THE HISTORICAL SHAPING OF THE FOUNDATIONS OF MATHEMATICS

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I would like to express my appreciation to the administration of Wheaton College for several ways in which they have helped with this Conference. For one thing, they have provided a Faith and Learning Seminar for Wheaton faculty during the summers, which has been a great help for many of us in our attempt to integrate Christianity with our discipline. For another, they have provided financial assistance for the expenses of this Conference. And finally, they have encouraged our efforts in several ways, as illustrated by President Armerding's willingness to share his time with us this evening.

It has been a source of wonderment for me to observe the reservations come in for this Conference, not only from mathematics faculty at Christian colleges, as we might expect, but also from undergraduate and graduate students, junior high and high school teachers, college professors at secular schools and individuals in industrial positions. An interest seems to exist in foundational issues of mathematics that was not present just a few years ago. I believe it would be helpful for many of you to gain a perspective on the goals of this conference if I would begin my remarks with a brief description of how this conference came into being.

Two years ago in April of 1975, I sent a letter to ten or fifteen Christian mathematicians with whom I was personally acquainted. The letter expressed my growing interest in the role that foundational studies could play in broadening my conception of mathematics, as well as imparting interesting and helpful perspectives to my students. I was asking if these individuals had a similar experience and if there were common interests here which we might profitably pursue. I received no negative replies. Either the individuals had made some faint efforts on their own, or else as novices to the foundations of mathematics, they were willing to learn. Perhaps even more important, was the need and enthusiasm expressed to pursue such investigations.

During the next year, several more letters were exchanged. A modest listing of articles was circulated, new names were added to the mailing list, and the idea of a conference was proposed. Then one year ago last April, we committed ourselves to this conference. I wish I could say that we chose this weekend purposely to coincide with the 200th anniversary of the birth of Gauss this Saturday. In reality, the Wheaton College calendar had no openings for conferences in the fall, traveling was out of the question in the winter (only a strange breed of person would attempt travel in the Chicago-St. Louis area in January), and so April, 1977, was chosen.

To the best of my knowledge, no one had done much speaking or writing on these issues, so we went to the volunteer system, assuring everyone that there would be no experts to make their first attempts look foolish. This helps explain why our program features three philosophers, one physicist, and 17 mathematicians from 15 different colleges. A secondary advantage to such diversity was to guarantee a fairly well attended conference, even if the participants were the only ones to come.

And so, here we are with much anticipation for a time of learning, a time of beginning new friendships, and possibly a time of evaluation of the direction of our professional goals. Perhaps a few words are in order as to our concept of the foundations of mathematics. At least initially, I do not mean the very technical aspect of foundational study, which presupposes an advanced degree in logic. Rather, I am thinking along the lines of E. W. Beth, when he stated in his book, Mathematical Thought, that "the primary aim of philosophical thinking is a relatively modest one; it consists in clarifying our concepts and deepening our interest."[1] Or again as the Encyclopedia Philosophy begins its section on the foundations of mathematics - "The study of the foundations of mathematics comprises investigations... that consist of general reflection on mathematics."[2]

And this is what I would hope we can begin to accomplish this weekend - general reflection, clarified concepts and deepened interest - not just in the philosophy and foundations of mathematics, but also in algebra and analysis, in geometry and topology, in application and in theory, in teaching and in research.

Most of the presentations of this conference will emphasize developments of the past one hundred years, since the philosophy of mathematics and the formal treatment of foundational issues are a product of the twentieth century. However, the ideas which have shaped the foundations of mathematics as we study them today go back over severa1 centuries. I would like to focus attention this evening on what I consider to be the two most significant developments in this shaping process.

The first of these two familiar developments is what we today call the discovery of non-Euclidean geometry, arising from the heritage of Euclidean geometry. The second development is the even more familiar discovery of the calculus and the implications of attempts to put it on a rigorous foundation. Though these two developments are almost completely unrelated to one another conceptually and historically, they do share a common concern with the infinite and a progression through two or three major formulations.

For geometry, the first formulation occurred with the writing of the Elements of Euclid about 300 B.C. For analysis, the first formulation was given by Newton and Leibniz during the late 1600's. Fo1lowing extensions, revisions and criticisms of these first formu1ations, each area underwent a second major formulation in the 1820's. The subsequent adjustments seemed to merge both areas into the axiomatic system stream which began to dominate mathematical development toward the end of the nineteenth century, and which continues unabated until the present day.

I would like to take a leisurely look at these two developments - tracing them through the stages of their formulation and indicating how they have been especially influential in bringing foundational issues into the open.

Before a theory can receive its first careful formulation, there is usually a groundwork laid by many individuals, whose efforts are largely uncoordinated and whose results are often fragmentary. The value of their work is usually seen only with hindsight - these are the giants to whom Newton attributed credit by his statement - "If I have seen further than others, it is because I have stood upon the shoulders of giants." The formulation occurs when one or two individuals come along who are able to separate the wheat from the chaff, and present a unified collection of results to be reworked and expanded by their successors.

In geometry, the groundwork stage was accomplished by a number of nameless individuals, such as the Pythagoreans, who generalized the tedious efforts of previous civilizations, as the Egyptian and Babylonian. In addition, we have the contributions made under the encouragement by Plato of such individuals as Eudoxus and Theaetetus.

Then around 300 B.C., the individual Euclid came on the scene to found the Alexandrian School of Mathematics and to give the definitive statement of the elements of mathematics of his day. His skill was not demonstrated so much by an original discovery, as it was in a masterful organization of previously known results. Beginning with a listing of 23 definitions, 5 postulates and 5 common notions, Euclid went on to build an edifice of 13 books consisting of 465 theorems on plane and solid geometry, on elementary number theory, on geometric algebra, and on the theory of proportion and incommensurable line segments. In his History of Mathematics, Howard Eves has this to say about the Elements of Euclid.

> "As soon as the work appeared, it was accorded the highest respect, and from Euclid's successors on up to modern times the mere citation of Euclid's book and proposition numbers was regarded as sufficient to identify a particular theorem or construction. No work, except the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influence on scientific thinking. Over a thousand editions of Euclid's Elements have appeared since the first one printed in 1482, and for more than two millennia this work has dominated all teaching of geometry."[3]

In reading biographical sketches of great mathematicians, how often it is the Elements which furnished the first introduction to mathematics or captured the individual's heart for mathematics. Bertrand Russell in his Autobiography says,

> "At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me that it was generally considered difficult, but I had found no difficulty whatever. This was the first time it had dawned upon me that I might have some intelligence. From that moment until Whitehead and I finished Principia Mathematica when I was thirty-eight, mathematics was my chief interest and my chief source of happiness."[4]

In calculus, we have the well-known instance of Archimedes in the century after

Euclid inscribing and circumscribing a circle with regular polygons of up to 96 sides in order to find an approximation for the number _. This method of exhaustion, as it is appropriately called, was a preview of the Riemann sums which would be used 2000 years later to approximate areas in a modern approach to integration.

As we know, Archimedes' work was about the only thing that came close to resembling the calculus until an outbreak of new efforts marked the first half of the seventeenth century. As one example we mention Kepler's laws of planetary motion which contained some hints of integration and which certainly provided a prod to develop the calculus as a tool for understanding celestial mechanics. Mention should also be made of Fermat's work in drawing tangent lines to curves, and the efforts of John Wallis and Isaac Barrow which gave Newton a good foundation to build upon. In fact, Barrow was the teacher of Newton at Trinity College who resigned his position in favor of Newton when he recognized the unusual intellectual gifts of his twenty-six year old pupil.

Archimedes was severely hampered in his quest for the calculus by the absence of several concepts which were very slowly accumulated over the centuries, but were available by the time of Newton and Leibniz. These include the discovery of the number zero and the positional base 10 notation by the Hindu-Arab culture in the 8th century A.D., and the introduction of letters to represent unknown values by the French mathematician Vieta in the late 16th century. We should also mention the Cartesian coordinate system presented by Descartes in his 1637 treatise on analytic geometry, which has been of great value in geometrically representing the algebraic curves under investigation in the calculus.

Two men, Isaac Newton and Gottfried Leibniz vied for the honor of being the first to express the general theory of the calculus. And indeed, both these men and their English and German countrymen did earnestly contest for this honor. Leibniz gained his edge by being the first to publish and also for choosing the more useful notation of differentials, rather than the fluxions of Newton. On the other hand, Newton seemed to be the first to conceive of the calculus, but was very hesitant to publish his work. In fact, it was only after a time lag of 20 years that the astronomer Halley was able to persuade Newton to write up his results, offering also financial support for the publication. When Newton finally did write up the Principia, he carried the implications of the calculus much further than did Leibniz, expressing both the general physical principles of motion and the universal law of gravitation.

When we say that Newton and Leibniz discovered the calculus, we mean that they discovered the result contained in the Fundamental Theorem of Calculus that the derivative and integral concepts, although very different in definition, are intimately connected as inverse processes. In addition, they developed a symbolism for expressing the derivatives and integrals of specific functions.

Admittedly the above comments furnish only a skeleton outline of the rich historical heritage that led to the first formulation of Euclidean geometry and the calculus. Nevertheless, we will now go on to consider the ways in which these two theories were modified, until they were reformulated with significant changes in the 1820's.

In the case of Euclidean geometry, this took rather a strange form. For some reason, attention was focused on the fifth of Euclid's postulates, which has come to be called the parallel postulate, having the familiar formulation due to Playfair in the late 1700's that through a point not on a given line, exactly one parallel line could be drawn to the given line. It became a widespread conviction that this was really a theorem, provable using only the first four postulates. However, all attempts to prove such a theorem failed. It often happened that someone believed they had a proof only to find later that they had unwittingly assumed a result equivalent to the parallel postulate.

To me, the most remarkable part of all is that so many people would have given so much concern to the fifth postulate of the Elements. For the issue at stake - is the parallel postulate an axiom or a theorem - scarcely seems urgent enough to occupy a central place of attention for more than 2000 years. Yet mathematicians, great and small, seemed unable to resist tackling this elusive problem. One wonders if it is simply due to chance, that interest in this question persisted from 300 B.C. through the Dark Ages, only to become stronger than ever in the eighteenth century. Or is it an example of the general providence of God, as He guides man to pursue those topics which will be fruitful in the revelation of knowledge?

Though this axiom vs. theorem mindset might seem to be sterile as far as advancing the knowledge of geometry is concerned, we can see with hindsight that these efforts to prove the dependence of the parallel postulate did really advance and expand geometric knowledge by furnishing theorems for the yet undiscovered non-Euclidean geometries.

Let us take a brief glance at some highlights of the 2000 year search for a proof of the dependence of the parallel postulate. Since most of the writings of the early Greek period have been long lost, it is fairly remarkable that the Elements has endured to the present day. For this, we should thank two Greeks, Theon and Proclus, who lived in the 4th and 5th centuries A.D. shortly before the light of learning was virtually extinguished during the Dark Ages. It was Theon's revision of the Elements which was the basis for most modern forms of Euclid's work. And it was Proclus who in his Eudemian Summary has given us virtually our only source for the period of Greek mathematics - leading up to the writing of the Elements.

As we approach the reawakening that was the Renaissance, the first mathematical book of significance to be printed was the Elements in Venice in 1482. And it seemed that the conviction that the parallel postulate was a theorem also survived the Dark Ages, as did the urge to prove this result. In 1733, the Italian priest Saccheri became the first to apply the method of proof by denial to this question. In his book, Euclid Freed of Every Flaw, Saccheri presented the three logical possibilities of no parallel lines, exactly one parallel line and many parallel lines. By showing contradictions under the hypothesis of no parallel lines or more than one parallel line, Saccheri hoped to derive the elusive proof. However, his hopes overcame his mathematical correctness, and he carelessly introduced his own flaws in his attempt to remove Euclid’s.

Lambert and Legandre furthered the spirit of Saccheri’s work through the use of the isosceles birectangle, trying to settle this parallel line axiom or theorem question once and for all. Finally, around the year 1825, three men - the great German mathematician and astronomer Gauss, the Hungarian army officer Bolyai, and the Russian professor Lobatchevsky - working independently of each other came to a surprising discovery: it had been so difficult to prove that the parallel postulate depended on the first four postulates because it was not true. The parallel postulate was indeed independent of the other postulates, to be assumed or denied according to the choice of the, individual geometer, without it being forced upon us by the nature of the real world. Thus, the essence of the discovery of non-Euclidean geometry is that Euclidean geometry becomes but one of many possible consistent geometries, anyone of which might be a valid model for the physical world.

The corresponding reaction to the first formulation of the calculus took quite a different turn than the introspective examination of the parallel postulate in Euclidean geometry. The remarkable insights of Newton and Leibniz were gratefully accepted and applied unquestioningly to all manner of problems in the physical realm. In recalling the familiar anecdote about military life - if it moves, salute it; and if it doesn't move, paint it - we might facetiously amend this to describe the reaction to this new tool of the calculus - if it moves, measure its rate of change, and if it doesn't move, measure its area.

We have already mentioned that Newton and Leibniz had many contemporaries who were also working on calculus concepts. Thus, when the so-called discovery of the calculus was made, numerous individuals had the background and ability to immediately work with these new ideas and to extend their usefulness. There was not a period of marking time which often follows a new discovery until others learn enough of the new field to further its development. The ones who contributed the most to the proliferation of early results were the brothers Jacques and Jean Bernoulli.

The most interesting fact about these two brothers is that they are but the most gifted two of a dozen members of the Bernoulli family who were prominent mathematicians and/or physicists, ten of them coming in a span of only three generations. These brothers became ardent disciples of the calculus of Leibniz and were responsible for a great deal of its early development and terminology. It was Jacques who suggested the name "integral" to Leibniz and who gave a proof for the divergence of the harmonic series. Jean is credited with the insight called L'Hopita1's Rule for finding limits of indeterminant forms and with beginning the study of the calculus of variations through his suggestion of the brachistochrone problem. But perhaps Jean's greatest claim to fame was that he taught the famous Euler.

It was Euler who dominated analysis throughout the eighteenth century. During a lifetime of 76 years, he divided his time between the St. Petersburg Academy in Russia and the Prussian Academy in Berlin. Despite the demands of a large family and the misfortune of total b1indness during his last 17 years, Euler wrote prolifically about all phases of mathematics, emphasizing especially a formalistic approach to the infinite processes of analysis, without sufficient attention to questions of convergence and mathematical existence. Thus, it is not surprising that a few nonsensical results would occur among the many amazing results of the calculus, which were being applied with an apparent magical touch to all manner of physical problems; for example, the work of Laplace in celestial mechanics and of Fourier in the conduction of heat.

A few individuals became concerned about this lack of rigor. A theologian, Bishop Berkeley, wrote the Analyst in 1734 which was an attack against fluxions or similar use of infinitisma1s in the calculus. No, one could refute what he said, but neither was he able to offer any suggestions for improvement. In 1754, d'A1embert recognized that a careful definition of what we now call the limit concept was needed to serve as a foundation for analysis, but he himself was unable to furnish it. A little later, Lagrange developed the calculus based on the power series representation of functions, thus avoiding the concepts of infinitesma1s and limits, as well as Euler's formalism.

As we move into the nineteenth century, we see Gauss publishing a careful memoir in 1812 on the convergence of the hypergeometric series. Finally, the Frenchman Cauchy delivered a series of lectures at the Ecole Polytechnique in Paris, which were written up in 1821 in the book Cours d'Analyse. This work became a landmark and the second major formulation of the calculus, being the first time that someone gave a carfu1 definition of the 1imit of a function and then used this to define the other related concepts of continuity, the derivative, the integral, and convergent sequences and series. In spirit, it is very much like our present-day calculus textbooks. Whether Cauchy did plagiarize many of his results from Bolzano, who was more or less isolated in Prague, is a question beyond our present scope of interest.

Both geometry and analysis would undergo further formulations than the two we have concentrated on. If time permitted, we could discuss the successful attempts by many to obtain a rigorous axiom basis for Euclidean geometry, or the extension of the concept of integration to the abstract spaces of the early 1900's, or the interesting reintroduction of the infinitesimal concept in the non-standard analysis pioneered by Robinson of Yale in the early 1970's.

However, our rehearsal of the historical development of non-Euclidean geometry and the calculus would not be complete without a consideration of the implications of these two familiar developments for present-day foundational studies. We must be careful not to overemphasize the importance of these two accomplishments in the 1820's. Yet it is undeniable that mathematics began to change in many significant ways in the years following the discovery of non-Euclidean geometry and Cauchy's formulation of the limit concept as a basis for the calculus. Let me now list 5 of these implications.

First, the question of truth. The discovery of the non-Euclidean geometries focused attention on the importance of axioms. The major concern would no longer be whether the theorems of a system expressed necessary truths about the physical world, but whether the theorems developed were free of contradictions. Consistency replaced truth as the primary concern of mathematics. This is not to say that there was no longer any concern whether mathematics could describe reality, but the question was now whether a mathematical system was a good model for the physical world, rather than being the unique characterization of it.

Several decades passed after 1820 before more than a few scattered individuals realized the implications of the non-Euclidean geometries. Then in the early 1900's, Albert Einstein used one of the non-Euclidean geometries as the model for the physical world in his famous theory of relativity. The popular form of this theory quickly captured the attention of the public, giving dramatic evidence that a non-Euclidean geometry could apply to a physical situation. Many believe the introduction of the concept of relativity into all areas of human knowledge and activity to be a consequence of non-Euclidean geometry.

It is essential to realize that the discovery of non-Euclidean geometries did not prove absolute truth did not exist. The mathematicians were only asserting that absolute truth could not be proved within their discipline. This does not abolish truth, but rather indicates it may transcend the highest level of human thought and effort. Unfortunately many disciplines reacted by denying the existence of absolute truth, asserting all things to be relative, whether they be in politics, ethics, or even theology.

The second implication is the axiomatic method. The long search into the true nature of the parallel postulate - independent axiom or dependent theorem - had uncovered a number of weaknesses in Euclid's original system. In particular, he failed to identify many necessary axioms. During the last part of the nineteenth century, men as Pasch, Pieri, and Peano undertook the challenge to provide a correct axiomatic formulation of Euclidean geometry. When David Hilbert finally achieved this goal in his book Foundations of Geometry published from lecture notes at the University of Gottingen in 1899-1900, it seemed that the beauty of his presentation and the authority of his person established the importance of the axiomatic method in modern mathematics.

Thus in addition to the postulates of Hilbert for Euclidean geometry, we have the postulates of Peano for the natural numbers, the set theory axioms of Zermelo and Fraenkel, and those of Russell and Whitehead for the predicate and propositional calculus. We also see the central role of axiomatics in the formalist program of Hilbert, in the significant results of Kurt Godel in the 1930's, and in the way that Paul Cohen settled the continuum hypothesis question in 1963.

The third on our list of implications is the proliferation of new developments. As the eighteenth century drew near its close, men as Lagrange were expressing a concern that virtually all mathematical questions were being answered and that there would soon be no new problems left to work on in the nineteenth century. But as Morris Kline stated in his book, Mathematics in Western Culture,

> "If the creation of non-Euclidean geometry rudely thrust mathematics off the pedestal of truth, it also set it free to roam...It now seems clear that mathematicians should explore the possibilities in any question and in any set of axioms as long as the investigation is of some interest...It is possible now to say with George Cantor that 'The essence of mathematics is its freedom.’".[5]

Nowhere was this more evident than in the proliferation of new structures in algebra. The quaternions of Hamilton in the 1840's furnished a non-commutative number system as well as being a forerunner to the tool of vector analysis.

The Boolean algebra of George Boole in the 1850's furnished the structure for Cantor's algebra of sets, for Russell's symbolic logic and for the structure of the modern computer. The matrix algebra of Cayley in the 1860's probably furnished the idea for Cantor to use infinite matrices in his diagonalization approach for proving some of his important results about infinite sets.

The fourth implication is the nature of the real numbers. Once Cauchy's program to base the concepts in calculus upon the limit definition was in place, it freed the gaze of analysts to focus more critically at some new and deeper problems. These include the concept of number itself, essentially unchanged from the time of the Greek Eudoxus, and the non-intuitive nature of analysis that permitted Weierstrass to find an example of an everywhere continuous, but nowhere differentiable function. The arithmetization of analysis program of Weierstrass to define and develop the nature and properties of number, along with Peano's postulates for the set of natural numbers, provided a framework for the important and frequent philosophical discussions on the nature and existence of number.

The fifth and final implication is the theory of the infinite. It is an interesting phenomenon that the infinite has often been the motivating factor whenever a significant advance has occurred in the development of mathematics. From the time of Euclid, we have the question of whether two lines will intersect if we only extend them farther out - "to infinity."

Then there are the paradoxes of convergence that the likes of Euler, Lagrange and Gauss wrestled with - as what is the value of the series l – l + l – l + l - l...? If the enumeration would ever terminate, we could answer 1 or 0. But if the summation involves an infinite number of terms, what then?

That "the whole is greater than the part" was obvious enough to Euclid that he placed it in the status of a common notion. But in the hands of Cantor, it became a definition - that a set was infinite if the whole was equivalent to a part, leading eventually to the endless chain of infinities that sparked the origin of the philosophies of mathematics. And it was in his study of the calculus, namely in dealing with the convergence sets of Fourier series, that Cantor first became interested in the question of set theory. The centrality of infinite sets in the subsequent development of the foundations of mathematics cannot be overemphasized.

I hope that my attempt to identify just the two familiar developments of non-Euclidean geometry and the calculus will not appear overly simplistic. Certainly other factors contributed to the foundations of mathematics, but it does seem that these two have played a unique role. Tomorrow we will examine some of their implications in greater detail, namely, the question of truth versus validity in mathematics, the existence question of mathematical objects, particularly numbers, the role of the axiomatic method, and the question of the infinite.

The ideal person we are looking for to work in this area would have to simultaneously be a Christian, a mathematician, a philosopher, a theologian and a logician. Since we do not have such universal men among us any longer, I would like to suggest in conclusion a few practical means we could take to improve our contribution as Christian mathematicians.

1. Attend a good conference on a Christian perspective on the Foundations of Mathematics.

2. Begin or continue a reading program on the historical and conceptual development of mathematics and the philosophy of mathematics. Such books are becoming increasingly available.

3. Sit in on a course in the history of philosophy or the philosophy of science at your school.

4. Try to introduce some of these new ideas into your teaching and into your dialogue with colleagues.

5. Try to put some of your ideas into writing so that they can be shared with others who are grappling with similar matters.

In the year 1930, David Hilbert was to retire from the University of Gottingen where he had served as a most worthy successor to the likes of Gauss and Riemann. His native city of Konigsberg decided to honor him at the fall meeting of the Society of German Scientists and Physicists. He chose the understanding of nature and logic as the topic of his farewell address.

In her excellent biography of Hilbert, Constance Reid devotes several pages to the ideas expressed in this talk. The electricity of the occasion comes through as Hilbert closes his address with these words expressing the conviction of his formalist goal - "We must know! We shall know”"

But we always know only in part, and within a few months, Kurt Godel was to publish his famous result which would assert that Hilbert and his formalist followers could not know everything they wanted.

While we can identify with this strong desire of Hilbert for knowledge; as Christians, we do not accept the love of knowledge as our primary motivation. Ours is better expressed by St. Paul, when he stated in II Corinthians 10, that we should "cast down reasonings and every high thing that exalteth itself against the knowledge of God, and bring into captivity every thought in obedience to Christ." May God be pleased to more fully realize this desire in each of us through the experiences of this conference.

Footnotes

1. E. W. Beth, Mathematical Thought,(Gordon and Breach, 1965), p. 124.

2. The Encyclopedia of Philosophy, Vol. 5, Paul Edwards, Editor (The Macmillan Company, 1967), p. 188.

3. Howard Eves, An Introduction to the History of Mathematics, 3rd Edition. (Holt, Rinehart and Winston, 1969), p. 114.

4. B. Russell, The Autobiography of Bertrand Russell, Vol. I. (Little, Brown and Company 1967), pp. 37-38.

5. Morris Kline, Mathematics in Western Culture, (Oxford University Press, 1953), p. 431.