called the law of large numbers. However, there are strings – $.10101010101010101010\ldots$, for example – that satisfy the law of large numbers (being half 1's and half 0's) but are not random. So Von Mises focused on substrings. If one picks the substring found in positions $\{1, 3, 5, \ldots\}$ of $.10101010101010101010\ldots$ one gets the string $.111111\ldots$, which is clearly not half 0's and half 1's. This, according to Von Mises, shows that $.10101010101010101010\ldots$ is not random. He then calls any selection process that can be described by a rule like "look at every other bit" an "acceptable selection rule." He then says that a random number is one for which all substrings selected by acceptable selection rules satisfy the law of large numbers. This was a good start to defining randomness for numbers but it didn't solve the problem because it provided no way to decide which rules were acceptable.

In 1936, Alan Turing (1912-1954) defined what has today come to be known as a Turing machine. Turing's goal was to develop an abstract, unambiguous formalization of the process of evaluating a mathematical function (such as $x^2 + 2x + 1$). While not a physical machine, Turing's "machine" was a careful description of a

* **This contribution is an excerpt from the paper, "Random Numbers and God's Nature" by James Bradley. The complete paper will be available in the book *Abraham's Dice*, edited by Karl W. Giberson, Oxford University Press, forthcoming Spring 2016.**

step-by-step process, what we commonly call an "algorithm."[1] Turing's concept heavily influenced the development of actual computing machines that took place in the decade following. Mathematicians and computer scientists generally regard Turing's efforts as successful; the "Church-Turing thesis" named for Alonzo Church (1903-1995) and Turing asserts that any operation that can be carried out on an actual computer can (at least in principle) be carried out on a Turing machine. Thus Turing machines provide an abstract setting in which one can ask and answer theoretical questions about what computers in principle can and cannot do. Von Mises wrote before the concept of Turing machine had been defined. Its introduction made it possible to say which selection rules are acceptable, namely the ones that can be formulated as a Turing machine.

Starting with the work of Per Martin-Löf in 1966, computer scientists have formulated a detailed theory of randomness for numbers; the theory extensively uses Turing machines. It includes numerous definitions of what it means for a number to be random, for what it means for one number to be more random than another, and many other nuances. The features of this theory that are of the most interest to us here, however, are:

1. Three definitions of "random" that have been shown to be equivalent;
2. The fact that using these equivalent definitions, it can be shown that, in a mathematically precise sense, almost all numbers are random.

The definitions are intuitively appealing and can be made mathematically rigorous – and when mathematicians formulate a concept in more than one intuitively appealing way and the definitions are subsequently shown to be equivalent, it reinforces the belief that they have successfully captured a significant idea. What follows is a brief, intuitive explanation of each concept; a more technical explanation can be found in the appendix. The three concepts are:

Irreducibility

Consider two bit strings: 10101010101010101010 and 01111110011110110111. The first has an obvious pattern; the second was generated by flipping a coin 20 times. The first can be generated by this algorithm:

Repeat 10 times: output '10'

The second, however, requires an algorithm like

Output '01111110011110110111'

That is, the second string cannot be reduced to one shorter and simpler than itself. The underlying intuition is that a string of n bits is random if any algorithm able to generate it requires at least n bits, i.e., the string is irreducible. Infinite strings are random if they cannot be reduced to finite expression.

Martin-Löf randomness

A string like 10101010101010101010 has an obvious pattern; a string like 01111110011110110111 does not. Of course, a string could look like it has a pattern near its beginning but then become patternless. Martin-Löf randomness captures the idea that a string which is not random has a finite pattern and maintains that pattern throughout a possibly infinite length. A random string is one that lacks a pattern.

Constructive martingales

Suppose a string of bits is revealed one bit at a time. The intuitive idea behind the martingale concept of randomness is that there is no betting strategy that would enable one to profit by predicting the next bit. That is, randomness defined in this way corresponds in a meaningful way with unpredictability.

[1] A precise definition can be found on the internet or in any introductory text on computing theory.

So, the underlying intuition behind these three concepts of random number is that a number is random if it is irreducibly infinite, has no finite pattern, and is unpredictable. The key point for us in this paper, however, is that random numbers are numbers. Thus:

> *If numbers have indeed existed in the mind of God from eternity, randomness is and always has been part of God's nature.*

I will explore the theological implications of this idea in the next section and the scientific implications in the section following that.

Theological implications

What does the idea that randomness is part of God's nature tell us about God?

First, let's consider what it does *not* tell us. The popular concept cited above is that randomness means not having a governing design, method, or purpose; without order; without cause. This popular concept is what makes the idea that randomness might be part of the divine nature seem strange or shocking. But algorithmic randomness is quite different from the popular concept of randomness and *is* informative about God's nature – even under a Christian theology which has always affirmed that God has designed the world, acts with method and purpose, and is orderly. Unlike popular notions of randomness, under the mathematical hypothesis of numbers being ideas in God's mind, the properties of random numbers are necessary properties of God's nature[2] and can enrich our understanding of God's infinitude. Before we can see what these properties add, though, we need to see how systematic theologians have historically understood divine infinitude. Here is a typical list of divine attributes [3, pp. 36-37] to provide a context for the analysis that follows:

The Nature of God

Divine sufficiency (primary and essential attributes of God inapplicable to creatures and not communicable to creatures)

- Uncreated
- Unity
- Infinity

The divine majesty (relational attributes of God displaying God's way of being present, knowing and influencing the world)

- Omnipresence
- Omniscience
- Omnipotence

[2]Note that this analysis does not simply give analogies taken from nature; rather it provides propositions that are necessary truths about God, subject to the limits of human language.

The Character of God

The divine thou (active and interpersonal attributes belonging to the divine-human relationship and analogous to personal experience);

- Incomparably personal
- Spiritual
- Free

The divine goodness (moral qualities intrinsic to the divine character)

- Holiness
- Goodness
- Compassion

In this taxonomy, infinity is one aspect of God's sufficiency; however, Christian thinkers' understanding of God's infinitude has varied over time and across religious traditions. The notion that God is infinite seems to have first appeared in Christian writings among early Gnostics. Augustine wrote that God is infinite in wisdom and is unbounded not in the sense of being suffused throughout space, but rather God is infinite "in another way" although he did not comment on what this way is [1]. Thomas Aquinas devoted Question Seven of the first part of his *Summa Theologica* to God's infinitude. He conceives of it as meaning that God is not limited in any way and is infinite in perfection in the sense that God's perfection cannot be diminished or increased. He wrote that God is unique in being infinite, that no bodily thing can be infinite, and there cannot be an actually infinite number. This latter notion originated in Aristotle's idea of "potential infinity" – for example, integers increasing without bound but not reaching a limit.[3] Some medieval scholars distinguished extrinsic and intrinsic infinity. The concept of "extrinsic infinity" was based on the integers continuing without limit; "intrinsic infinity" was based on the notion of a finite space being infinitely divisible. They suggested that God's infinitude is intrinsic not extrinsic; this seems to be a way to affirm God's infinitude while avoiding the notion that God is infinite in extent – but this was ambiguous about the relationship of intrinsic infinitude to God's nature. John Calvin's concept of God's infinitude was "beyond our senses." Many theologians have pointed out that God's infinitude is not separable from other attributes – it is part of what it means for God to be omniscient, omnipresent, and omnipotent. Some pointed out that God's infinitude, when applied to time, is God's eternality; when applied to space, is God's omnipresence. Herman Bavinck emphasized that God's infinitude applies to character attributes as well as sufficiency and majesty and in this way is quite different from a quantitative notion of infinity [2, pp. 159-160]. The principal common theme, however, that runs through these notions is "without limit." And the etymological basis of infinity is 'unlimited.'

I can see two ways that the idea of divine randomness can enrich our understanding of God's infinitude: (1) it can serve a pedagogical role by providing images that enable us to form clearer concepts of divine infinitude, thereby enriching our worship and (2) it can introduce aspects of divine infinitude that had not been previously noted.

[3] Following the work of Georg Cantor, mathematicians today would say that Aquinas was incorrect. Not only are there actually infinite numbers, there are infinitely many of them of infinitely many different sizes. For Aristotle, numbers were quantitative aspects of physical things. Aquinas seems to have used this Aristotelian concept in saying there cannot be an actually infinite number, although he does not explicitly mention Aristotle when he says this. This is one place where Christian Platonism enjoys a decided advantage over Aristotelianism. Seeing numbers as ideas in God's mind removes the conflict Aquinas saw between God being uniquely infinite and there being actually infinite numbers – actually infinite numbers can exist because they participate in the divine infinitude. For a discussion of Cantor's work and its theological implications, see [4].

(1) The integers provide an image that many theologians have used to illustrate God's infinitude. The concepts of randomness discussed here follow in that tradition – an infinite number that is irreducibly random is one that cannot be described by repetition of a finite string; a Martin-Löf random number lacks a pattern that can be generated by any (necessarily finite) algorithm. Both of these concepts provide images of the idea that God cannot be described in terms of any finite thing – no complete description of God is possible.

(2) The martingale definition of randomness introduces an aspect of the divine nature that I have not seen discussed in connection with the doctrine of divine infinitude, namely the element of surprise or mystery. Random numbers are mysterious, such that no matter how many bits of one have been revealed, the next bit is still unpredictable. Saying that such numbers exist in the mind of God provides an image of the idea that one may indeed understand aspects of God truly and may learn more, but never come to the point where there are not further aspects of God that are surprising. Put differently, no matter how much knowledge one has of God, God's mystery remains unfathomable.[4]

In summary, discussions of divine infinitude that are informed primarily by the image of the integers lead one to the idea that God is unlimited, but not much more. A comprehension of divine randomness extends this understanding, nuances it, and enriches it.

References

[1] S. Augustine. *Confessions*. Grand Rapids, MI: Christian Classics Ethereal Library, Book VII, Chapter XIV.

[2] H. Bavinck. *Reformed Dogmatics, Volume Two, God and Creation*. Baker Academic, 2004.

[3] T. C. Oden. *Classic Christianity, A Systematic Theology*. HarperOne, 1992.

[4] C. Tapp. Infinity in Mathematics and Theology. *Theology and Science*, 9(1), February 2011.

[4]When presenting this concept at a recent conference, one conferee commented, "My tradition has always focused on God's covenantal faithfulness, but you are asking us to see God very differently." I found it quite heartening that the analysis of divine randomness opened a new understanding of God for this person.

Mathematics Without Apologies: A Portrait of a Problematic Vocation by Michael Harris—A Review

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Abstract

The subtitle of *Mathematics without Apologies* by Michael Harris is a "Portrait of a Problematic Vocation," and that problematic vocation involves pure mathematics. What do pure mathematicians do, and why should they do it? Harris critiques the usual answers of truth, beauty, and potential applications as he gives a contemporary revision of G.H. Hardy's *A Mathematicians Apology*. Described as a post-post-modern book, *Mathematics without Apologies* provides a cultural, sociological, philosophical and psychological landscape of the profession and the life of a mathematician. We will explore whether a romantic view of the profession satisfactorily answers the questions and how it might compare to a Christian perspective.

In this world of economic demands, what justifies pure mathematics? The subtitle of *Mathematics without Apologies* [5] is "Portrait of a Problematic Vocation." That problematic vocation is a pure mathematician. Harris finds the common justifications of beauty, truth, and usefulness as misleading. He is more interested in how mathematics is actually done rather than in the usual narratives justifying mathematics. His contention is that before we can decide on the "ought" of mathematics, we have to focus on the "is."

Harris is engaging and provocative. He describes mathematical research as not so much solving problems but as challenging perspectives and raising more questions. His book does the same. He does not systematically argue from first principles but instead connects an eclectic array of historical, philosophical, sociological, and popular cultural references with mathematics. The myriad of citations and ideas provide additional trails to explore. Part autobiographical, Harris' writing integrates his own mathematical experiences as examples. There are chapters on "How to Explain Number Theory at A Dinner Party." He includes witty dialogues between a number theorist and a performing artist about the nature of mathematics. It is difficult to keep up, but there are many rewards for those who persevere.

His approach is very postmodern. Harris states up front that he will not arrive at definite conclusions and wonders if mathematics needs to be justified at all. He does not claim to be objective, and he uses his personal story to inform his description of the actions and practices of the profession. Part 1 of *Mathematics in a Postmodern Age* [6], written and edited by ACMS members, aptly describes his approach. Rather than the modernist concerns of the TRUE and the BEAUTIFUL to inform the law-like structure of the universe, Harris explores the PRAGMATIC, the METAPHORICAL, ONE's INDIVIDUAL FREEDOM, COMMUNITY DISCUSSION, and MUTUAL UNDERSTANDING. As suggested in *Mathematics in a Postmodern Age*, a postmodern would claim we should take into account the social and cultural influences that are bound to be part of any thinking process. We "should be guided more by how we actually conduct our cultural practices and not by idealized constructs that may not exist" [6].

Likewise, Harris is not overjoyed with the idealized constructs of philosophers and sociologists whose claims are divorced from mathematical practice. They often ignore or misrepresent the field of mathematics as an afterthought to the scientific process. The philosophy of Mathematics is often grounded in addressing truth claims presupposing an "outsized logical formalization" which matters little to research mathematicians. In other words, the norms of mathematical practice do not line up with philosophical norms. Instead, Harris is interested in the attention and participation in the discipline's tradition and its orientation to the future.

Participating in the discipline's tradition involves the hierarchy of problems and standing in research mathematics. To determine standing, Harris appropriates the term *charisma* as defined by Max Weber. Although the term may be similar to *prestige*, *status*, or *standing*, Harris prefers *charisma* for its subjective, romantic, and not altogether rational associations. It is a quality which endows the bearer of charisma with recognizable authority, genuineness, and exceptional qualities so that others then act accordingly. While religious leaders Jesus, Mohamed, and Buddha possess Weber's charisma, mathematicians with charisma determine the ground rules for the social understanding of the discipline as well as the value judgments that organize its contents. What is specific to pure mathematics is that it must create its own value system since the meaning is not attached to truth or a corresponding physical reality. Those with charisma set the research programs and determine the important problems. Those "acting accordingly" participate in the programs and solve these guiding problems. Mathematicians are free to pursue any problem of his or her own choosing, but the value and charisma which follows from a conjecture or solution very much depends on how the claim complies with the standards set by those with charismatic authority. Thus, pure mathematics possesses *socially constructed freedom*.

Be that as it may, the professional autonomy of the field is being challenged by the increasing pressures of university budgets and government demands for commercial applications. The title to Chapter 3, "Not Merely Good, True, and Beautiful" tips off his views on the usual rationalizations for pure mathematics.

One of the usual justifications for pure mathematics is its potential for applications. Harris calls this the "Golden Goose" argument. Because the potential benefits are so unpredictable, Powerful Beings should fund research based on curiosity alone. To Harris, it "is not only dishonest but also self-defeating to pretend that research in pure mathematics is motivated by potential applications." Not only are pure mathematicians not oriented in this way, to bow to Powerful Beings and to focus on potential applications hampers research and hinders motivation with undue pressure on the researcher. The curiosity of problems leads to theory and not the other way around. Furthermore, the exchange may be a Faustian bargain. While departments and mathematicians may receive additional funding, there are compromises of autonomy and ethics. Writing an entire chapter on the role of mathematics in the 2008 financial crisis, Harris describes a financial world rocked by misunderstandings of the mathematical model, unethical behavior, or both. Meanwhile, money was coming into mathematics departments as they were training more financial quants. Harris does not have problems with the financial equations but with the mindset of the economic agent as it rams the presumed moral lessons of public economic policy into popular consciousness even though these lessons are not always in society's best interests.

A second justification for pure mathematics is that it is beautiful. G.H. Hardy's *Apology* [4] made beauty the first test of mathematics and claimed that there is no permanent place for ugly mathematics. Following suit, mathematicians often compare their work to art and its beauty. What is ironic is that many contemporary artists are not concerned with beauty. Moreover, while mathematicians insist they are artists, most artists want to keep their distance from associations with formulas and equations. Meanwhile, contemporary mathematical research is developing too-long proofs with too-long definitions so that comprehension is only obtainable by specialists. Harris proposes that when mathematicians refer to beauty, they really mean pleasure. Discovering mathematical results provides a particular type of mathematical pleasure. Lacking the language to describe the nuances of pleasure, mathematicians fall back on the term beauty.

The third justification is that mathematics deals with truth. The problem is that proofs are becoming increasingly too elaborate to be fully understood by an individual mathematician. Harris advises that we should get out of the habit of assuming that mathematics is about being rational as understood by the philosophers of Mathematics. Proof is not the culmination but rather a confirmation of intuition. The answers are less important than how they change the way we look at the questions. The more important challenge is how to arrive at a common language. How can one formulate or even solve a problem if the ideas to formulate that problem have not been invented?

As hinted at in the Chapter 7 title, "The Habit of Clinging to an Ultimate Ground," the preoccupation with the foundations of mathematics to provide certainty is a distraction and unattainable. Each new mathematical discovery leads to new questions so that full understanding will never come. Whereas foundational questions deal with ultimate truth, Harris sees foundations as a metaphor and not amenable to mathematical concepts. Foundational questions are a nuisance and a restraint to the imagination. They are too closely tied to Western metaphysics, and they ignore other traditions of thought. Rather than looking back, we should look forward to other principles of classification that can advance knowledge.

If utility, truth, and beauty are not the justifications, what is? Desiring to communicate the values and emotional investment needed for mathematics, Harris tries to reflect the intentions of mathematicians. He provides a composite portrait referencing culture and a diversity of contemporary mathematicians.

For example, do mathematicians see themselves in cinema? Are all mathematicians misfits on the verge of madness losing their minds with lofty thoughts, as depicted in recent Hollywood movies? Wanting to counter the stereotypes of mathematicians, Berkeley mathematics professor Edward Frenkel co-directed, co-wrote, and starred in the short film *Rites of Love and Math* [2]. In the movie, a mathematician finds a mathematical equation for love. While this formula can bring great happiness to the world, this formula can also be used as a weapon against humanity. The protagonist and his lover realize that he will be caught and killed by the evildoers, but he wants his formula to live on. In one final act of passion, he tattoos the formula on her body rather than destroying the equation.

In providing various interpretations to this and other films, Harris introduces the concept of mathematician as a romantic– not as a lover but as one marked by the imaginative appeal to subjective qualities and freedom of form. Reine Graves, co-writer and co-director of *Rites of Love and Math*, views mathematics as one of the last areas where there is genuine passion not dominated by economics as in art and cinema. Mathematical historian Amir Alexander sees the practice of higher mathematics as inseparable from the tales of mathematician as romantic hero. For Alexander, the romantic hero is "a doomed soul whose quest for the sublime leads to loneliness, alienation, and all too often early death" [1]. Galois is the quintessential example in terms of his exaggerated life, but more to the point, he participated in the shift away from the Enlightenment's focus on the natural world to create a higher reality based on the laws of mathematics.

Harris does not see mathematics as a religious experience but uses religious and transcendental language to describe the mathematical process. What is noteworthy is that Harris points to experiences similar to ones that have been described by ACMS members as being experiences of worship. The sensation of the mathematician working on his or her formula is analogous to a feeling of love. It is a feeling of transcendence which cannot be described within a formula but is perhaps analogous to the use of infinity to represent what is incomprehensible to the human mind. Harris points to Loren Graham's book *Naming Infinity* and to the Russian Orthodox theologians who see love, knowledge, and truth as essentially personal [3]. The act of knowledge is a relationship, a kind of friendship, between the one who studies and the one who is studied. Just as in the doctrine of name worshipping– repeating the names of God brings one into His presence—naming the mathematical objects brings the mathematics into existence. To translate into Russian the word "truth" in Jesus' phrase, "I am truth," one can use *istina* which carries the sense of genuineness or the word *pravda* which deals with law, justice, and rules. The Russian Orthodox theologians used *istina* as did Frenkel's Mathematician in *Rites*. In closing the chapter on mad/martyred/mathematicians in cinema, Harris states, "Our readiness to sacrifice our minds and bodies to our vocation is the ultimate proof that what we are doing is important."

In line with the mathematician as a romantic with an emotional investment in his or her craft, Harris proposes a vision of the good mathematical life. Mathematics is a **relaxed field** not subject to the pressures of material gain and productivity. It values contributions to a coherent and meaningful **tradition** carrying on a dialogue with human history and past achievements with an orientation to the future. Mathematical tradition exhibits universality as it welcomes contributions from all nationalities. At a fundamental level, mathematical research is a pursuit of **pleasure**. Thus, the mathematical life is more closely tied to human flourishing. If one asks for the usefulness of mathematics, the same question could be asked about the use of philosophy, music, or a newborn baby. Taking a cue from Jeremy Bentham, society should provide the greatest happiness to as many people as possible. Mathematics should resist economic compromises which move them away from the pursuit of pleasure to a stressed life.

In critiquing Harris' depiction of the mathematical good life, it should be recognized that the romantic ideal is an appealing one. The romantic acknowledges life's shortcomings (such as there is no ultimate truth to explain existence) and then tries to transcend the world by pursuing one's own constructions of guidelines or truth. In mathematics, the romantic recognizes the fruitlessness of foundational truths and instead removes more and more veils only to find more veils waiting for him or her. Mathematics is not pursued for its applicability, perhaps leading to abuse, but for the enjoyment it brings. It involves play, which is characteristically human. One's imagination can roam free without being bound to the necessities of existence. It is a life which emphasizes dignity over importance. The romantic ideal provides an individual authenticity and seeming purpose while still allowing individuality and freedom.

The challenge in maintaining ethical behavior while being true to oneself is how one determines which is which without an outside standard. The romantic speaks to a culture which wants freedom without difficult responsibilities. For the Christian mathematician, Harris' portrayal contains many elements of a Christian vision of human flourishing. The yearning to know and discover is part of being human. Avoiding temptations to desire control over the world or others for selfish gain is admirable. However, Harris' vision lacks ultimate purpose. Pursuing pleasure can turn into self-indulgence if it lacks the larger meaning that Christianity may provide. The vocation for Christians is to work with all joy serving God and not exclusively ourselves. A Christian does not have to believe that mathematics provides ultimate truth and certainty, but there is a need to reconcile our work to a triune God. Determining the *telos* is not easily done and takes much effort and wisdom. Still, our relationship with mathematics does not exist solely for the sake of mathematics. The process of doing mathematics can inform ourselves and others about God.

In the larger culture, Harris' critiques of the usual justifications need to be taken seriously. The great challenge is arguing for pure mathematics in today's climate clamoring for practicality and tangible economic results. As a professor at a liberal arts institution, I side with the idea that the liberal arts are intended to be liberating, to develop character, and to develop the internal person. To exclusively view something in terms of practicality and usefulness rather than inherent self-worth should be resisted. Without an agreed-upon philosophical foundation, it is difficult to make these arguments. There will need to be alternate explanations for the convenient associations of true, beautiful, and good which is part of a much larger discussion.

In this review, I have only scratched the surface of what is in the book. One meets many contemporary mathematicians with stimulating perspectives. *Mathematics without Apologies* introduces a variety of characters, ideas, and metaphors to unpack. There is literary criticism, pondering of the role computer proofs, category theory as a philosophical foundation for mathematics, a critique of *A Mathematician's Apology*, the mathematician as trickster, Hausdorff as the model for the modern mathematician, and much more. The book does not have to be read cover to cover so that one can pick out the chapters to read without losing much.

I recommend reading the book if only to move beyond a modernist mindset. While Harris may not provide the appropriate answers, he does reveal the challenging questions being asked today moving the discussion beyond those asked 60 years ago.

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Designing for Mistrust

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Abstract

The 2014 ACM North Central Region programming contest contained a problem about a group of v bandits who want to use multiple locks to seal their treasure and distribute keys in such a way that no group of less than m bandits can open all the locks. The problem asks for an algorithm that will determine the number of locks needed for any set of parameters (v, m).

I will present an analytic solution that produces a minimum number of locks, a recurrence relation solution, and a constructive algorithm that can print out a table showing the locks and which subset of bandits hold keys for each lock. Each table forms a balanced incomplete block design (BIBD). The parameters of the BIBD can be uniquely determined from v and m.

Locked Treasure

One of the problems from the 2014 ACM North Central Region programming contest was the following (with a change of variable names) [1].

Problem (ACM North Central Region)**.** *A group of v $(1 \leq v \leq 30)$ bandits hid their stolen treasure in a room. The treasure needs to be locked away until there is a need to retrieve it. Since the bandits do not trust each other, they wanted to ensure that at least m $(1 \leq m \leq v)$ of the bandits must agree in order to retrieve the treasure.*

They have decided to place multiple locks on the door such that the door can be opened if and only if all the locks are opened. Each lock may have up to v keys, distributed to a subset of the bandits. A group of bandits can open a particular lock if and only if someone in the group has a key to that lock.

Given v and m, how many locks are needed such that if the keys to the locks are distributed to the bandits properly, then every group of bandits of size at least m can open all the locks, and no smaller group of bandits can open all the locks?

For example, if $v = 3$ and $m = 2$, only 3 locks are needed — keys to lock 1 can be given to bandits 1 and 2, keys to lock 2 can be given to bandits 1 and 3, and keys to lock 3 can be given to bandits 2 and 3. No single bandit can open all the locks, but any group of 2 bandits can open all the locks.

Terminology

Assume that the values of v and m have been given.

Definition 1 (Conforming Solution)**.** *A set of locks and a distribution of keys for the locks is called a* conforming solution *if every set of m bandits can open every lock and no smaller set of bandits can open every lock.*

Definition 2 (Mistrust Lock; Mistrust Design)**.** *A lock is called a* mistrust lock *if there is a group of exactly $m-1$ bandits who* do not *have a key for that lock, and all of the other $v-m+1$ bandits* do *have a key for that lock.*

The unordered collection of all $\binom{v}{m-1}$ distinct mistrust locks is called a mistrust design. *This design will be denoted by $M_{v,m}$.*

Example 1 ($M_{5,3}$)**.** *In the chart below, the rows represent bandits, the columns represent the locks. Bandit i has a key for lock L_j if there is an X in ith row, jth column. The chart shows the mistrust design $M_{5,3}$.*

$M_{5,3}$	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
1	X			X		X	X		X	X
2		X			X	X		X	X	X
3			X	X	X		X	X		X
4	X	X	X				X	X	X	
5	X	X	X	X	X	X				

It will be convenient to mention two important theorems about the binomial coefficients $\binom{n}{r}$. Recall that for $n \geq 0$ and $0 \leq r \leq n$, $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$ and $\binom{n}{r} = 0$ if $0 \leq n < r$. The theorems are easy to prove algebraically and also by using simple combinatorial proofs.

Theorem 1. *For all nonnegative integers n and r with $r \leq n$*

$$\binom{n}{r} = \binom{n}{n-r}$$

Theorem 2 (Pascal's Theorem)**.** *For all positive integers n and r with $r \leq n$*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Proofs of both theorems can be found in [2].

An analytic solution

A solution can be found by considering subsets of bandits of size $m-1$. For each distinct subset, D_i, of size $m-1$, create a lock for which none of the bandits in that set have a key, but for which every other bandit *does* have a key. There are $\binom{v}{m-1}$ such subsets. Denote the lock associated with subset D_i as L_i. Thus, D_i consists of all the bandits who *do not* have a key to L_i.

Proposition 1. *At least $v-m+1$ bandits must each have a key for any lock which is part of a conforming solution.*

Proof. If fewer bandits each *have* a key, then at least $v-(v-m) = m$ bandits will *not* have a key. That means there is a set of m bandits who cannot open the lock, so the lock is not part of a conforming solution. $\square$

Definition 3 (Minimal Conforming Solution). *A set of locks and a distribution of keys for the locks is called a* minimal conforming solution *if it is a conforming solution, no smaller set of locks is a conforming solution, and every lock has exactly $v - m + 1$ keys.*

Proposition 2. *A minimal conforming solution will not have any repeated locks (locks with keys distributed to the same set of bandits).*

Proof. Nothing is gained by repeating a lock.

□

Theorem 3 (Mistrust Designs). *Suppose a set of v bandits wish to use the mistrust design $M_{v,m}$ to seal their treasure. Then*

- *Every subset of m of the bandits is able to open all the locks*
- *No subset of $m - 1$ or fewer bandits will be able to open all the locks.*

Proof. Notice that no lock has more than $m - 1$ bandits who do not have a key to that lock. This means that in any subset of m bandits and for each lock, at least one bandit will have a key to the lock.

Now consider any set of $m - 1$ bandits. That subset will be the associated subset D_i for some lock L_i. None of those bandits can open lock L_i. Clearly, no subset of fewer than $m - 1$ bandits will fare any better than a subset of size $m - 1$.

□

Theorem 4 (A Minimal Set Of Locks). *Suppose a set of v bandits wish to use multiple locks to seal their treasure in such a way that every subset of m of the bandits is able to open all the locks, but no smaller subset can do so. Then*

- *It is possible to ensure this with a set of $\binom{v}{m-1}$ locks.*
- *No set of fewer than $\binom{v}{m-1}$ locks will accomplish this.*

Proof. The first claim can be achieved by using the mistrust design $M_{v,m}$ (see Theorem 3.)

Suppose now that a set, L, of fewer than $\binom{v}{m-1}$ locks will be used. Proposition 1 ensures that at least $v-m+1$ bandits must each have a key for any chosen lock. Thus, at most $m - 1$ bandits will not have a key for any lock.

For each lock $L_i \in L$, denote by N_i the subset of bandits who *do not* have a key for that lock. So $|N_i| \leq m-1$ for all i. Since there are fewer than $\binom{v}{m-1}$ locks, there must be at least one subset of bandits having $m - 1$ members that does not equal N_i for any i. (If $|N_i| < m - 1$ it cannot equal any subset of $m - 1$ bandits. Even if $|N_i| = m - 1$ for all i, there are not enough locks to include every subset of $m - 1$ bandits.) Denote one such missing subset as T. The bandits in T can open every one of the locks in L, since at least one of the bandits in T is missing from N_i for all i. That is, for all i, at least one of the bandits in T has a key for L_i.

Since a subset of $m - 1$ bandits can open all the locks in L, L cannot be a conforming solution to the problem. The failure arose from the assumption that L has fewer than $\binom{v}{m-1}$ locks, so any conforming solution must contain at least $\binom{v}{m-1}$ locks.

□

Corollary 1. *For a given v and m, the only minimal conforming solution is a mistrust design.*

Proof. Any conforming solution must contain $\binom{v}{m-1}$ locks. If any of those locks have more than $v-m+1$ keys, it will be more efficient to use a mistrust design, which has only $v-m+1$ keys per lock. □

A solution using a recurrence relation

It is possible to use a recurrence relation (and dynamic programming) to calculate the number of locks needed in a mistrust design. Since the number of locks in a mistrust design can be expressed with a binomial coefficient, it seems reasonable to try a recurrence relation patterned after Pascal's Theorem. The following definition and theorem show that this approach does work. As a bonus, the recurrence relation provides a constructive algorithm for actually printing the mistrust design as a table.

Definition 4 ($L(v, m)$)**.** *For $v, m \geq 1$, define $L(v, m)$ by*

$$\begin{aligned} L(v,m) &= 0 \text{ if } m > v \\ L(v,1) &= 1 \\ L(v,v) &= v \\ L(v,m) &= L(v-1,m) + L(v-1,m-1) \text{ for } v,m \geq 2 \end{aligned}$$

Theorem 5 ($L(v, m)$ is the number of locks in a mistrust design)**.** *The number of locks in the mistrust design $M_{v,m}$ can be calculated using the recurrence relation $L(v, m)$. That is, $L(v,m) = \binom{v}{m-1}$ for all $v, m \geq 1$.*

Proof. The three base conditions are easy to see. If more than v bandits are needed to open all the locks, then the task is not possible. If every bandit should be able to open all the locks, then only one lock and v keys are needed. If all v of the bandits are needed to open all the locks, then v locks, each having only 1 key will be needed. (See the first example after this proof.)

A two-way induction will be used to show that $L(v,m) = \binom{v}{m-1}$, the number of locks in $M_{v,m}$. Notice that

$$L(v,1) = 1 = \binom{v}{1-1} = \binom{v}{m-1}$$

$$L(v,v) = v = \binom{v}{v-1} = \binom{v}{m-1}$$

Suppose for all w with $1 \leq w < v$ and any m with $1 \leq m \leq w$ that $L(w,m) = \binom{w}{m-1}$. Then

$$\begin{aligned} L(v,m) &= L(v-1,m) + L(v-1,m-1) \\ &= \binom{v-1}{m-1} + \binom{v-1}{m-2} \\ &= \binom{v}{m-1} \end{aligned}$$

by Pascal's Theorem. □

Example 2 ($M_{3,1}$ and $M_{4,4}$)**.** *The locks and the key distributions for $M_{3,1}$ and $M_{4,4}$ are shown here.*

$M_{3,1}$	L_1
1	X
2	X
3	X

$M_{4,4}$	L_1	L_2	L_3	L_4
1	X			
2		X		
3			X	
4				X

The recurrence relation provides a constructive algorithm for producing the table of key assignments for the locks. The table for $M_{v,1}$ is a column of v Xs. The table for $M_{v,v}$ consists of a v by v table with Xs on the main diagonal (and nowhere else). Otherwise, assume that the tables $M_{v-1,m}$ and $M_{v-1,m-1}$ can be constructed. Then the table for $M_{v,m}$ can be built in the following manner. Rename the locks for $M_{v-1,m-1}$ using the subscripts $\binom{v-1}{m-1}+1,\ldots,\binom{v}{m-1}$.

$M_{v,m}$	$L_1 \cdots L_{\binom{v-1}{m-1}}$	$L_{\binom{v-1}{m-1}+1} \cdots L_{\binom{v}{m-1}}$
1		
$\cdots$	$M_{v-1,m}$	$M_{v-1,m-1}$
$v-1$		
v	X X $\cdots$ X	

Notice that bandit v only holds keys to the first $\binom{v-1}{m-1}$ locks. If any $m-1$ of the first $v-1$ bandits try to unlock all the locks, by the design of $M_{v-1,m}$ and $M_{v-1,m-1}$, they will not have enough keys to unlock the lefthand subset of locks but they can open the righthand subset of locks. Adding bandit v to such a subset will allow all the locks to be opened.

Similarly, any subset of $m-1$ bandits that includes bandit v will be unable to open all the locks since bandit v does not have any keys to the second collection of locks and the other $m-2$ bandits do not have enough keys among them to unlock the collection. Adding one more bandit will enable all the locks to be opened.

Example 3 ($M_{6,3}$ from $M_{5,3}$ and $M_{5,2}$)**.** *The constructive algorithm can be illustrated by creating $M_{6,3}$ from $M_{5,3}$ and $M_{5,2}$. Each of those mistrust designs can be created from smaller mistrust designs. (It might be of interest to divide the chart even further to see this.)*

$M_{6,3}$	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}
1	X			X		X	X		X	X	X		X	X	X
2		X			X	X		X	X	X		X	X	X	X
3			X	X	X		X	X		X	X	X		X	X
4	X	X	X				X	X	X		X	X	X		X
5	X	X	X	X	X	X					X	X	X	X	
6	X	X	X	X	X	X	X	X	X	X					

From Mistrust Designs to Balanced Incomplete Block Designs

Example 4 ($M_{5,3}$ in a different format)**.** *The following chart lists the locks for $M_{5,3}$. Under each lock is the subset of bandits that hold keys for that lock. The bandits are denoted as thief 1 through thief v: $t_1, \ldots, t_v$.*

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
t_1	t_2	t_3	t_1	t_2	t_1	t_1	t_2	t_1	t_1
t_4	t_4	t_4	t_3	t_3	t_2	t_3	t_3	t_2	t_2
t_5	t_5	t_5	t_5	t_5	t_5	t_4	t_4	t_4	t_3

This arrangement is an example of a combinatorial design called a balanced incomplete block design. In the formal definition below, the blocks correspond to the locks in Example 4 and the varieties correspond to the bandits.

Definition 5 (Balanced Incomplete Block Design)**.** *A* balanced incomplete block design*, abbreviated* BIBD*, is a combinatorial design consisting of a finite collection of finite sets (called* blocks*), each consisting of a finite number of elements (called* varieties*). The boundary conditions a BIBD must satisfy are expressed in terms of five parameters, commonly expressed as the 5-tuple of positive integers, (v, b, r, k, λ).*

The parameter v represents the number of distinct varieties; the parameter b represents the number of blocks. Every variety is required to be in exactly r blocks, and every block must contain exactly k varieties. Finally, every pair of distinct varieties must appear in exactly λ common blocks.

A combinatorial design which meets these conditions is often referred to as a (v, b, r, k, λ)-design.

A (v, b, r, k, λ)-design with $k = v$ and $r = b$ is called trivial.

Notes:

1. The first four parameters are positive: $v \geq 1, b \geq 1, r \geq 1, k \geq 1$.
   The parameter λ is nonnegative: $\lambda \geq 0$.

2. A trivial BIBD consists of b identical blocks,each containing every variety.

3. The design in Example 4 is a $(5, 10, 6, 3, 3)$-design.

Example 5 (A $(7, 14, 6, 3, 2)$-design)**.** *Here is a $(7, 14, 6, 3, 2)$-design using the set of varieties $\{0, 1, 2, 3, 4, 5, 6\}$. There are $\binom{7}{3} = 35$ variety subsets of size 3, but only 14 of them are represented as blocks.*

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
0	0	0	0	0	0	1	1	1	1	2	2	2	2
1	1	3	3	5	5	3	3	4	4	3	3	4	4
2	2	4	4	6	6	5	6	5	6	5	6	5	6

An alternative method for displaying a BIBD is to use an incidence matrix.

Definition 6 (The Incidence Matrix of a BIBD)**.** *Let D be a (v, b, r, k, λ)-design with varieties $\{t_1, t_2, \ldots, t_v\}$ and blocks $\{L_1, L_2, \ldots, L_b\}$.*

The incidence matrix *of D is the v by b matrix, M, where*

$$m_{ij} = \begin{cases} 1 & \text{if } t_i \in L_j \\ 0 & \text{otherwise} \end{cases}$$

Example 6 (The incidence matrix for $M_{5,3}$)**.** *If the varieties (bandits) in $M_{5,3}$ are denoted by $\{t_1, t_2, t_3, t_4, t_5\}$, then the incidence matrix for that mistrust design is:*

$$\begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{cccccccccc} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Compare this to the previous representation as:

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
t_1	t_2	t_3	t_1	t_2	t_1	t_1	t_2	t_1	t_1
t_4	t_4	t_4	t_3	t_3	t_2	t_3	t_3	t_2	t_2
t_5	t_5	t_5	t_5	t_5	t_5	t_4	t_4	t_4	t_3

The incidence matrix makes it easy to see some very basic necessary conditions for the existence of a BIBD.

Theorem 6 (The Parameters of a BIBD)**.** *Let D be a balanced incomplete block design with parameters (v, b, r, k, λ). Then*

$$bk = vr \qquad \text{and} \qquad r(k-1) = \lambda(v-1)$$

Proof. Let M be the incidence matrix for the (v, b, r, k, λ)-design. Both equations will be proved using combinatorial proofs that count 1s in M.

The first equation is verified by counting all the 1s in M two different ways.

Since there are b blocks, each containing k varieties, each of the b columns of M will contain k 1s, for a total of bk 1s. On the other hand, each of the v varieties is in r blocks, so each of the v rows of M contains r 1s, for a total of vr 1s. Therefore, $bk = vr$.

The second equation is verified by counting the 1s in a submatrix of M. Start by choosing any variety, t. Delete the row of M that corresponds to t and delete every column that corresponds to a block that does not contain t. Now count the 1s in the matrix, M_t, that remains.

Since t is in r blocks, there will be r columns in M_t. Each of those columns will contain $k-1$ 1s (since the 1 in t's row has been removed). On the other hand, t is in λ common blocks with each of the $v-1$ other varieties. So each of those varieties contributes λ 1s to M_t. Consequently, $r(k-1) = \lambda(v-1)$.

□

Theorem 7 (Mistrust Designs as BIBDs)**.** *A mistrust design is a*

$$\left(v, \binom{v}{m-1}, \binom{v-1}{m-1}, v-m+1, \binom{v-2}{m-1}\right) \text{ - design.}$$

Proof. v The number of varieties is v, the number of bandits.

b A mistrust design was defined by creating all $\binom{v}{m-1}$ subsets of $m-1$ bandits and for each subset associating a lock. Thus there are $b = \binom{v}{m-1}$ blocks.

k The subsets of $m-1$ bandits are the ones who do not hold a key to the associated lock. That means there must be $k = v-(m-1) = v-m+1$ who do have keys to a given lock. So there are $k = v-m+1$ varieties in every block. The design consists of *all* $\binom{v}{m-1} = \binom{v}{v-m+1} = \binom{v}{k}$ subsets of size k.

r How many blocks will contain each variety? Consider a variety t. Once t has been chosen, we still have $v-1$ varieties from which to chose the other $k-1 = (v-m+1)-1 = v-m$ varieties in the subset (block). So t will be in a subset of size $v-m+1$ exactly $\binom{v-1}{k-1} = \binom{v-1}{v-m} = \binom{v-1}{m-1}$ times (using Theorem 1). Therefore $r = \binom{v-1}{m-1}$.

λ If we choose a pair of varieties, there will be $k-2 = (v-m+1)-2$ other varieties with them in a block. There are $v-2$ other varieties from which to choose those $k-2$ varieties, so the original pair of varieties will be together in $\binom{v-2}{k-2} = \binom{v-2}{(v-m+1)-2} = \binom{v-2}{m-1}$ subsets of size $v-m+1$. Consequently, $\lambda = \binom{v-2}{m-1}$. □

Example 7 (Some BIBD parameters from Mistrust Designs)**.** *The following table lists the BIBD parameters for several mistrust designs. The parameters were calculated by using Theorem 7:* $(v, b, r, k, \lambda) = \left(v, \binom{v}{m-1}, \binom{v-1}{m-1}, v-m+1, \binom{v-2}{m-1}\right)$.

v	m	v	b	r	k	λ
6	3	6	15	10	4	6
6	4	6	20	10	3	4
8	4	8	56	35	5	20
10	3	10	45	36	8	28
10	6	10	252	126	5	56

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Software Engineering I: Teaching Challenges

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Abstract

The term software engineering can be traced to the late 1960's in response to large-scale, software development problems. Since then it has evolved as a discipline, both within industry and the academy. There have been distinct educational successes: "Standard practice" has matured (and found its way into more textbooks), the ACM and IEEE Computer Society have published curriculum guidelines, computer science programs commonly offer at least one software engineering course, and software engineering degrees (undergraduate or graduate) are more common. However, software engineering still presents a challenge. The term itself has no consensus definition; software applications and development environments are significantly varied; and development practices are often unique to their problem domain. This paper describes these challenges and some possible responses, with some thoughts related to an existing Software Engineering I course.

Introduction

Here we assume that we are dealing with a single, introductory Software Engineering I course (that is not necessarily part of a software engineering degree). Since the term software engineering has multiple meanings, we will also assume the IEEE definition:

> "The application of a systematic, disciplined, quantifiable approach to the development, operation and maintenance of software; that is, the application of engineering to software." [1]

Also, we assume that an undergraduate degree in software engineering is defined by the Accreditation Board for Engineering and Technology (ABET) [17], which has accredited only twenty-one programs (See [20] that describes the University of Rochester's program.) An accredited software engineering degree should not be confused with what the U.S. Department of Education's calls a "degree in computer software engineering" [3], of which 106 are listed on their website (including some two-year degrees and some for-profit organizations).

Challenges

1. Terminology

The term "software engineering" has no real consensus definition. Software companies routinely refer to their programmers as software engineers; job listings often associate "software engineer" with entry-level programming jobs with minimal degree requirements; and, as mentioned above, the U.S. Department of Education assumes a definition that conflicts with that of the IEEE and ABET-accredited programs. Consequently, this causes confused expectations among students, academic administrators, practitioners, employers, and software engineering instructors.

2. Software Variety

Software applications and development environments are significantly varied. Consequently, a single software engineering course cannot address the full variety of application domains nor the variety of development environments, much less the variety that will exist in the future.

3. Development Practices

Process models, tools, frameworks, and languages are often unique to a particular problem domain and they may fall in or out of favor. So, it is not possible to choose the "most likely environment" that students will encounter after they graduate, much less the environment of the future. Furthermore, it is not reasonable for the instructor (or the academic department) to frequently change process models, tools, frameworks, or languages.

Responses to Challenges

1.Terminology

You cannot change the definitions used by the U.S. government (the Labor Department and the Education Department, in particular). Nor can you alter the language in company job notices. But you can affect the definition within your department, your school, and among your students.

It is advisable to adopt a single, standard definition of software engineering, e.g., the IEEE definition mentioned above, within your sphere of influence. This does not imply that other definitions are not possible, but it acknowledges that a single definition can avoid unnecessary confusion and that the definition has satisfied a recognized standards organization.

A discipline involves standard terminology plus a collection of "how to's". Therefore, it is important to emphasize a consistent and coherent set of software engineering terminology, especially within the course. Students, in particular, tend to discount the significance of definitions, but software engineering relies heavily on clear communication, which in turn requires consistent terminology.

Choose a text that uses terminology consistently, has a coherent software engineering philosophy, and pays attention to the history of the discipline. A discipline is based on principles and practices that develop over time, along with its philosophy.

2. Software Variety

Emphasize a problem domain that interests you and can serve as the basis for several projects over time. This allows you to become more of an expert and it provides administrative stability from semester to semester. If you change problem domains too frequently, it creates extra overhead and forces you to spend time on what is new rather than what is foundational. This does not mean that you do the same project each semester, but it does mean that one project may build on another (which is a good experience for the students) and that you are "creating useful capability" within a problem domain.

Also, try projects that are not confined to information technology, because an expanding body of software involves hardware devices and control systems – what could be called "engineering applications". You may have to do some of the coding yourself, depending on the application, but that would provide a more real-world flavor since students would not implement everything from scratch. Such projects can be more interesting, your students will expand their job prospects, and it can be done with fairly inexpensive hardware. [1]

[1] See Sparcfun [8] and similar vendors.

The infrastructure that you choose (e.g., programming language, operating system, framework, IDE, source code control system, and hardware platform) will be determined, to a large extent, by what precedes your course in your curriculum and what is feasible within your department. Fortunately, open-source software is readily available. Though, what to choose and when it should change will still be an issue. The infrastructure can change over time, but resist changing too much at once and avoid making major changes each year. Whatever you do should be both feasible and sustainable.

3. Development Practices

It is important to be aware of current development practices, which means reading, trying out new things, and periodically altering the course. But there must be a balance between change and stability. Change increases overhead and too much change can cause chaos. Yet, too much stability causes stagnation.

Students should understand the reasons for process models, the options available, and the criteria for choosing for a given situation. There is a spectrum of choices [11].

A process model should be chosen for the group project that is appropriate and tailored for those circumstances. Students should come to realize that no one process model is always appropriate and that any process model should usually be tailored. For example, Scrum will likely not be the same from one setting to another.

We have been using iterative development methods since the late 1990's, when we used *UML and Patterns* [19], even before the text began to mention agile. Lately we have used Scrum-specific references and have incorporated some Scrum concepts (e.g., user stories, product owner, sprint) within the project. However, the course schedule (i.e., MWF) does not easily allow the daily standup; the team members are not really "experts"; the team does not have as much autonomy as they would in a company setting; and my role of instructor is not always compatible with my role as Scrum product owner.

There are several misunderstandings concerning agile that are hard to displace. Students often think that agile allows the development team free reign and removes the need for any documentation or a plan, which is a misreading of the Agile Manifesto [2]. Also, organizations or individuals may treat agile (e.g., Scrum) as a magic box that will produce "better software, faster", or the customer may not realize that agile demands more of them, in many ways, compared to a "command and control" setting. See [11] for an insightful discussion of agile methods.

The Essence of Software Engineering [18] attempts to move above "competing" methods to describe basic elements of any development methodology, i.e., the SEMAT kernel. Their work appears promising.

Software Engineering I

Trying to accomplish "everything" within a 3-hour course is not possible, so, you must choose topics wisely, focusing on what is "foundational". Trying to cover everything in a typical software engineering text will not allow enough time for a project. Also, it is important for students to realize the difficulties of software development, via a small group project.

Structure

We have been using a general software engineering text [12] to provide terminology and structure, making sure that the terminology is "standard" (as much as possible) and that the text is not biased with respect to a particular development method. We then supplement with shorter references (e.g., "Why Software Fails" [15], "Scrum: A

Breathtakingly Brief and Agile Introduction" [21]). For example, Braude and Bernstein [12] uses terminology that is consistent with Fairley [16], one of the texts used in the Software Project Management course. [2]

A Canvas [4] quiz is due at the start of each chapter, which highlights important terms and concepts. Questions are mostly True/False and fill-in-the-blank to reduce grading time. Each quiz can be taken multiple times prior to the due date (when it is recorded), and students can revisit any quiz to study for an exam. Repetition is helpful.

Two in-class, closed-book exams are worth 40%, Canvas open-book quizzes (and written assignment) are 15%, the group project is 20% and the comprehensive final is 25%. The final is a good time to review concepts, especially those from earlier in the semester. There is a temptation to forgo the final when there is a project, but the project cannot capture everything that is in the course and it is important for each student to demonstrate that they understand the "software engineering mindset".

Class time during the last four weeks of the semester are spent in the lab working on a group project, with four or five people per group.

The Project

The four-week project typically has four iterations (e.g., Scrum sprints). During the Spring 2015 semester the four sprints took 5, 10, 9, and 7 days, respectively. Sprint 1 lays the groundwork and should be short. Sprints 2 and 3 represent the bulk of the implementation and sprint 4 contains system testing and polishing. Students receive feedback from both the "instructor" and the "product owner" after each sprint, which can be confusing since I play both roles.

The project does not try to replicate a "real" project. Rather, it is a "guided" experience, not a simulated work environment. Students are not ready for a "real project" in an introductory course. However, they are ready for "experiences" that convey the difficulties of working in a group and to be exposed to sound design and process methods.

A guided project means that the architecture, the artifacts, the basic design and the code structure are, to a large extent, dictated. For example, the structure of the domain classes and their respective sets are standardized, the format of the test output is specified, and the schedule is pre-determined. The reason: The project is short and there are many concepts to illustrate. Also, our capstone course is based on a "real" project, suggested and supervised by a third party.

The centerpiece of the project is a domain model (i.e., a UML class diagram lacking methods) and a high-level architecture, both specified by the instructor. The domain model gradually evolves into a design class diagram (consistent with Larman [19]). Students formulate user stories from an initial list of functional requirements, are given an IBM Rhapsody project [3] that contains the UML domain model, and they then create use cases and related sequence diagrams for a small number of user stories.

This approach allows for some agile (i.e., Scrum) concepts such as sprints, a product backlog, sprint backlogs, and self-organizing teams. But there are no daily stand-ups (due to the nature of a three-hour, MWF course), I cannot really play the role of product owner for multiple project groups (last semester there were nine), and the product backlog does not change as much as it could in a real setting. However, there is enough "reality" for students to realize that communication is important and that configuration management is necessary when code is shared.

We have successfully used Trello [10] to manage product and sprint backlogs. I create a Trello "board" with a specific structure for each group and then invite students to join their respective group. This allows me to

[2]R. Fairley is a senior member of the IEEE who has chaired several software standards committees.

[3]IBM Rhapsody [7] is free to universities.

manage the product backlog (which I control), each group can manage their sprint tasks, and I can see how things are progressing during the sprint as tasks are moved from “Doing” to “Done”.

A peer review occurs after each sprint, which encourages students to improve their evaluation by the end of the project. Each student receives a score at the end of a sprint that is the product of their peer review (as a percentage) and their group score. The peer review asks the following questions, each of which uses a 5-point scale:

- Do they do what they are asked to do?
- Do they actively seek to help?
- Are they congenial?
- Do they communicate well?
- Do they produce high-quality work?
- Do they show up when expected?

Avoid

Here are some things that I try to avoid during the project.

- Allowing a third party to define the project (and dictate technology), because there is not enough time for the extra overhead in an introductory course.
- Evaluating a project by only looking at the output. Good design is usually “under the hood”.
- Allowing a superhuman team member to single-handedly carry the project over the finish line.
- Using a source control system, such as Subversion [9] or Git [6], without an established configuration control procedure.
- Allowing test output that can be easily faked, e.g., “Test successful”.
- Allowing each group to choose how much to do for each iteration (which would be difficult to track with multiple groups).
- Allowing each group to choose their own project problem (which would create too much administrative overhead).

Concepts to Emphasize

Concepts emphasized throughout the course include the following.

- Software development is a human enterprise that requires cooperation and communication.
- Change must be intentionally managed throughout the software life cycle, not simply during development.
- Decisions should be recorded, not simply conveyed.

- Configuration management is a process, not simply a tool.
- Coupling and cohesion influence quality.
- Purpose and strategy are different.
- Verification compares a work product with a standard and validation determines whether the work product has value.
- There is no "silver bullet" for the "Werewolf" [14] or the "tar pit" [13] that is software development .
- Any development model should be tailored to your circumstances.
- If someone inherits your project you should have provided enough information to support it.

Recommendations

- Use hardware devices to allow for interesting projects with minimal cost. For example, lately we have used RFID readers with a patient tracking system (within a doctor's office), a timing system for marathon runners, and a room access system.
- Coordinate early with your IT support staff. Last-minute requests are unreasonable, both for them and for you.
- Employ a three-layered architecture, i.e., presentation layer, logic layer, and data layer, with a facade object containing all logic-layer capabilities for the presentation layer.
- Use Subversion or Git to manage the software configuration during the project from within the Eclipse [5] IDE.
- Limit the number of students per project group to 4 or 5. In a real project more would be reasonable, but communication overhead becomes a significant burden when the group goes beyond five in a class setting.
- If there are multiple project groups, it is advisable to have each group work on the same problem. Multiple problems with multiple groups create far too much overhead.

Comments

The Computer Science department at Baylor University has a software engineering "track" (not a software engineering program), which is a restricted form of the BSCS degree. In addition to the BSCS requirements (which include SWE I and SWE II), the track requires Software Project Management, Software Quality Assurance, Engineering Economics, and two semesters of calculus-based physics. SWE I students have already taken two semesters of C++ programming, data structures, algorithms, and software systems (i.e., machine language). In addition, a one-credit Java course is a co-requisite for SWE I.

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Pressure and Impulse in Student Learning: What I Learned from Teaching Physics

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Abstract

In the fall of 2014, a one-semester break between the departure of my department's sole physics professor and the arrival of his replacement afforded me the opportunity to teach a first-semester, calculus-based physics class. The thirty-year gap between the introductory physics courses I had taken myself and this course I was to teach, combined with a two-week notice prior to the start of the semester, put me in the interesting position of learning alongside my students. Wading through an unfamiliar text, trying to understand publisher-produced lecture slides, learning from and getting frustrated with online homework, entering review sessions fearing what questions might be asked-all these things gave me a much clearer understanding of and empathy with the student experience and prepared me to make effective changes in other courses I teach.

Long before I had finally abandoned the hunt for enough adjuncts to cover my college's fall physics offerings, my immediate supervisor (the Dean of the Faculty) had been encouraging me to take on one of the courses; she wanted first-hand input on the difficulty of teaching physics from someone whose teaching she was familiar with. She presented this plan in the following winsome way, putting to good use her PhD in psychology: "I have an idea, but you're not going to like it."

As it turns out, my own thinking had been along the same lines, but I was a bit hesitant: my connections with physics since my undergraduate days were pretty minimal. Many years of teaching third-semester calculus did give me confidence with the vector material, but I'd had conversations with a string of physics colleagues whose responses to my teaching suggestions were usually along the lines of "Kim, teaching physics is not like teaching math."

Right off the bat, the physics text gave me some insight into that statement, noting helpfully, in a section on problem-solving, that "Physics is not mathematics," because "math problems are clearly stated, such as 'What is $2 + 2$?' " [2, p.19]. Apparently, to a physicist, if you're doing math, you're doing computation.

If you're doing physics, however, you're solving a word problem: as the same text noted in the next paragraph [2, p. 19], "The translation from words to symbols is the heart of problem solving in physics." I found this claim very interesting, particularly since, with the exception of using the word "math" in place of "physics," this is exactly what I say to mathematics students.

Physicists also may have a more restrictive view of reasoning. One question, in the publisher-provided slides, asked for the force on the fulcrum in the middle of a balanced system, instructing students that "answering this requires reasoning, not calculating" [1, p.91]. Now, it seems to me that to get from the input of 1000 N to the result of 2000 N, one must do a bit of calculation. Thinking about torque, about the fact that the system is in balance, and about the relative locations of key pieces of the set-up are certainly helpful, but those ideas can only do so much on their own. Besides, since when is calculation not in the reasoning toolkit?

Applying rules of logic to this chain of statements (math is calculation, calculation is not reasoning) led to a conclusion I immediately rejected and, I confess, poked fun at for the rest of the semester.

This insight into one possible view of mathematics from the vantage point of physics is not one of the key lessons I took from my brief foray into the world of physics teaching, however. Instead, what I learned was that I knew almost nothing about what students really go through when struggling with difficult material.

I did at least go in knowing that I would need regular feedback if I was to have any chance at success with this unfamiliar course, so I gave students multiple opportunities to provide anonymous comments. For the most part, I got pretty standard stuff:

- Work more examples.
- Quizzes encourage me to keep up.
- The pace is too fast/just right/too slow.
- The quick-check problems help me see whether or not I really understand.

Quick-check problems were multiple-choice problems provided in the notes, like clicker problems, though I used the low-tech "hold up the correct edge of a labeled notecard" response method rather than clickers. My colleagues in biology and physics have long used such questions, but they were new to me. My own experience in working through the notes showed me how helpful these questions were as I tried to gauge my own understanding, so I wasn't surprised when students felt the same way.

Other comments were a bit more surprising, particularly a claim by many students that they were demoralized by group work. In my undergraduate experience, group work was usually boring, because I seemed often to be slowly explaining to others what I saw immediately myself. Perhaps because of this, I never thought of it as demoralizing. Many students, however, said they didn't like working in groups to solve problems they hadn't seen before. Several specified that it was because there was always someone in the group who saw what was going on before they did, and this made them feel stupid; eventually they quit trying and just waited for the faster thinkers to explain things to them. They did find it helpful, though, when I gave them time to work through a tough problem on their own, then had them discuss it in small groups, and then went over it with the whole class, so I worked more of this "think-pair-share" type of approach into our classroom time.

Other lessons learned (or re-learned, or maybe understood deeply for the first time):

(1) I learned about the pressure of learning:

- You can't skim a text written in an unfamiliar language. This had never sunk in for me before, since I never actually read the text when I was a student. As an undergrad, I'd look for examples similar to problems I had to work, and I read through those. In grad school, I looked for definitions and theorems that involved the objects I was working with, and I read through those, but I can't say I spent any time on the prose of the texts.
- You can read the text and still not know the material.
- You can follow everything you read and still not know the material.
- You can work through all the online homework until your answers are correct and still not know the material.
- You don't want to learn these lessons during an exam (or worse, for me at least, during an ill-prepped exam review session).

(2) I learned about the effect of impulse:

- Unprepared test-takers are panicky test-takers.
- Panicky test-takers are impulsive.
- The impulses of panicky test-takers rarely move them in the right direction.

Now, it's not like I didn't already know these things. In fact, I've been sharing these insights with my students for years. But I never really faced them as a student myself, so I couldn't speak with much authority, and I think my students could tell. In this class, however, *they believed me*. I told them how many hours I spent on the homework and how carefully I had to work through the text and the slides to understand the concepts. After the first 'disastrous' review session, I apologized and admitted that I had fallen into all the traps I tell my students to avoid. I acknowledged the pressure of having to perform while being tested and the frustration of going back to your room afterward and being able, with the pressure off, to work in 5 minutes all the problems that had stumped you. I told them that they needed to work *a lot* of text problems, until they got to a point where they'd worked their way through so many different scenarios that their problem-solving became as much instinct as reasoning. And from that point on, most of them did. This was never the case in any course I had previously taught.

So, what are the main lessons I took from this experience? First, multiple-choice quick-check questions are great! They help students see, immediately, whether they really understand a concept or are just following someone else's logic. I've always asked questions while I teach, and I'm pretty good at giving students time to think through them, but putting a multiple-choice question on a slide gives students time to digest the question and also to think not only about why their choice is correct but also about why the options they rejected are incorrect. I learned to wait until each student had committed to an answer before I went on; that got most of them to really think it through instead of just waiting for the quick students to blurt something out. Such questions also give you a good read on students' comprehension: if nearly all of them have the correct response, you can more quickly talk through what makes the incorrect responses incorrect and what makes the correct response correct; if you're seeing a number of different answers, you can have students try to convince those around them that they are correct and allow themselves to be convinced by the valid arguments of others, with many opportunities to recognize invalid reasoning as well.

Second, and most important, really: If you have a chance, teach outside your area of expertise. Physics was ideally suited to reconnecting me (or perhaps really connecting me for the first time) with the student experience. I suspect that many future college professors find, as I did, that it is pretty easy to breeze through their undergraduate courses, getting good grades without really committing to the material. Not all students have that ability, and I had to abandon my reliance on it in order to understand the material well enough to face students' questions. Teaching outside my comfort zone gave me authority from which to speak about effective study habits, interaction with the text, and use of resources. It took a lot of work and challenged me in ways I had never been challenged before. And I would love to do it again.

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God: One

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Abstract

The ontology of mathematical objects has been of interest for millennia. I focus on the ontology of the number one in relationship to the ontology of God.

Introduction

Let ONE represent "the essence of what 'the number one' is." I focus on the cardinal nature of the number one as opposed to its ordinal nature. The primary question is "What is ONE?" First, consider:

> "There is no physical entity that is the number 1. If there were, 1 would be in a place of honor in some great museum of science, and past it would file a steady stream of mathematicians gazing at 1 in wonder and awe." [3, Introduction]

Why would mathematicians be in wonder over the number one?

ONE in Mathematics

ONE is foundational to counting, which is foundational to much of mathematics. This is nicely shown by Doron Zeilberger in his Fundamental Theorem of Enumeration: ([4, p. 550])

$$|A| = \sum_{a \in A} 1.$$

With some philosophical hand waving (which I admit deserves more attention, but that will be suppressed for now), ONE can be seen as foundational to much of the mathematical spaces that mathematicians work within. Just starting with ONE, we immediately have a unit $\{1\}$. From this we can imagine the existence of the unit and the non-existence of the unit: $\{0, 1\}$. But, imagining ONE twice, ONE three times, and so on brings us to $\mathbb{N}$. Bringing in the concept of symmetry of an anti-ONE, an anti-ONE twice, etc., gives us $\mathbb{Z}$. But then considering the set of all integers over non-zero integers, $\mathbb{Q}$, and the set of all Dedekind cuts, $\mathbb{R}$, (recalling that each Dedekind cut is an infinite set of rational numbers), we quickly imagine ordered pairs, triples, etc. to then have $\mathbb{R}^2$, $\mathbb{R}^3$, etc. So ONE is important to mathematics. But, is ONE important to Jesus?

ONE in Theology

Mark 12

In Mark chapter 12, someone asks Jesus "Of all the commandments, which is the most important?" Jesus famously responds with loving God and loving neighbor. But, the first part of Jesus' response in verse 29 is

> "The most important one," answered Jesus, "is this: 'Hear, O Israel The Lord our God, the Lord is one.' "

Jesus is quoting one of the most important pieces of scripture for the Jews, from Deuteronomy 6:4. My understanding is that this is commonly not seen as God being numerically one, so differing from ONE, but I still see it as reasonable to ask about how "One God" is related to ONE. But, first, where else in the Bible is it common to use the word "one?"

More than One?

In Genesis 2 we have two (husband and wife) becoming "one flesh." But then in Ephesians 5 Paul informs us that this is a mystery actually referring to Christ and the Church. In traditional Christian Theology we have God as trinity[1]: "Three in One." How are these connected to ONE?

The Ontology of ONE

Trinity

Does the concept of the trinity lead us to ONE and THREE as eternal concepts? Or does it lead us to conclude that we must leave the relationship between God and numbers a mystery?

One God

Was there one God before God created the number one? I like asking this question to my students. Here are some responses attempting to honor God's sovereignty:

ONE is...

- part of a "continuous creation," so distinct from God [5, p.71]
- uncreated, so part of the nature of God in a mysterious way [1]
- not "real" in a Platonic sense [2, p. 230-235]
- ...wait!...Is "one" the same in "one God" and "number one?" [Ibid.]

Let's explore this last question.

Necessity

Is ONE necessary in any universe God may create? I claim "yes" since one universe would be recognized by God.

But what if God had chosen not to create anything? Is ONE necessary then?[2] Perhaps another way to put it is: Must God's eternal unique existence correspond to ONE? In light of the trinity, I am not confident in making a claim either way. And, in fact, in light of the oneness of Christ and the church, there seems to be some things concerning ONE that may be beyond our understanding. Or perhaps ONE should be seen as a rather distinct idea from many other ideas that are represented by the word "one."

[1] I thank William Lindsey for pointing out the importance of including the trinity in this presentation

[2] A conversation with Jim Bradley sparked this question. Thank you Jim Bradley for the opportunity!

Conclusion

ONE is important to mathematics. "One God" is important to Jesus, but we must be careful with how this relates to ONE, especially in light of the trinity and the sovereignty of God. May our questions concerning ONE, and with mathematics in general, increase our wonder of the ultimate one, God.

Acknowledgements

I give a special thanks to my wife Karen for helping me sort my thoughts.

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Experiencing a Paradigm Shift: Teaching Statistics through Simulation-based Inference

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Abstract

For decades, introductory statistics has been taught as an application of formulas, making use of normal and other distributions, and relying heavily on algebraic skills of students, in short, emphasizing mathematical thinking. More recently, several textbook author teams have published statistics textbooks that place an increased emphasis on simulation and randomization methods as the way to motivate statistical reasoning (e.g., inference) leading to a decreased emphasis on the algebraic manipulation of formulas and theory-based approximations to sampling distributions (e.g., [3]; [7]). This paper describes simulation-based inference curricula more fully, reports on the necessary steps towards the implementation of such an approach, and provides both qualitative and quantitative comparisons of this new pedagogical approach with a more traditional approach. Appropriate justification of the approach to teaching and learning statistics is also provided, along with providing an overview of recent trends to shift to this approach in statistics courses taught at the high school, junior college, and university levels across North America, including a number of Christian colleges and universities affiliated with the ACMS.

Introduction

If you have taught an algebra-based introductory statistics course (e.g., the AP statistics equivalent course) during the past few decades, then perhaps the following description is familiar. Topics normally proceed from descriptive statistics to probability theory, from sampling distributions and the Central Limit Theorem to inferential statistics focused on hypothesis tests and confidence intervals. The variety of different tests and intervals normally includes a single mean, two means, ANOVA, a single proportion, two proportions, Chi-Square, and correlation/regression. Typically, these courses make extensive use of formulas, tables of values for t, z, χ^2, and F distributions, graphing calculators, and/or any of a wide array of statistical packages (e.g., SPSS, Minitab, Excel, R). In short, this standard introductory statistics course consists of a collection of theory-based techniques and is often taught from a perspective of an application of mathematical formulas rather than as a course in statistical thinking.

A Challenge to this Paradigm

George Cobb [2] offers a sharp critique of this familiar version of the introductory statistics course. Among other issues, he notes that many of these theory-based techniques have validity conditions that are not satisfied in many real data contexts. Furthermore, Cobb notes that some of these techniques have been modified over the years to overcome some of the challenges posed by these validity conditions. As an example, the “Plus

4" method for computing confidence intervals for a single proportion was developed in the 1990s as a way to overcome the need for a minimum number of successes and failures in the sample data. Cobb refers to examples such as this one as a "cathedral of tweaks" and a futile effort to eliminate this shortcoming of theory-based methods. In these situations, Cobb posits that, at best, students will memorize technical rules, losing sight of the big picture of how to think statistically.

As an alternative, Cobb and a larger team of statistical educators and researchers proposed a new approach to the teaching, learning, and doing of inferential statistics. This paradigm makes extensive use of simulation, both tactile and computer-based via applets developed by Beth Chance and Allan Rossman (see [1] for more details on the rationale for these applets. Access them at http://www.rossmanchance.com/ISIapplets.html). The applets use a combination of resampling and randomizations to produce large scale simulations based upon the sample data. An experimental p-value based upon these simulations can be easily computed, and this value will be approximately equal to the p-value using the comparable theory-based approach, assuming that the validity conditions are satisfied. Interestingly, when these two p-values do not match, the fault lies with the validity conditions but the simulated p-value can still be used. In the past few years, at least two textbooks have been written that make extensive use of simulation and randomization, including appropriate applets (cf., [3]; [7]).

Highlights of the New Paradigm for Teaching and Learning Statistics

By designing an introductory statistics course using the resources described above the teaching and learning paradigm may differ in several important ways. First, an overview of the scientific method as applied to statistical analysis is possible from the beginning of the course using a discussion of inference on a single proportion as an initial case study with inferential statistics following closely thereafter. Roy and her colleagues [5] make the pedagogical case for introducing p-values in this manner during the first week of such a course. Essentially, students have good intuition in this situation and can relatively easily understand the core logic of inference of simulation with a single proportion without getting stuck in technical definitions (e.g., sampling distribution; binomial distribution; normal approximation to the binomial, etc.) The remainder of the course allows students to deepen their understanding of the overall process. P-values and overarching themes such as significance, estimation, generalization, and causation are investigated. Descriptive statistics, including means, medians, standard deviations, five number summaries, dotplots, boxplots, correlation, and least squares regression lines, emerge in a "just in time" manner as the concepts are needed to perform related statistical tests. Similarly, those probability concepts necessary for understanding the inferential statistics are introduced as the need arises.

A brief example from the [7] textbook will illustrate the general approach. Like essentially all other examples and explorations in the book, this example is taken from actual research. Two dolphins, Buzz and Doris, were placed in a pool but separated by a curtain. Doris was instructed to communicate with Buzz who, in turn, pressed one of two buttons. A correct result by Buzz was followed by fish being fed to both dolphins. In a total of 16 attempts, Buzz answered correctly 15 times. Initially, students are asked whether this data constitutes evidence that the dolphins are communicating, and most of the students agree that this is the case. When asked how they would justify this finding, the discussion eventually leads to the need to simulate what would happen if Buzz was guessing, by using an ordinary coin flipped a total of 16 times, repeatedly. A class set of results normally suffices to convince students that an outcome of 15 (or more) successes is an extremely rare event, assuming that Buzz is merely guessing. At this point, the one proportion applet is introduced and students use their own smart phones or laptops to simulate Buzz guessing for a total of 1000 or 10,000 sets of 16 flips, noting the proportion of repetitions in which 15 or more successes are observed (see Figure 1).

Notice that students must eventually learn to interpret this simulated p-value as the likelihood of obtaining 15 or more successes out of 16 attempts assuming that the null hypothesis (i.e., Buzz is guessing) or chance

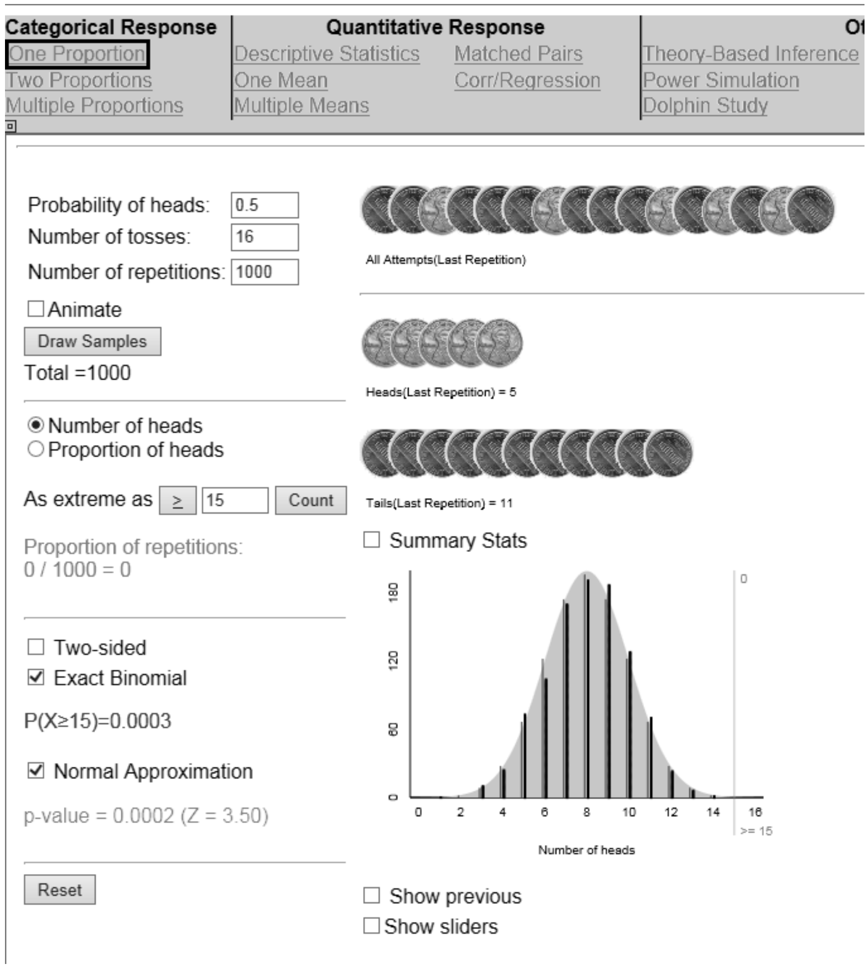

Figure 1: One Proportion applet to model initial Buzz and Doris Example

model is true. An extension of this example involves a follow-up experiment in which Buzz was successful only 16 out of 28 times. This time the applet provides a nice visual image for the simulated p-value and students readily conclude that these data do not provide strong evidence that the dolphins are communicating. As a final note on this example, the researchers later acknowledged that they stopped feeding the dolphins at some point during this latter study so the results are hardly surprising (see Figure 2). Note that the applet also provides the exact p-value using the binomial distribution along with an estimate using the
normal distribution which is not accurate due to the discrete nature of the data. Other topics in the introductory statistics curriculum can be similarly addressed using simulation, randomization, and corresponding applets.

More Applets and Potential Uses in an Advanced Probability and Statistics Course

When an advanced course in probability and statistics is offered at Trinity Christian College, the need to develop probability theory, probability distributions, expected values, and variances, limits the time spent on inferential statistics to the final five or six weeks of the course. For this reason, there is typically not sufficient time to develop the Buzz and Doris and similar one proportions tests in great detail. However, other topics normally covered in this portion of the course can be enhanced by using the applets and demonstrating how simulation and randomization provide an excellent approximation to the theory-based p-values.

One such example is the matched pairs test. Although this test often poses some initial confusion for students,

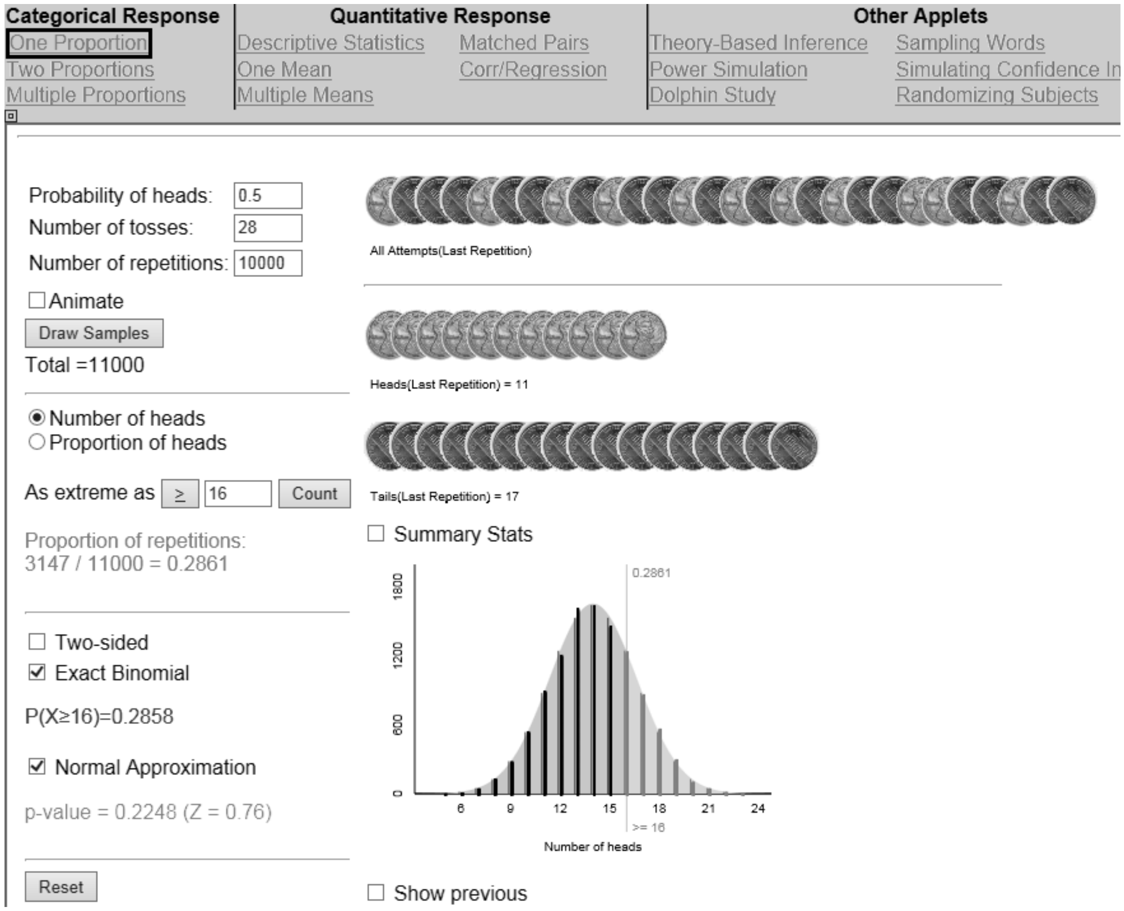

Figure 2: One Proportion applet to model follow-up Buzz and Doris Example

the applet makes the reason for the pairing of the data values clear visually and highlights how this method, when applied correctly, dramatically reduces the variability in the sample. Simulation can then be done to produce an approximate p-value (see Figure 3). It is important to note that applets are also provided to perform the theory based test for one mean (as in the case of the matched pairs test), two means, one proportion, and two proportions. For other tests, including multiple proportions, multiple means, and the slope of the regression line, the theory-based method is included as an option within the simulation applet. In the Tintle et al. text, [7] sections at the end of each chapter provide an overview of the theory-based method, including a discussion of the necessary validity conditions.

A second example is correlation and regression. In this area, the applet provides a helpful interpretation of the Least Squares Regression Line. It allows students to experiment with many possible lines through the data, and it provides a comparison of the total squared error in each case, both numerically and graphically (see Figure 4). As before, simulation can be performed to produce an approximate p-value.

Finally, students often find it difficult to correctly interpret the meaning of confidence intervals. Typically, the student will claim that there is a 95% probability that the actual mean will fall inside a 95% confidence interval for μ when, in fact, this is a nonsensical statement since the interval either did or did not succeed in trapping the parameter value, and we typically do not know because μ is unknown. The Simulating Confidence Intervals applet provides an opportunity for students to construct 100 different 95% confidence intervals for the mean. Typically, all but around 5 or so of these intervals will include the correct value (which must be known for this illustration) (see Figure 5).

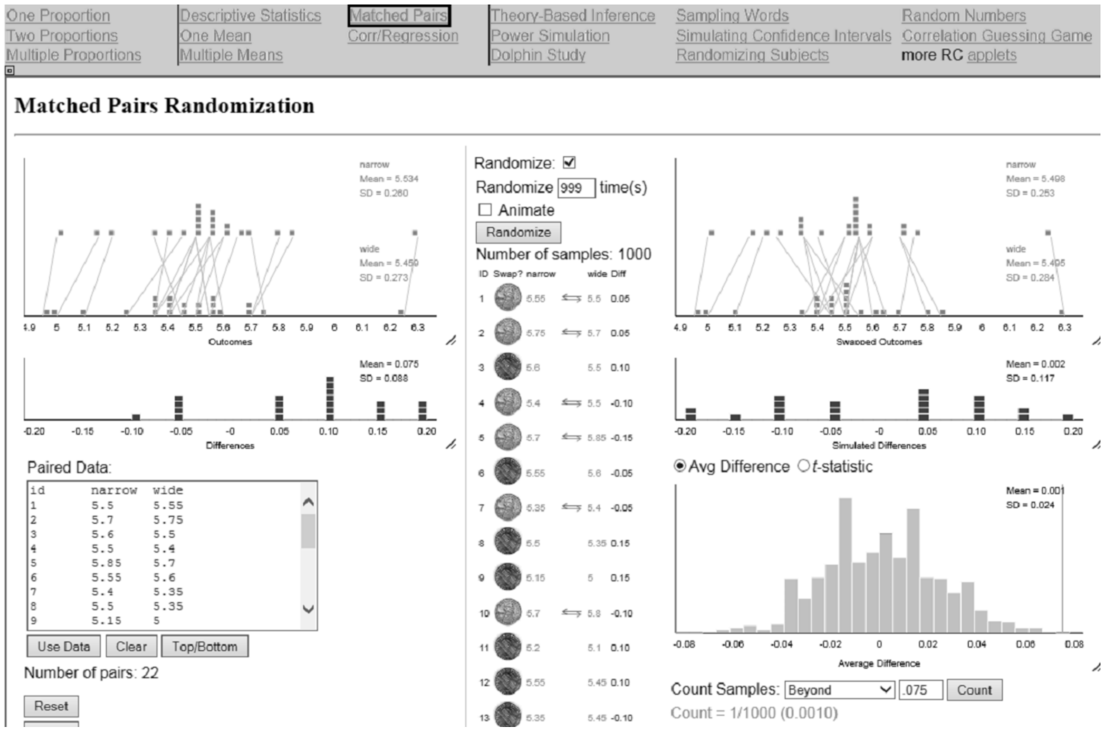

Figure 3: Matched Pairs Applet

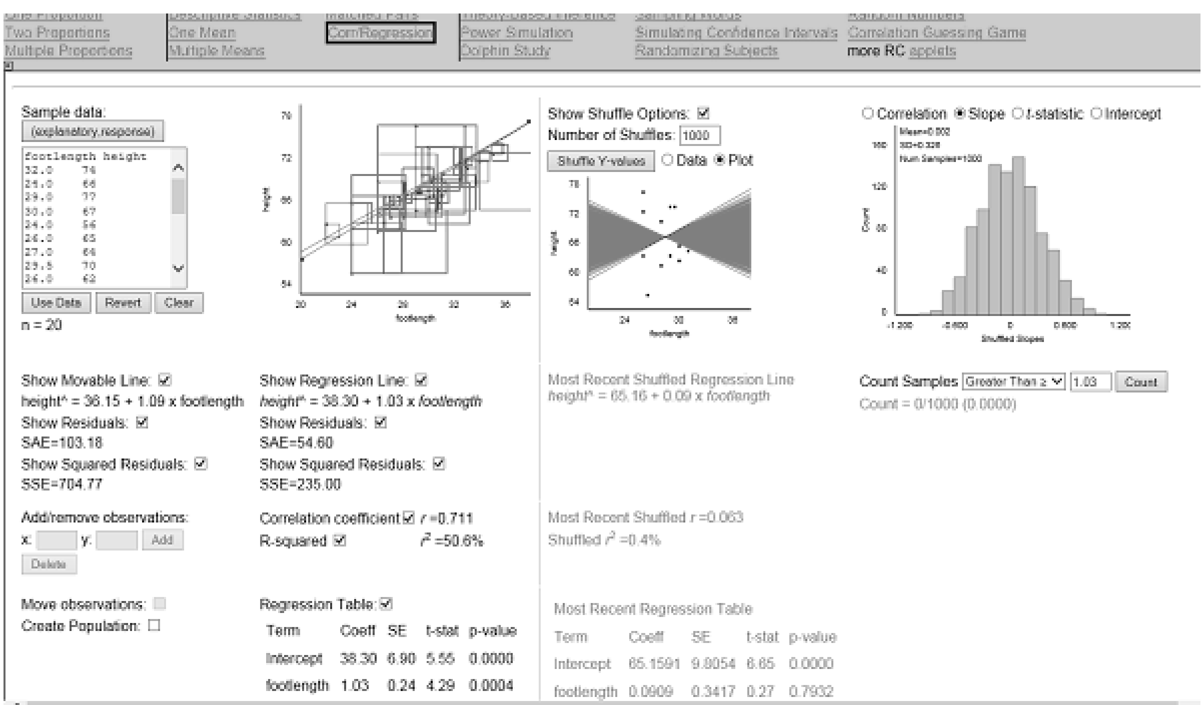

Figure 4: Correlation and Regression Applet

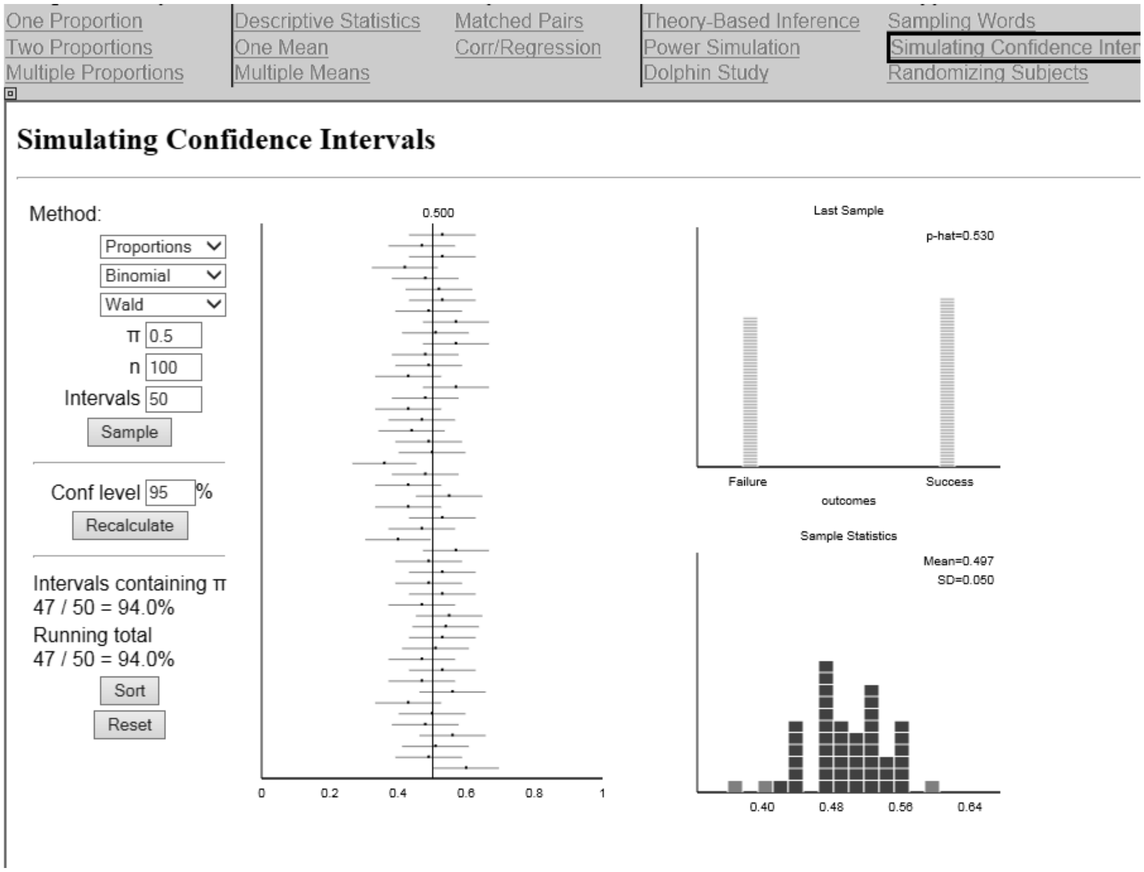

Figure 5: Simulating Confidence Intervals Applet

Results of Teaching and Learning Introductory Statistics Using Simulation-Based Methods

Tintle and his colleagues [11] report on quantitative results comparing conceptual knowledge of statistical concepts by students taking courses using this new paradigm and those taught using the older and more familiar paradigm. A 40-item multiple choice instrument, based upon the Comprehensive Assessment of Outcomes in Statistics (C.A.O.S.) was administered at three different times: during the first week of the course, during the last week of the course, and four months following the completion of the course. For 33 of these 40 items, both groups of students performed similarly. However, for 6 of the remaining 7 items, the students who learned using simulation-based methods outperformed their counterparts in the more familiar statistics courses. Not surprisingly, one of these 6 items assessed a student's knowledge of a p-value. These findings have now been replicated at multiple institutions [9]. Perhaps more importantly, there was less of a decline in performance four months following the course for the students in the simulation-based course [10]. That is, these students retained more of their accumulated conceptual knowledge of statistics as they enrolled in courses in their major that required them to apply this understanding. Swanson, VanderStoep, and Tintle [6] document similar findings related to student attitudes towards statistics and the learning of new statistical concepts.

The first author of this paper used a well-known traditional text [4] during the fall of 2013 before switching to a text using simulation and randomization [7] during the three subsequent semesters. Overall, the percentage of students earning grades of C or better in the course was similar for both approaches. Like in the traditional course, some students struggled in the simulation-based course. However, the focus of these struggles were less algebraic and arithmetic in nature and more connected to the statistical concepts such as causation and generalization. Equally important, the use of real data and explorations promoted a greater level of student engagement in the simulation-based course.

Conclusion

If you are currently teaching introductory statistics using the old paradigm, then we encourage you to try a simulation-based approach. You may also learn more by reading Cobb's article [2], or by reading a more recent article related to the impact of simulation ideas on the entire undergraduate statistics curriculum [8], or by visiting the simulation-based inference blog where numerous statistics educators post on their experiences using simulation-based inference in their classes. Although it would be helpful to augment your current course with the use of some of these freely available applets, you will derive the greatest benefit and personal satisfaction if you accept the challenge to join the new teaching and learning paradigm. Enrolling in a 2-4 day intensive workshop on these approaches would also greatly enhance your chance of success in this effort. Please contact any of the authors for additional information.

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The Mathematics of Evolution

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Abstract

Like many universities, Lee University has a non-major's mathematics course for liberal arts students. The course typically includes a potpourri of topics: logical thinking, scientific notation, linear functions, estimation, and probability. At Lee, we have found a way to conclude the course that applies these varied topics to an issue designed to engage student interest and promote critical thinking. We have developed a series of three lessons on the mathematics of evolution. This paper includes a sampling of the topics included in those lessons.

One of the fundamental questions of life is **Where did life come from?** Many people believe that man is the product of a random natural process called **evolution**. Millions of years ago, "simple" life forms appeared on the earth. Over time, these simple forms gradually became more complex. Eventually, man appeared.

What does mathematics say about the hypotheses and equations that provide the scientific background for the theory of evolution?

Radiometric Dating

Since evolution is dependent on the earth being millions of years old, we begin by asking, "How old is the earth?" Estimates of the age of the earth vary widely from 6,000 years to 4.5 billion years. Those who hold to the recent creation of the earth are sometimes called "young-earth" creationists and those who subscribe to a more ancient origin are said to believe in an "old-earth." There are highly trained scientists (and devout Christians) in both groups. Most attempts to date the age of the earth are based on properties of radioactive decay. For example, fossils can be dated with Carbon-14 and igneous rocks with Potassium-40. These techniques **assume** that the rate of decay has been constant through all time. Indeed, over the last hundred years, the rates have appeared to be constant. But can we conclude that they have always been the same?

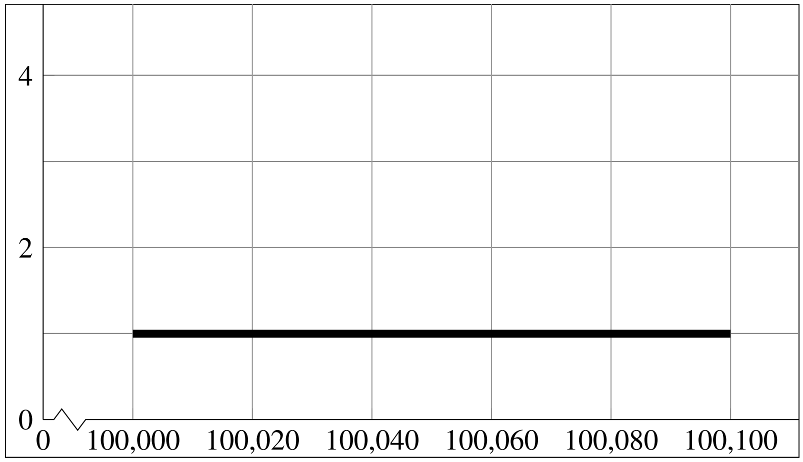
Extrapolation: Using the Present to Measure the Past

Suppose for the moment that the earth is 100,000 years old. Let t represent time in years since the formation of the earth and suppose we measure some physical quantity $Q(t)$ that may or may not change with time. For example, $Q(t)$ might be the ratio of the half-life of ^{14}C at time t to the half-life of ^{14}C measured in the year 1900 A.D. If $Q(t)$ is constant, then the half-life of ^{14}C would be constant, too. We consider three possible functions that might model the behavior of Q over time.[1]

$$f(t) = \frac{t^2+t}{t^2+1}, \quad g(t) = 1, \quad \text{and} \quad h(t) = \frac{t^2+1}{t^2+t}.$$

Which of these three functions do you think is graphed below, where we have shown the portion of the graph from $t = 100,000$ to $t = 100,100$?

[1] For the sake of simplicity, we have chosen functions that have values near 1. By taking multiples of these functions, we could make them approach any finite value.



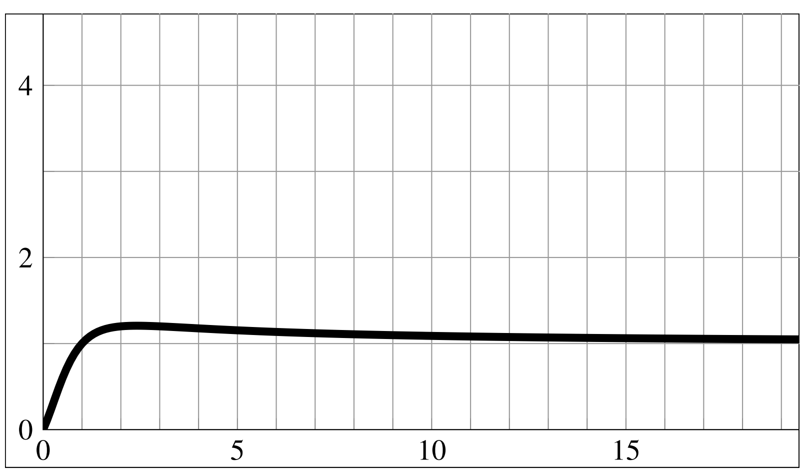
Graph for $t = 100,000$ to $t = 100,100$ Years

Actually, the graph could be any one of these three functions since they are indistinguishable over the range of $t = 100,000$ to $t = 100,100$ as shown. Here is a table of the values for the three functions, where we have truncated the numbers to nine decimal digits:

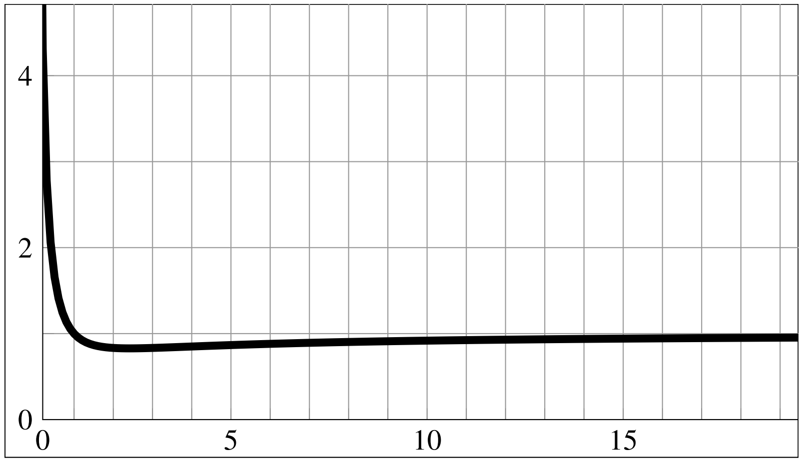
Time in years	$f(t) = \frac{t^2+t}{t^2+1}$	$g(t) = 1$	$h(t) = \frac{t^2+1}{t^2+t}$
$t = 100,000$	$1,000\,009\,999$	$1.000\,000\,000$	$0.999\,990\,000$
$t = 100,050$	$1,000\,009\,994$	$1.000\,000\,000$	$0.999\,990\,005$
$t = 100,100$	$1,000\,009\,989$	$1.000\,000\,000$	$0.999\,990\,010$

As we progress through the 100 years, the function $f(t)$ is a tiny bit more than 1 and decreasing just slightly. The function $g(t)$ is constant at 1. The function $h(t)$ is a tiny bit less than 1 and increasing slightly. We observe that all three functions are virtually constant and their differences are so small they cannot be seen. But what happens to the functions $f(t)$ and $h(t)$ for values of t close to zero? Are the functions "constant" there?

Here are the graphs of the functions near $t = 0$:



Graph of $f(t) = \frac{t^2+t}{t^2+1}$ near $t = 0$



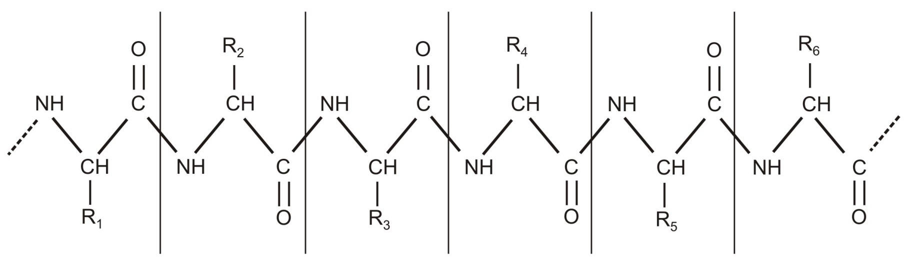
Graph of $h(t) = \frac{t^2+1}{t^2+t}$ near $t = 0$

As we go from right to left, the f function increases slightly and then turns and falls abruptly to 0. The h function decreases slightly to a minimum point and then rises without bound. So what appears to be constant in the present is far from constant in the past.

These graphs illustrate the danger of taking an apparent trend that appears in a relatively short period of time and extrapolating that trend over a great many years. Since we cannot know the initial conditions at time $t = 0$, there is no way of knowing for certain that what appears to be constant now has always been constant in the past. And without that knowledge, there is no way to be sure of the age of the earth.

The Origin and Progression of Life

Proteins are the basic building blocks of life. They consist of long strings of amino acids.



There are 20 kinds of amino acids that combine together in a specific order for each protein. Some proteins contain more than 20,000 amino acids, but the probability of even a small protein developing by random chance is so small it is hard to distinguish it from zero. For example, pancreatic ribonuclease is a small protein made up of a string of 127 amino acids. Could it have evolved by means of a sequence of random mutations in the DNA that controls the production of protein in the cell?

Almost all mutations are either neutral or harmful to an organism's ability to reproduce [4]. But let's be generous and suppose that 1 out of every 10 mutations to the DNA in a cell is beneficial in two ways:

- It moves the DNA one step closer to enabling the cell to produce pancreatic ribonuclease. That is, the protein synthesized by the DNA has one more amino acid in agreement with pancreatic ribonuclease.
- It alters the DNA in such a way that the cell is not destroyed by natural selection.

Since pancreatic ribonuclease requires the exact ordering of 127 amino acids, its production could be accomplished by a sequence of 127 beneficial mutations, each of probability $1/10$. This gives a combined probability of $(1/10)^{127} = 10^{-127}$ or 1 out of every 10^{127} trials. Let's see how long this might take, given a most favorable environment.

Example. Produce a Simple Protein by Mutations
Suppose that the 5.5×10^{15} square feet of the earth's surface is covered entirely by mutating cells with one billion (10^9) cells per square foot. And suppose that each cell undergoes a random mutation at the rate of one mutation every second, with $1/10$ of those mutations being beneficial (as described above). How long would we expect it to take for one of the cells to develop the ability to synthesize pancreatic ribonuclease?

We need on average to have 10^{127} trials to generate one cell capable of synthesizing pancreatic ribonuclease. Since it requires 127 mutations for each trial and mutations occur at one per second, it will take 127 seconds for a cell to go through its 127 mutations. Then, if it is unsuccessful, it can start over again with a new trial.[2] Thus, worldwide there are

$$(5.5 \times 10^{15})(10^9) = 5.5 \times 10^{24} \text{ trials every 127 seconds}$$

or, on average, $\frac{5.5\times10^{24}}{127} \approx 4.33 \times 10^{22}$ trials a second. Since we need 10^{127} trials, this will take

$$\frac{10^{127}}{4.33 \times 10^{22}} = \left(\frac{10}{4.33}\right)\left(\frac{10^{126}}{10^{22}}\right) \approx 2.31 \times 10^{104} \text{ seconds.}$$

Since there are 3.16×10^7 seconds in a year, we would expect it to take

$$\frac{2.31 \times 10^{104}}{3.16 \times 10^7} = \left(\frac{23.1}{3.16}\right)\left(\frac{10^{103}}{10^7}\right) \approx 7.31 \times 10^{96} \text{ years.}$$

This is trillions of trillions times the maximum age of the universe: 14×10^9 or 14 billion years.

Of course, pancreatic ribonuclease is of little benefit to the creature that has randomly generated the enzyme unless a fully functioning pancreas (which is far more complex than the enzyme) emerges simultaneously. We pursue this further in the next section which looks more closely at how the DNA code works.

The Genetic Code

Every human life begins as a single cell: a fertilized egg. This first cell divides into two, the two into four, and so on. Soon the cells begin to differentiate: some into heart muscle, some into arms and fingers, and some into eyes or ears. Over the next few months, hundreds of bodily systems are constructed in a precise way at a precise time. And all the information necessary to direct the construction and operation of these systems is somehow coded in the DNA of that first cell. How is that possible? How can so much information be compressed into such a small space?

Part of the answer to that question is found in the design of the DNA code itself. DNA consists of long twisted strands of four nucleotides abbreviated **A**, **T**, **C**, and **G**. The primary DNA code groups these nucleotides into triplets called codons. There are 64 ($4^3 = 64$) possible codons, and each one encodes for one of the 20 amino acids used in the synthesis of proteins. It turns out that most sequences of DNA are poly-functional. That

[2] To simplify the computations, we assume that whenever a cell is destroyed by natural selection, it is replaced by a new cell in the next set of trials so that the population remains constant at one billion cells per square foot.

is, they can encrypt multiple overlapping codes.[3] For example, the codon **CTG** encodes for the amino acid Leucine. But when read backwards (**GTC**) it encodes for Valine.

$$\begin{array}{ccc} & \longrightarrow & \text{Leucine} \\ & \textbf{CTG} & \\ \text{Valine} & \longleftarrow & \end{array}$$

If the **CTG** codon is followed by the **CCG** codon (Proline) and the first nucleotide is skipped, then the first codon would become **TGC** (Cysteine).

$$\cancel{\textbf{C}}\underline{\textbf{TGC}}\textbf{CG} \quad \longrightarrow \quad \text{Cysteine}$$

To illustrate the power and weakness of a poly-functional code, we look at an example of a mathematical code that has five levels of meaning. Suppose we take as our encrypted message (M) the first 16 digits in the decimal expansion of π:

$$M = \ 3\ 1\ 4\ 1\ 5\ 9\ 2\ 6\ 5\ 3\ 5\ 8\ 9\ 7\ 9\ 3$$

and we use Translation Table 1.

	0	1	2	3	4	5	6	7	8	9
0	B	□	S	□	S	T	□	□	□	A
10	K	M	S	D	O	W	O	I	I	R
20	N	□	L	T	D	V	I	E	H	E
30	R	□	A	S	X	□	□	L	I	C
40	V	S	F	□	F	□	Y	N	E	K
50	A	Q	O	V	E	E	N	G	E	L
60	M	I	W	□	□	Z	U	S	A	I
70	□	H	G	D	A	I	J	O	E	U
80	E	D	L	S	H	T	U	□	Y	G
90	R	□	P	□	C	O	S	R	N	E

Table 1

When we write the digits in M as pairs of numbers we get the following translation for level one:

(Level 1)	31	41	59	26	53	58	97	93
	□	S	L	I	V	E	R	□

If we reverse the digits in M and group them in pairs, we get the Level 2 message:

[3] Some of the additional codes are based on reading the codons backwards or starting the translation at a different nucleotide, as illustrated here with Leucine, Valine, and Cysteine. Recent research has found other codes that are not based on consecutive codons of nucleotides. (See [7] and [8].)

(Level 2)	39	79	85	35	62	95	14	13
	C	U	T	□	W	O	O	D

To get the Level 3 message, return to the original message M. Begin by dropping the first three digits and the last five digits:

$$1 \quad 5 \quad 9 \quad 2 \quad 6 \quad 5 \quad 3 \quad 5$$

Now replace each digit by a number pair that is the sum of the digits up to (and including) that digit. For example, the first pair is 1 (written as 01 so that there are two digits). The second pair is $1 + 5 = 06$. The third pair is $1 + 5 + 9 = 15$, etc.

(Level 3)	01	06	15	17	23	28	31	36
	□	□	W	I	T	H	□	□

The coding for Level 4 is more complicated. Start with the numbers from Level 3:

(Level 3)	01	06	15	17	23	28	31	36

Separate the digits from 0 to 9 into two groups: Those digits that have a pointed / flat top (1, 4, 5, and 7) and those digits that have a curved top (0, 2, 3, 6, 8, and 9). Then permute the digits within each group as follows:

$$1 \longrightarrow 7 \longrightarrow 4 \longrightarrow 5 \longrightarrow 1 \quad \text{and} \quad 0 \longrightarrow 8 \longrightarrow 2 \longrightarrow 9 \longrightarrow 6 \longrightarrow 3 \longrightarrow 0$$

This generates the message for Level 4.

	01	06	15	17	23	28	31	36
(Level 4)	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$
	87	83	71	74	90	92	07	03
	□	S	H	A	R	P	□	□

To obtain the fifth level, begin with the original message M. Drop the first two digits and the last six digits:

$$4 \quad 1 \quad 5 \quad 9 \quad 2 \quad 6 \quad 5 \quad 3$$

Now replace each digit by a number pair that is twice the sum of the digits up to (and including) that digit. For example, the first pair is $(2)(4) = 08$. The second pair is $(2)(4+1) = 10$. The third pair is $(2)(4+1+5) = 20$, etc. If the number is over 99, subtract 100 and write what is left.

(Level 5)	08	10	20	38	42	54	64	70
	□	K	N	I	F	E	□	□

The message is now complete. The first level says "sliver." And the next four levels describe how a sliver might be obtained: "cut wood with sharp knife."

Suppose that a "mutation" occurs in the original message M. For example, suppose the fourth digit is changed from a 1 to a 3. Call this message M'.

$$M' = \ 3\ 1\ 4\ 3\ 5\ 9\ 2\ 6\ 5\ 3\ 5\ 8\ 9\ 7\ 9\ 3$$

Now use the same translation table and the same instructions at each level to decode the mutated message. In Level 1, when we group the digits in pairs, the second pair changes from 41 to 43. This changes the "S" to a blank. And “sliver” changes into “liver.”

(Level 1)	31	43	59	26	53	58	97	93
	□	□	L	I	V	E	R	□

For Level 2, reverse the original M' digits and pair them:

(Level 2)	39	79	85	35	62	95	34	13
	C	U	T	□	W	O	X	D

For Level 3, drop the first 3 and the last 5 digits from M':

3 5 9 2 6 5 3 5

Then replace each digit by the sum of the digits up to that point:

(Level 2)	03	08	17	19	25	30	33	38
	□	□	I	R	V	R	S	I

For Level 4, permute the digits from Level 3:

$$1 \longrightarrow 7 \longrightarrow 4 \longrightarrow 5 \longrightarrow 1 \quad \text{and} \quad 2 \longrightarrow 9 \longrightarrow 6 \longrightarrow 3 \longrightarrow 0 \longrightarrow 8 \longrightarrow 2$$

	03	08	17	19	25	30	33	38
(Level 4)	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$
	80	82	74	76	91	08	00	02
	E	L	A	J	□	□	B	S

And finally, for Level 5, drop the first two and last six digits from M':

4 3 5 9 2 6 5 3

Then replace each digit by twice the sum of the digits up to that point:

(Level 5)	08	14	24	42	46	58	68	74
	□	O	D	F	Y	E	A	A

At the first level, the message has changed from “sliver” to “liver.” This potentially makes some sense and might be considered an increase in the level of information. A liver (the human organ) is certainly more complex than a sliver. But the message contained in the other four levels has become unintelligible:

CUT WOXD IRVRSIELAJ BS ODFYEAA

It certainly tells us nothing about how to make a liver.

While a poly-functional code is very powerful in that it can compress a lot of information into a small space, it is also very weak when it comes to allowing beneficial mutations. A potentially desirable change in one level will inevitably destroy information at the other levels. So the random creation of new organs and biological systems (evolution) cannot be accomplished by mutations and natural selection.

In fact, the mathematics suggests that instead of the biological world evolving upward, it is slowly devolving to a lower level. That is, the genetic clock is running backwards, not forwards. Indeed, this is what biological scientists are now finding [1, 2, 3, 5, 6]. For example, Dr. James Crow of the Genetics Laboratory at the University of Wisconsin writes,

> Since most mutations, if they have any effect at all, are harmful, the overall impact of the mutation process must be deleterious. [2, p. 8380]

And after careful analysis he concludes that "the decrease in viability from mutation accumulation is some 1 or 2% per generation."[4]

If evolution cannot even preserve the genetic information we currently possess, it certainly could not have "created" this information in the first place! There may be philosophical reasons for holding to the validity of the theory of evolution, but from a mathematical perspective, the evidence points strongly in the opposite direction.

Note: For information on obtaining a digital file of the complete set of lessons, please contact the author.

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[4] While Dr. Crow concludes that mutations are driving us backward, not forward, he is not a prophet of doom. He is optimistic that advances in technology (particularly in molecular biology) will enable future generations to overcome their increasing genetic weaknesses. (See [2, p. 8385].)

The Math Olympian, by Richard Hoshino. A Review

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Abstract

Richard Hoshino is a tutor at Quest University in B.C., and an ACMS member. He wrote the book *The Math Olympian* "in his spare time" and it was published in February 2015. We highlight some of the mathematics, character development, and existential themes of the book, noting also the author's interest in the liberal arts.

The Math Olympian is centered around Bethany McDonald, a Grade 12 student in Nova Scotia, and the five questions Bethany needs to solve in order to qualify to represent Canada at the International Math Olympiad. The plot uses flashbacks a la *Q&A* (a.k.a. *Slumdog Millionaire*) to draw out Bethany's story and allow her to see how she has grown into the young lady and mathlete she has now become. Each of the five multi-chapter vignettes also provides her with a key clue or motivation needed to approach the current problem.

In the course of the story's vignettes, Bethany, from Sydney, N.S., finds her best friend, Grace, who is from Vancouver, B.C. Bethany is being raised by a single mother, and has a father she's never met. Grace's mother and brother died in a car accident, leaving her father Ray, a pastor, to single-parent her. Grace and Bethany meet at a Math Camp and their friendship grows as they work towards their joint goal of making Canada's team for the International Math Olympiad.

The International Mathematical Olympiad (IMO) is the world championship of mathematical problem-solving for high-school students. Nearly one hundred countries are invited and can send their top six 'mathletes.' These students are scored according to their results on various mathematics contests, such as the setting of this book, the Canadian Mathematical Olympiad. Mathematical Olympiads consist of a series of very difficult questions. Many math professors may not be able to solve even one of these problems, since they require many hours of practice, which train and develop one's creative problem-solving skills and proof-writing techniques. Working through these math problems is a very large part of the novel; Hoshino does a good job of walking the reader through the thought processes that can go on in these problems.

In addition to the competitive nature of these mathematical contests, Bethany works both against and together with her peers, including her main high-school rival, Gillian. Throughout her journey, Bethany learns about doing mathematics as a team and writing the contests because they are fun, enjoyable, and challenging. This positive perspective helps her deal with her competition anxiety, which she finds she has inherited from her mother. Hoshino also presents the negative effects of competition through the lives of Gillian and her mother, whose own story of failure and redemption parallels Bethany's. Throughout the book, Bethany sees both of them struggle with the unhealthy pressure and fear of failure and learns to confront those issues in herself.

The author, Richard Hoshino, made a conscious choice to have the main characters, and many of the supporting characters, female. Another strong character, Bethany's high-school Math teacher, belongs to the Membertou First Nation tribe. From the Q&A section at the end of *The Math Olympian*, Hoshino notes,

> I want to challenge the common stereotype that mathematics can only be done by boys, nerds, and Asians (i.e., people like myself). I want *The Math Olympian* to reveal how, with inspired mentorship, anyone can succeed in mathematics and develop the confidence, creativity, and critical-thinking skills so essential in life. Through my involvement with math outreach programs at Dalhousie University, I met young women like Bethany all throughout Nova Scotia. It is my hope that Bethany's story will inspire high school students, girls especially, to participate in math contests and math outreach activities, take mathematics courses in college or university, and pursue a future career in mathematics to tackle the complex challenges of the 21st century.

The book includes at least one Christian character, Grace's father, Mr. Wong. Bethany learns much from him, both about working hard and believing in herself, but also about seeing a bigger picture, even in the context of grief and great disappointment. Discussions that allude to Jesus' parable of the pearl of great value and Pascal's wager also cover the idea of axioms and the nature of truth. Bethany also learns from Grace's struggle with faith and mathematics; She is first introduced to faith when she meets Grace, who believes that Bethany's life experiences are not a result of "lucky coincidences" but of God. Soon after, Grace's own faith is challenged by her math peers and mentors and by the presentation of the problem of evil. Together Bethany and Grace learn that mathematical development involves challenging traditional ways of thinking and questioning the current authorities, as seen in the discovery of non-Euclidean geometries. This puts Grace in conflict with her ?unscientific? faith, and this struggle is brought to the forefront through arguments or discussions between Grace, who has rejected her old faith, Grace's father, who holds strongly to his faith, and Bethany, who is undecided in terms of faith. Their discussions tease out the differences between the nature of faith and the nature of mathematics. Faith is presented as outside of science, outside of the realm of certainty. Comparisons are made between between axioms and faith (or lack of faith), and Bethany sees the inevitability of these foundational truths. "After a certain point, you need to stop doubting and decide what is truth!"

Hoshino's treatment of this issue was focused in a few chapters in the centre of the book, but it comes up many times throughout the book as Bethany asks harder questions about the usefulness of math and doing math. Is math the ticket to good decision-making? The book lets the questions remain to be puzzled over; we found ourselves engaged in the issue, but we don't feel like we were being given the answer or, even worse, being beaten over the head with the "right" answer. There is one instance where cliché may be used in a way that made us wince a bit, but overall, this topic is presented well.

Near the end of the book, Bethany's answer to the question, *What have you learned from studying mathematics?* does a good job of putting many of these things together in a coherent statement appropriate for a thoughtful high-school student. Her response is both humble and honest.

Hoshino uses his supporting characters to introduce a variety of mathematical philosophies on the nature and purpose of mathematics. Bethany's many mentors come from different philosophical backgrounds, ranging from an extreme utilitarian view ("it empowers you with the tools to influence social change") to a view almost in line with Hardy's claim that in all of his mathematics, he'd "never done anything useful." The tension in the different views shows up early: Bethany's early mathematics hero asks her, "Why does math need an application for it to be worthwhile?" while her first tutor ensures that every lesson and technique he shows her is then exemplified in a real-world situation. These different views are presented in positive lights, giving the reader a taste of various mathematical philosophies and letting them form their own opinions. In addition to the purpose of mathematics, the author also focuses on the beauty of mathematics. Bertrand Russel's view of a "cold and austere" beauty of mathematics is presented, along with the celebratory "sense of wonder" that one can feel while doing mathematics. Along with her struggle over the role of mathematics and the possibility of truths found outside rational thought, the question of how her love of math can be useful to her once her IMO dreams are finished is one that underlies much of the book.

The Math Olympian is book is primarily for high school students interested in math and those students' teachers. The professor reviewer believes that it succeeds at the teacher level and is already looking at ways to incorporate it (a novel!) into his introductory university-level calculus class. The book contains many problem-solving hints (e.g., break a problem into parts to make it simpler, look for patterns by considering simpler cases, sometimes making a problem harder makes it simpler, learn to ask the right questions, etc.). It gives strong rationale as to why we should learn how to write mathematics, especially proofs. As noted above, Hoshino's work also presents existential questions about God, faith, life, environmentalism and more in the context of mathematics. The plot is engaging and the explanations of some unorthodox mathematical approaches are done well overall. Richard Hoshino is a veteran of these math Olympics competitions, having brought home a silver medal for Canada at the IMO in 1996. His descriptions of life preparing for and writing these tests comes across as quite realistic. Our only issue is Bethany's incredible (at least to us) ability to write immaculate proofs under intense time constraints, although she is noted several times in the book as having extraordinary skills in this area.

A (female) high-school student who was also given the book to read commented that "the characters were great" and "the set up with the question and the answer spanning or bookending the chapters helped it to feel … suspenseful, because I wanted to know how she solved them." Overall, the book is "a really unique idea; not like anything I've ever read." On the other hand, she cautions that the "math got complicated; if you don't like math as much as I do, it would be super tedious."

Is this book worth reading and keeping on your bookshelf or in your classroom for your high-school or undergraduate students to read? We believe so. However, it is worth noting that this book is set entirely in Canada; it is unfortunate that it will likely be this "deficiency" that could make it less attractive to readers south of the 49th Parallel.

Finally, we'd also like to note the positive light given to undergraduate liberal arts institutions. Both reviewers received their undergraduate degrees at a liberal arts university and the professor now teaches at one. We agree with the assertion made in the novel that strong math students can rightfully choose a liberal arts and sciences university and not need to make apologies for that decision. At the end of the book, Bethany and Grace need to decide whether to attend the large comprehensive university, St. FX University, or Quest University, the small, private liberal arts university. Does Bethany end up at the liberal arts institution in the end? You'll have to read the book. . . .

Ten Mathematicians Who Recognized God's Hand in their Work

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The fear of the LORD is the beginning of wisdom, and knowledge of the Holy One is understanding. (Prov. 9:10)

Abstract

"Whoever is moved by *faith* to assent to [the Christian religion], is conscious of a continued miracle in his own person, which subverts all the principles of his understanding, and gives him a determination to believe what is most contrary to custom and experience." (Scottish philosopher David Hume (1711-1776))

Evidently Hume's cynical pronouncement did not apply to Euler, Cauchy, Cantor, and other profound thinkers who believed God had commissioned and equipped them to glorify Him in their pursuit of truth through mathematics - And based on their extraordinary achievements the principles of their understanding do not appear to have been subverted too badly! Leading mathematicians of past generations commonly affirmed that God created and sovereignly rules the universe and that He providentially sustains and nurtures His creatures. Despite Hume's assertion, history teaches us that faith often informed rational inquiry and vice versa. In many cases Christian commitment stimulated intellectual activity; sometimes mathematical understanding led to spiritual insight. In this paper, ten of history's most influential mathematicians express the role faith in God and religious conviction played in their work in their own words.

The essay for each mathematician includes an overview of his/her accomplishments, a brief description of his/her faith and convictions, and one or more quotations. I did not summarize or paraphrase the eloquent and moving statements of these gifted and godly practitioners of the art of mathematics. I proceed in chronological order.

1. Nicholas of Cusa (1401 – 1464)

Nicholas of Cusa (also Nikolaus Cusanus), a cardinal of the Roman Catholic Church in the Holy Roman Empire (Germany), was one of the greatest scholars and theologians of the fifteenth century. A true Renaissance man, Nicholas made significant contributions to law, philosophy, art, science, mathematics, astronomy, medicine, and theology, and served as an administrator, diplomat, scientist, and jurist as well as a cleric. Mathematically and scientifically, Nicholas was a catalyst for later discoveries in a number of key areas: calendar reform, calculus and set theory, and heliocentricity and the noncircular orbits of planets. Nicholas's writings reveal him to be an orthodox Christian immersed in praise to God for His transcendence and for His providential care, and to Jesus Christ His Son for His mediatorial and redemptive work. According to Nicholas, God transcends all understanding and is eternal and immutable [15, pp. 28,109], and that nothing will happen except according to God's providence [15, p. 118]. He sees in Christ the union of human and divine natures:

> In you, O Jesus, since you are human son, human filiation is most profoundly united to divine filiation, so that you are deservedly called son of God and human son. [15, p. 274]

> For the sins of humanity this sinless Jesus must suffer and die, and be raised from the dead on the third day (Selected Spiritual Writings, pp. 184, 188); God then saves the sinner through faith [in Christ] above all reason. [15, p. 197]

Nicholas's mathematical thought greatly impacted his understanding of God, as the following quote demonstrates.

> All our wisest and most divine doctors concur that visible things are truly images of invisible things and that from creatures the Creator can be seen in a recognizable way as if in a mirror or in an enigma. But the fact that spiritual things, unattainable by us in themselves, may be symbolically investigated rests on what we have already stated. For the way to the uncertain is possible only by means of what is presupposed. But all sensible things are in a continual instability because of the material possibility abounding in them. However, where such things are considered, we perceive that those things, such as mathematicals, which are more abstract than sensible, are very fixed and very certain to us, although they do not entirely lack material associations, without which no image of them could be formed, and they are not completely subject to fluctuating possibility. Proceeding in this way of the ancients, we agree with them in saying that since our only approach to divine things is through symbols, we can appropriately use mathematical signs because of their incorruptible certitude. [15, pp.100-102]

2. Johannes Kepler (1571 – 1630)

One of the truly great early modern scientists was the German astronomer and mathematician Johannes Kepler. His laws of planetary motion were foundational to Newton's law of universal gravitation and revolutionized the field of astronomy; his method of smallest divisions paved the way for the discovery of infinitesimal calculus [12, pp. 146,383]; his genius for geometry led to his groundbreaking work in optics; and his enthusiastic support of Copernicanism led to its eventual acceptance by the scientific community and the world. In addition, Kenneth Howell claims it was Kepler who first searched systematically for physical causes of celestial phenomena and whose mathematical application achieved a degree of accuracy previously unknown [22, p.109]. Kepler regarded himself as a lifelong Lutheran, but he was unable to subscribe to his church's official confession. He took exception to the beliefs that Christ's body was omnipresent and that His body and blood were present with and permeated the bread and wine during the Lord's Supper, a viewpoint called consubstantiation. He favored the Calvinist position that Christ was spiritually but not bodily present with the elements during the sacrament [23, pp. 369-382]. As a result of holding this conviction he would suffer ongoing persecution from his church. Kepler unapologetically expressed his praise of the Creator and His wonders as revealed in His two books, Scripture and Nature, particularly the heavens, in several of his scientific works and in his personal letters. The following quote illustrates his belief that mathematical reasoning illuminates our understanding of God:

> For He Himself has let man take part in the knowledge of these things and thus not in a small measure has set up His image in man. Since He recognized as very good this image which He made, He will so much more readily recognize our efforts with the light of this image also to push into the light of knowledge the utilization of the numbers, weights, and sizes which He marked out at creation. For these secrets are not of the kind whose research should be forbidden; rather

> they are set before our eyes like a mirror so that by examining them we observe to some extent the goodness and wisdom of the Creator. [12, p. 381]

Kepler reveals in the statement below that his ultimate desire was to glorify God in his work:

> I had the intention of becoming a theologian. For a long time I was restless: But now see how God is, by my endeavors, also glorified in astronomy. [8, p. 31]

3. Blaise Pascal (1623 – 1662)

One of the most profound thinkers of the 17th century was the French mathematician, physicist, philosopher, and theologian Blaise Pascal. He (along with Pierre de Fermat) is credited with developing modern probability theory; he devised Pascal's triangle as an efficient means of obtaining the binomial coefficients; in exploring the geometry of cycloids Pascal employed methodology that anticipated Newton and Leibniz in their development of calculus; he made important discoveries regarding fluids (e.g., Pascal's Law), pressures, and vacuums; and he invented the hydraulic press, the syringe, and a primitive calculating machine. Transformed by a dramatic religious experience in 1654, Pascal devoted the rest of his life to studying and writing theology and philosophy, producing such influential works as The Provincial Letters and Pensees (Thoughts). (See [6].)

To Pascal Christianity was both rational and spiritual. He observed that

> The Christian religion has as many signs of certainty and of evidence as the things which are received in this world as the most indubitable. [10, p. 326]

Yet reason alone was insufficient for a true understanding of God:

> It is the heart which perceives God and not the reason. That is what faith is: God perceived by the heart, not by the reason. (Pensées, 424 - [31])

He belonged to a sect of Roman Catholicism known as Jansenism, which emphasized man's depravity, God's sovereignty in salvation and sanctification, and the importance of pious living. One must understand the truths of the Gospel, i.e., Christi's divinity and humanity, His death on behalf of sinners, His resurrection, and His offer of salvation to all who believe upon Him. But understanding must be accompanied by heartfelt belief, which is possible only by God's initiative. And this faith must be characterized by action—a life of prayer, self-sacrifice, and service to the poor and needy [31].

Though Pascal would vehemently deny that mathematical reasoning produces faith, he contends by his celebrated Wager that having faith is reasonable:

> Belief is a wise wager. Granted that faith cannot be proved, what harm will come to you if you gamble on its truth and it proves false? If you gain, you gain all; if you lose, you lose nothing. Wager, then, without hesitation, that He exists. (Pensées, 233 - [31])

4. Gottfried Wilhelm Leibniz (1646 – 1716)

The German mathematician and philosopher Gottfried Wilhelm Leibniz substantively enhanced so many fields of knowledge that he has been dubbed the last universal genius. Along with Sir Isaac Newton, he is responsible for developing the infinitesimal calculus, and it is his notation that has survived to the present day. The binary system, invaluable to modern computer science, owes its origins to Leibniz. The fields of physics, biology, medicine, linguistics, law, politics, and theology have also greatly benefited from his expertise. His contributions to philosophy are considerable - he was one of the seventeenth century's leading proponents of rationalism, and his works laid the foundation for important branches of modern philosophy. Optimism, the view that God created the best of all possible universes, and monadology, the belief that elementary and indivisible entities (monads) comprise the spiritual realm, are attributed to Leibniz [27].

Leibniz emphasized God's supremacy in the universe, especially as the only perfectly rational Being. God is all-wise, all-powerful, and benevolent; He employed divine mathematics to maximize goodness in creating and sustaining the best of all possible worlds: All in all that method of creating a world is chosen which involves more reality or perfection, and God acts like the greatest geometer, who prefers the best construction of problems [32, p. 26]. He endowed man, whom He created in His image, with sufficient reason to apprehend Him and His works. Though no substitute for faith, reason must serve as a foundation to faith, lest it be superstition [25, pp. 487-496].

The following quotes demonstrate that Leibniz's mathematics informed his understanding of God and vice versa:

> The sovereign wisdom, the source of all things, acts as a perfect geometrician, observing a harmony to which nothing can be added. True physics should in fact be derived from the source of the divine perfections. It is God who is the ultimate reason of things, and the knowledge of God is no less the beginning of science than his essence and his will are the beginning of beings. It sanctifies philosophy to make its streams arise from the fount of God's attributes. Far from excluding final causes and the consideration of a being who acts with wisdom, it is from these that everything must be derived in physics. [13, p. 157]

> God is all order; he always keeps truth of proportions, he makes universal harmony; all beauty is an effusion of his rays. It follows manifestly that true piety and even true felicity consist in the love of God, but a love so enlightened that its fervor is attended by insight. [26, p. 51]

> Since therefore it is by the nature of things that God exists, that he is all-powerful, and that he has perfect knowledge of all things, it is also by the nature of things that matter, the triangle, man and certain actions of man, etc., have such and such properties essentially. God saw from all eternity and in all necessity the essential relations of numbers, and the identity of the subject and predicate in the propositions that contain the essence of each thing. [26, p 242]

5. Colin Maclaurin (1698 – 1746)

The greatest Scottish mathematician of the eighteenth century, Colin Maclaurin defended and developed Newton's method of fluxions (calculus) and made important discoveries in algebra, geometry, actuarial science, and mechanics, particularly the gravitational attraction of ellipsoids. His Treatise of Fluxions brought organization and rigor to Newton's calculus, especially regarding the fundamental theorem of calculus, integration theory, optimization, and summation of infinite series. He (along with Leonhard Euler) discovered the Euler-Maclaurin formula, an important relationship between integrals and sums, and the Integral Test for convergence of a series. Though not the first to approximate functions by power series, he worked so extensively in this area that the Taylor series centered at zero bears his name. His pioneering work in actuarial science benefited the widows and orphans of Scottish professors and pastors. (See [2].)

A Presbyterian raised in a clergyman's home, Maclaurin believed God to be the Author and Governor of the universe, the first and supreme Cause, and the Lord and Disposer of all things ([2, pp. 3, 382], [35, p. 7]) The perfection and multiplicity of design and the fulfillment of messianic prophecies provided him with compelling evidence of the existence and attributes of God:

> The plain argument for the existence of the Deity, obvious to all and carrying irresistible conviction with it, is from the evident contrivance and fitness of all things for one another, which we meet with throughout all parts of the universe. . . The admirable and beautiful structure of things for final causes, exalt our idea of the Contriver: the unity of design shows him to be One. The great motions in the system, performed with the same facility as the least, suggest his Almighty Power, which gave motion to the earth and the celestial bodies, with equal ease as to the minutest particles. The subtility of the motions and actions in the internal parts of bodies, shows that his influence penetrates the inmost recesses of things, and that He is equally active and present everywhere. The simplicity of the laws that prevail in the world, the excellent disposition of things, in order to obtain the best ends, and the beauty which adorns the works of nature, far superior to anything in art, suggest his consummate wisdom. The usefulness of the whole scheme, so well contrived for the intelligent beings that enjoy it, with the internal disposition and moral structure of those beings themselves, show his unbounded goodness. [2, p. 381]

He viewed the prophecies of Daniel 9 (seventy weeks representing 490 years) to be one of the best proofs of Christianity [30, p. 19]. To Maclaurin, nature is God's handiwork and studying His works (via science and mathematics) is tantamount to studying Him:

> But natural philosophy is subservient to purposes of a higher kind, and is chiefly to be valued as it lays a foundation for natural religion and moral philosophy; by leading us, in a satisfactory manner, to the knowledge of the Author and Governor of the universe. To study nature is to search into his workmanship; every new discovery opens to us a part of his scheme. And while we still meet, in our enquiries, with hints of greater things yet undiscovered, the mind is kept in a pleasing expectation of making a further progress; acquiring at the same time higher conceptions of that great Being, whose works are so various and hard to be comprehended. [2, p. 3]

6. Johann Bernoulli (1667 – 1748)

Johann Bernoulli, along with his older brother Jacob, disclosed to the world the utility of the infinitesimal calculus by bringing it to bear upon a variety of problems in the sciences and applied mathematics. His ability to integrate functions by antidifferentiating them enabled him to be the first to solve certain differential equations and sum infinite series. An optimization problem he proposed in 1696 (that he and others eventually solved) founded the branch of mathematics known as the calculus of variations. He also made valuable contributions to the science of mechanics, particularly, hydrodynamics [3].

Following in the footsteps of previous generations of Bernoullis, Johann was an orthodox Calvinist. His grandparents had been Huguenots (members of the Protestant Reformed church of France) and had migrated from Antwerp, Belgium, to Basel, Switzerland, to avoid persecution by Roman Catholics. Later on in life, in response to an unjust accusation of religious heresy, he passionately defended his faith: All my life I have professed my Reformed Christian belief, which I still do [33, pp. 27-28]. It is clear that Bernoulli's pursuit of mathematics appreciably informed his understanding of God and His attributes, and his desire to glorify Him:

> Nowhere is God's power and wisdom more evident than in the study of his works, and none is better equipped for this study than the philosopher and mathematician, who tries to fathom both

> the nature and character of God's works. They are much to be ridiculed who scoff at philosophy and mathematics pretending the latter are of no advantage in matters of the greatest importance. [33, pp. 28-29]

> God the almighty grant that all this turns only to the glory of his name. [3, p. 93]

7. Leonhard Euler (1707 – 1783)

The breadth and depth and impact of Leonhard Euler's works establish him as one of the truly great mathematicians of all time-mathematics historian Morris Kline ranks him at the highest level with Archimedes, Newton, and Gauss [24, p. 401]. His output was colossal – he was far and away the most prolific writer in the history of (mathematics) [21, p. 433]. Euler's genius brought new and powerful results to nearly every branch of mathematics-analysis (calculus, differential equations, calculus of variations), algebra, classical and analytic number theory, complex variables, Euclidean and differential geometry, topology, graph theory, and combinatorics. Especially noteworthy has been his enrichment of analysis: There are few great ideas pursued by succeeding analysts which were not suggested by Euler, or of which he did not share the honor of invention [11, p. 247].

Other fields have benefited from the Swiss mathematician's superlative industry and insights, most notably physics but also engineering, navigation, and business. The discoveries and advances he made in several branches of physics, including mechanics, astronomy, electricity and magnetism, light and color, hydraulics, optics, acoustics, and elasticity, were considerable-Clifford Truesdell regarded him as the dominating theoretical physicist of the eighteenth century [34, p. 106]. Euler proclaimed the God of the Bible to be omnipotent, designating Him the Almighty and the Divine Omnipotence [20, p. 5]; omniscient, speaking of the infinite wisdom of the Creator and His Most consummate wisdom [20, pp. 395, 401]; and omnipresent, declaring that His power extends to the whole universe and to all the bodies which it contains. . . God is everywhere present [20, p. 409]. Euler declared that the world is the work of his infinite might and wisdom [19, II] and that everything has been created in the highest perfection [20, p. 390]. Euler's scientific expertise deepened his reverence for God as Creator, Sustainer, and Ruler of the universe. He asserted that the immensity (of space and the heavenly bodies) is the work of the Almighty, who governs the greatest bodies and the smallest [20, p. 5]. In stark contrast to the deists, he argued that the universe is no mere machine but is

> infinitely more worthy of the almighty Creator, who formed it. The government of this universe will, likewise, ever inspire us with the most sublime idea of the sovereign wisdom and goodness of God. [20, p. 382]

Euler was particularly intrigued with vision and the structure of the eye:

> the eye alone being a masterpiece that far transcends the human understanding, what an exalted idea must we form of Him who has bestowed this wonderful gift, and that in the highest perfection, not on man only, but on the brute creation, nay, on the vilest of insects! [T]hough we are very far short of a perfect knowledge of the subject, the little we do know of it is more than sufficient to convince us of the power and wisdom of the Creator.
>
> We discover in the structure of the eye perfections which the most exalted genius could never have imagined. [20, pp. 198, 187]

As a Calvinist, Euler insisted that God sovereignly foreordains all events without violating man's free will; that man is fallen in nature and is utterly incapable of saving himself; and that God is the architect of man's salvation in Christ:

> It is therefore a settled truth that Christ is risen from the dead: since this is such a marvel, which could only be performed by God alone, it makes it impossible to cast any doubt on the divine sending of Christ into this world. Consequently, the doctrine of Christ and of his apostles is divine and since it is directed toward our true happiness, we can therefore believe with the strongest confidence all the promises which have been made in the gospel regarding this life as well as the one to come, and view the Christian religion as a divine work aiming at our spirituality. But it is not necessary to elaborate further on all this, since each one who is convinced only once of the resurrection of Christ cannot doubt any further the divinity of Holy Scripture. [19]

Euler further argued that

> The holy life of the apostles, and of the other primitive Christians, appears to me an irresistible proof of the truth of the Christian religion. [20, p. 507]

8. Maria Agnesi (1718 – 1799)

Though chiefly remembered today for a curve that bears her name, Italian mathematician, philosopher, and linguist Maria Agnesi initially achieved fame for authoring Analytical Institutions, the first comprehensive calculus textbook. Praised for its organization and clarity, this work synthesized results from a number of mathematicians, notably Newton and Leibniz. Born into a wealthy Milanese family, Maria was a child prodigy who learned seven languages by the age of 13. Being the eldest of 21 children (her father was married three times), she was expected to teach her younger siblings - the mathematics textbook mentioned above was originally intended for their instruction. In a commentary she wrote on l'Hôpital's calculus text, she discussed the curve which is now named the witch of Maria Agnesi, the word witch resulting from a mistranslation of an Italian word meaning to turn as in to turn a sail [1]. Agnesi's life was characterized by religious zeal, submission to authority, and self sacrifice. An orthodox Catholic, she earnestly sought to emulate Christ in His piety and sufferings, and to deny the flesh and earthly pleasures [28, p. 673]. In obedience to her father, she mastered abstract topics in philosophy and the sciences and regularly conversed in them with his erudite house guests. Upon his death she abandoned these activities in favor of caring for the underprivileged women of Milan. She devoted the rest of her life and means to serving the poor, the infirm, the orphans, and the elderly, eventually dying in poverty [29, pp. 145-147], [1]. Maria's scholarly endeavors were intended solely to glorify God, and clearly they enriched her understanding of Him and His attributes.

> Man always acts to achieve goals; the goal of the Christian is the glory of God. I hope my studies have brought glory to God, as they were useful to others, and derived from obedience, because that was my father's will. Now I have found better ways and means to serve God, and to be useful to others. [29, p. 145]

> Holiness does not consist in doing great and admirable works, but in doing every thing, however small, with sublime intentions; intentions of love, and abundance of sanctifying grace. [14, p. 217]

> In truth, the principle on which you [Bertucci] founded your doctrine, that is the law of uniformity that God proposed as his way to operate, is an extremely just principle, and this uniformity in God's way to operate can be seen more and more as one makes progress in the knowledge of natural things. And one should not be afraid that this uniformity stands in opposition to the variety of things, which form the beauty of the universe, and instead the variety is more beautiful and admirable in the midst of this uniformity. [14, p. 192]

9. Augustin-Louis Cauchy (1789 – 1857)

One of the most prolific writers in the history of mathematics (second only to Leonhard Euler in the sheer volume of his works), the French mathematician Augustin-Louis Cauchy made major contributions to complex function theory, analysis, algebra, number theory, and mathematical physics. One of the pioneers of complex analysis, he developed Cauchy's integral theorem, Cauchy's integral formula, the concept of residues, the residue theorem, and the Cauchy-Riemann equations. By introducing rigorous standards in calculus regarding limits, continuity, and tests for convergence of series, he was one of the founders of modern analysis. As an algebraist, he initiated the study of permutation groups, and as a number theorist, he proved Fermat's polygonal number theorem. In physics, he made important discoveries in the wave theory of light and the mathematical theory of elasticity, and his argument principle has benefited control theory in engineering. He also generalized the theory of determinants [5]. An orthodox Roman Catholic, Cauchy passionately defended and practiced the Christian faith throughout his lifetime. In a pamphlet he published he declared:

> I am a Christian, that is, I believe in the Divinity of Jesus Christ, with Tycho Brahe, Copernicus, Descartes, Newton, Fermat, Leibnitz, Pascal, Grimaldi, Euler, Grudin, Boscowich, Gerdil; with all the great astronomers, physicians, geometricians of past ages... My convictions are not the 'result of inherited prejudices', but of profound examination... It gave me great pleasure to find all the nobility and generosity of the Christian faith in my illustrious friends. [7]

Closely aligned with the Jesuits, he strictly observed their practice of charitable deeds and service to the poor. He and other members of the Institut Catholique, established to promote Catholic higher education in France, regularly prayed that God would bless their pursuit of truth and that He would be glorified in their work [9, pp. viii, 180].

Cauchy believed that his faith inspired his mathematical/scientific achievements, and conversely, that his work enhanced his spiritual illumination:

> The Christian religion is so highly favorable to the advancement of the sciences and to the development of the most noble faculties of our intelligence. [9, p. 216]

> The great crime of the last century was that of wanting to raise nature itself up against its very Author, of desiring to set creatures in a state of permanent revolt against the Creator and even to arm the sciences against God Himself, the sciences whose only real aim must be the search for truth... In many instances the science of numbers and analytical methods can help us to discover the truth or, at the very least, how to recognize it. [9, pp. 218-219]

10. Georg Cantor (1845 – 1918)

Brilliant in his ability to abstract, German mathematician Georg Cantor revolutionized mathematics by developing modern set theory. Formerly primitive and limited in its utility, set theory now provides a generalized framework in which components of a subject are unified and clarified, and in which powerful problem-solving machinery can be brought to bear. This transformation was a result of Cantor's investigations of well-ordered sets, one-to-one correspondences, countable and uncountable sets, transfinite numbers (representing degrees of infinity), cardinal and ordinal numbers and their arithmetic, and the continuum hypothesis. Results of fundamental importance he was able to prove include Cantor's theorem (the cardinality of the power set of a set is greater than that of the set); the countability of the rational numbers (using his famous diagonal argument) and the algebraic numbers; the uncountability of the real numbers and the transcendental numbers; and the existence of Cantor's set, which is uncountable but has measure zero [4].

As an orthodox Lutheran, Cantor viewed God as all wise, all powerful, infinite, and perfect in all His ways. He believed God had blessed him with profound insights concerning infinity and had commissioned him as a prophet to proclaim this message to others. Asserting the existence of the actual infinite brought him fierce opposition from both mathematicians and theologians, the latter group decrying his apparent violation of God's position as the unique and supreme infinity of the universe [17, pp. 120-124, 143-145, 229, 238-239]. Cantor pointed out that God's status as the Absolute Infinity enhanced rather than diminished His infinitude and the extent of His nature and dominion, and showed more effectively His perfections [18, pp. 45-46].

The following quotes express Cantor's fervent hope that his work would glorify God and edify His church:

> It would please me best if my work would be of benefit to the Christian philosophers dearest to my heart, to the philosophia perennis' [lasting philosophy]. [25, p. 534]

> Every extension of our insight into the origin of the creatively-possible therefore must lead to an extension of our knowledge of God. [25, p. 535]

> I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. [17, p. 124]

> From me, Christian philosophy will be offered for the first time the true theory of the infinite. [16, p. 107]

Conclusion

Despite the contemporary view that God and religious faith play no part in serious intellectual inquiry, history has repeatedly demonstrated that quite the opposite is true. As the quotes in this paper illustrate, many leading mathematicians proclaimed the greatness of God and His works, and readily acknowledged their utter dependence upon Him for initiative and insight. Rather than substantiating David Hume's claim, discovery and advancement in mathematics and the sciences produced in the subjects of this paper a greater appreciation for an all-wise, all-powerful, and benevolent Creator, who has enlightened the human mind and spirit so that He may more fully be known and worshiped. May today's Christian mathematicians and laypersons alike not follow the world's lead in applauding man's accomplishments apart from God, but instead learn from our illustrious forebears that He alone enables our endeavors to succeed and He alone is worthy of praise.

> Unless the LORD builds the house, they labor in vain who build it. (Ps. 127:1)

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Mystery of the Infinite: Developing a Mathematically Based Summer Scholars Program

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Abstract

Since the summer of 2013, the Templeton Honors College, at Eastern University, has been conducting a Summer Scholars Program for high school students. In summer of 2014, two mathematicians were invited to design a mathematics based course for the Summer Scholars Program [5]. The resulting course included a rigorous study of discrete mathematics, as well as a variety of lectures and extracurricular activities integrating faith and philosophy with mathematics. This paper will give an overview of the program and the new course, highlighting successes as well as noting some of the challenges.

Introduction

Imagine teaching a college level Discrete Mathematics course in nine days. As if that were not challenging enough, imagine teaching the course to high school students, while also incorporating robust discussions of how faith and mathematics intersect. As crazy as it might sound, I helped design and implement a program doing just that in summer of 2014. In these pages, we will look at this program, beginning with an overview of the broader Summer Scholars Program at the Templeton Honors College. From there we will consider specifics of both the mathematics and the faith integration. I will share with you the outcomes, to include some of the student feedback. I will also highlight some of the challenges of designing and running such a program. Finally, we will look at how the program is continuing to develop. It is my hope that this material will be a useful resource to anyone interested in developing a similar program, and I invite you to share any ideas or wisdom that you might have.

Overview of the Program

Founded in 1999, the Templeton Honors College (THC) provides a home for undergraduate students at Eastern University who are interested in rigorous education that acknowledges the supremacy of Christ in all things. The college is named for Drs. John and Josephine Templeton, who have supported the college and its programs generously. The THC uses a great books curriculum, employs exemplary teachers who invite students into scholarship, and operates on a small cohort model.

In the summer of 2013, THC faculty and staff launched a new program, this time aimed at high school students. In its first summer, the Summer Scholars Program welcomed 28 high school students, from rising juniors to recent graduates, for a nine-day residential program hosted on Eastern's campus. The course was titled "The Examined Life: Knowledge, Wisdom, Virtue, Calling", and the program was designed to provide an authentic Templeton Honors College experience. Students read Plato's *Theaetetus*, Euripides *Bacchae* [4], and excerpts from Cicero's *Orator* [2], discussing the texts with THC faculty. Students were joined and assisted by current THC students, who served as TA's throughout the program. Successful students were able to earn three college credits, corresponding to a course offered at Eastern through the THC.

Throughout the course, students also had a chance to visit Independence Hall (Eastern is located in a Philadelphia suburb), to see a stage adaptation of C.S. Lewis' *The Great Divorce*, and to hear lectures from several other faculty speakers. Each day of the program began and ended with devotional time, and students were encouraged in everything they did to consider how the subjects they were studying related to their faith.

Students were enthusiastic about the program, with many students expressing interest in returning the following summer. As such, THC faculty and staff quickly began planning a new course that could be offered, setting on the theme "Citizenship: on Earth as it is in Heaven?" At the same time, they began to consider whether it would be possible to offer a second course, perhaps even a mathematics course.

Mystery of the Infinite - Specifics of the Course

As it turns out, one member of the THC faculty, Dr. Walter Huddell, is a mathematician. Dr. Huddell is also the chair of the Mathematics Department within Eastern's College of Arts and Sciences. When approached by THC leadership about developing a course for the Summer Scholars Program (SSP), Dr. Huddell was enthusiastic, and he soon invited me to join him in the work. We were given the task of adapting an existing mathematics course to the SSP format, and we settled on our Discrete Mathematics course.

At Eastern, Discrete Mathematics serves as a bridge course, giving students an introduction to proof writing and higher level mathematics. This choice, therefore, avoids topics that many students take in high school, while also avoiding Calculus as a prerequisite. We also recognized that the course offers a number of excellent opportunities for connecting the mathematics to philosophy and theology. As we explained some of the mathematics to the THC faculty and staff, they were particularly interested in the topic of cardinality, and we ultimately settled on "Mystery of the Infinite" as a title for the course (it was suggested that this would be better for advertising than "Discrete Mathematics").

Now, trying to fit a full semester of Discrete Mathematics into a 9 day program is not an easy task. What follows is a list of topics that we generally cover:

- Introductory Logic - Statements, Truth Tables, Logical Equivalences
- Techniques of Proof - Direct, Contradiction, Biconditional, Cases
- Set Theory - Basic Notation, Subsets, Set Operations, Proving Set Properties and Laws
- Relations - Definitions, Examples, Properties of Relations
- Functions - Definitions, Surjections, Injections, Bijections
- Cardinality - Countable and Uncountable Sets
- Equivalence Relations - Definitions, Quotient Sets, Modular Arithmetic
- Mathematical Induction

To even attempt to cover this amount of material, we recognized that the students would have to get started before they even arrived at Eastern. There was some precedent for this, as students in the 2013 SSP were expected to complete some reading and writing prior to the residential program. As such, we sent students a significant amount of material covering introductory logic and the basics of set theory. They then had about a month to read through the material and to complete a exercises based on the readings.

Once students arrived, we spent the first day of the program getting to know them and reviewing the preprogram material, answering any outstanding questions and trying to ensure that they had a good foundation for rest of the course. From that point onward, we began to work through the rest of the material, generally giving three hour-long lectures each day. Each lecture was immediately followed by a problem session, recognizing that you can only truly learn mathematics by doing mathematics. Students also had evening problem sessions to continue processing the material they were learning.

During the course, students were expected to complete and submit homework on each topic that we covered, and we had TA's (again drawn from current Eastern students) grade the homework. We also gave two exams during the program, essentially corresponding to a midterm and a final. Finally, the students were able to visit the Museum of Mathematics in New York City (a welcome change of pace to the intensive lectures and problem sessions, and perhaps a bit more fitting for a mathematics course than a visit to Independence Hall).

Going Beyond the Mathematics

As with the other SSP courses, we ultimately wanted to help students deeply connect their faith to the subjects they were studying. Therefore, we built a number of other components into the program. In addition to some discussion of integration during lectures, each day began and ended with devotional time. Further, most evenings included a lecture or special activity that dealt explicitly with integrating faith into mathematics (or academics as a whole).

One of the highlights of the evening activities was a visit to Eastern's observatory and planetarium, where our resident cosmologist, Dr. David Bradstreet, discussed the integration of faith and science. On another night, students watched *The Proof* [3], which details Andrew Wiles journey in solving Fermat's Last Theorem; following the movie, students discussed themes with an Eastern alumnus. On other nights we invited theologians and philosophers to discuss topics such as mystery in mathematics and faith, and Gödel's Incompleteness Theorems. Finally, to break things up a little bit, we spent one evening with students playing a variety of board games (especially games emphasizing logic and strategy).

In addition to scheduled events during the residential part of the program, we asked students to do some reading and writing at the intersection of mathematics, philosophy, and theology. At the end of the residential program, we handed students copies of Flatland and a couple of chapters from *Mathematics Through the Eyes of Faith* [1], along with prompts for essays. These readings and writings served as a sort of capstone to the program, giving students the chance to process and respond to some new material in light of all that they had learned during the residential portion of the program.

Outcomes and Feedback

All told, 19 students attended the Mystery of the Infinite course, and everyone survived. Of those students, 14 passed the course with a C or better (with more A's than any other grade). Of those who did not achieve a C, most were students who struggled from the beginning of the course with the speed and with the transition to abstract material. We ultimately counseled some of these students to forget about the grade and focus on getting as much out of the program as was reasonable for them (counsel that was uniformly well received).

Perhaps a better indicator of the success of the program, however, is the feedback left by students. The most common remark in the feedback was that students would have liked more time to process the material; this was not too shocking, as it was clear by the end of the residential part of the program that even the best students were exhausted. However, each student was also very positive about the program overall. For example, see the comments below:

- "The workload and pacing were enough to push everyone to their limits, and was done well."

- "Before the program I was a little unsure if I was going to enjoy it, but after I feel like I don't want to even leave, I had such a fantastic time."
- "Before I was worried about a systematic learning experience, however it proved to be very personal and hands-on."
- "The learning atmosphere was amazing!"
- "We learned about math and how God was related. It was more fun than I expected."
- "I thought it would be more rigid. It was not, I felt very supported and encouraged when I was confused."
- "This was a lot more interesting than what I was used to."
- "It was an amazing experience being surrounded by other Christians and getting to learn more about theology and math related to Christianity."

A further measure of the success of the Summer Scholars Program, as a whole, is its proficiency at recruiting students. Of the approximately 60 students who have participated during the first two years of the program, 14 have ultimately enrolled at Eastern (keep in mind that many of the students who have participated are only now entering their senior year of high school, so the proportion is likely to increase further).

Challenges

To any who are interested in developing a similar program, we can now confidently say that it is not a small undertaking. Indeed, we encountered numerous challenges along the way (some of which are still being worked out); these challenges can be broadly classified as length of program, cost, a text, and personnel.

As noted above, the length of the program is a feature which significantly affects the student's experience. Several factors went into settling on a 9 day residential program. The first of these is cost, which must be kept low enough to attract students. We were able to obtain some outside funding for scholarships, but students still paid approximately $1,800 for the program; as this included all expenses (including the trip to the Museum of Mathematics) and most students earned 3 college credits, this cost is not unreasonable, but it does still limit the pool of applicants. To run the program for a longer period would have necessitated raising costs further.

On the other end of things, the length of the program was shaped by the amount of material to be covered. Even with nine days, we were hurried to complete the course. We also had to be very careful to plan the number of contact hours with students so as to ensure that state mandated minimums were being satisfied.

Moving on, we quickly realized that we would need to provide students with a text. Again, cost constrained us here, and we ultimately decided to write our own text. In many ways, this turned out to be a blessing, as we were able to integrate a number of faith perspectives and philosophical asides holistically into the text. At present, we have 129 pages of TeX-ed material, complete with a large number of exercises, and we plan to continue refining the material over the coming years (I plan to use it for my Discrete Math class in the spring of 2016). Ultimately, we hope that such a text might find a market within faith-based schools.

With regard to personnel, we were blessed with some fantastic administrative support and leadership. They took care of many of the logistical details (advertising, enrollment, dorms, meals, transportation, etc), and helped the students to feel at home within the first moments of their arrival at Eastern. We were also able to recruit some fantastic teaching assistants from the ranks of Eastern Math majors (and even a recent alum). The TA's spent nearly every waking hour with students, while also managing to prepare for problem sessions and grade homework. Their importance to the program cannot be overstated.

Finally, and still on the topic of personnel, the program also requires a large commitment from the faculty. My colleague and I put untold hours into planning the course and (especially) writing the text. During the residential portion of the program we put in full days with the students, sometimes staying late into the night. Even after the completion of the program, we had to grade exams and essays. Despite this work, monetary compensation was minimal (there is some hope that pay will grow as more students are attracted to the program, and that workloads will decrease as the program grows more established). All that said, it was a fantastic experience to work with these young students, giving them a glimpse into how rich mathematics can be.

Future Development and Conclusion

With the overall success of the Mystery of the Infinite course last summer, we are planning to continue offering the course on a biannual basis (a number of students participate in the SSP for two summers in a row, so we want to keep the offerings fresh). That said, we've learned quite a bit even in the first incarnation, and we intend to continue developing the course. In particular, we see significant room for improvement in the intensity of the program, in our staffing, and in the text.

As noted in the previous section, most of our students found the program to be very intense and would have liked more time. One possible change, therefore, would be to run the program for more days, provided that we can keep the cost low enough to continue attracting students. Alternately, we may adjust the syllabus; in particular, the integrity of the course could probably be maintained without covering equivalence relations (from my conversations with faculty at other institutions, this topic is often omitted from a Discrete Math course). More radically, we could also choose to drop the explicit connection to our Discrete Mathematics course, instead offering students general credit. This would have the disadvantage of likely not transferring to schools other than Eastern, but it would let us relax expectations quite a bit and focus on making the program interesting and enjoyable. Apparently, this approach was taken in a similar program at Taylor University, and the following years actually saw a (somewhat counterintuitive) boost in student interest and attendance.

With regard to staffing, Dr. Huddell and I are cognizant that we are two white males, and we don't want to send false signals about who is qualified college level mathematics. As such, we are hoping to recruit another professor to add some diversity to the program faculty. We also plan to continue investing time and energy in the textbook. With more time, we expect to expand up on the faith integration, to refine and add to the exercises, to provide solutions to at least some of the exercises, and, possibly, to add more content.

Moving beyond the particular Mystery of the Infinite course, we see an opportunity to develop more mathematics courses for the Summer Scholars Program. For example, a course on the basics of Non-Euclidean geometry would be a very interesting follow up for students who have recently taken a high school course in Euclidean geometry (and, the development of Non-Euclidean geometry again offers a host of interesting points for faith integration). We also see some potential in courses on Dynamical Systems or Knot Theory.

All in all, the Mystery of the Infinite does seem to have been a success, and it looks like it will continue to improve in coming years (I'm excited to see where it goes!). That said, I close with a word to the reader. If you have been involved in developing or running similar programs and would like to share ideas, we would be grateful to learn from your experience and to share more of ours—please get in touch. On other end of things, if you are working to get a program started, we would again welcome you to contact us with any questions. Finally, if you know of any students who might be interested in participating in the program, please send them our way! Information about the Summer Scholars Program can be found at templetonhonorscollege.com/summer, or by contacting the author.

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Preparing Students to Read a Calculus Textbook

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Abstract

Consider the exercise of reading the textbook before class. While most educators agree that this practice leads to better learning, too often students enrolled in a calculus class do not find pre-class reading a valuable use of their time, and their commitment to doing so fades. Why is this? As instructors, we hope that these students will be well-versed in the fundamental concepts of the subject by the time they prepare for their final exam, but as they progress through the course and encounter new concepts, they may not be ready for the technical language of the standard calculus textbook. Further, their conceptual understanding of the subject matter—*why is it important?, how is it relevant?, does this connect to something I already understand?*—is probably not well developed. As a result they may not be ready for an explanation that includes precise terminology, presupposes a student's interest in the end application, and fails to make explicit ties to prior knowledge. This talk will describe an alternate approach to reading a calculus text that places its reading after the lecture. The main focus of this talk will be the pre-lecture reading assignment and activities that are not intended to replace the reading of the calculus text but simply displace it to after the lecture.

Introduction

In this paper I will consider the simple exercise of reading a Calculus textbook. In particular, I will describe an alternate approach to reading the textbook which places the reading after the lecture. In reality, the focus of the paper will not really be on reading the text, but on changes made to the delivery of course material which allow for a later reading of the textbook. I will begin by motivating the need for change. After offering reasons why I believe that change is beneficial, I will describe the changes that I made to the delivery of my calculus one course with an emphasis on the materials that I have written to prepare students for lecture.

Motivation for Change

I have been teaching calculus for almost thirty years and for the first time this past year I encouraged my students to wait until after the lecture to read the textbook. Prior to this past year, I had always encouraged my students to read the textbook before they came to class. In class I lectured on that material and encouraged students to complete practice problems after the lecture. This seemed like a reasonable approach, and it worked well for me when I was a student, but is it an effective approach for all or at least most students? That is, is it reasonable for me to expect that my students are well prepared for class, having read the text and maybe even practiced a few problems before arriving? These are good questions, but I believe that there is another question that preempts these questions and others like them: Is it really true that the textbook was written with this purpose in mind?

To answer this question, consider the perspective of the textbook author. One of the first and most critical steps an author takes in writing a book is to identify his or her audience. Who will read the book? What will be their frame of mind when they do so? What should they retain from the reading? What is the reading level of the reader? For the author of a calculus textbook, these can be complicated questions that are difficult to answer. In fact, when you consider everyone who may eventually use a calculus textbook, it is unlikely that the author's approach will satisfy all the potential readers. Personally, I have read calculus textbooks from at least four different perspectives: 1) as a student brand new to the subject, 2) as a student who has recently learned some of the material and is progressing to more advanced ideas within the course, 3) as a student who is a few years removed from the course material and in need of a review of certain key ideas that have recently resurfaced in a different context than when I first encountered them, and 4) as a faculty member preparing to teach my own calculus course with greater interest in the details, derivations, and nuances of a topic than I had as a student.

As a calculus instructor, my target audience consists of the first two categories in the above list. Generally speaking, I teach students who are brand new to the subject of calculus and hope that by the time they take the final exam they are well versed in the fundamental concepts of the subject and ready to dig in more deeply. In other words, my goal as an instructor is to help students progress from category one into category two. I believe that these are two distinctly different audiences and therefore they should not be lumped together into a one-size-fits-all writing approach as is done by most textbooks.

In light of this, consider the idea of a pre-lecture reading assignment. The focus for such an assignment should be on those students who are brand new to the subject matter. These students may be at an appropriate reading level, but not necessarily in the discipline and certainly not as it relates to the topic that is to be introduced. Their handle on the vocabulary that they will encounter in a first reading of the textbook is most likely weak. Further, their conceptual understanding of the subject matter–*why is it important?, how is it relevant?, does this connect to something I already understand?*–is also probably not well developed. In other words, they are not ready to encounter an explanation that includes precise terminology, presupposes a student's interest in the end application, and fails to make explicit ties to prior knowledge. Yet this is precisely what students face when they read a calculus textbook without any prior instruction. They just are not ready for it. This can lead to frustration and in the end a lack of motivation to complete the reading assignment. This is unfortunate, because a week later, after some instruction, these same students most likely would be ready to read their calculus textbook. But right now–prior to any instruction–they are a different audience and their learning would benefit if the assigned reading recognized this distinction. In contrast to the technical language of a textbook, I believe that a good pre-class reading assignment will pay particular attention to the ideas of scaffolding and student motivation.

Let's first consider the idea of scaffolding. The human brain is a fantastic creation, and its ability to store and recall information is amazing. When learning something new, the brain tries to comprehend the subject in terms of its relationship to something else already known and well understood. It ties that new knowledge to existing knowledge. It is natural for instructors to relate details to other details because their understanding of the subject is mature enough to make these connections. Students, however, do not have such a mature view of the subject matter and are often not capable of seeing how the details of the course fit into the big picture. Therefore, students need a constant reminder of the connection between new details and the big picture, lest they begin to focus on the details and lose the ability to identify the key concepts of the subject. Often it is the key concepts and ideas that we really hope students will retain while accepting the reality that their memory of precise details will fade in time. A good pre-class reading assignment will provide the means and context to help students make these connections (hence the scaffolding metaphor) and hence increase their motivation to learn the material well.

The idea of contextual learning also enhances student motivation. Students are usually best motivated when they have a desire to know the material and have some understanding of its significance. If they cannot answer for themselves, *why does this matter?*, their motivation for learning drops off considerably. Again, while many instructors will nod in agreement with the statements submitted here, the fact is that the focus in a calculus course is often more on the technical aspects of the course than reinforcement of why it matters. This question is not a frequent focus point in many technical courses. As such, student motivation for the subject and enthusiasm for (what they perceive to be) a parade of insignificant details suffers.

In light of these observations, I have written a primer that introduces the student to each topic in Calculus 1. I think of this as the *Reader's Digest* version of the subject. It is written on a level that is reasonably accurate and sophisticated, but is intended for a lay audience that is new to the subject. This will provide the proper setting to introduce the key ideas and concepts of the topic, without necessarily dealing, yet, with the precise vocabulary and nuances. In the context of college teaching, the classroom lecture/practicum and follow-up textbook reading will emphasize the details and technical vocabulary and methods.

In this *Reader's Digest* version of the text, each topic is introduced with a one to three page reading assignment followed by a ten to fifteen minute computational exercise. The primer attempts to accomplish several objectives, although not necessarily each objective with each topic. These objectives include:

- To introduce the topic to be covered in lecture
- To effectively tie the new topic to ideas already developed in the course
- To emphasize the significance of the subject in a broad context
- To form a computational foundation for the discussion to be had in lecture
- To make students think about and do mathematics that is a prerequisite for the course

In the following section, I give an example of one of the pre-lecture assignments, including both the reading and computational components of the exercise.

An Example Reading: Finding the Length of a Curve

Measuring distance traveled by an object is an important process in physics. For example, the amount of fuel necessary for a rocket to reach the moon is dependent on the distance that rocket travels. When an object travels along a straight line, the formula known as the "distance formula", which is a direct result of the famous Pythagorean Theorem, can be used to easily calculate the distance traveled. This formula is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_2, y_2) and (x_1, y_1) are two points in a plane that represent the starting and ending points respectively of the object. When an object does not travel along a straight line, but a curved path, the calculation is a bit more complicated. In fact, unless the curved path is a circle or some other geometric entity for which a formula for arc length is known, the best that we can do without calculus is approximate the distance traveled. However, with calculus, we are able to get an exact value for the distance traveled. How does calculus make this happen?

Generally speaking, calculus uses some pre-calculus concept or calculation (in this case, the distance formula) and through the limit process that we will learn about during the next couple of weeks, exchanges what would have been an infinite number of pre-calculus calculations for just one calculus calculation. In some sense then, many of the calculus calculations that we will perform in this class are just the result of applying the limit process to some pre-calculus formula. In this course, we will study two very important problems related to the idea of distance traveled and the rate at which that distance traveled is changing (in other words, the velocity). They are the slope of the tangent line problem (differentiation) which is closely related to the pre-calculus idea of the slope of a line, and the area under the curve problem (integration), which is closely related to the pre-calculus idea of the area of a rectangle. While the slope of a line and the area of a rectangle may seem to have nothing in common with distance and velocity, in the coming weeks we will see that they are in fact closely related.

In class, we will model the calculus process by showing you how to find the length of a curve. In particular, we hope to ask and answer the question of what distance a home run ball has travelled. We will use the equation: $y = \frac{(x+4)(400-x)}{400}$, $0 \leq x \leq 400$ as a model of the path that our home run ball has taken. Prior to class, do the following:

1. Read the following comments by William Jenkinson in an article he wrote for the Home Run Encyclopedia regarding the conversation over the longest home runs ever hit.

   > It should be noted that those regular references over the years to 500- and 600-foot home runs were born out of scientific ignorance, misinformation, or even deliberate exaggeration. The most common cause for overstatement has been the basic misconception about the flight of a batted ball once it has reached its apex. Seeing great drives land atop distant upper-deck roof[s], sportswriters observing the occurrence from a press box would resort to their limited skills in mathematics without any regard for the laws of physics. Perhaps the ball had already flown over 400 feet, whereupon it was interrupted in midflight at a height of 70 feet above field level. Awed by such a demonstration of power, the writers would then describe the event for posterity as a 500-and-some-foot home run. With the guidance of our scientific brethren, we know that once a batted ball has reached its highest point and lost most of its velocity, it falls in a rapidly declining trajectory. The aforementioned fictional home run could have been reported at 550 feet in a prominent newspaper, and re-created at that length by historians for years thereafter, when in fact it traveled about 100 feet less. Hyperbole has always been part of the phenomenon of long-distance home runs, and this factor must also be considered [2].

2. Answer the following questions:

   (a) What is a name for the curve that we are using to model the path that our home run ball has taken?

   (b) Sketch a rough graph of the path.

   (c) Based upon Jenkinson's comments above, explain why the curve we are using to model the path taken by the ball is probably not a good choice.

   (d) The distance that we hope to measure is not the same as the distances used to describe the lengths of home runs in the reading above. Explain the difference.

3. Consider the path described by $y = \frac{5}{x}$, $1 \leq x \leq 5$. First, find a lower and upper bound to the length of this path. Estimate the length of the path [3].

Changes to the Delivery of Course Material

Moving the reading of the calculus textbook to after the lecture and replacing it with a pre-lecture activity that consists of both reading and computation allowed me to rethink my approach to delivering course material during the lecture. In the past, I made little effort to explicitly connect my lecture to the textbook reading. This approach most likely did not motivate my students to read the textbook. However, since the main objective of the pre-lecture assignment is to prepare students for lecture, it is important that students are actually doing the pre-lecture assignment. In order to motivate students to do the reading and computations in the pre-lecture assignment, I restructured my lectures to ensure that they begin where the pre-lecture assignment left off. In this section I will first describe some of the changes that I made to the structure of my calculus lecture and then I will illustrate what that might look like for the pre-lecture assignment that is described in section 3 of this article.

In order to maximize class time with my students, I encourage them to arrive early (I teach an 8:00 class, so calculus is the first class of the day for all of my students). Prior to class, I ask students to place their computational work on the board. Often there are multiple problems to consider, so several students can make contributions to the pre-class work. Unfortunately, students are often reluctant to place their work on the board and need some encouragement before they are willing to do so. I try to create an environment that encourages students to contribute by emphasizing that I am not necessarily looking for the *right* answer to a question, but simply work that allows the class as a whole to begin a discussion centered on the topic of the day. I also suggest that sometimes mistakes can be very helpful in developing a mature understanding of a topic. Of course it helps that students know that their pre-class contributions contribute to their class participation grade.

I begin class by asking the class as a whole to critique the work that has been placed on the board. Together, the students and I correct any mistakes that have been made and complete any work that was not placed on the board by a student. Once we have discussed the questions that were assigned in the pre-lecture reading assignment, we transition to the main theme of the day. During this transition period, I often extend what students were working on in the pre-lecture assignment and use it to motivate the topic of the day. I usually set aside the first ten to fifteen minutes of class to discuss the pre-lecture assignment.

At this point I would like to revisit the pre-lecture assignment that is described in section 3 above and make a few observations. My goal for the lecture that accompanies this pre-lecture assignment is to introduce students to calculus and in particular the two main problems of calculus: the tangent line problem, and the area under the curve problem. To accomplish this, I ask students to consider how they might measure the length of a curve. The pre-lecture reading reminds students how to find the distance between two points and rephrases the length of a curve question in terms of the distance that a home run ball travels.

Prior to class, it is my hope that a student will sketch $y = \frac{(x+4)(400-x)}{400}$, $0 \leq x \leq 400$, the curve that models the path of the home run ball. This gives me a chance to briefly review quadratic equations and their graphs. I try to connect this to the reading assignment by asking why this parabola is not necessarily a good model for the path that a home run ball might take. I then turn my attention to finding the length of $y = \frac{5}{x}$ on the interval $1 \leq x \leq 5$. In the pre-lecture assignment, students are asked to find bounds for the length of this curve on the given interval. This question is open-ended and allows me to transition to the main goal for this particular lecture.

After determining appropriate bounds for the length of this curve, I continue class by asking how we might approximate the length of the curve. We consider breaking the interval $1 \leq x \leq 5$ into four subintervals and constructing four line segments each with endpoints on the curve itself. After calculating the sum of the lengths of these four line segments, we turn to the computer and repeat the exercise for 100 line segments. I then use this work to briefly introduce the tangent line problem and the area under the curve problem.

Conclusion

My goal in writing a primer that includes both pre-lecture reading and computational exercises is not to replace the calculus textbook, but to prepare beginning students to read their textbooks. I agree with Herbert Simon, one of the founders of the field of Cognitive Science, who said,

> Learning results from what the student does and thinks. The teacher can advance learning only by influencing what the student does to learn [1].

The calculus textbook is a tool that I use to try to influence my students to learn. Among other things it provides my students with additional instruction, examples, and problems to practice in order to develop the skills required to do calculus.

Unfortunately, most calculus textbooks are not written with just the beginning calculus student in mind. Therefore, a pre-lecture assignment that requires reading the textbook may include vocabulary, definitions, and a context that are inappropriate for students who are new to the material. Such students may become discouraged by these reading assignments and eventually stop reading (i.e. doing). By moving the reading of the textbook to after the lecture and providing students with reading and computational assignments that are more appropriate for their pre-lecture understanding of the material, my hope is that my students will not only complete the pre-lecture assignments, but in the process become better-prepared to read their calculus textbook.

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Math, God and Politics—A Fight over Geometry in 19$^{\text{th}}$ Century Italy

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Abstract

In 1839, the leader of the synthetic school of geometry, Vincent Flauti, issued a polemic, reminiscent of the Renaissance public challenges (most famous was Tartaglia and Fiori's over cubic equations) to the analytic school, headed by Fortunato Padula [9]. Three geometric problems were proposed, all carefully chosen to guarantee a victory for the synthetic school. The judges were from the Royal Academy of Sciences, men also favorable to the synthetic method. Why then did the analytics take up this challenge, and who were the real victors? To understand this we need to look back to the period in Naples beginning with the Neapolitan Enlightenment and trace the evolution of these two schools, their leaders and their goals both for mathematics and for the future of Naples and Italy.

Synthetic vs. analytic approach to geometry

The synthetic school had formed around Nicola Fergola (1753-1824) a leading Neapolitan mathematician. The work and teaching of the synthetics centered on pure geometry. Its approach to solving problems was specific, "every problem required a different geometric construction, and thus a geometer required intuition along with skill, knowledge, and experience, which could be gained only by long training" [9]. The analytic method on the other hand was general. Every problem could be put into an equation. Each problem could then be solved, mechanically, using the same steps. Nothing was left to the intuition of the geometers. This method was seen as 'easy', 'mechanical' and 'easily learned' [9].

The analytics and the synthetics both shared the same heritage of mathematics. Their leaders had studied the writings and developments of leading European mathematicians. However, how they chose to view and apply this information varied according to their social, political and religious interests. It led them to hold different 'images' of mathematics and to thus practice mathematics in different ways.

Fergola's early influences

Antonio Genovesi (1712-1769), who held the chair of political economy at the University of Naples, was the primary initiator of the Neapolitan Enlightenment [14]. He advocated the cultivation of an educated and informed public, who also had the freedom to express its opinions, which could open up the kingdom's intellectual life to the issues of human welfare being discussed elsewhere in Europe. His objectives were intellectual and educational; they were not immediately political. He saw philosophy as coming to the aid of rulers, indirectly guiding the reforming activity of the government [5]. The government appeared willing to accept philosophy's offer of assistance and Enlightenment and reform intersected during the 1770s and 1780s. Some of these reforms took place in the economy, the feudal system, and the power of the Church.

The focus was also on empiricism. Genovesi was concerned with demystifying the various disciplines, including the sciences. He viewed these disciplines as the means for the cultural and economic progress of the country rather than sources of eternal truths. Social problems should take preeminence over theoretical ones [9].

Nicola Fergola, (1753-1824) was born in Naples. His father wanted him to study law, as this was one of the profitable professions of the time. Fergola studied at the Dominican school, influenced there by the religious values of the Dominicans, including the admonition to live simply. A deeply religious man, he continued with the ascetic values he learned from the Dominicans in his youth.

Fergola attended the University of Naples where Genovesi chaired the department of political economy. His interest lay in mathematics, but as there was little advanced mathematics taught at the university (the main emphasis was medicine and law), he was forced to become mainly a self-taught mathematician. He had access to a private library, which aided his self-education that contained the works of the leading European mathematicians. His readings encompassed classical Greek texts, almost all the major texts of the 17th century including Galileo, Cavalieri, Descartes, Huygens, Newton, Jacob Bernoulli, as well as all the great writers of the 18th century, in particular, Euler, D'Alembert, Daniel Bernoulli, Clairaut. He also studied many works of Lagrange.

Genovesi appreciated the synthetic method and advocated the ordering of knowledge in a systematic, methodical way. He suggested composing science according to the synthetic method and advocated the axiomatic method of Euclid. Also drawn to Euclid's methods, Fergola saw *The Elements* as the epitome of a solid foundation for learning mathematics while also aesthetically pleasing. He proposed to use it as a model for his teaching of calculus and mechanics and also for issues relating to the existence of God and miracles. (See [5].)

Fergola's school

In 1770 the *Liceo del Salvatore* (School of the Savior) hired Fergola to teach. The following year he opened his own private studio where he taught advanced mathematics. Such private schools were not unusual in Naples. They often provided basic training in various disciplines but often also offered a superior level of preparation for the university, and for technical careers. Fergola's school quickly acquired a good reputation and many of the brightest students studied there. One of these students was Annibale Giordano, a mathematical prodigy, who in 1786, at the age of 16, presented to the Royal Academy of Sciences in Naples an article entitled "Continuation of the same subject" which brought him a monthly stipend from the school. The following year he wrote an article on the problem of Cramer and which was published in *Memori of the Accademia* and gave him a certain notoriety throughout Europe. (See [5].)

Fergola produced several mathematical articles during the 1770s and 1780s. During this time Fergola entertained fairly broad interests. He taught and did research in both synthetic and analytic geometry. He praised the works of Lagrange (a leading proponent of algebraic and analytic mathematical method) and D'Alembert. Fergola and his school thrived during this period. His school generated most of the scientific production in Naples regarding mathematics. However, by 1786 Fergola's interests begin to shift toward the synthetic method. He showed particular interest in the "forgotten method of the ancients" ([9]) and what he considered their elegant, intuitive and certain solutions. He began to see analytical methods as being unreliable, and lacking secure logical foundations.

In 1789 Fergola finally achieved the rank of full professor at the *Liceo del Salvatore* from the rank of deputy. That same year his prodigy, Giordano, received a professorship at the Military Academy. However, what looked like the beginning of a promising academic career for Giordano took a radical turn when Giordano joined the reformists to promote Jacobin ideas.

Jacobin Science and mathematics

The Neapolitan Jacobins believed that science should speak to the real needs of the people; a view of science in opposition to that of one "pure" or contemplative. The leaders of this school in Naples were Carlo Lauberg, a scientist, and Giordano. Lauberg had applied for the same professorship as Giordano, but Fergola opposed his appointment because he viewed him as morally reprehensible (Lauberg was a defrocked priest). (See [9].) Fergola advocated for the appointment of Giordano instead.

Lauberg and Giordano opened a school and wrote textbooks advocating the analytic method. A strong distinction began to emerge between the goals of the synthetic and analytic schools. The synthetic method focused on single, disconnected truths. Only the gifted could find new geometrical truths. The analytics presented universal methods that permitted the development of scientific knowledge in the modern era, including the spheres of social and political sciences. They wrote their textbook with the goal of being useful for their country. To them, math was a universal language, which every rational being could comprehend. They believed in the mathematization of human sciences, and that social and political order could be built on basis of the new theorems of the social sciences of politics and economics. (See [9].)

With the French Revolution in 1789 (and the execution of the king and queen in 1794) the European monarchy's support for reform began to wane ([14]). The Revolution provoked a conservative reaction against the ideas of the 18th century European enlightenment and, since the Enlightenment thinkers had greatly valued science, there was also a reaction against applications of science in various aspects of life because those applications could lead to materialism and ultimately to political radicalism [1]. French Revolutionary ideas about religious toleration, civil equality, freedom of the press, and liberal ideas about education clashed with the church's view that it had the responsibility for the spiritual, moral and educational welfare of the people [12]. And while the governments had been anti-clerical in the 1770s and 1780s, this changed in the 1790s. The Church now became the main ally of the monarchy in its opposition to Jacobin tendencies and liberal ideas.

Political and Religious Changes in the 1790s

During the 1790s a Reactionary Catholicism took root in Naples. It aimed at a return to a theocratic society based on an idealized image of the Middle Ages [9]. Its proponents saw themselves as defenders of a venerable and natural conception of the world, which had first come under attack in Renaissance, beginning a moral and political corruption that had led to the Protestant Reformation and later to political revolutions. These Conservatives oversimplified much of history, inventing much of this interpretation and their "venerable tradition," in order to provide credibility and authority for their arguments. The synthetic school with its emphasis on pure mathematics and suspicion of the applied analytical approach, became the "scientific component" of the Reactionary Catholic movement. Fergola, as leader of the synthetic movement, and his school, flourished during this time period.

In 1796 the French armies invaded Italy and the era of the Italian Republics commenced. Though short-lived, this era saw the beginning of societal transformations that would challenge the ancient regime in Italy, whose power lay in the nobility, the religious orders and the papacy. The separate Italian principalities transformed into centralized, bureaucratic autocracies. Feudalism was abolished, and fiscal and financial administrations were centralized. Vast tracts of Church land were sold to pay for these changes. Privatization of land was promoted to improve agriculture and develop new industries were developed. These changes accelerated economic development but also heightened the social costs of modernization [3].

In January 1799 a revolutionary agitation shook Naples and brought about the proclamation of a Jacobin republic as well as the abolition of feudalism. Fergola left for the countryside during this time period to avoid the conflict. Six months later the Neapolitan Republic was militarily defeated by the Bourbon allies and with the help of the British. There followed a period of mass executions and exile of the survivors. Many of these victims were young, well-educated men and women with some training in the scientific disciplines. Almost a generation of intellectuals and professionals in Naples was destroyed [9].

After the Republican defeat Fergola, a loyalist, returned from his countryside sojourn. The King rewarded the loyal and conservative ecclesiastics and professors by giving them positions of power in dioceses, universities, and military and naval academies. The King also turned to them to reorganize the public education system and cultural institutions. Reformists, on the other hand, were viewed with suspicion and are held responsible for the revolution and for social disorder in general.

The analytic method in mathematics was associated with social change whereas the synthetic method viewed mathematics as a separate entity to be studied for its beauty and logical structure. By 1800 the preeminence of the synthetic methods became even more pronounced in teaching and research of Fergola's pupils. Mathematics had become synonymous with synthetic and descriptive geometry (also taught and developed in Fergola's school). Thus, it is ironic, that though Fergola's awareness of mathematical developments in Europe and his sharing of this knowledge with his pupils, had been responsible for bringing Naples into the European mathematical community, now, with his focus on synthetic geometry and its deemphasizing the role of analytical geometry, popular in the rest of Europe, "Naples, was, once again, out of Europe." (See [9, p. 691].)

Religiosity of Fergola and the Catholic Enlightenment

Fergola's 1789 appointment as full professor at the *Liceo del Salvatore* (School of the Savior) was a royal appointment and came with the condition that he publish his lectures. In 1792-1793 Fergola published his lectures on Newton's *Principia*. These lectures are essentially an apologetic reading of the *Principia*, where Fergola thinks of a God who orders, and orders the heavens with geometric laws [5]. Fergola was particularly interested in the study of forces "for it permits us to discover 'the laws of the universe' and 'the deep knowledge of who rules and sustains it.' Indeed, according to Fergola, the source of all the forces acting in the universe is the 'Hand of the Living God"' [9, p. 693].

Fergola also applied his synthetic method to religious themes. In 1804 he wrote *Theory of Miracles* in which he argued for the existence of miracles in general and in particular the miracle of the liquefaction of the blood of St. Januarius, the patron saint of Naples. (The 'miracle' of the liquefaction of coagulated blood appears to be peculiar to the region of Campania in Italy). In it he "argues for the spirituality of the soul and the nonmaterial origin of human thought. He rejects materialism, in its many variants, because empirical reality is too complex to be reduced to material causality" [9, p. 692]. He wanted this work to be a geometric discussion of religious matters [5].

In his private life, Fergola continued to live as an ascetic. As a leader of a school he saw his responsibility as going beyond teaching mathematics, but also "to help the pupils in their spiritual growth, in order to make them good Christians and good citizens." He was concerned with his students' moral behavior and their religious and political ideas. "Mathematics, as it was practiced in Fergola's school, was a 'spiritual science,' a science that brings its practitioners very close to the mind of the Almighty. It was also a powerful resource in the fight against atheism and materialism." (See [9, p. 692].)

In his teachings we also begin to understand how he saw the distinction between synthetic and analytic geometry. For Fergola synthetic geometry was the language of God. The true mathematician contemplates the world around him and subsequently the geometrical character through which it is written, in order to understand eternal truths. Synthetic geometry with its emphasis on intuition and reasoning, completely independent of matter, was more certain, and more valuable than analytic geometry. The expediency of analytic geometry for applied mathematics made it a useful instrument, but certainty was not attainable in analytic geometry. Certainty was limited to pure mathematics. We also begin to see a portrayal of what a true mathematician is—an ascetic, a contemplator of geometrical truths. He is not concerned with the applications of math to political or social problems. He is part of ancient tradition that went back to the Middle Ages when the church, the monarchy and mathematics formed a harmonious alignment. The idea that the synthetic school was a continuation of an ancient tradition was actually a myth perpetuated by Fergola. It was rather, a new phenomenon, a reaction to the secularization of science and society.

Reactionary Catholics used Fergola's writings to argue for the fallible nature of empirical knowledge and the limits of scientific investigation. Fergola's *Theory of Miracles* was written at the bequest of Bishop Colangelo, who was interested science, philosophy and history. At the same time when Fergola was working on this essay, Colangelo also published a book on the progress of science. He attacked the autonomy of scientific practice from religion. "Colangelo claims that only a scientist who is also a good Christian can achieve relevant scientific results" [9], p. 695. (This claim resurfaced in Ventura's 1824 eulogy of Fergola in which Ventura stated that "only a Christian can penetrate mathematical truth and investigate empirical reality properly" [9, p. 697]). Colangelo drew on Fergola's work in *Theory of Miracles* in his propositions regarding how God rules the universe. God is characterized as an absolute monarch, whose will is the source of all natural law. A miracle is then an event that breaks some natural law. If God can break his own laws to allow miracles to happen, then cannot the king, if he is the source of sovereignty, also "break" his laws? This argument took on political significance two decades later when King Ferdinando I 'legitimately' abolished the constitution he had granted to his kingdom.

Napoleonic Era

At the beginning of the 19th century, Italy was a collection of principalities under the rule of various European powers, along with the Marches, Umbria, and the Papal States, which were under papal rule. No constitutions protected the rights of individuals. Customs and practices were local, as were weights and measures and currency. The different parts of the country were isolated geographically; there were no railways connecting the different regions. The Church was in charge of the educational system and of ministering to the poor. The ecclesiastics and the aristocracy held power. There was no real middle class. The professional class consisted mainly of lawyers and physicians. All this was to change under Napoleonic rule.

In 1806 Napoleon was crowned King of Italy in Milan. The following year he established his brother, Joseph, as King of Naples. Thus began the decade long period of French rule that was to change the foundations of Italian life and precipitate the unification of the Italian peninsula.

Napoleon had a passion for uniformity and for centralized administration. Decisions previously made at the local level were now made by central authorities. Fiscal and commercial reforms were instituted; weights, standards and currency were standardized, and internal customs barriers were removed. The French penal, civil and commercial codes were introduced, the feudal system abolished, taxation systems were reorganized. The political and economic power of the Church was under attack and many ecclesiastical lands were sold. New religious liberties were granted, with equal rights being given to Jews.

Modernization brought by the French benefited some, but not all. Resistance to change was also strong, for to many it meant higher taxes and military conscription [4]. There was also resistance to standardizing weights and measures for how could an equal sized plot of mountainous land be considered as equally sized to fertile land? According to Lucy Riall, "This struggle between those who embraced change and those who resisted it, and between the manifest need for reform and the need to control its consequences, was to characterize political debate in Italy during the decades following the fall of Napoleon" [13, p. 10].

First controversies between analytic and synthetic schools

These societal changes were reflected in the mathematical community and led to the first real controversies between the analytic and synthetic schools. At this point Fergola and his students held the leading mathematics chairs in the universities, essentially controlling the teaching of math in the university and also heavily influencing mathematics education in the high schools. But a new system of education, and a type of "mathematician" began to emerge which not only threatened this dominance, but the role mathematics was to play in society.

In 1808 the Royal Corps of Engineers of Bridges and Roads was established. Placed under the Ministry of the Interior, the Royal Corps had jurisdiction over roads, bridges, dams, monuments and public buildings, as well as ports and channels [5]. It was a modernizing force in service to the central government as opposed to the local authorities. The Royal Corps provided new infrastructures, which facilitated economic and technological changes. Its effects were also felt in the educational system, for it taught a different type of mathematics. Engineers now assumed a higher social status on par with professions of medicine and law. The training of engineers required new higher institutions of education and a different kind of mathematical training.

In 1807 the government issued a decree prescribing that all textbooks used in the public schools had to be approved by the Ministry of Internal Affairs [5]. A commissioner was appointed to oversee the compilation of these textbooks. Nicola Fergola served as one of the members of this commission as he had considerable didactic experience in both public and private education. The task of drawing up a complete course of mathematics was given to two of Fergola's students, one of which was Vincenzo Flauti (1782-1863). Flauti would become the face of the next generation of Fergola's school. The commission wanted textbooks that would organize all the necessary mathematical knowledge needed by students in secondary schools in a systematic way. It was to be an integrated curriculum in which arithmetic, geometry, algebra, differential and integral calculus, etc. were no longer treated as isolated and independent, but united and connected. For many mathematicians the unity of the course was to be ensured by the notion of abstract or general quantity, from which they built the analysis or algebra. It was necessary that the method used was independent from any particular case, and the algebra alone had this character because one could include in its generality any quantity without considering the specifics and therefore it was not subject to peculiarities of chance. These methods were then applied to other mathematical disciplines, including the geometry. This was the approach of members of the analytic school, closely linked to the ideas of Euler and Lagrange. Others, however, opposed this idea of algebra as a center of mathematics. Fergola and the synthetics wanted to ensure that this unitary presentation of mathematics would be based on the synthesis and the Euclidean method. They endeavored to design a course based on this paradigm. (See ([5].)

In March of 1811 the *Scuola di Applicazione di Ponti e Strade*, and in August 1811, the Polytechnic School were created, based on the model of the *Ecole Polytechnique* in France. Both were to provide the basic scientific training for engineers. These schools were structured differently than the universities. Their curriculum was radically different, with focus on the applied mathematics needed for building technological and economic reforms. Entrance was based on an exam designed to test mathematical knowledge.

One disadvantage of the French model of centralized decision-making was that the decision makers often did not have first hand knowledge of local conditions. The government turned to the engineers, and the supposedly neutrality of their scientific judgments to be these local experts [11]. The engineer was seen as speaking the voice of Reason, and his understanding of natural and social reality gave him morality superiority in the eyes of the reformers, many of them now returning from exile. Appointment to the Corp of Engineers was based on a merit system of promotion alien to social life in the ancient regime. It opened up opportunities for men from the middle level of the bourgeoisie to acquire a new status within society. However, the rise of the engineer's status and authority in society meant that traditional authorities: the landed aristocracy, the church, local communities, etc., lost their status as decision-makers. This naturally created feelings of jealously and created social tension between the engineers and the local authorities.

This social tension led to a controversy between the members of Fergola's school who had recently had control of teaching mathematics in the university (and therefore heavily influenced mathematics education in the high schools) and those who were associated with the Corps of Bridges and Roads and the *Scuola di Applicazione*. The controversy was never exclusively academic [7] Animosity between the two groups went beyond technical mathematical arguments; there were accusations of 'moral depravity' and corrupting the minds of young students. Scientific, educational, political and academic career issues intermingled with the future understanding of how mathematics was to be done. To understand the reason for these polemics, let us take further look at the premises behind each of these schools.

The synthetic school believed that mathematics was an intellectual activity practiced to uplift the human spirit [7]. It did not require practical applications, was not done for technical preparation useful for any profession. The teaching of mathematics should focus on rigor and deductive reasoning for these practices would help individuals learn to develop their intellectual capacity, becoming inventive and creative. Fergola's school saw analysis as contaminated by empirical considerations and in need of more rigorous foundations. Synthetic teaching required long training, it was based on discovering and grooming the natural talent of exceptional students; typical product of this system was the child prodigy [11].

The analytic method was heavily influenced by the great French analysts of the 19th century. It supporters were seen as progressives, young 'outsiders' aware of current mathematical developments, and supporters of reform [9]. The supporters of the analytic method described it as a natural way of reasoning. This method was based on an assumption that all complex problems could be broken down into their elementary components, which could be solved by techniques of algebraic analysis and calculus using mechanical manipulation of symbols. With proper training any student could master its techniques because it was simply the natural way the human mind worked [11].

The controversies between the two schools began formally in 1810 with a criticism of Fergola's work by Ottavio Colecchi, a Dominican philosopher and teacher of calculus at the *Scuola di Applicazione*. Colecchi reproached Fergola for putting too much emphasis on pure geometry and being dismissive of advances made by modern analysis [9]. Colecchi's attack may have been precipitated by the fact that the Fergoliani school at this point had complete dominance of the universities and the Academy of Sciences. Also Flauti had been charged with the charge of writing the new math textbooks for the schools and the educational ideas of Fergola and his synthetic methods were certain to underline these books. In 1811 the *Scuola di Applicazione* was to open, followed by the *Scuola Politecnica e Militare* in 1812. These schools did not fall under the control of the Fergoliani, and their supporters wanted to ensure a different model in teaching and research.

In 1811 Fergola's school responded with the publication of a brochure that had the stated purpose of showing the value its school had in giving rise to innovative scientific production, mainly in the school of geometry, and to a lesser extent, in the field of analysis [5]. It also contained a defense of the cultural value of geometric research conducted with synthetic methods, stating that is was a matter of national pride to develop the ancient geometry. It claimed Neapolitan mathematicians had not had the chance to cultivate research activities, or because of the absence of a suitable cultural climate, or too many commitments, been unable to develop the ancient geometry. But with the emergence of the figure of Fergola and his school, this ignominious void was fulfilled. With the publication of this brochure the controversy finally exploded.

This exchange of diatribes was just the first in a series of accusations between the two schools, which would culminate in the 1839 disfida. Each school responded to changes happening in Naples at the turn of the century. The analytics and the engineers saw mathematics as an instrument of modernization. It brought a hope for technological innovation and a democratization of science itself. Doors were opened to a new bourgeois society made up of equals, who could play a new role in society while earning a living. More conservative mathematicians with strong religious ties such as Fergola and Flauti, considered the synthetic approach as the most "appropriate response to what they perceived as a broad cultural and moral crisis of European civilization" [11, pp.23-24].

Bourbon Restoration 1815

In 1814 Napoleon was defeated and in 1815 with the Congress of Vienna most of the Italian States were almost entirely under foreign dominance. However, the Restoration governments did not dismantle the excellent administrative machinery Napoleon had left in the Kingdom of Italy. They adopted, and adapted many of the French administrative and legal innovations and continued the modernizing efforts set in motion by the French. They recognized the value of new roads, canals and railways for creating trading opportunities. The rulers looked for ways to promote economic growth without making concessions for political or cultural freedoms. The Restoration governments were interested in investing in education, but not in any taught subjects that could lead to political upheaval [2]. Science seemed a safe subject, and the study of science, especially applied science and engineering, acquired a social prestige [8]. The school of Fergola continued to dominate the university and Academy of Sciences, but the *Scuola di Applicazione*, after some initial difficulties, continued to be a major player in Restoration Italy. These difficulties were due to the return to power of those most threatened by engineering activities, i.e. the local political conservatives who were now back in power. The autonomy of the school was restricted. A branch of Bridges and Roads replaced the Corps of Engineers. Staff was reduced and provinces were allowed some local oversight on projects.

Fergola by this time was fading. He suffered from mental illness and his religious conservatism now approached bigotry and a kind of religious delirium [5]. It was time for the next generation to take the lead. The two main leaders of the Neapolitan mathematical establishment during the years 1820-1840 were both students of Fergola. One was Vincenzo Flauti who now became the successor and interpreter of Fergola. The other was Francesco Tucci (1790-1875). Tucci became the leader of the analytical camp. In 1813 Tucci was appointed a professor at the Polytechnic School and later as professor at the Military College where he became the director. Again, it was an argument over the value of algebra in solving geometric problems that appears to have brought about the break between Tucci and his former teacher. Tucci, as a professor in the Scuola di Applicazione, was concerned with concrete problems his engineers faced, and ways to train and prepare them to solve those problems. To him, this could be done best using the methods of analysis and algebra.

Flauti at this time continued to work on his textbooks. But he also devoted much time to the publication of the work of Fergola. It became an objective of the school of Fergola to develop a myth around the man Fergola, his influence as a mathematician and a defender of the Christian faith. Fergola was primarily a teacher. While he was a devout Catholic and loyalist, he was temperate in his viewpoints. But with Flauti taking the helm, the synthetic school became more restrictive in its views. Flauti was also more interested in using the synthetic approach in academic battles [5]. However, in terms of real scientific output, little was being produced by the synthetics during these years [5]. It seemed like all of its efforts were being used up in perpetuating a myth about the importance of the school, and also in attacking the work of the analytics. On the other hand, with the exception of Tucci, the analysts also had little scientific production. Their focus was using mathematics as a tool for engineering.

In 1820 a revolution took place in Naples that resulted in Ferdinand I pledging to grant a constitution. However, he then revoked this constitution in 1821 (recall the argument in Fergola's *Theory of Miracles* where a sovereign ruler can override his own laws). (See [9].) In this political crisis the *Scuola* became a target for conservative politicians. The director of the school had to defend the unique role of the school for the training of engineers, and that this was different than the university training required for other mathematicians and scientists.

Fergola died in 1824 and one of his pupils, Ventura, gave an impassioned eulogy stressing Fergola's ascetic life and used him as proof that it was possible to be a good mathematician and a good Christian. He went on to state that only a Christian could penetrate mathematical truth; in the hands of atheists mathematics becomes a tool for the destruction of society.

Carlo Rivera became the director of the *Scuola di Applicazione* in 1826. Rivera worked to restore the authority and autonomy of the Corps. Rivera realized that in order to implement technological and economic throughout the kingdom, there needed to be standardization of weights and measures and a single decimal system. This went against traditional practices of local areas having their own measurements for different kinds of items. Measurements need differed from region to region. Once again it was centralization versus local autonomy, standardization versus variability [11]. Also at stake was tax reform for the multiplicity of existing local measures offered landowners the ability to evade taxes [5].

Ceva Grimaldi, a member of the aristocracy and a future prime minister of the Kingdom of the Two Sicilies headed up the anti-standardization campaign. Grimaldi defended the interests of the provincial elites, landowners, and private contractors. He repeatedly attacked Rivera and his engineers, calling into question the validity of the training the engineers received at the *Scuola di Applicazione* as well as their accomplishments.

Both sides waged a public relations campaign to win over adherents. Rivera repeatedly extolled the economic and technological accomplishments of his engineers: land reclamation, creation of a modern fishing industry, establishment of factories, development of new villages, etc. He wrote both rhetorical pieces and technical reports extolling the virtues of his engineers, and defending their work. In 1835 he organized a special exhibition at the *Scuola di Applicazione* to celebrate the corps [10]. By having his students give lectures on a variety of technical and mathematical topics, and by displaying their work, which included plans of bridges and ports, he highlighted the valuable training provided by the school. Rivera also demonstrated the value of the coordinated group effort of the corps as opposed to the isolated genius, which was characteristic of the synthetic school.

Grimaldi, and the conservatives, waged their own public relations campaign. Not only did they question the training of the engineers and engineering's ability to solve social problems with their mathematics, they criticized the need for such societal changes. They promoted a mythical description of a happy and utopian Neapolitan countryside, one that didn't need the innovations of the engineers. In a historical essay on public works dating back to the Middle Ages (appeals to the Middle Ages was common by conservative factions throughout the Risorgimento), Grimaldi derided the notion that civilization in southern Italy began with the Corps of Engineers. (See [11].) In his pre-French description of Naples, everything functioned well and in harmony—peasants and artisans lived together, cared for by the aristocracy and the church; the roads and waterways were well managed, The enemy was not the king, or the church, or the existing order, but whatever was 'foreign' or 'abstract'.

The 1839 Challenge

The controversy between the analytic school (the engineers) and the synthetics (conservatives and Flauti) escalated during the 1830s. Several incidents toward the end of the 1830s precipitated the 1839 challenge. One was the standardization of weights and measures that was scheduled to go into effect in 1840. In 1838 Grimaldi wrote an essay against the reform of weights and measures. This essay opened with a long scientific introduction by Vincenzo Flauti, Fergola's heir and leader of the synthetics. In it Flauti stated that mathematical abstractions are useless for addressing issues related to the public welfare. He used the standardization of weights and measures as an example of one of these useless measures. (See [11].)

In 1838 Fortunato Padula (1815 - 1881), a graduate and later director of the School of Engineers published a book entitled "Geometric problems resolved by algebraic analysis." In it he showed how to translate geometric problems into analytical language and how doing so simplified the solution process. His goal was to show the superiority of the analytical methods used by the *Scuola di Applicazione* [5]. Flauti wanted to respond to the work of Padula. However, this wasn't easy given the current scientific production of the synthetic school. By the late 1830s most people had abandoned the synthetic value of research and Flauti himself was not publishing any new research. He realized there was a need for young scholars who could give it new vigor. He saw in Nicola Trudi (1811-1884) a new young genius who could revive the glories of the synthetic school [5]. Trudi had recently found a very elegant, purely geometrical, solution to the problem of inscribing three circles in a triangle under specified conditions (a Malfatti circle problem). His solution was published in the Proceedings of Petersburg. Flauti decided to issue a challenge to the analytics using this problem, recently solved so elegantly by Trudi, as one of the challenge items. In doing so Flauti achieves two important goals: it would bring his young protégé Trudi to the attention of mathematicians, establishing Trudi in the research world. It also would raise the prestige of the synthetic school [6]. Flauti chooses two additional problems. The first of these involved inscribing in a given circle a triangle whose sides pass through three data points. Annibale Giordano, another protégé of the Fergola school, using synthetic methods, had solved this problem back in 1786. The third problem was a problem from projective geometry: to inscribe in a given pyramid four spheres that touch each other and touch the faces of the pyramid. Choosing problems that had been solved by the best students of the synthetic school, along with a problem in projective geometry, a field of research of the synthetics, seemed to guarantee a win for the synthetic school. It also helped that the judges were members from the Academy of Sciences, which was under the control of the synthetic camp.

If Trudi was the principal researcher and star of the synthetic school then Fortunato Padula was his counterpart in the analytic school. In addition to his textbook he had written several works on geometry and mechanics. It was Padula who responded for the analytics to Flauti's challenge. Padula published a booklet, dedicated to Carlo Rivera that opened with a historical piece in which he criticized and rejected all the theoretical assumptions of the synthetic school. He pointed out the limitations of the synthetic method, specifically that the synthetic study of geometrical abstractions did not introduce students to the modern developments in mathematics. He then pointed out the advantages of the application of mathematics can have to arts and industries. (See [9].) He solved the first two problems and stated that the third problem was unprovable; it was over determined. While one could solve problems like this, he claims that "we are not interested" in this kind of work [9]. Padula's solutions were rejected by the Academy of Sciences and Trudi and the synthetic school were declared the winners. Ironically, this 'win' ultimately became a defeat for the synthetic school. It turned out that Trudi's solutions were obtained using analytic methods; Trudi in fact was a master of analytic geometry. However, there was a temporary renewal of the synthetic school, thanks to Trudi. Projective geometry became the main focus of his research. He also studied algebra, combinatorics and differential geometry. He did little research into the study of synthetic geometry.

Both Padula and Trudi's work attracted the attention of European mathematicians [5]. Jacobi and Steiner, during their 1844 visit to Naples, praised both Padula and Trudi's work. Poncelet did the same in 1866. At the 1845 Italian Congresses of Science meeting in Naples, both Padula and Trudi lectured. Trudi spoke on algebraic equations, Padula on the theory of fluids. Mathematicians from all over Italy, as well as several other countries in Europe, attended the meeting. Naples was back in Europe; its mathematical isolation was at an end.

The story of Flauti and Trudi in many ways mirrors the story of Fergola and Giordano. The mentor takes great delight in his pupil, promoting his accomplishments, supporting his research, even when that research took on different focus than that of the teacher. But in the end both protégés turned their back on their teachers for a combination of political and mathematical reasons. Giordano joined Lauberg and the Jacobins. He saw the value mathematics could play when applied to social and natural science. Trudi's defection was longer in coming. For the two decades following the public challenge of 1839, Flauti supported and encouraged Trudi's work in projective geometry [6]. He considered Trudi as a member of the school of Fergola and Trudi's writings on projective geometry an integral part of the school's production. In doing so, Flauti showed a willingness to expand the interests of the school of Fergola beyond the geometry of the ancients or that of synthetic geometry. Of course Flauti himself, with his study of descriptive geometry, had similarly expanded and renewed the mathematics practiced by the synthetic school. The synthetic school, under Flauti, continued to dominate the Academy of Sciences and the universities, but its influence began to wane. (See [5].)

The events of 1860 with the annexation of the Kingdom of the Two Sicilies to Piedmont, leading to the unification of Italy, profoundly changed Neapolitan society, and consequently its academic world. The old power groups reorganized themselves in a different way and wound up being incorporated into the new Italian state. Old mathematical disputes between the university and the *Scuola di Applicazione* modified as the division between the work of the Corps of Engineers and that mathematics taught in the university began to close. There was an growing awareness of the link/close relationship between university teaching and research. (See [5].)

The events of 1860 also changed the relationship between Flauti and Trudi [6] Flauti strongly opposed the annexation of Naples to Piedmont. While the rest of the population was celebrating the annexation, Flauti closed the balcony of his house in protest. He didn't stop there; he went on to write several pamphlets vehemently criticizing the reform laws of the University of Naples. These reforms were a continuation and expansion of the university reforms begun in 1850. Following the 1860 annexation and unification, Flauti was able to retain the title of professor emeritus, and Trudi his position as a university professor. However, when it came time to appoint members of the new Academy of Sciences which was to replace the deleted Academy of the Bourbons, while Flauti, despite his many criticisms of the university reforms, was consider for membership, in the end, he was not chosen. Trudi was chosen [5]. It became apparent that it was not enough to be a good mathematician, political correctness was also required. Those associated with the Bourbon ruling class were offered the chance to join the new regime. Trudi chose to do so; Flauti did not. Trudi and Flauti parted ways. "Trudi became an Italian mathematician. Flauti, however chose to die a Neapolitan." (See [6].) Flauti and Trudi did more than part ways. Flauti, feeling betrayed by Trudi, denounced him, accused him of ingratitude, hurling insults at him. Trudi's attitude was one of forgetting. In his subsequent writing he remembered the contributions of mathematicians such as Euler and Crelle but failed to mention those of Fergola [5]. The adjective "Bourbon" took on a derogatory connotation and Flauti was definitely a Bourbon. In order to become an Italian mathematician, Trudi needed to break with his past. Flauti and the school of Fergola became for him a "damnatio memoriae."

Lessons

What can we learn from this period of Italian mathematics and its central players? One of the first lessons is that changes in society have a definite impact on mathematics, how it is viewed and used, as well as the definition and role of a mathematician. The controversies between the synthetic and analytics school appeared to be about the right way to do mathematics, but underlying this were major worldview assumptions about how truth is revealed, how mathematicians are to interact with society, the purpose of doing and teaching mathematics, and how to respond to changes in our natural world. For Fergola and his followers, mathematics and faith were definitely intertwined. Mathematics for them was a way to meditate on the wonders of God and his creative order. But mathematics became a tool to wage war against the changes they saw occurring in the world around them. In the early years Fergola was not only aware of the work of Lagrange and his analytical methods, Fergola advocated for these methods. That changed when these methods took on a political and social role that threatened the established order—the role of the church in society, the aristocratic power structure, and the local autonomy, which allowed for decisions to be made by people who understood the needs of the people in their care. He, and his followers, saw the implications the analytical methods were having in society. Many advocates for the analytical methods saw their abstraction, and their separation of the mathematics from the geometric construction as a new way of looking at mathematics and its applications in society. Social and natural problems could be mathematized and solved. Gone was the spiritual element of mathematics; it was replaced by a sense of man's ability to solve all the problems in the world around him—social, economic, technological—with the help of mathematics. It was natural that Fergola would react to this. But he and his followers did it in such a way to set up a false dichotomy between the two kinds of mathematics. There wasn't a 'right' kind of mathematics, it was what the mathematics represented and how it was used. Fergola's school also used mathematics to defend its worldview. By making claims that only a Christian could understand and produce good mathematics, or using mathematics to justify certain religious or political practices, they used mathematics to promote their worldview, just as the analysts did.

It is interesting to note that Fergola and Flauti were both willing to expand their view of acceptable mathematics when it came to the work of their protégés. Fergola continued to support the work of Giordano, even after Giordano joined the analytics; Flauti continued to support the work of Trudi even when Trudi's work focused more on analytic and projective geometry. For both Fergola and Flauti this was probably due to the natural pride they had in the accomplishments of their students. For Flauti it also appeared to have been a desire to have the name of the Fergola school be glorified, so he 'saw' Trudi's work as a continuation of the Fergola tradition even though the mathematics Trudi did was different. In the end it wasn't about the mathematics itself, but what it represented to each party. After the 1839 challenge, the importance of the synthetic school waned and most mathematicians abandoned its methods. Even the memory of the school was lost until late in the 19th century when Gino Loria, a mathematical historian, read a few lines about the school in some writing of Michael Chasles [5].

Looking back on the history of mathematical controversies allows us to reflect, with some detachment, on decisions mathematicians make when dealing with changes in the world around them. Society and the natural world are changing today, at probably even a faster rate than in the 19th century. Mathematics plays a role in how we respond to those changes. It is important that we be clear about what the issues are, and how mathematics actually applies to those issues. Rather than make false dilemmas (such as the right way to do geometry), mathematicians from different perspectives would be better off dialoguing and working together to find common ground. Of course, there may be times when it is necessary to take a firm stand, but these should be articulated clearly and not hidden behind false polemics.

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Parables to a Mathematician

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Abstract

Jesus frequently used parables in His ministry, usually short narratives illustrating the outcomes of people's choices. In John 3:12 and Matthew 13:10-15, He explained that one reason was to be sure that people who genuinely wanted to understand His message would be able to do so. Since most of His audience was familiar with an agrarian economy, Jesus spoke extensively of wheat, fish, trees, wine, debt, tenants, lamps, etc. Many people have speculated on parables Jesus might have used had He lived in a different society. This non-scholarly (but hopefully thought-provoking) talk will propose parables targeted toward groups of mathematicians with various levels of Christian background.

1. Parable of the Lecturer

A professor entered her large lecture hall and began to illustrate the details of hypothesis testing to her General Statistics class. Some of her students were busy texting throughout the hour, cradling their phones with the time-honored logic "if she can't actually see my phone, she won't know why I'm staring between my thighs for 50 minutes." At the end of the hour, they left, blissfully unencumbered by the details of P-values. Other students took notes and later started the homework, but then realized the only thing they knew about Type I and Type II errors was that they are both bad. When the exam came, they wrote little and scored little. Still others were enthused about the content and finished the homework assignment, but later in the semester when their extracurricular activities pressed in, they had no more time to study. When the exam came, they wrote much and scored little. But some students thoughtfully listened, read, practiced, and organized. On the exam, some got a B+, some an A-, and some an A. Anyone who hears this should pay attention.

2. Parable of Convergence of a Series

To introduce the ratio test for the convergence of infinite series, a calculus professor asked the class to discuss in groups the convergence of $\sum_{n=1}^{\infty} \frac{n^{400}}{2^n}$. One group reported divergence by the n-th term test. When questioned, they displayed a calculator graph of the sequence $a_n = \frac{n^{400}}{2^n}$ for $1 \leq n \leq 100$, showing monotonic increase. The professor suggested trying even larger values of n; the group then reported that (a_n) is monotonic for n as large as 300, at which point their calculator had a numerical overflow. Overhearing this discussion, several other groups switched their answers to divergent. I tell you, don't be deceived. Don't be surprised when "evidence," even when provided by "experts," arises that my word is not true. Test this "evidence" thoroughly and objectively. Anyone who hears this should pay attention.

3. Parable of No Pop Quizzes

Rebecca was doing very well in her discrete math class, which met on Mondays, Wednesdays, and Fridays. One Friday, at the beginning of a unit on logic, the professor announced there would be a pop quiz next week.

Rebecca reasoned that the quiz wouldn't be on Friday since if the quiz hadn't happened by Wednesday, the students would all know it would be on Friday. Since the quiz would then fall on either Monday or Wednesday, by the same reasoning it wouldn't be on Wednesday. But then it wouldn't be a surprise on Monday either, so there could be no quiz at all. Rebecca choose not to study and was very unpleasantly surprised when she did poorly on the pop quiz on Monday. I tell you, watch, for I will return at a date when you don't expect me. Those who are watching will be blessed. Anyone who hears this should pay attention.

4. Parable of "Whose Paper?"

Three theoretical physicists, Jill Strake, Rafael Ebberle, and Miranda Collet, collaborated on a paper in which certain mathematical properties were derived for spacetime wormholes. The paper was actually a direct continuation of earlier results from Strake incorporating some new ideas from Ebberle. Collet was a senior colleague of Ebberle who participated in several helpful but brief discussions with the other two. The paper was accepted by a mathematical journal, and a review copy was sent to the authors listing them as Collet, Ebberle, and Strake. Strake, believing that she should be listed first as the lead author, contacted the editor who explained that authors' names were always listed in alphabetical order as a matter of editorial policy. Strake requested an exception, explaining that the majority of content was due to her and that her tenure review committee might make incorrect inferences from the citation, but the editorial board declined to adjust. "Since you submitted a paper with three names on it," the editor said, "presumably you all had a significant part in generating the content. We are publishing your paper as you hoped we would and doing you no disservice by following our policy." I tell you, the last will be first and the first will be last. Don't become so fixated on your own career that you lose the wonder of discovery of your Father's world. Anyone who hears this should pay attention.

5. Parable of the Journal Review

The kingdom of heaven is like a mathematical journal's review process which selects the submitted articles meeting the journal's publication standards and rejects the rest. It will be like this at the end of the world.

6. Parable of the Groundbreaking Proof

The kingdom of heaven is like a famous conjecture outstanding for 350 years, which when a mathematician became obsessed with it, he worked in complete isolation for seven years, then announced a proof with joy. I tell you, let your life be guided by what is truly important, for you will become like the thing into which you invest your time.

7. Parable of Standardized Testing

The kingdom of heaven is like the secondary school mathematics curriculum into which state legislatures gradually inserted a variety of standardized tests until the whole curriculum was permeated with assessment.

8. Parable of the Old and New

No one uses classical Euclidean geometry to study the shape of DNA molecules, otherwise the problems would be completely intractable. Rather, to solve these new problems, we use newer concepts from topology and knot theory. Likewise, don't rely totally on human reasoning to explain the Father's unearned love for you and His plan for creation. Logic has its place, but now that you are born again, let the Spirit guide your thinking.

9. Parable of the Proofreader

Someone who hears my words is like a mathematician who double checks his proof of a theorem before working on its corollary. When the referees reviewed his paper, they recommended publication it because it was carefully written, based on prior good work, and showed creativity. But the one who does not hear my words is like a mathematician who submits a paper that will not withstand logical examination. When the referees reviewed his paper, it was rejected out of hand. I do not promise you that life will be easy—in fact, I promise that life will be hard. Heaven and earth will pass away, but my word will not pass away. Be sure to invest your life in things that will remain when the power centers of this world crumble.

10. Parable of the "Obvious""Facts"

Brandon, a talented freshman who unfortunately got a 2 on the AP Calculus exam, was beginning to share his boredom in his university Calculus I course. After the professor used the disk method to prove the formula for the volume of a sphere, Brandon commented that obvious results should not need formal proof. The professor invited Brandon for coffee after class. During the course of conversation, the professor asked, "If I have an infinite number of dresser drawers, each with at least one sock in it, is it possible to select exactly one sock from each drawer?" "Sure," replied Brandon. "OK," the professor continued, "and if I cut a solid ball apart and put the pieces back together into another solid ball, must the volume remain the same?" "Sounds right," replied Brandon a little less confidently. The professor then explained that the Banach-Tarski Paradox shows that both of these obvious facts can't be simultaneously true. I tell you, be alert for oversimplifying issues—think them through with the Spirit's guidance. And when you have the opportunity to express your opinions, express them with courtesy and meekness, realizing that if even your perceptions about mathematics can be badly inaccurate, your opinions about spiritual matters may also be flawed. Anyone who hears this should pay attention.

11. Parable of an Axiom Choice

There are many spirits gone out into the world; consider carefully what you accept as absolute truth, for your beliefs will build upon themselves and shape your whole world. Learn a lesson from geometry. A Euclidean carpenter mindlessly squares up foundation walls using 3-4-5 right triangles, fits together rectangular bricks, and builds bridges across angles by starting in the interior and building straight out to the sides of the angle. On the other hand, a hyperbolic carpenter measures areas of triangles with a protractor, measures distances in "absolute length units," and simply accepts without question that all roofs sag. These two carpenters could never build together—each would believe the other is a madman. Such will be the fate of all your human relationships if you become deceived into thinking your own opinions are canonical.

Conclusion

I found the writing of these parables to be extremely difficult. When I reflected about why this was, I realized a number of reasons: my mind kept starting with the mathematics rather than with the spiritual principle (is this a math conference for Christians or a Christian conference for mathematicians), I personally have few relationships with people having both mathematical sophistication and Christian naiveté, storytelling is de-emphasized in western academic culture, getting the right length is essential to a good parable, it is hard to decide how much humor to incorporate, and it would be nearly sacrilegious to contextualize some parables (such as the prodigal son).

The *Best* Religious Calendar

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Abstract

We show that the phases of the moon loosely cycle with short period 19 years and long period 160 years. Counterintuitively, the variation in the short period is over five times larger than the variation in the long period. We give a rationale as to why this ratio is so large involving what we call the *signature* of a real number and the *continued fraction* of a real number.

Hooray! It's Hanukkah![1] It's Easter![2] It's Ramadan![3] It's Diwali![4] It's Durin's Day![5] It's Tét![6] When next will such a day arrive on the same date with respect to the seasons, and, in particular, the Gregorian calendar?

The *Quest* or answer to this question will be that integer p years for which the standard deviation between new moons p years apart is minimal, provided p is not too large. Does such an integer exist? Isn't the moon's cycle chaotic?

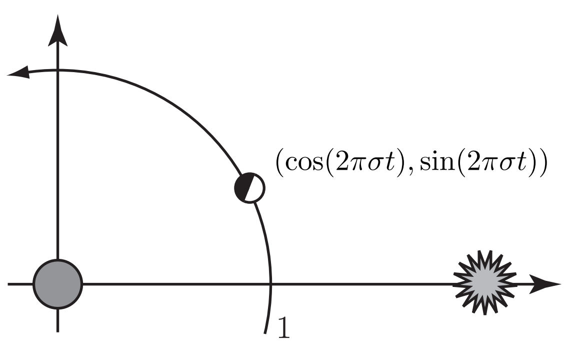
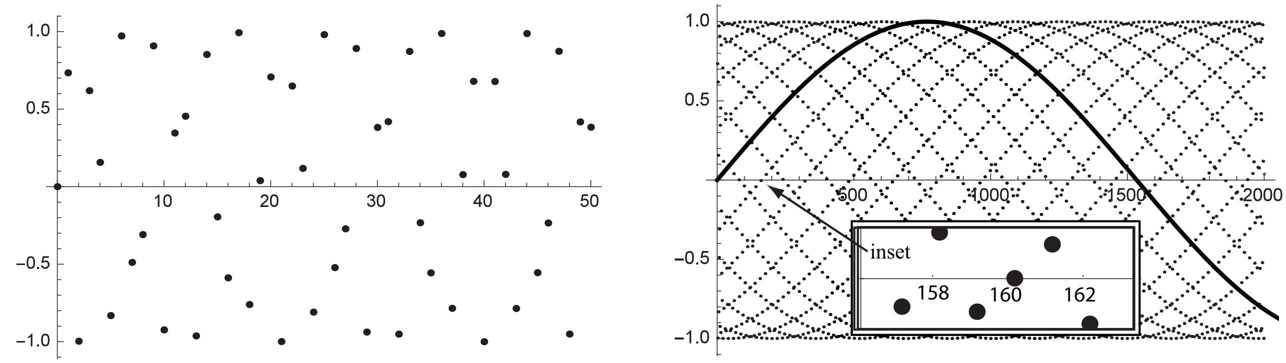


Figure 1: The earth-moon-sun model

A Simple Earth-moon-sun Model

The question of the frequency at which the moon cycles is rooted in the earliest days of mankind. One difficulty with characterizing the moon's motion is that it involves the three body problem. After Isaac Newton derived Kepler's laws from first principles assuming an inverse square law of gravitation, he focused on the earth, moon, and sun, so as to determine where the moon would be at any time and ultimately gave up, saying to Edmund Halley that

[1] A Jewish holiday starting on the 25^{th} day of the lunar month Kislev, where the beginning of each month is a new moon.
[2] A Christian holiday on the first Sunday after the full moon following the spring equinox.
[3] A month long Muslim fasting holiday starting with the new moon that initiates the lunar month of Ramadan.
[4] A Hindu holiday whose zenith is the new moon between mid-October and mid-November.
[5] A fictional realm of Middle Earth holiday starting on the first day of the last moon of autumn [7].
[6] Also known as the Chinese New Year, which is usually the second new moon after the winter solstice.



(a) A smattering of dots

(b) Connecting a strand of dots

Figure 2: The moon's signature, $\mathcal{S}_\sigma$

> The three body problem had "made his head ache, and kept him awake so often, that he would think of it no more." (Steven Strogatz [6, p. 160])

Although the mean time it takes the moon to complete one circuit of the earth with respect to the sun is about 29.53 days (the *synodic* period), the exact time varies up to about 7 hours from this mean. Indeed, in 1887, Henri Poincaré showed the futility of searching for an analytic lunar cycle formula, that the very pattern is one of chaos. Of course, we can dynamically number crunch and extend our predictions of the earth's and the moon's position to a reasonable degree of tolerance arbitrarily far into the future as NASA has done, logging the dates of the four quarters of the moon over a 6000 year period, [2].

In spite of Poincaré's observation—which might doom the very foundation of our Quest—we model the motion of the earth, moon, and sun, using the simplest non-trivial model imaginable and hope that analyzing such a model leads us nearer our Quest. Today, from NASA websites, the moon's mean period around the earth is $T_m \approx 27$ days, 7 hours, 43 minutes, and 11.5 seconds $= 2\,360\,591.5$ seconds (the *sidereal* period) and the earth's period around the sun with respect to the relatively fixed starry background is $T_s \approx 365$ days, 6 hours, 9 minutes, and 9.5 seconds $= 31\,558\,149.5$ seconds. Even though we have long since abandoned a geocentric model of the solar system—to keep the corresponding dynamics simple—we imagine the sun as going around the earth with period T_s. Thus the relative angular velocity σ of the sun to the moon is $\sigma = T_s/T_m \approx 13.368747$.

Again, to keep the dynamics simple, we fix the sun as well as the earth in place so that the relative angular velocity of the moon around the earth is $\sigma_0 = \sigma - 1 \approx 12.368747$, a value agreeing with our experience of having twelve moons (months) per year. With respect to the earth and the sun under these assumptions, the moon's position with respect to time t is *simple harmonic* motion given by

$$(\cos(2\pi\sigma t),\, \sin(2\pi\sigma t)) \tag{1}$$

wherein the moon is unit distance from earth's center. To analyze when new moons reprise in our system, it is sufficient to study only the second component of (1), $w(t) = \sin(2\pi\sigma t))$, because a new moon occurs at each of w's roots when the graph of w crosses the t-axis from negative to positive. For any positive real number α, we say that α's *signature* $\mathcal{S}_\alpha$ is

$$\mathcal{S}_\alpha = \{(n,\, \sin(2\pi\alpha n))|n \in \mathcal{N}\},$$

where $\mathcal{N}$ is the set of integers, first defined in [4] and [5]. In terms of our model, $\mathcal{S}_\sigma$ logs how far the moon is from being new or full at precisely n years later than the initial time, $t = 0$ years. When we peek at $\mathcal{S}_\sigma$ over a short period of years, such as 50 years, we see no real pattern in the arrangements of the signature's dots, as shown in Figure 2a, but when we take a long view over, say, 2000 years, the dots seem to sort themselves into 19 sine-like strands, with successive strands being 160 year translates of each other, as shown in Figure 2b, in which we have connected adjacent dots along the strand containing the origin. That is, if time z years is the date of a new moon, then $z \pm 19$ years are also close to being new moon dates because $w(z) \approx 0$ and $(z \pm 19,\, w(z \pm 19))$ are adjacent strand points with $(z,\, w(z))$, which means that $w(z \pm 19) \approx 0$. Furthermore, $(z + 160,\, w(z + 160))$ is on the strand adjacent to $(z,\, w(z))$'s strand and $w(z + 160) \approx 0$, which means that at time $t = z + 160$, the moon should almost be new.

That is, our Quest answer has dwindled to two options, $p = 19$ years or $p = 160$ years. Which is the better value in our model?

Checking the Simple Earth-moon-sun Model Against NASA Data

The observation that the moon's phases cycle with period 19 years dates at least to the fifth century BC Athens astronomer Meton[7] who championed a new calendar based upon a 19 year cycle. Indeed the Babylonians probably knew about this 19 year cycle as well. And all of the world's lunar holidays, including Easter, are on a 19 year cycle. Did they/we get it correct?

To determine this toss-up, we contrast the 19 year cycle and 160 cycle using data available from NASA's website [2]. We say that a *short span* is an ordered pair of new moon dates approximately 19 years apart, and a *long span* is an ordered pair of new moon dates approximately 160 years apart with the first components being less than the second components.

Table 1 is a listing of 30 non-overlapping short spans ranging from 1480 to 2099 wherein each span contains precisely four leap years. The asterix marking the year 1600 in the table's first column serves to alert the reader of the 1582 calendar change from the *Julian Calendar*—in which every fourth year is a leap year—to the *Gregorian Calendar*—in which every fourth year is a leap year *except* at century years non-divisible by 400, called *deficient* centuries. That is, year 1700 is a deficient century, but not year 2000. The central column of this table gives the time difference in hours, *modulo* 19 years, between the short span dates: second component minus first component. Thus, for example in the first row of the table, the difference between 3 November 1499 and 2 November 1480 is 24.63 hours. The mean and standard deviation of these 30 *short span differences* are $\bar{x}_1 \approx 16.16$ and $s_1 \approx 7.79$ hours, where each short span contains exactly four leap years.

Table 2 is a listing of 30 non-overlapping long spans ranging from year 2000 BC through year 2881 AD. In the first column some years have asterisks attached, such as 1601* and 1761**. The notation n* means that the long span $(n,\, n + 160)$ contains exactly one deficient century. For example, between 1601 and 1761, the year 1700 is deficient. The notation n** means that the corresponding long span contains exactly two deficient centuries. For example, between 1761 and 1921, both 1800 and 1900 are deficient. Thus, the time differential within a single-asterisk-long span must be reduced by 24 hours, and within a double-asterisk-long span by 48 hours. The central column of this table reflects this adjustment. Note also that no long span in the table contains the year 1582, the year the Gregorian Calendar supplanted the Julian Calendar in our time reckoning.

The mean and standard deviation of the *long span differences* of these 30 long spans are $\bar{x}_2 \approx 25.62$ hours and $s_2 \approx 3.31$ hours. The ratio of s_1 to s_2 is about 2.35; equivalently, the ratio of their variations is over 5.5. In trials of 30 alternate short and long spans, this ratio waxed higher at times. Does our simple model anticipate a variation ratio this high, or is this ratio inflated because of chaos?

[7] Meton is immortalized in Aristophanes' play *The Birds* in which he makes a brief appearance as a comic circle-squarer accoutred with compasses and straight-edges [1, p. 304].

Table 1: NASA's New-moon Dates 19 Years Apart

α	hour lapse	$\alpha + 19$ years
2 Nov 1480 @ 8:48	24.633	3 Nov 1499 @ 9:26
22 Oct 1500 @ 23:23	15.683	23 Oct 1519 @ 15:04
11 Oct 1520 @ 15:54	8.117	12 Oct 1539 @ 0:01
29 Oct 1540 @ 20:50	18.667	30 Oct 1559 @ 15:30
19 Oct 1560 @ 6:49	24.000	20 Oct 1579 @ 6:49
5 Nov 1600* @ 22:53	8.150	6 Nov 1619 @ 7:02
25 Oct 1620 @ 13:55	21.983	26 Oct 1639 @ 11:54
15 Oct 1640 @ 4:10	22.183	16 Oct 1659 @ 2:21
3 Nov 1660 @ 1:06	11.300	3 Nov 1679 @ 12:24
22 Oct 1680 @ 11:55	11.317	22 Oct 1699 @ 23:14
12 Oct 1700 @ 10:16	24.017	13 Oct 1719 @ 10:17
31 Oct 1720 @ 11:42	21.883	1 Nov 1739 @ 9:35
20 Oct 1740 @ 16:35	8.150	21 Oct 1759 @ 0:44
9 Oct 1760 @ 1:36	−8.600	8 Oct 1779 @ 17:00
27 Oct 1780 @ 17:10	24.167	28 Oct 1799 @ 17:20
18 Oct 1800 @ 8:58	18.717	19 Oct 1819 @ 3:41
6 Nov 1820 @ 0:08	8.067	6 Nov 1839 @ 8:12
25 Oct 1840 @ 8:59	15.567	26 Oct 1859 @ 0:33
14 Oct 1860 @ 14:38	24.517	15 Oct 1879 @ 15:09
2 Nov 1880 @ 15:55	18.533	3 Nov 1899 @ 10:27
23 Oct 1900 @ 13:27	7.200	23 Oct 1919 @ 20:39
12 Oct 1920 @ 0:50	19.667	12 Oct 1939 @ 20:30
30 Oct 1940 @ 22:03	24.633	31 Oct 1959 @ 22:41
20 Oct 1960 @ 12:02	14.350	21 Oct 1979 @ 2:23
9 Oct 1980 @ 2:50	8.733	9 Oct 1999 @ 11:34
27 Oct 2000 @ 7:58	19.667	28 Oct 2019 @ 3:38
16 Oct 2020 @ 19:31	23.633	17 Oct 2039 @ 19:09
4 Nov 2040 @ 18:56	14.250	5 Nov 2059 @ 9:11
24 Oct 2060 @ 9:25	8.900	24 Oct 2079 @ 18:19
13 Oct 2080 @ 2:44	22.800	14 Oct 2099 @ 1:32
average lapse:	16.160	st.dev: 7.79 hours

Table 2: NASA's New-moon Dates 160 Years Apart

α	hourly lapse	$\alpha + 160$ years
5 Nov 2000 BC @17:15	26.117	6 Nov 1840 BC @19:22
8 Oct 1840 BC @6:34	29.717	9 Oct 1680 BC @12:17
9 Oct 1680 BC @12:17	30.067	10 Oct 1520 BC @18:21
10 Oct 1520 BC @18:21	29.333	11 Oct 1360 BC @23:41
11 Oct 1360 BC @23:41	28.017	13 Oct 1200 BC @3:42
13 Oct 1200 BC @3:42	26.300	14 Oct 1040 BC @6:00
14 Oct 1040 BC @6:00	23.900	15 Oct 880 BC @5:54
15 Oct 880 BC @5:54	21.283	16 Oct 720 BC @3:11
16 Oct 720 BC @3:11	19.783	16 Oct 560 BC @22:58
16 Oct 560 BC @22:58	20.150	17 Oct 400 BC @19:07
17 Oct 400 BC @19:07	22.233	18 Oct 240 BC @17:21
18 Oct 240 BC @17:21	24.850	19 Oct 80 BC @18:12
19 Oct 80 BC @18:12	26.917	20 Oct 81 @ 21:07
20 Oct 81 @ 21:07	28.283	22 Oct 241 @ 1:24
22 Oct 241 @ 1:24	29.233	23 Oct 401 @ 6:38
23 Oct 401 @ 6:38	29.667	24 Oct 561 @ 12:18
24 Oct 561 @ 12:18	28.850	25 Oct 721 @ 17:09
25 Oct 721 @ 17:09	26.717	26 Oct 881 @ 19:52
26 Oct 881 @ 19:52	24.167	27 Oct 1041 @ 20:02
27 Oct 1041 @ 20:02	21.817	28 Oct 1201 @ 17:51
28 Oct 1201 @ 17:51	20.350	29 Oct 1361 @ 14:12
29 Oct 1361 @ 14:12	19.917	30 Oct 1521 @ 10:07
25 Oct 1601* @ 23:11	22.883	27 Oct 1761 @ 22:04
27 Oct 1761** @ 22:04	25.567	30 Oct 1921 @ 23:38
30 Oct 1921 @ 23:38	27.433	1 Nov 2081 @ 3:04
1 Nov 2081** @ 3:04	28.317	4 Nov 2241 @ 7:23
4 Nov 2241* @ 7:23	28.917	6 Nov 2401 @ 12:18
8 Oct 2401* @ 3:10	27.700	10 Oct 2561 @ 6:52
10 Oct 2561** @ 6:52	26.283	13 Oct 2721 @ 9:09
13 Oct 2721 @ 9:09	23.967	14 Oct 2881 @ 9:07
average lapse:	25.62	st.dev. 3.31 hours

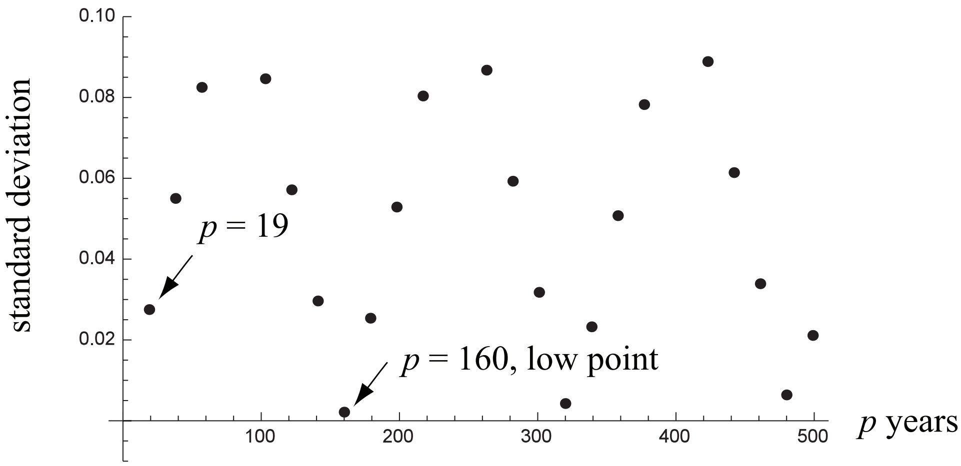


Figure 3: Standard deviation, $\sqrt{1-\cos(2\pi\sigma p)}$

Expected Value of the Variation in New Moon Dates p Years Apart

To calculate the variation between new moon years given by our earth-moon-model, recall that the *average value* of a function g over the interval $(0,\, T)$ is

$$\frac{\displaystyle\int_0^T g(x)\,dx}{T}. \tag{2}$$

Let $f(t,\, p) = w(t+p) - w(t) = \sin(2\pi\sigma(t+p)) - \sin(2\pi\sigma t)$. Since the period of $w(t)$ is $\frac{1}{\sigma}$, the average value of $f(t,\, p)$ with respect to t is 0 by (2) for any given integer p. And so the average value of the square of $f(t,\, p)$—the variation we seek—is $v(p)$,

$$v(p) = \sigma \int_0^{\frac{1}{\sigma}} \Big(\sin(2\pi\sigma(t+p) - \sin(2\pi\sigma t)\Big)^2\, dt = 1 - \cos(2\pi\sigma p). \tag{3}$$

Figure 3 is a graph of the standard deviation of the differences in moon positions p years apart as p ranges from 1 to 500. In particular, the standard deviation is lowest at $p = 160$, and is not quite so low at twice and thrice this value. Indeed, since $w(19) \approx 0.0275$ and $w(160) \approx 0.0021$ then $\frac{w(19)}{w(160} \approx 12.90$. If the ratio of the standard deviations of the difference in moon displacement from new and full phase 19 and 160 years apart in our model is positively correlated with the ratio of short and long span differences with respect to NASA's data bank, then our simple earth-moon-model has indeed anticipated the standard deviation at $p = 160$ years being significantly less than at $p = 19$ years in NASA's data. The fact that it does so is fairly remarkable, considering that the moon's position in time is chaotic! That is, our humble earth-moon-sun model is fairly powerful.

Continued Fractions and Precession of the Earth

At this point, our reaction might be, "Of course the standard deviation is less at 160 than at 19, because $\frac{59}{160}$ more closely approximates the fractional part of $\sigma \approx 12.368747$ than does $\frac{7}{19}$.

These two denominators, 19 and 160, arise from the *continued fraction* representation for σ,

$$\sigma = [12; 2,\, 1,\, 2,\, 2,\, 8,\, 12,\, 1,\, 9,\, 4,\, 1,\, 2,\, \ldots],$$

whose *convergents*—the partial expansions of this representation—approach σ, whose third, fourth, fifth, and sixth convergents of $\sigma - 12$ are

$$\frac{1}{2+\frac{1}{1+\frac{1}{2}}} = \frac{3}{8}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{7}{19}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{8}}}}} = \frac{59}{160}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{8+\frac{1}{12}}}}}} = \frac{715}{1939}. \tag{4}$$

A second reaction to these results might be, "The average difference between long spans from Table 2 is $\bar{x}_2 \approx 25.62$ hours. Why so far from 0?"

Some of $\bar{x}_2$'s seemingly inflated value is due to chaos <u>and</u> to the *precession* period of the earth wherein earth's new year rotates around the zodiac in about 25800 years. So we could modify σ by somewhere around $\frac{1}{25800} \approx 0.0000387597$, and re-analyze our model accordingly. Alternatively and more easily—we exploit the recursive nature of the convergents in (4) and realize that the convergent $\frac{59}{160}$ is the *general mediant*, $f(n)$, of its two preceding convergents,

$$f(n) = \frac{3+7n}{8+19n} \tag{5}$$

where $n = 8$. Observe that $f(0) = \frac{3}{8}$ the third convergent of $\sigma - 12$ and $f(n)$ converges monotonically to $\frac{7}{19}$, the fourth convergent, as $n \to \infty$. Therefore, chaos and precession may have caused the relative angular velocity value σ of the moon to stray from $f(n)$ where $n = 8$. Since precession should increase σ's value which in turn means that n should decrease, this modified angular velocity has new fifth convergent either $f(7) = \frac{52}{141}$ or $f(6) = \frac{45}{122}$. Consulting NASA's data for medium spans of length 141 years and 122, we find that 141 years is the more agreeable result. For a cycle length of 141 years, Table 3 gives the mean as about 8 hours with a standard deviation of 2.51 hours. Therefore 141 bests a cycle of 160 years.

Table 3: NASA's New-moon Dates 141 Years Apart

α	hour lapse	$\alpha + 141$ years
14 Oct 3 @ 6:09	6.98	14 Oct 144 @ 11:08
29 Oct 7 @ 20:18	12.07	30 Oct 148 @ 8:22
15 Oct 11 @ 13:14	5.33	15 Oct 152 @ 18:34
31 Oct 15 @ 14:24	9.92	1 Nov 156 @ 00:19
16 Oct 19 @ 21:28	6.68	17 Oct 160 @ 4:09
2 Nov 23 @ 4:58	7.52	2 Nov 164 @ 12:29
18 Oct 27 @ 8:23	8.98	18 Oct 168 @ 17:22
3 Nov 31 @ 15:39	5.80	3 Nov 172 @ 21:27
19 Oct 35 @ 23:00	11.43	20 Oct 176 @ 10:26
4 Nov 39 @ 23:40	5.02	5 Nov 180 @ 4:41
average lapse:	7.97	st.dev: 2.51

Conclusions

The answer for our Quest of when a lunar holiday will next arrive is a toss-up between 19, 141, and 160 years, with 19 losing the standard deviation battle, yet winning a measurement test that truly matters, namely, a time lapse within man's expected lifespan. So until the human race can more than double its life expectancy of about 70, 19 beats both 141 and 160, and is the winner by default! Nevertheless, from this little study we make three observations.

- Sometimes a simple model may unveil a phenomenon's structure, such as uncovering the numbers 19 and 160 and 141—as well as remind us about pertinent related mathematical tools, such as continued fractions, (for which [3] is a good resource).
- Sometimes a simple model may suggest an important natural definition, such as the definition of a number's signature. In particular, $\mathcal{S}_\sigma$ is an explicit visual representation of the denominators of the fourth and fifth convergents of σ, and in fact, implicitly contains the rest of them as well—but that is another story.
- Even though a phenomenon's chaotic nature may dissuade us from further investigation, this model suggests that an attractor might exist within the earth-moon-sun system which somehow yanks the system into long range regularity.

Finally, we close with an advisory-alert. Since the moon is currently receding from the earth at a rate of about 3 cm/year, the angular velocity of the moon is decreasing while earth's angular velocity about the sun more or less remains constant. As time goes on the 19 year cycle of the moon will fade, and eventually wax into an 11 year cycle. The good news for our lunar holidays: we need not change our calendars for at least another four hundred thousand years.

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The Remarkable Mrs. Somerville

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Abstract

As a woman growing up in the late eighteenth century, Mary Somerville (1780-1872) was denied access to most formal education and getting a university education was completely out of the question. Yet her interests in nature, science, and mathematics, coupled with an intense curiosity and tenacious desire to learn led her to eventually be known and respected by scientists, mathematicians, and intellectuals in both Britain and France. She is one of the important woman in the history of mathematics, even though she did not publish original work. However, she was a talented writer, producing several significant works, including *Mechanism of the Heavens*, a translation and amplification of Laplace's great work, and, at the end of her life, a series of *Personal Reflections*. Reading through her reflections gives an interesting glimpse into her personality, her opportunities for social networking, and some of what motivated her work. This paper introduces Mary Somerville and provides glimpses into the circumstances that led her to write her first book, *Mechanism of the Heavens*.

As is true for nearly every woman in the history of mathematics, Mary Somerville (1780-1872) was a most remarkable person. Denied both a formal early education and the opportunity to attend university, she diligently nurtured her passion for mathematics and science to become the author of important books [4, 5, 6] and earn a respected place among the scientific elite of her era [2, 1, 7]. Her broad interests also included painting and music, and she must have been a lively conversationalist; a popular dinner guest, she tells about a number of social engagements, both at home and abroad, in her personal reflections. Her friends in Britain included John Herschel, Thomas Young, Michael Faraday, William Whewell, Ada Lovelace, and Charles Babbage, among others, and when she was abroad she conversed with people like Laplace, Poisson, and even Lafayette. She had occasion to meet Princess Victoria and was present at Victoria's Coronation. In short, from rather plain beginnings, she became a person with significant intellectual standing and accomplishment.

Fortunately, Mary Somerville gives us a glimpse into her life through a collection of personal reflections published shortly after her death [3]. Although there has been a great deal of scholarship published about her, she is still a stranger to many people. Therefore, in what follows I would like to offer a brief introduction, based on her own words, to her early life, leading up to the publication of her first book. I hope you will find her story interesting enough to inspire you to learn more about her life and accomplishments.

Although Mary Somerville is often referred to as a mathematician, she actually produced very little original mathematics. Her major scientific contributions consist of four substantial works primarily designed to introduce science and mathematics to a broad, general audience. It is helpful to keep in mind that these works were written to disseminate scientific results, often requiring the use of technical mathematics, by a person who was essentially an amateur mathematician/scientist, all while she was also running her household, raising and helping to educate her children, and dealing with all the uncertainties and challenges that life brings. Although we cannot consider any of these works in detail, we will draw significantly on her fifth major work, a series of personal reflections, published shortly after her death.

Her first book, *Mechanism of the Heavens* [3], was published in 1831 and intended as a translation and amplification of Laplace's Mécanique Céleste. Because this text was written for a general audience but requires a substantial knowledge of mathematics (including many results in the calculus) and science, she begins with a seventy page Preliminary Dissertation. Kathryn Neeley [2] points out that as Mary Somerville was writing this book she was confronted with the significant problem of how to make the material both relevant and interesting to the reader (page 101). Neeley describes how, "These seventy pages exemplified the abilities and qualities of her writing that were to be her greatest strengths throughout her career-the capacity to help the reader visualize the view of the natural world that emerged from physical science, an ability to associate that view with elevated feeling and beauty, a capacity to achieve clarity and precision without providing overwhelming amounts of detail, and an ability to connect esoteric theories to problems of everyday life. In the process, she created a powerful interpretive frame for science and articulated the intellectual foundations on which the remainder of her scientific work and writing would be built." The book was a resounding success and was quickly adopted as a textbook at Cambridge.

Her second work, *On the Connextion of the Physical Sciences* [4], followed shortly after the first, appearing in 1834. As Richard Holmes writes in The Observer, she surveyed what then existed of contemporary science, chemistry, physics, and astronomy, and described the unity of their underlying principles and methodology. He goes on to say that, "As a result of a positive review by William Whewell, FRS, the future master of Trinity College, Cambridge, the inclusive term "scientist" was coined. Amazingly, the word had not yet existed before 1834." The book had 10 editions and Holmes claims that it shaped ideas in science for the next half century.

Referring to her third book, *Physical Geography* (1848), Neeley states that, "Her objective was to discern the most important connections and thus to determine the principles that shaped the land and explained the distribution of animal and vegetable life on land and water [5]." Her final work, *On Molecular and Microscopic Science*, was published in 1869. She started writing the book in her 79$^{\text{th}}$ year and it gave her an opportunity to return to a discussion of the material world, the place where her initial interest in science was awakened.

The final work on our list, *Personal Reflections*, also appeared in 1873 [7], but it is, as the title indicates, a personal history and not a scientific work. The edition that is most readily available was edited by her daughter, who included a number of helpful comments and explanations. This book, being much more personally oriented than the others, provides interesting details about her thoughts and activities and gives her opportunity to insert opinions and commentary about both events and people she meets throughout her life. It also provides a glimpse into the scientific community in the nineteenth century.

Along with details about her life, Somerville also offers interesting, even if relatively unimportant asides. Descriptions of dinner parties and trips to the theater are included, as well as personal commentary about other people, some of which is amusing as well as insightful. For instance, she describes an aging professor who kept trying to dye his hair until it eventually acquired a purple hue, while also encouraging Mary to bring young women around to see his experiments. Despite his heroic efforts, he never succeeded in attracting any young admirers. Through such reflections we are given insights into her life and personality. While she is generally gracious in her descriptions of others, she also comes across as a person with strong opinions and principles, even if they are contrary to the expectations of society. This can be seen in an incident that occurred after her first husband died. She says, "I forgot to mention that during my widowhood I had several offers of marriage. One of the persons whilst he was paying court to me, sent a volume of sermons with the page ostentatiously turned down at a sermon on the Duties of a Wife, which were expatiated upon in the most illiberal and narrow-minded language. I thought this as impertinent as it was premature; sent the book back and refused the proposal." (See [7, p. 87].)

Her reflections also contain many incidents where she is either encouraged or criticized. It is interesting to note that encouragement came primarily from men, often prominent men who have established reputations for scholarship. On the other hand, most of the criticisms about her adopted life come from other women and often because she is not following a typical role for a woman in her society.

Mary Somerville's early life was spent in Burntisland, a small coastal town in Scotland, located across the Firth of Forth from Edinburgh. In her early years she was apparently given little parental direction or schooling and spent many hours by herself, exploring the coast and acquiring an appreciation for nature. Her father, a naval officer who eventually rose to the rank of Admiral, was away from home for long periods. She describes how the freedom she had as a child would soon change. "When I was between eight and nine years old, my father came home from sea, and was shocked to find me such a savage. I had not yet been taught to write, and although I amused myself reading the "Arabian Nights," "Robinson Crusoe," and the "Pilgrim's Progress," I read very badly, and with a strong Scottish accent; so, besides a chapter of the Bible, he made me read a paper of the "Spectator" aloud every morning, after breakfast; the consequence of which discipline is that I have never since opened that book" [7, p. 19]. Mary is soon sent off to boarding school for some formal education, but she is utterly wretched. Not only feeling out of place academically but also out of her social class, she returns home after one year.

Throughout the history of her early childhood, Mary often comments on the issue of education. She is especially critical about the lack of opportunity for most girls and young women, but also indicates a dissatisfaction with the prevailing attitude in society toward educating girls. In bad weather, when she was trapped indoors, Mary took to reading on her own, but even this was criticized. For instance, "My mother did not prevent me from reading, but my aunt Janet, who came to live in Burntisland after her father's death, greatly disapproved of my conduct. She was an old maid who could be very agreeable and witty, but she also had all the prejudices of the time with regard to women's duties and said to my mother, "I wonder you let Mary waste her time in reading, she never shews (sews) more than if she were a man" [7, p. 27].

At the age of 13, Mary is sent back to school to practice writing and learn some basic arithmetic. She also begins to play the piano, initiating a lifelong interest in music. During that same year, primarily through a visit to her aunt and uncle, she begins to see an opportunity to take up more formal learning on her own. She describes her uncle, Dr. Somerville, who is also the father of her future husband, as a kindly man and the first person to really approve of her thirst for knowledge. Somehow she finds the courage to tell him about her desire to study Latin and to her surprise, he heartily encourages her. She describes this as one of the happiest periods of her childhood.

Throughout her reflections, Somerville inserts comments related to beliefs or positions she would hold later in life. For instance, she recalls a visit to another uncle, this one in Edinburgh, where she is able to attend school. She states that she made great progress in her writing and arithmetic but soon forgot it for lack of practice. It was there that she began to form her own opinions, even if they were opposed to those held by her family. A common theme is the discrimination against women in education. She relates, "From my earliest years, my mind revolted against oppression and tyranny, and I resented the injustice of the world in denying all those privileges of education to my sex which were so lavishly bestowed on men" [7, p. 45]. In spite of her lack of educational opportunity, an unusual event involving number puzzles would kindle Mary's interest in mathematics.

In her writings, Mary gives the impression that she would not be a teenager who enjoys attending tea parties and other formal events. However, it was at a "required" tea party that Mary met a Miss Ogilvie, a woman who impressed Mary with her sewing. After Mary asked about it, Miss Ogilvie referred her to a fashion magazine which, unlike such magazines today, also included a variety of number puzzles. Mary recounts "At the end of a page I read what appeared to me to be simply an arithmetical question; but on turning the page I was surprised to see strange looking lines mixed with letters, chiefly X's and Y's and asked; "What is that?" "Oh," said Miss Ogilvie, "it is a kind of arithmetic; they call it Algebra; but I can tell you nothing about it." And we talked about other things; but on going home I thought I would look if any of our books could tell me what was meant by Algebra" [7, p. 46].

Given her natural curiosity, Mary wanted to learn more about this exotic "arithmetic" but could find nothing about algebra in the books available to her. She decided to look further and eventually was able to acquire copies of Euclid and of Bonnycastle's *Algebra*, getting what she "so long earnestly desired" [7, p. 52]. Since she couldn't just drop her responsibilities at home, she had to give up sleep to study, that is, until the servants told about her using more than her share of candles. As a result, her candle was taken away as soon as she went to bed. Nevertheless, she persevered and found ways to continue studying this exciting new subject.

At the age of 24, Mary married Samuel Grieg, a man who does not share her passion for mathematics or science. This marriage lasts for three years until Grieg's death. Curiously, she doesn't mention him again in her reflections, but it's clear that her life is then changed. For one thing, she is now a mother with the responsibility of young children, but she also appears to have the benefit of at least some financial independence, allowing her to pursue her studies more openly. "I was much out of health after my husband's death, and chiefly occupied with my children, especially with the one I was nursing; but as I did not go into society, I rose early, and, having plenty of time, I resumed my mathematical studies. By this time I had studied plane and spherical geometry, conic sections, and Fergusson's 'Astronomy.' " [7, p. 77].

Having made impressive progress studying mathematics, she decides to start reading Newton's *Principia*, but encounters difficulties. Perhaps realizing she needed help, Mary makes the acquaintance of Mr. Wallace, a mathematics teacher at a military college, who would eventually become a professor of mathematics at the University of Edinburgh. Wallace agrees to correspond with her about solving problems and probably provided some needed motivation and support along the way. She recounts, "At last I succeeded in solving a prize problem! It was a diophantine problem, and I was awarded a silver medal cast on purpose with my name, which pleased me exceedingly" [7, p. 78].

This encouragement seems to have inspired her to keep working, and Wallace assists her by providing a reading list of books in higher mathematics, including Lacroix's *Algebra*, as well as his *Differential and Integral Calculus*, Euler's *Algebra* and his *Isoperimetrical Problems* (in Latin), Lagrange's *Theory of Analytical Functions*, and other works by authors like Biot, Poisson, and Laplace. Although this list seems daunting, Mary approached the challenge of reading these books with great enthusiasm. "I was thirty-three years of age when I bought this excellent little library. I could hardly believe that I possessed such a treasure when I looked back on the day that I first saw the mysterious word "Algebra," and the long course of years in which I had persevered almost without hope. It taught me never to despair" [7, p. 79].

Mary is delighted with these books and pursues her studies with vigor, even though there is criticism at home: "I was considered eccentric and foolish, and my conduct was highly disapproved of by many, especially by some members of my own family, as will be seen hereafter. They expected me to entertain and keep a gay house for them, and in that they were disappointed. As I was quite independent, I did not care for their criticism" [7, p. 79].

Continuing her work, Mary embarks on another classic, and decides to begin reading Laplace. Along the way she makes the acquaintance of Professor Playfair, who will become another helper and encourager. She characterizes Playfair as "gravely cheerful, perfectly amiable, and though he was respected and loved, could be a severe critic."

Not surprisingly, Laplace must have given her quite a challenge, as she appears to briefly question her ability, but soon regains the confidence to continue. "I had now read a good deal on the higher branches of mathematics, but as I never had been taught, I was afraid that I might imagine that I understood the subjects when I really did not; so by Professor Wallace's advice I engaged my brother to read with me, and the book I chose to study with him was the Mécanique Céleste. Mr. John Wallace was a good mathematician but I soon found that I understood the subject as well as he did. I was glad, however, to have taken this resolution, as it gave me confidence in myself and consequently courage to persevere. We had advanced but little in this work when my marriage to my cousin, William Somerville (1812), put an end to scientific pursuits for a time" [7, p. 82].

Her second husband, William Somerville, the son of her encouraging uncle, appears to have been an ideal match for Mary. He was a physician who was a member of the Royal Society and engaged in scientific circles. Thus Mary was given more access to the scientific community. By all accounts, they had a compatible and happy marriage, and William was extremely supportive of Mary's pursuing her interests and continuing her work. This was evident even before they were married. 'I had been living very quietly with my parents and children, so until I was engaged to my cousin I was not aware of the extreme severity with which my conduct was criticized by his family, and I have no doubt by many others; for as soon as our engagement was known I received a most impertinent letter from one of his sisters, who was unmarried, and younger than I, saying she "hoped I would give up my foolish manner of life and studies, and make a respectable and useful wife to her brother.' I was extremely indignant. My husband was still more so, and wrote a severe and angry letter to her; none of the family dared to interfere again" [7, p. 87].

In 1816, William Somerville is appointed a member of the Army Medical Board and the couple moves to London, putting them at the center of the scientific and intellectual community. For instance, soon after arriving in London they made the acquaintance of Sir William and Lady Herschel, who were very kind to them. Their son, John, who was closer to Mary's age, had recently concluded a very successful career as a student in Cambridge. He would become a very important, supportive, and close friend to Mary for many years.

Not long after they were married, the Somervilles made a trip to Switzerland, stopping in Paris on the way. While in Paris, she and William are entertained by a notable list of important people. She relates having a long conversation with, among others, Laplace and seems especially taken with his wife.

The Somervilles continued enjoying life in London, where they were actively involved in society and seemed to know everyone of importance, although Mary doesn't provide many details about how these relationships were formed. For instance, she mentions that they frequently went to see Mr. Babbage, while he was making his calculating machines. They also continued making trips to the continent. After returning from one of these trips, Dr. Somerville received a letter from Lord Brougham who was the head of the Society of Diffusing Useful Knowledge. Brougham indicates that the Society wanted Mary to author an annotated translation of the *Mécanique Céleste*, which would be "a description of that divine work as will both explain to the unlearned the sort of thing it is-the plan, the vast merit, the wonderful truths unfolded or methodized-and the calculus by which all this is accomplished, and will also give a somewhat deeper insight to the uninitiated" [7, p. 161]. Essentially he is asking Mary, who has yet to publish anything of importance, to translate and amplify a substantial book. She eventually agrees and the result is her first book, *Mechanism of the Heavens*.

Although no doubt flattered by the proposal, she must also have been intimidated by this challenge. She was asked to translate Laplace's work, one that required sophisticated mathematics, into a book suitable for a general audience. Along with making the text accessible, she would also have to devise necessary diagrams to illustrate the text, diagrams that were not included in the original. Of course, this writing project would be taken on in addition to her regular responsibilities of raising children and running a household. In the end, she agrees to make an attempt, provided that should her manuscript be unacceptable, it would be put into the fire. Realizing the unique opportunity that was presented to her, she concluded "Thus suddenly and unexpectedly the whole character and course of my future life was changed" [7, p. 162].

Aided by her talent for writing to a general audience, as well as her singular ability to focus on the task at hand, she produced a work of around 700 pages It was a great success, no fires were needed. Somerville received congratulatory letters from the leading figures of the day including John Herschel, with whom she apparently consulted during the writing, William Whewell, and George Peacock, whose comments were typical of those she received. Peacock writes “I consider it to be a work which will contribute greatly to the extension of the knowledge of physical astronomy, in this country, and of the great analytical processes which have been employed in such investigations. It is with this view that I consider it to be a work of the greatest value and importance. Dr. Whewell and myself have already taken steps to introduce it into the course of our studies at Cambridge, and I have little doubt that it will immediately become an essential work to those of our students who aspire to the highest places in our examinations” [7, p. 171].

She obviously appreciated these positive responses. Regarding Peacock’s letter, she writes “I consider this the highest honour I ever received, at the time I was no less sensible of it, and was most grateful. I was surprised and pleased beyond measure to find my book should be so much approved of by Dr. Whewell, one of the most eminent men of the age for science and literature; and by Dr. Peacock, a profound mathematician, who with Herschel and Babbage had, a few years before, first introduced the calculus as an essential branch of science into the University of Cambridge” [7, p. 172].

With the success of *Mechanism of the Heavens*, Mary Somerville was established as a legitimate member of the highest ranks of British scientists, and the groundwork was laid for additional books. She had come a long way from her early struggles fighting the educational discrimination she felt in her early years, and it’s gratifying to see that she was recognized and encouraged by people she highly respected.

Mary continues in her writing and active social life in London until her husband suffers a serious illness and the family moves to Italy in 1838, in hopes that a warmer climate would be beneficial for him. After William died in 1860, Mary moved her family to Naples, where she died in 1872. She is buried at the English cemetery in Naples.

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Physical Activity in a Theory of Computing Class

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Abstract

Physical activity breaks, sometimes called brain breaks, are beginning to gain attention among K-12 teachers as a way to keep their students alert and engaged in the classroom. In the Fall 2014 semester, faced with the task of teaching an introductory course in Theory of Computing in a once-a-week, $2\frac{1}{2}$-hour format, I decided to try incorporating physical activity into my own classroom. Time is precious in the college classroom, so any physical activities have to be directly related to the course material. I will describe some physically active exercises that I used in the classroom to teach students about regular expressions, finite automata, and other theoretical concepts. During the semester, I found that these exercises helped students to have fun and to stay connected to the material, even at the end of this long, late-night class. I also found that the exam averages and the overall course average were higher in Fall 2014 than they had been during the previous four years of teaching this night class. This invites further experimentation with the technique in future semesters.

The Challenge

In the Fall semester of 2014, I was assigned to teach a course called Foundations of Computer Science. In the Rowan University curriculum, this is an introductory course in the theory of computing: students learn to use regular expressions, finite automata, context-free grammars, and elementary symbolic logic.

The challenge of the course was the time slot: In most Fall semesters, my section of Foundations of Computer Science is held once a week, from 6:30-9:00 p.m. The students have probably just had dinner, after a long day of work and classes, and they are now settling in to study mathematics for $2\frac{1}{2}$ hours. Even the most dedicated students will have trouble holding their concentration for that long.

Physical activity breaks have started to become popular in some K-12 classrooms as a way to increase students' attention. The idea is to have students physically get up and move around during classroom time, so that the physical energy will lead to increased mental energy. Lengel and Kuczala [1] and Sladkey [2] are among the many authors who have designed activities for elementary and secondary classrooms.

I wanted to try this in my own classroom. Contact hours in a university setting are precious and few, however, so any activities we do should be related directly to the content. With this in mind, I developed a few physical exercises to use in the Fall 2014 Foundations of Computer Science class. The students had fun with them, were better able to pay attention during class, and showed improved mastery of the concepts by the end of the semester.

The Activities

Activity 1: Catch!

This exercise uses two or three small, soft balls, such as Koosh balls, that are easy to throw and catch, but soft enough that they won't do damage if someone misses a catch.

Pose a question for the students to think about, then throw a ball out into the classroom. Whoever catches the ball tries to answer the question. If the student is obviously stuck and can't come up with the answer, then after a short time tell the student to throw the ball to someone else in the class; the new student then tries to answer the question. Repeat until the question has been answered, or until it's clear that the students are lost and that the professor needs to take a few minutes and talk about the question and its answer directly.

This works best with short questions that require attention from the students but that do not involve lengthy problem-solving. Examples from the Foundations class included:

If $L = \{ab, c\}$ *and* $M = \{a, ab\}$, *what is* LM*?*

Tell me one string that's accepted by the grammar that's currently on the board.

What states can you reach from state 0 on a Λ*-transition, in the finite automaton that's currently on the board?*

This activity is both physical and mental: It requires students to look for a ball in the air, catch it, and throw it, waking up their bodies; it also requires figuring out the answer to a question, waking up their minds.

Activity 2: Find Your Match

In this family of exercises, two or more different descriptions of an assortment of formal languages are written on pieces of paper, which are then distributed to the students. The students try to find the other students in the class who are holding the pieces of paper that describe the same language as the paper they are holding.

Variation 1: Regular Expressions and Languages

Write regular expressions on green pieces of paper, and write corresponding strings from their languages on yellow pieces of paper. Give half the students green pieces of paper, and the other half yellow. Tell the students to get up, walk around, talk to each other, and find the student with the piece of paper that matches theirs. For example, the student with

b* a

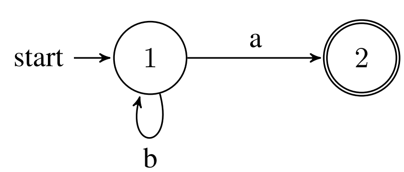
and the student with

IN: bbba, a   OUT: baba, bb

should walk around until they find each other.

Variation 2: Finite Automata, Regular Expressions, and Languages

As before, write regular expressions on green pieces of paper, and strings from their languages on yellow pieces of paper. Additionally, include blue pieces of paper with nondeterministic finite automata on them, such as this one:



Distribute the green, yellow, and blue pieces of paper, and tell the students to walk around until they find the two pieces of paper that match theirs.

What Worked Well

These activities require that the students in the class be willing to try something that is lighthearted and out of the ordinary. I asked the class at the beginning of the semester whether they would be willing to try some experimental physical exercises during the semester, and they agreed.

We found that the exercises did help the students to be more attentive, especially as we moved into the second hour of the evening. Besides the physical alertness, we also found that the exercises lightened the mood of the class. Throwing a ball around the room is a silly thing to do, after all, as is walking around asking your classmates what's on their papers. The students had fun, and this made learning easier.

I also found that playing catch with the ball eased the pressure of being called on. If a student is stuck trying to figure out an answer, then never mind: toss the ball to someone else, and they'll help you out. The random element of the ball toss means that no one is really being singled out, and being able to throw the ball means that no one has to figure out a hard question alone.

Occasionally, students started to get lazy with throwing and catching the ball: they would ask for a volunteer to help with answering a question, and gently hand the ball to the volunteering student. This makes the activity much less physical, and thus less useful. On these occasions, I switched to asking the students to throw the ball back to me after attempting to answer, and I then tossed the ball back out into the class to be caught by a new student. Usually, this was enough to raise the physical activity level, while still keeping the mood light.

Grade Improvement in Fall 2014

I have taught the Foundations of Computer Science class every Fall semester since 2010, in this once a week, $2\frac{1}{2}$ hour format. While I have always incorporated active elements in class, such as interactive problem-solving and students presenting homework solutions on the board, Fall 2014 is the first semester in which I used the physical exercises described in this paper. I had expected to find that students would enjoy the class more when we incorporated physical activity, and I did indeed see this. However, I was pleasantly surprised to find that the students' average semester grades also increased in Fall 2014:

Semester	Mean Semester Grade
Fall 2010	83.8
Fall 2011	79.9
Fall 2012	81.9
Fall 2013	83.5
Fall 2014	**86.7**

The mean semester scores increased several points over the scores in the previous semesters. I found a similar increase in the students' average exam scores. This is an encouraging preliminary result, and I plan to repeat the exercises in upcoming semesters to see if the pattern holds.

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A Triune Philosophy of Mathematics

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Abstract

What is mathematics and is it discovered or invented? The Humanist, Platonist, and Foundationalist each provide answers. But are the options within the philosophy of mathematics so limited? Rather than viewing and describing mathematics in a mutually exclusive manner, each of these approaches includes components of truth from a greater triune philosophy of mathematics. This paper will briefly outline existing philosophies and then introduce an inclusive triune paradigm through which to explore fundamental questions about mathematics.

Introduction

My parents were hippies who were leery of traditional education. So out of a desire to both protect and also encourage questioning they put me into alternative public schools. These weren't edgy enough and so they allowed me to homeschool junior high into high school. This put me on a fast track and I began community college during what would have been my junior year of high school. I jumped right into calculus and worked my way through differential equations. After two years I transferred to The Evergreen State College to continue my alternative education with an interdisciplinary liberal arts degree studying political science, literature, and mathematics. With such an eclectic background, I didn't have a clear direction following my bachelor's degree so I went on to graduate school in mathematics thinking, "If this doesn't work out, I can do something else later." While a graduate student I was given the opportunity to teach and my career path suddenly became clear. I was hired by Highline College right out of graduate school where I became the youngest tenured faculty member in College history.

While this makes me sound smart, it really means that I had a lot of growing to do as an educator and colleague. But I was in a supportive environment and by my eighth year I was firmly established as a teacher, in service, and in professional growth, and I generally felt that I knew my professional direction. In 2008 I attended a talk by a colleague [2]. The talk itself was on polling and statistics and not related to this paper. However in the midst of the lecture my coworker said, "I think math was invented by people, not discovered."

Is math discovered or invented? In all my non-traditional education as well as traditional community college and graduate studies and then continuing into the first eight years of my professional work, I don't once recall having asked myself the question, "Where does math come from?" But while this was the first time these ideas had ever registered in my mind, I have come to realize over the last seven years of study that I had subconsciously adopted a framework for understanding mathematics. As John Synge said, "[E]ach young mathematician who formulates his own philosophy — and all do — should make his decision in full possession of the facts. He should realize that if he follows the pattern of modern mathematics he is heir to a great tradition, but only part heir." [9, p. 166] This certainly encapsulates my mathematical journey.

As I have come to have "full possession of the facts", I've learned that there are three main ways to explain the origin of mathematics. Within each of these broad categories there is a spectrum of nuance. Others have written compelling descriptions of this, but allow me to outline using broad strokes so that I may synthesize the field. The three broad views are as follows.

Foundational Philosophies		
Intuitionism	Logicism	Formalism

Figure 1: The spectrum of Foundational philosophies

Foundational philosophies: Mathematics is developed from axioms and definitions using logic

Humanistic philosophies of mathematics: Mathematics is invented by humans who are the source of math

Mathematical Platonism Also called 'mathematical realism', this view holds that mathematics exists 'out there' to be discovered; perhaps owing its existence is to God, but perhaps not

While some readers may recognize or be able to articulate their philosophy of mathematics, others may resonate with my story in that I was years into my career as a professional without realizing that I even had a view. I believed mathematics devoid of presupposition without even having the vocabulary to articulate my own presupposition about the field. So as I clarify the basic views available for later synthesis, I encourage you to ask yourself where these views match your training, intuition, and pedagogy.

The Foundational Philosophies

The first paradigm is that math is logic — this is the basis of the foundational philosophies: intuitionism, logicism, and formalism. If you do research on the philosophy of mathematics, these three views are described over and over again to the point that they nearly define the field.

The intuitionists such as Kronecker and Brouwer held that humans create the axioms of logic/mathematics and that we then manipulate these axioms to construct the theorems of mathematics in a constructivist manner. Because it stems from our work, the intuitionism shows existence by demonstrating a formula/algorithm/recipe to explain how each entity may be constructed. Because of this, intuitionists rejected proof by contradiction as well as the existence of an actual infinity. For them the source of mathematics was decidedly human. Or as Kronecker famously wrote, "God made the integers, all else is the work of men." [11, p. 19]

The logicists movement was begun by Frege, reached its height with Russell and Whitehead, and concluded with Gödel. They felt that the axioms of logic were self-evident truths that were known intuitively to the logician. They accepted the rules of logic a priori. In their effort to make solid their foundation, they held that some axioms were self-evident that are not so evident. Certainly the axiom of choice is on this list. Of the foundational camps, logicism was the most fully developed. For the logicist, the source of mathematics was beyond the human experience, self-evident, and discovered (albeit by a select few).

The formalists led by Hilbert were perhaps the largest group. They did not concern themselves with the source of the axioms but worked from these using every clever device they could devise. They had no issue with contradiction or infinity. Hilbert referred to math as a meaningless game. [1, p. 21] The formalist didn't have a strong opinion about where mathematics comes from; after all, it didn't matter anymore than the source of Chess or Monopoly.

Mathematical Humanism

The second paradigm is mathematical humanism: all mathematics is somehow human in nature/origin. Unlike the foundational philosophies, the subcategories are not as clearly defined. In part this is because mathematical humanism is more current and thus hasn't had as much time to mature. The spectrum within mathematical

Humanistic Philosophies		
Biology & Brain	Language	Social Construction

Figure 2: A spectrum of humanistic philosophies

humanism that I will discuss ranges from a biology-brain model, to language, and ends with social constructivism. Of these, the idea that math is a language is probably the oldest while social constructivism seems most dominant among educators.

According to authors Lakoff and Nunez, our ability to perform abstract reasoning is biological. [8, p. 347] Mathematics is ultimately grounded in experience. [8, p. 49] It is effective because mathematics is a product of evolution and culture. [8, p. 378] Mathematics doesn't have an independent existence. It is culture dependent and only exists through grounding metaphors. [8, pp. 356, 368] Consequently the philosophy of mathematics is the realm of cognitive science and not the domain of mathematicians. [8, p. xiii] Where does mathematics come from? For these philosophers, the source of mathematics is biological and evolutionary and thus serves only an evolutionary purpose... which is to say it has no intended purpose.

Perhaps the most commonly held humanistic philosophy of mathematics is summed up in the phrase, "Mathematics is the language of science." This originates with Galileo who wrote: "[The universe] cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it." [3, p. 4] Today this phrase is most often used *outside* of university math departments because it defines mathematics through its applications and universities produce pure mathematicians (more akin to the formalists of the foundational movement). The basic premise of the view is that mathematics is invented as a way to describe discoveries in the natural world. Math isn't monolithic and unchanging because language changes. The strength of this view is that it seems to explain the perceived transcendence and beauty in math by tying it back to science.

Mathematics is something people do according to Reuben Hersh. [5, p. 30] The philosophy of mathematics is the study of what mathematicians do. [5, p. xii] The emphasis of social constructivism is on *practice*. As a practice there has been an evolution of mathematical knowledge. [5, p. 224] This extends even to including proof itself. [5, p. 6] As such, mathematics is a social construction. It draws on conventions of language, rules, and agreement in establishing truths. Mathematical knowledge and concepts change through conjectures and refutations. The focus is on creation rather than the justification knowledge. [5, p. 228]

Mathematical Platonism

The third paradigm is mathematical Platonism (or mathematical realism) and is loosely based on the Plato's theory of forms and divided line. There is less explicitly written supporting Platonism (and much against). However many mathematicians are Platonists although not aware of it. Some know it and are reluctant to admit it because it seems mystical. Unlike the foundational philosophies and mathematical humanism, there is less written on the subtlety and nuances of Platonism. Thus the spectrum that I am about to describe is of my own creation (that is, unless I happened to get it from some abstract realm).

As given in *Principia,* Russell practiced what I dub, "finite mathematical Platonism." He began from a short list of self-evident "discovered" axioms. Then mathematics was built/created from these few eternal building blocks. This is a similar approach to that taken by Euclid. Finite Platonism gels nicely with the axiomatic

Mathematical Platonism/Realism		
Finite Axioms	Countable (All Truth)	Uncountable (Kitchen Sink)

Figure 3: How much lies within the Platonic realm?

method. It probably isn't a stretch to claim that the optimism following the "discovery" of Newton's laws stemmed from this same view: namely that the universe could be described by just a few simple laws. Today physicists are searching for the theory of everything ... a few simple statements to describe all. In this view, very little is required of the mathematical form (which only contains a few statements) and much of the human mathematician (who massages the few givens into the body of mathematics). So where does mathematics come from? Well it begins with a few eternal truths and then is created by the mathematician.

The most widely held version of realism holds that all mathematical truth is found "out there" (in the Platonic form). This includes triangles, pi, and the golden ratio. It also includes every real number (yes, I know they are uncountable) and proofs big and small (think the proof of Pythagorean theorem vs. that of Fermat's Last Theorem). That is every true mathematical statement (regardless of how useful or elegant) can be discovered in the form. One author, Lakoff, calls this the romance of mathematics. [8, p. xv]

Finally, the philosopher Alvin Plantinga affirmed what I call an uncountable mathematical Platonism. As a theist, he holds that math exists in the mind of God. He sees all mathematical entities as uncreated necessary beings whose existence is affirmed by God's nature. For Plantinga, God affirms the existence of all propositions, states, and possible worlds. But God affirms the truth of only some. [10, p. 143] This is a kitchen sink view. That is everything exists "out there" to be discovered—true and false. But not all is true.

Mutually Exclusive Models?

In my reading, most authors want you to choose one and only one paradigm. They often use an elimination argument to justify their position. For example, a typical humanist's argument might be summarized: formalism is dead and Platonism requires God. Thus the only option remaining to us is humanism. The problem is that this assumes that (a.) the discussed options are disjoint, (b.) that all the options are being considered, and (c.) that the premises are correct. I am going to focus on the first assumption: namely that the options are disjoint.

At first glance, Platonism, humanism, and the foundational views seem mutually exclusive (disjoint). If math is discovered "out there" then it can't originate within us. If it comes from within us, then it isn't a game we play, and certainly a meaningless game sounds nothing like the eternal truths of an ideal realm. But perhaps this is a false trichotomy.

Consider the often told Indian parable of blind men trying to describe an elephant. One blind man feels a serpent, another a tree, and a third a spear. While these seem very different, we know that the legs, trunk, and tusks of an elephant are all part of the same animal. Could our seemingly mutually exclusive views of mathematics simply be appendages of a greater and more inclusive truth?

What I am proposing is a triune philosophy that envelops and includes much of a wide swath of the paradigms discussed. Whereas before, we saw three world views, each with its own nuances, now we envision mathematics on a higher dimension.

The key to this view is quite simple. Namely that the center of each of the three views represents the strength of the position. I dare say that many would agree that mathematics is a logical language we speak to describe abstract or immaterial truths.

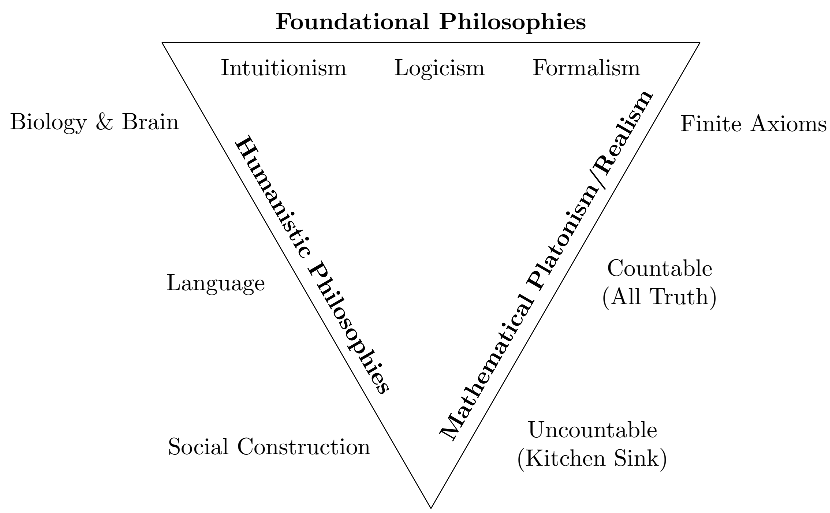

Figure 4: A synthesis of views

To understand this view, it is insightful to think about what each paradigm sees as its greatest ideological adversary. We see this by comparing the center of each edge with its opposite vertex.

Example 8. *Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history.*

Authors Lakoff and Nunez explain their view and make it very clear who/what they view as their opposition. "Mathematics as we know it is human mathematics, a product of the human mind. Where does mathematics come from? It comes from us! We create it . . . Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history." [8, p. 9]

They take on Platonism very directly. They write, "Human beings can have no access to a transcendent Platonic mathematics, if it exists. A belief in Platonic mathematics is therefore a matter of faith . . . There can be no scientific evidence for or against the existence of a Platonic mathematics . . . therefore human mathematics cannot be part of a transcendent Platonic mathematics, if such exists. [8, p. 4]

Whether you accept their argument or not, it should be clear that they see the primary counterargument to their biology & brain explanation for the origin of mathematics as what they dub "The Romance of Mathematics." As they write, it's the stuff of movies like 2001, Contact, and Sphere. But while it initially attracted them to mathematics, they are now more enlightened. [8, p. xv]

Example 9. *Formalism vs. mathematics as the language of science—the debate between pure vs. applied mathematics.*

The distinction we make between pure and applied mathematics is relatively recent. Are we standing on the shoulders of mathematicians or physicists – a good argument can be made for both. Prior to 1900, one can make the broad generalization that there was some pure mathematics but no pure mathematicians. But around the time when the foundational philosophies were being developed, this distinction was drawn. As G.H. Hardy wrote, "Pure mathematics is on the whole distinctly more useful than applied." [4, p. 134] Taking this one step farther, the father of Formalism, David Hilbert is quoted as saying, "Mathematics is a game played according to certain simple rules with meaningless marks on paper." For the formalist, mathematics was certainly a language.

However, it wasn't a language intended to communicate information outside of mathematics. Rather than being the language of science, mathematics was the language of mathematics.

The author Morris Kline wrote that most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. [7, p. 278] Today mathematicians and physical scientists go their separate ways ... mathematicians and scientists no longer understand each other. [7, p. 286] Under the influence of formalism and the other foundational philosophies, mathematicians no longer speak the language of science.

Example 10. *Mathematics is fallible and a social construction.*

The social constructivists reject the narrow definition that math is logic. For example, the humanist Reuben Hersh is concerned with the edifice that remains in university mathematics departments. He believes that his philosophy recognizes the scope and variety of mathematics, fits into general epistemology and philosophy of science [note, science and not mathematics], is compatible with practice – research, application, teaching, history, calculation, and mathematical intuition. He also rejects certainty and indubitability as false and misleading. [5, p. 33]

The opposition is clear: It's the foundational philosophies (primarily formalism as the most dominant view). Proof in particular is the opponent of this view. Hersh writes, "The trouble is, 'mathematical proof' has two meanings. In practice it's informal and imprecise. Practical mathematical proof is what we do to make each other believe our theorems. Theoretical mathematical proof is formal. It's transformation of certain symbol sequences according to certain rules of logic." [5, p. 49] The only reason to believe in mathematics is—it works! [5, p. 213] There is no infallibility. [5, p. 215]

Hersh wants to redefine mathematics as fallible and a social construction. As such he must take on the establishment. And the power brokers in mathematics hold the foundational view that math is logic and as such is pure and unchanging.

Before further sharing what I call the triune philosophy of mathematics, it's important to recognize that ideas have consequences, and that this remains a truism in the philosophy of mathematics as it is elsewhere. The way we answer "Where does math come from?" impacts research and education. Given my own background as an educator, I'd like to say few words on education.

Example 11. *The philosophy of mathematics and its influence on education.*

Nicolas Bourbaki is the collective pseudonym under which a group of mathematicians wrote a series of books with the goal of grounding all of mathematics in set theory. Their approach is similar to that of the formalists. The manifesto of Bourbaki has had a definite and deep influence. In secondary education the new math movement corresponded to teachers influenced by Bourbaki. "The devastating effect of formalism on teaching has been described ... [5, p. 238]" through books like *Why Johnny Can't Add*. [6]

Today we can see the influence of each major paradigm through the competition between teaching through a discovery method, cooperative learning, and skill based manipulations. Given the massive fiscal investment in mathematics education in the U.S., finding the perfect pedagogy is somewhat of a holy grail. But what if our issue is having too limited a view on mathematics? While I don't claim to be an expert on human cognition or learning, I postulate that a pedagogy that incorporated aspects of all three major philosophies would be more attractive to the next generation of students. If you will, it's almost a philosophical parallel to teaching to multiple learning styles.

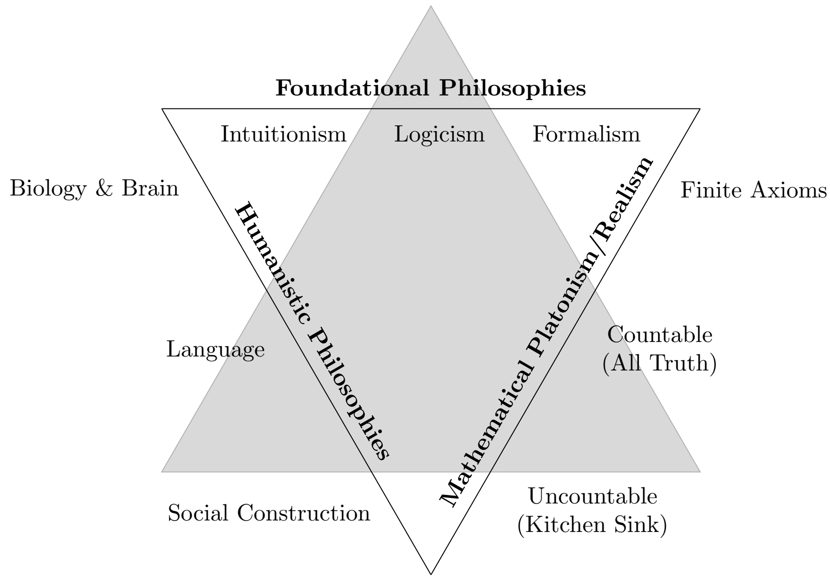

Figure 5: A triune philosophy of mathematics

A Synthesis of Views

If the three major branches of the philosophy of mathematics are not mutually exclusive, it is possible that a broader, more inclusive, philosophy of mathematics exists. Is mathematics invented or discovered - yes. I'm proposing a view that incorporates the strengths of each paradigm but that comes with some ambiguity - what I am calling, "A triune philosophy of mathematics."

Just to clarify, this essay isn't intended to end a discussion but rather to begin a conversation. What is good? What arguments are logically sound? What passes the experiential sniff test? This conversation is going to force us to go much deeper into the details than this paper has allowed. And as Whitehead and Russell learned full well, the devil is in the details.

For me, this was the image that first opened my eyes to a triune philosophy of mathematics. It incorporates the greatest strengths of each paradigm inside a single figure. There is a common practice of mathematics between philosophies. That is, our old friends from calculus and algebra haven't changed—the integral and derivative are calculated the same way whether discovered, invented, or based on the axioms of logic.

One powerful aspect of this model is that it gives a place to look for counterarguments. That is, the center of each edge of the original triangle is strong while its vertices are potential weaknesses. As we look to certain places to find counter examples to prove/disprove a mathematical claim, this gives us a direction to look to substantiate/discount philosophical arguments.

Bear in mind that the distinctions I am making are tentative. That is the vertices of the new solid triangle (triune math) could shift to include more/less of the gray triangle. For that matter, one might argue that these are not triangles at all, but that there are many more sides on each figure. But while I acknowledge that this is a legitimate objection and requires serious consideration, it isn't my hypothesis.

For the humanist, the source of truth must come from within the cosmos. The Platonist says math resides outside the material world. The foundationalist says that math is from self-evident axioms and doesn't bother to justify their existence. Going back to the parable of the blind men and the elephant, there was a clear source for higher knowledge (namely the elephant). If this triune philosophy more fully describes the nature of mathematics, then it too is likely grounded in a greater rationale.

I believe mathematics is firmly grounded in the triune God of orthodox Christianity. Following in the footsteps of Kepler, Newton, Euler and countless others, I believe that there are aspects of mathematics that go beyond the physical world:

1. mathematics is logical and self-existent because it is part of the nature of a logical and self-existent God
2. humans create and speak the language of mathematics as image bearers of one who walked among us
3. we can discover eternal transcendent truth because the spirit of God speaks to each one of us

For some the very mention of God may be enough to discredit this whole triune philosophy of mathematics. For others, the selection of a specific God may be too much. I accept this critique but challenge you: Is there any existing philosophy of mathematics that fully describes the marvel and practice of mathematics? If not, could there be a greater elephant in need of description? If so, what is its size and shape?

Conclusion

So far as I know, this triune philosophy of mathematics is a new idea (perhaps discovered, perhaps invented). This essay marks the first time it has been shared in print. It's quite possible that I will soon find out the importance of tenure as this could be the last essay I ever write. Jests aside, I anticipate next steps in two directions. The first is in answering the likely objections that this paper will receive. The second is in fleshing out the details wherein the truth most likely lies.

Is it worth it? Yes, ideas have consequences and we have gone too many years under the allusion that mathematics is a field devoid of presuppositions. This introduction to a triune philosophy of mathematics should bring this out in the open. Something needs to change in mathematics and I propose that it is how we view and understand where mathematics comes from.

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Home Primes and Foreign Primes

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Abstract

Home primes and foreign primes are produced by a simple recipe that blends prime factorizations with recursion. The *home prime* of a positive integer n is formed by concatenating the prime factors of n in nondecreasing order. If the resulting integer is prime, then we have found the home prime of n. If not, then we repeat the process as many times as needed to obtain a prime. For instance, $35 = 5 \cdot 7$. After concatenation, we have $57 = 3 \cdot 19$, which is followed by $319 = 11 \cdot 29$, which is followed by 1129, which is prime. Thus, the home prime of 35 is 1129. To obtain the *foreign prime* of a positive integer n, we form the next integer by concatenating the prime factors of n in nonincreasing order. For example, starting with $35 = 7 \cdot 5$, we next consider $75 = 5 \cdot 5 \cdot 3$, followed by $553 = 79 \cdot 7$, followed by 797, which is prime. Thus, 797 is the foreign prime of 35. In this talk we give some results about home primes and foreign primes for integers $n < 100$. As one might expect from the arbitrary nature of the concatenation process, there are few easily discernible patterns.

Home Primes

What is a home prime?

Number theory abounds with simply stated, accessible questions. Are there infinitely many primes? Are there infinitely many twin primes? Can every even integer be expressed as the sum of two primes? Now we add the following question: Does every positive integer have a home prime?

We find the *home prime* of a positive integer n by concatenating the prime factors of n as many times as needed to obtain a prime number. Before giving a formal definition, we illustrate with $n = 15$. The prime factorization of 15 is $15 = 3 \cdot 5$. Concatenate the prime factors to obtain 35. Since 35 is composite, we repeat the process. The prime factorization of 35 is $35 = 5 \cdot 7$, so we obtain 57 after concatenation. Since 57 is composite, we factor again: $57 = 3 \cdot 19$. After concatenation, we obtain 319. Since 319 is composite, we factor again: $319 = 11 \cdot 29$. After concatenation, we obtain 1129. Since 1129 is prime, we have found the home prime of 15.

Here is a summary of what we did:

- Start with $n = 15$
- $15 = 3 \cdot 5 \to 35$
- $35 = 5 \cdot 7 \to 57$
- $57 = 3 \cdot 19 \to 319$
- $319 = 11 \cdot 29 \to 1129$
- 1129 is prime in 4 iterations

We are now ready to consider a more formal definition of a home prime:

Definition. *The* ***home prime*** *of a positive integer $n > 1$ is the prime obtained by the following algorithm:*

1. *Is n prime?*
   - *If yes, then stop. You have reached the home prime of n.*
   - *If no, then continue.*
2. *Concatenate the prime factors of n in nondecreasing order.*
3. *Return to step 1 and repeat with the number formed in step 2.*

To further illustrate the challenges involved in calculating home primes, we present two more examples of a home prime calculation.

The home prime of 45

To calculate the home prime of 45, we must account for repeated prime factors in the concatenation step of the algorithm. Note also that we require 6 steps to find the home prime of 45, whereas only 4 steps are required to find the home prime of 15.

- Start with $n = 45$
- $45 = 3^2 \cdot 5 \to 335$
- $335 = 5 \cdot 67 \to 567$
- $57 = 3^4 \cdot 7 \to$ 33,337
- 33,337 $= 17 \cdot 37 \cdot 53 \to$173,753
- 173,753 $= 239 \cdot 727 \to$ 239,727
- 239,727 $= 3 \cdot 41 \cdot 1949 \to$3,411,949
- 3,411,949 is prime in 6 iterations

The home prime of 56

Our first two examples of calculating the home prime of an integer used odd integers. Now we present an example of calculating the home prime of an even integer.

- Start with $n = 56$
- $56 = 2^3 \cdot 7 \to 2227$
- $2227 = 17 \cdot 131 \to$ 17,131
- 17,131 $= 37 \cdot 463 \to$ 37,463
- 37,463 is prime in 3 iterations

Properties of home primes

It may seem from these examples that home primes do not possess much structure. The concatenation process splices prime factors together in an arbitrary way. Furthermore, this process depends on the prime factorization of an integer. The distribution of primes within the integers does not fit an easily described pattern, so we do not at first expect to find structure in the home primes. Nevertheless, using our examples, we may describe two elementary properties of home primes.

Primes are home primes

The home prime of any prime integer n is n. For instance, 2 is the home prime of 2, 3 is the home prime of 3, and 71 is the home prime of 71. When n is prime, the home prime generation process terminates without any concatenation.

Descendant property

If h is the home prime of an integer n, then h is also the home prime of any integer that was created during the home prime generation process. For instance, in the first example above, 1129 is the home prime of 15, but it is also the home prime of 35, 57, and 319.

Home prime history and records

Home primes were first introduced by Heleen [2] in 1990 as *family numbers*. In a follow-up article in 1997, Heleen [3] presented the results of his calculation of the home primes of positive integers less than or equal to 1000. At the time, there were only two primes less than 100 for which he was unable to determine home primes. These integers are 49 and 77. Note that since $49 = 7 \cdot 7$, once we find a home prime for 49, it will also be the home prime of 77. It remains an open question whether or not 49 and 77 have a home prime. The latest result [1] shows that after 119 iterations, we arrive at an integer with 252 digits. Elliptic curve factoring has been used to get to this point, and similarly powerful factoring algorithms will be needed to finally resolve the question.

Herman and Schiffman's article [4] in *Mathematics Teacher* first attracted the author's attention to home primes. They show several ways to use the study of home primes in middle school and high school mathematics classrooms. In particular, they list several research questions for students to solve. The interested reader may consult [4] for more information.

Both Heleen and Herman and Schiffman display a table of home primes for integers less than 100. The home prime in this range with the largest number of digits is 80. Its home prime requires 31 iterations of the home prime generation process, and the resulting home prime has 48 digits. By contrast, the surrounding integers display home primes of much smaller lengths. Since 79 is prime, its home prime is 79. The home prime of 81 requires 9 iterations and has 15 digits. Just as the primes yield little discernible structure, so do the home primes.

Foreign Primes

Now we turn our attention to a modification of the algorithm that is used to generate the home prime of an integer. Instead of concatenating the prime factors in nondecreasing order, we concatenate them in nonincreasing order.

Definition. *The **foreign prime** of a positive integer $n > 1$ is the prime obtained by the following algorithm:*

1. *Is n prime?*
   - *If yes, then stop. You have reached the home prime of n.*
   - *If no, then continue.*
2. *Concatenate the prime factors of n in nonincreasing order.*
3. *Return to step 1 and repeat with the number formed in step 2.*

To illustrate the calculations involved, we will consider the integers that we used as examples above.

The foreign prime of 15

First, consider 15. Instead of factoring $15 = 3 \cdot 5$, we write $15 = 5 \cdot 3$. After concatenation, we have 53. Since 53 is prime, 53 is the foreign prime of 15. As above, we may illustrate this process using fewer words:

- Start with $n = 15$
- $15 = 5 \cdot 3 \to 53$
- 53 is prime in 1 step

The foreign prime of 45

As above, for our second example, we find the foreign prime of 45. Notice that 45 has repeated prime factors in its prime factorization. It also provides an example of a case in which we have to use the foreign prime generation process more than once (unlike what is true for 15).

- Start with $n = 45$
- $45 = 5 \cdot 3^2 \to 533$
- $533 = 41 \cdot 13 \to 4113$
- $4113 = 457 \cdot 3^2 \to$ 45,733
- 45,733 $= 83 \cdot 29 \cdot 19 \to$ 832,919
- 832,919 is prime in 4 steps

The foreign prime of 56

When we attempt to find the foreign prime of 56, something strange happens.

- Start with $n = 56$
- $56 = 7 \cdot 2^3 \to 7222$
- $7222 = 157 \cdot 23 \cdot 2 \to$ 157,232
- 157,232 $= 317 \cdot 31 \cdot 2^4 \to$ 317,312,222
- 317,312,222 = 99,721$\cdot 43 \cdot 37 \cdot 2 \to$9,972,143,372
- $\cdots$

Notice that at each step, after concatenation, we have an even integer. Since the smallest prime factor of any even integer is 2, every iteration of the foreign prime generation process produces an integer that ends in a 2. Hence, all such integers are even. That means that the algorithm never terminates. Hence, 56 does not have a foreign prime.

Properties of foreign primes

Using the above examples as motivations, we may make some elementary observations about foreign primes.

Primes are foreign primes

When p is prime, the concatenation step of the foreign prime generation algorithm is not needed. Thus, the foreign prime of p is p.

Descendant property

As we noted for home primes, when we find the foreign prime of an integer n, we have also found the foreign prime of all integers that were obtained during the foreign prime generation process. For example, 832,919 is the foreign prime of 45, but it is also the foreign prime of 533, 4113, and 45,733.

No even composite integer has a foreign prime

In our third example above, we showed that 56 does not have a foreign prime. In fact, there are infinitely many integers that do not have a foreign prime.

Theorem 1. *Suppose n is an even composite integer. Then n does not have a foreign prime.*

Proof. Suppose n is an even composite positive integer. Then the prime factorization of n must contain at least one 2. When the primes of n are concatenated in nonincreasing order, the resulting integer n' ends in 2. Thus, n' is also even. It follows that the prime factorization of n' also contains at least one 2. Hence, the process for generating foreign primes never terminates. □

Contrast this result with the unsettled question about whether or not 49 and 77 have home primes. There may be a proof that 49 and 77 do not have home primes, or we may have to wait for more powerful computer factoring programs that will show that 49 and 77 do have home primes.

Foreign primes for $2 \le n \le 100$

Table 1 gives the foreign primes for all integers $n \le 100$. It was generated using Maple 17. Notice that we do not include any prime integers since we know that every prime integer is its own foreign prime. For a similar reason, we do not include any entries for even composite integers. The largest number of iterations needed to produce a foreign prime is 19. This amount of work is needed for the 30-digit foreign prime of 85. This record-setting foreign prime calculation requires fewer repetitions of the foreign prime generation algorithm than the record-setting home prime calculation for integers less than or equal to 100. It appears that the existence of a home prime for n is independent of the existence of a foreign prime for n. For example, every even composite integer in this range has a home prime but not a foreign prime. For a second example, consider 49 and 77. We do not yet know whether or not 49 and 77 have a home prime, but they certainly have a foreign prime, namely 311,173.

Future research

There are many questions about foreign primes that may be the subject of future research projects. For example, how far can we extend the list in Table 1 above? We know that even composite integers do not have foreign primes. Are there any other families of integers that do not have foreign primes? The distribution of foreign primes does not seem to follow a pattern. Is there a way to classify integers that are foreign primes? Suppose n has both a home prime and foreign prime. What is the relationship between these two primes? It is hoped that this introduction to the study of home primes and foreign primes will motivate the reader to seek answers to these questions and others.

Acknowledgements

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Table 1: Foreign Primes for all Integers $n \leq 100$

n	Number of iterations	Foreign prime of n
9	2	113
15	1	53
21	1	73
25	7	1,035,533
27	2	3,733
33	1	113
35	3	797
39	2	197
45	4	832,919
49	6	311,173
51	1	173
55	6	1,035,533
57	1	193
63	1	733
65	2	5,333
69	1	233
75	2	797
77	5	311,173
81	2	101,113
85	19	261,991,759,117,813,943,443,761,273,497
87	1	293
91	1	137
93	1	313
95	3	41,113
99	5	4,914,373

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[3] J. Heleen. Family numbers: Constructing primes by prime factor splicing. *Journal of Recreational Mathematics*, 28(2):116–119, 1997.

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Conference Photographs

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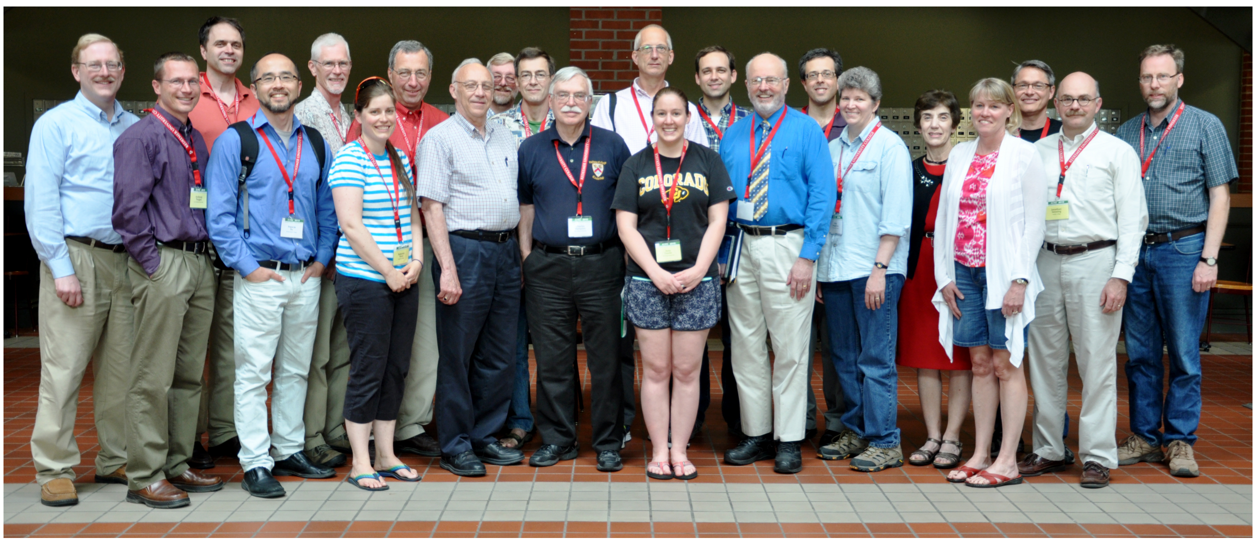

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