THEISM & MATHEMATICAL REALISM

John Byl
Trinity Western University
byl@twu.ca

1. Introduction

What does mathematics have to do with God?

Not much, according to most people outside of this association. It is, therefore, quite surprising how often non-theists, when discussing the foundations of mathematics, refer to God. Many contemporary philosophers of science perceive theism to be the basis for classical mathematics and mathematical realism, both of which are therefore found to be objectionable.

The aim of this paper is to briefly examine the close connections between God and mathematics. These connections, I believe, could be more fully exploited by Christians.

2.1 The Classical Christian View on Mathematics

Is mathematics a mere human invention? Or is it an exploration of an already existent realm? That is the basic question.

Historically, most mathematicians have believed that mathematical truths such as "2+5=7" exist independently of human minds, being universally and eternally true. Mathematicians believe they are discovering properties of, say, prime numbers, rather than merely inventing them. This view of mathematics dates back to Pythagoras (500 BC) and Plato (400 BC). It is often called “Platonism”. In order to avoid association with other features of Plato’s philosophy, I shall refer to the notion that mathematical truths are objective as mathematical "realism".

Mathematics has some profound philosophical implications. Bertrand Russell, certainly no friend of theism, concluded from his study of the history of Greek philosophy that "Mathematics is...the chief source of the belief in eternal and exact truth, as well as in a super-sensible intelligible world" (Russell: 37). This is so, Russell argues, because of the abstract nature of mathematical concepts. For example, geometry deals with exact circles, but no physical object is exactly circular. This suggests that exact reasoning applies to ideal, rather than physical, objects. Furthermore, numbers appear to be time-independent. Hence mathematics seems to deal with an ideal, eternal world of pure thought.

But where and how do such mathematical entities exist? The existence of eternal, ideal mathematical thoughts seems to require the existence of something actual in which they exist.

The early theistic philosophers Philo (ca 20 BC - ca 50 AD) and Augustine (354-430) placed the ideal world of eternal truths in the mind of God. Augustine argued that mathematics implied the existence of an eternal, necessary, infinite Mind in which all necessary truths exist. He asserted that we all know time-independent truths about logic (e.g, A = A) and mathematics (e.g., 2+2=4). But changing, material things can't cause fixed, eternal truths. Nor can finite human minds, since our thinking does not make them true but is, rather, judged by them. Thus truth must derive from something non-material that is superior to the human mind. Mathematical truths must depend on a universal and unchanging source that embraces all truth in its unity. Such a Truth must exist and is by definition God (Geisler: 154).

Thus arose the classical Christian view that mathematics exists in the mind of God, that God created the universe according to a rational plan, and that man's creation in the image of God entailed that man could discern mathematical patterns in creation. Mathematics was held to be true because of its supposed divine origin. The notion of a rational Creator and creation, espoused vigorously by Kepler, Galileo, Newton, and many other scientists, was a major factor in motivating the scientific revolution.

2.2. The Demise of the Classical Christian View

Ironically, the very success of mathematical models in physics led to the demise of the classical Christian view. First, the clockwork universe of Newtonian physics no longer seemed to need a God to run it. Second, naturalist models contradicted the supernatural events related in the Bible, thus undermining its authority. Such developments eventually induced many scientists to banish God from their worldview.

However, a godless worldview left mathematics without a solid foundation. If mathematics were not God-given, why should it be true? Consequently, an extensive quest was made for grounding mathematics on an indubitable set of self-evident axioms. Some initial successes included proofs of the consistency of first-order logic and geometry. Unfortunately, after 1931, when Gödel showed that, for more sophisticated mathematics, no axiomatic basis could be proven consistent within that system, the quest was generally abandoned. Thereafter the soundness of mathematics had to be accepted largely on faith.

2.3. Naturalism and Realism

Nowadays many philosophers of mathematics reject mathematical realism, primarily because of its perceived connection to theism.

For example, Reuben Hersch, in his recent book *What is Mathematics really?* (1997), finds that "Recent troubles in philosophy of mathematics are ultimately a consequence of the banishment of religion from science" (Hersch: 122). He concedes that "Platonism...was tenable with belief in a Divine Mind...The trouble with today's Platonism is that it gives up God, but wants to keep mathematics a thought in the mind of God" (Hersch: 135). "Once mysticism is left behind...Platonism is hard to maintain". (Hersch: 42)

Similarly, Yehua Rav (1993) comments:

> "Whereas the quarrel about universals and ontology had its meaning and significance within the context of medieval Christianity, it is an intellectual scandal that some philosophers of mathematics can still discuss whether whole numbers exist or not." (Rav: 81)
>
> "There are no preordained, predetermined mathematical "truths" that just lie out...there. Evolutionary thinking teaches us otherwise."(Rav: 100)

Once theism is dropped, it is difficult for realism to explain (1) where objective mathematical truths exist and (2) how we have access to them. Mathematical realism is plausible, it seems, only within a theistic worldview.

2.4. The Advantages of Realism

In spite of what philosophers of mathematics might believe, most mathematicians remain realists. In the last century realism has been explicitly defended by a number of outstanding mathematicians, including Georg Cantor, Kurt Gödel, G.H. Hardy, and Roger Penrose. For example, Roger Penrose (1989: 95, 112) writes that "Like Everest, the Mandelbrot set is just there" and "There is something absolute and 'God given' about mathematical truth". Likewise, Hardy believed that "mathematical reality lies outside us...our function is to discover or observe it...the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations" (Hardy: 123-4).

Realism has a number of distinct advantages over the rival view that mathematics is merely a human invention. Consider the following factors:

A. Realism is very effective and fruitful. The notion that there is a mathematical universe waiting to be explored provides a powerful incentive for research, much more so than a mere dabbling in arbitrary inventions of the mind. Thus, even if one really did not have a realist view of mathematics, it might be beneficial to pretend in it.

B. Realism explains the universality of mathematics. The notion of an objective mathematical world explains why mathematicians widely separated in space, time, and culture find the same mathematical theorems and ideas.

C. Gödel 's incompleteness theorem, which proves that not all mathematical truths can be derived from a finite set of axioms, seems to imply that (i) our grasp of mathematical truth extends beyond our construction of mathematical systems and (ii) the class of mathematical propositions is essentially, actually infinite. Realism can supply the vast infinities of objects that mathematics requires - more objects than any finite mind could construct.

D. Realism is indispensable for science. Modern physics is so heavily dependent upon mathematics that its theories could hardly be even stated without mathematics. Since science deals with real objects, it would seem that mathematics must also deal with real objects. This applies even more so for those embracing a realist view of scientific theories. How can scientific theories be true unless the underlying mathematics is also true?

E. Finally, realism explains the applicability of mathematics to the physical world. If mathematics is merely a human invention, why is it that relatively simple mathematical theories yield such accurate representations of the physical world? Sophisticated theories, such as relativity or quantum mechanics, can be aptly summarized in just a few small mathematical equations and their logical implications. The amazing success of physics is largely due to its basic mathematical nature. This suggests that the physical world reflects the same mathematical structure that mathematicians explore. Eugene Wigner, a Nobel prizewinner in physics, in a famous article entitled "The Unreasonable Effectiveness Of Mathematics", commented on the amazing applicability of complex analysis to quantum mechanics:

> "It is difficult to avoid the impression that a miracle confronts us here, quite comparable to the...miracles of the existence of laws of nature and of the human mind's capacity to divine them" (Wigner: 8)

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve" (Wigner: 14).

2.5. Three Worlds - Three Mysteries

However, as we noted, realism without theism poses problems. Thus Roger Penrose, who favors a naturalistic mathematical realism, is left with a number of baffling loose ends. He concludes his book *Shadows of the Mind* (1994) by remarking on three worlds and three mysteries.

The three worlds are the physical world1 wherein we live, the mental world2 of our conscious perceptions, and the platonic world3 of mathematical forms. The three mysteries are how these three apparently quite diverse worlds can mutually interact.

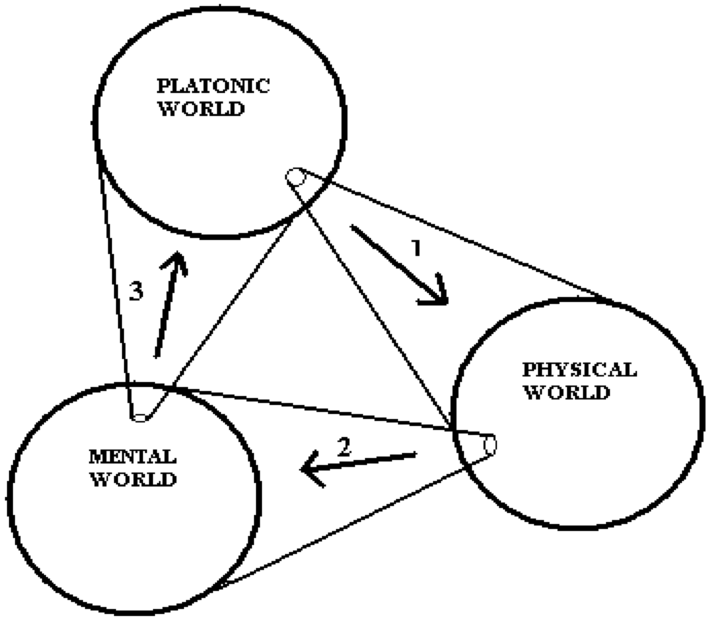

Figure 1. Three Worlds and Three Mysteries

1. Mathematics and Matter

Mystery 1 concerns mathematics and matter. Why are mathematical laws so important in the physical world? World1 seems to mysteriously emerge from world3 or, perhaps more correctly, from only a small part of world3.

Science is based on the assumption that the universe has an underlying rational structure. But why should that be so? And why should that mathematical structure be sufficiently simple that man is able to discern it?

Further, the ideal, mathematical world contains all possibilities, as pure abstractions. Of these infinite possible worlds, only one is actualized in our physical world. How was the choice made? How was it actualized? How can we bridge the gap between the ideal world of necessary truths and the actual, contingent physical world, where just one of these ideal worlds takes on concrete form?

2. Matter and Mind

Mystery 2 is how the physical world1 of matter can give rise to perceiving and thinking beings, as well as to their mental world2. How can matter give rise to consciousness? How can configurations of atoms generate purpose, values and emotions?

3. Mind and Mathematics

Mystery 3, which brings us full cycle, is how mind, supposedly evolved from matter, nevertheless has access to the eternal truths of world3.

Charles Darwin already wondered whether minds produced through the blind mechanism of natural selection could be trustworthy. The evolutionary biologist J.B.S. Haldane worried:

> "If my mental processes are determined wholly by the motions of atoms in my brain, I have no reason to suppose my beliefs are true...and hence I have no reason for supposing my brain to be composed of atoms".

Many naturalists deny that mind has any influence on matter; it is generally deemed to be no more than a reflection of our brain states. But how, then, can mind invent anything, let alone deep mathematical concepts that are applicable to the real world?

Penrose considers the platonic world to be primary - its existence being almost a logical necessity - and the other two worlds to be its shadows (Penrose 1994: 417). The mathematical aspect of the physical world implies it has been designed, that mind controls matter, and that mind is prior to matter. This is opposite to materialism, which sees mind as a mere impotent residue of matter. Penrose has no solution to these three mysteries, which seem very difficult, if not impossible, to resolve within a naturalist framework. Indeed, the most profound philosophical questions are precisely those regarding the origin and interactions of the three worlds.

Of course, Penrose's three mysteries are easily resolved within the classical theistic view of mathematics. To summarize our argument thus far: the nature of mathematics as necessary, eternal universals suggests the existence of an eternal Mind in which they reside. The existence of a mathematical structure to the universe suggests its creation by a rational Creator, who freely chose which possible world to actualize.

3. Theistic Ontology

We now consider the other side of the coin: What does theism – particularly Christianity - have to say about mathematics?

3.1. God and Logic

We note first the close relation between logic, which is basic to mathematics since it concerns truth and consistency, and God.

The biblical God presents Himself to us as a God of truth and faithfulness: "God who never lies" (Titus1: 2); "the faithfulness of the Lord endures forever" (Ps.117: 2); "thy word is truth" (John 17:17). Since God's identity is always the same, the logical law of identity must be eternally valid. God is truth, not falsehood; God means what he says, not its contradiction. Hence the law of non-contradiction holds.

Arthur Holmes (1977) states:

> "...A is not non-A...because God made it so, and because God Himself is God, and not non-God...It derives from God and his unchanging fidelity."(Holmes: 88)

In a similar vein, John Frame (1987) writes:

> "Does God, then, observe the law of non-contradiction? Not in the sense that this law is somehow higher than God himself. Rather, God is himself non-contradictory and is therefore himself the criterion of logical consistency and implication. Logic is an attribute of God, as are justice, mercy, wisdom, knowledge". (Frame: 253)

In short, the very nature of God implies the eternal validity of the basic laws of deductive logic (i.e., the laws of non-contradiction, identity, and excluded middle).

3.2. God and Number

God has also a numerical nature. Vern Poythress argues that, since the Father, Son, and Holy Spirit (3) are eternal, so is number. Poythress asserts that the plurality of this world - the works of God - finds its basis in the plurality of the fellowship of the Trinity; the unity and inner consistency of the world are based on the oneness of God. Hence the Trinity resolves the ancient problem of reconciling the "one" and the "many" (Poythress: 179).

Further, the Bible depicts God as performing the mathematical operations of numbering stars (Ps.147: 4) and hairs (Matt.10: 30), measuring the waters in the hollow of his hand, and meting out heaven with the span (Is.40: 12).

3.3 God's Knowledge and Wisdom

The Bible tells us that "the Lord is a God of knowledge" (I Sam.2:3); God knows everything (I John 3:20). His knowledge encompasses all things past, present, and future (Is.48:5), as well as all things possible, even the thoughts of man (I Cor.3:20; I Chr.28:9). God's knowledge is infinite (Ps.147:5). Since God's omniscience is complete, including even future events and possibilities, God's knowledge corresponds to an *actual* infinity of thoughts, rather than an ever-growing *potential* infinity. Among other things, such infinite knowledge embraces all possible mathematical propositions.

God's wisdom as reflected in his work of creation. By wisdom the Lord made all his works (Ps.104:24), including the heavens (Ps.136:5) and the earth (Prov.3:19). From the infinity of mathematical possibilities, God has chosen to incorporate only a small subset in the contingent, physical universe, which might have been different. A personal God who created the world by a free act of will bridges the gap between the *necessity* of the conceptual realm and the *contingency* of the physical realm.

3.4. God's Nature and Mathematics

This brings us to some difficult theological questions. Orthodox Christianity maintains that only God is self-existent and that He is the creator of everything else. How, then, are abstract, necessary objects related to God?

Alvin Plantinga (1980) argues that, since God is necessarily omniscient, necessary propositions (e.g, 7+5=12) are necessarily always known to God, who thus affirms their existence. Abstract objects of logic and mathematics are ideas in the mind of God. Although God has no control over these abstract necessarily existing ideas or properties, they pose no threat to God in any way. They are not agents that oppose God but, on the contrary, they depend on God to affirm their existence. Plantinga writes:

> "According to Kronecker God created the natural numbers and men created the rest...Kronecker was wrong on two counts. God hasn't created the numbers; a thing is created only if its existence has a beginning, and no number ever began to exist. And secondly, other mathematical entities (the reals, for example) stand in the same relation to God...as do the natural numbers. Sequences of numbers, for example, are necessary beings and have been created neither by God nor by anyone else. Still, each such sequence is such that it is part of God's nature to affirm its existence". (Plantinga 1980: 142)

Plantinga believes that in exploring mathematics one is exploring the nature of God's rule over the universe ... and the nature of God Himself. He concludes "Mathematics thus takes its proper place as one of the loci of theology." (Plantinga 1980: 144).

Roy Clouser, on the other hand, contends that we can't know anything about God's essential nature. He believes that God has taken on logical and numerical characteristics only for the sake of covenantal fellowship with us. God could have taken on quite different characteristics, had He wanted to. According to Clouser (1991), God accommodates Himself to our creaturely limitations; God's uncreated, unrevealed being is unknowable to us.

Plantinga and Clouser agree that God upholds the realm of objective mathematical laws. The question is whether these laws are really necessary and whether they essentially apply to God. This raises deep, subtle questions pertaining to God's essential nature. For example, it seems implausible that God's unrevealed being would be inconsistent with his revealed being and that the laws of identity and non-contradiction would not apply to God's unrevealed being. God certainly gives us no hint of that in his revealed Word.

It must be stressed that God is the ground of all being. God upholds everything, even all possibilities, establishing what is possible and what is necessary. Necessary truths are those that are true in all possible worlds. Is it possible for God to have established other possibilities than those He did establish? In a deep sense, no, for only those possibilities exist which God upholds. Necessary truths are ordained as such by Him who determines all necessity. Necessary truths by no means diminish God's sovereignty. On the contrary, the omnipotence of God is most dramatically illustrated by the fact that God establishes and upholds even whatever is possible and whatever is necessary.

4. Theistic Epistemology

How does finite man come to know eternal mathematical truths? This is the main problem associated with mathematical realism. The theistic solution is as follows.

The Bible tells us that man was made in the image of God (Gen.1:26-30; I Cor.11:7), with the ability to rule God's creation (Gen.1:28). This image includes not only righteousness but also rationality and creativity. The ability to do mathematics seems to be innate in human minds. This involves the capacity for abstract thought, as well as the ability to discern and symbolize. As such, it is intimately connected to linguistic ability. Our minds have been formed in such a way that they can readily handle abstract thought, symbolic representation, and logical manipulation.

Alvin Plantinga comments:

> "God has ...created us with cognitive faculties designed to enable us to achieve true beliefs with respect to a wide variety of propositions - propositions about our immediate environment, about our own interior lives, about the thoughts and experiences of other persons, about our universe at large, about right and wrong, about the whole realm of abstracta - numbers, properties, propositions - ... and about himself." (Plantinga 1993: 201)

Furthermore, the Bible indicates that the mind contains a number of innate, self-evident truths that are directly impressed on the mind. These include a sense of deity (Rom.1:18-32), a sense of right and wrong (Rom.2:15), reasoning ability, and even a sense of eternity: "He has put eternity into man's mind" (Eccl.3:11 RSV).

In practice, our basic, innate, intuitions seem to yield direct knowledge of:

(1) The laws of deductive logic, such as the law of non-contradiction.
(2) Discreteness, the ability to distinguish between objects. This is closely related to counting, involving the notion of natural numbers, and collecting, involving the notion of sets.
(3) Continuity, the ability to distinguish size and shape of spatial objects.

These basic intuitions form the basis of logic, algebra, and geometry.

The Fall profoundly diminished the image of God in man. How did this affect his mathematical ability? John Calvin distinguished between our natural and supernatural gifts. The natural gifts include soundness of mind and uprightness of heart; the supernatural gifts include the light of faith and righteousness (Calvin: II, ii, 12). After the Fall the natural gifts were corrupted whereas the supernatural gifts were completely taken away. Calvin distinguished between man's faculties (i.e., reason and will) and the use to which they were put. Their exercise is now put to opposing God rather than serving Him. Man still has an innate awareness of God but he willfully suppresses it (cf. Rom.1).

Thus man's ability to reason is still functional. Fallen man is still able to discern the laws of logic and to do valid mathematics. He may make mathematical mistakes but these can be corrected. Calvin affirmed that also unbelievers could acquire such knowledge:

> "What shall we say of all the mathematics? Shall we esteem them the delirious ravings of madmen? On the contrary...we shall admire them because we shall be constrained to acknowledge them to be truly excellent. And shall we esteem anything laudable or excellent which we do not recognize as proceeding from God? ...Let us learn from such examples how many good qualities the Lord has left to the nature of man..." (Calvin: II, ii, 15)

Of course, human mathematics also has a cultural component: mathematical knowledge and concepts are passed on through the generations.

5. Theistic Justification of Mathematics

Thus far we have argued that basic mathematical concepts are innate in man, even after the Fall. What about more advanced mathematics? How closely does *human* mathematics resemble *divine* Mathematics? How much of human mathematics can we accept as definitely true? How will a Christian mathematics differ in content from secular mathematics?

Mathematicians differ about which mathematical concepts and methods are valid. One's choice will depend largely on one's presuppositions, driven in turn by one's worldview. Here, then, a Christian perspective can in principle make a substantial difference.

5.1. Mathematics from the Bible

Some people have tried to derive mathematics from the Bible. Gordon Clark (1968) claimed that the Bible, in its frequent usage of logical arguments (e.g, I Cor.15: 12-50 or Matt.12: 25-29), presupposes the validity of all the laws of deductive logic. Logic is necessarily involved in any communication of the word of God: to proclaim the Word as opposed to what contradicts it (I Tim.1: 3ff; 2 Tim.4: 2ff). Similarly, the Bible has numerous instances of arithmetic operations - addition, subtraction, multiplication, and division. J.C. Keister (1982) claims that all the axioms of arithmetic are illustrated in Scripture. For example, he argues that "there shall be five in one house divided, three against two, and two against three" (Luke 12:52), states that $3 + 2 = 2 +3$ and illustrates the more general arithmetic axiom $a + b = b +a$.

Such biblical examples certainly support the validity of our arithmetic and logic. However, one must be careful in drawing general conclusions from a limited number of specific cases. Moreover, this method yields at most only a very small subset of mathematics.

5.2 Classical Mathematics and Theism

A better approach might be to ground the truth of mathematics on the attributes of the biblical God: His infinity, omniscience, and omnipotence, as well as His logical and numerical nature. Classical mathematics is based on the ideal operations of such a God.

The constructionist mathematician Errett Bishop notes:

> "Classical mathematics concerns itself with operations that can be carried out by God...You may think that I am making a joke...by bringing God into the discussion. This is not true. I am doing my best to develop a secure philosophical foundation...for current mathematical practice. The most solid foundation available at present seems to me to involve the consideration of a being with non-finite powers - call him God or whatever you will - in addition to the powers possessed by finite beings." (Bishop 1985: 9)

Bishop himself rejected classical mathematics and urged a constructive approach to mathematics:

> "Mathematics belongs to man, not to God...When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself". (Bishop 1967)

5.3. Permissible Logic and Proofs

What sort of mathematical operations is Bishop referring to? First, classical mathematics is based on two-valued logic. Any well-posed mathematical proposition is either true or false. Thus, for example, Goldbach's Conjecture (i.e., that any even number can be written as the sum of two primes) is either true or false, even though we may not yet know which it is. This is called the logical Law of Excluded Middle.

Constructionists insist that a proposition is neither true nor false until we can construct an actual, finite proof. Hence they object to the Law of Excluded Middle and, also, to proof by contradiction, which is based on it. For example, a classical proof of the existence of transcendental numbers simply notes that the reals are uncountable and the algebraic reals are countable. Hence, the number of non-algebraic (i.e., transcendental) reals must be uncountable, even though we haven't identified a single specific transcendental number. Contrariwise, the constructive approach requires the actual construction of a specific transcendental number.

Constructionism entails the rejection of a good many results of classical mathematics, to the extent that it fails to support the mathematics needed in modern physics. For example, Geoff Hellman (1993, 1997) contends that it is impossible to reformulate quantum theory without resorting to the Law of Excluded Middle. More sophisticated mathematical concepts are required to prove important theoretical results in physics. These include infinite Hilbert spaces in quantum mechanics, the Hawking-Penrose singularity theorems in general relativity, and renormalization in Quantum Electrodynamics. It would seem that believing our best theories in physics implies accepting also their presumed mathematical ontology. This involves, at the very least, several levels of infinite sets (e.g, naturals and reals).

For theists, mathematics exists independent of human minds. Hence a mathematical entity need not be explicitly constructed in order to exist. God surely knows whether any proposition is true or false. Thus theism validates two-valued logic, as well as both direct and indirect proofs.

5.4 Infinity

The concept of infinity is the key to the philosophy of mathematics. Theological considerations led to medieval philosophers to postulate an actual infinity. Today, however, humanism views infinity with suspicion.

Augustine considered actual infinity to be one of the mathematical entities that existed in God's mind. He wrote:

> "Every number is known to him 'whose understanding cannot be numbered' (Ps.147: 5). Although the infinite series of numbers cannot be numbered, this infinity of numbers is not outside the comprehension of him 'whose understanding cannot be numbered.' ...every infinity is, in a way we cannot express, made finite to God." (Augustine: xii, 19).

God's infinite knowledge implies an actual infinity of thoughts, including, at the very least, the infinite set N of natural numbers {1, 2, 3...}. Actual infinity can thus be considered to exist objectively as an actual, complete set in God's mind. Therefore theism justifies the Axiom of Infinity, which posits that infinite sets exist.

The Axiom of Infinity is of crucial importance since certain other fundamental axioms of Zermelo-Fraenkel set theory are concerned with operations involving infinite sets. These are the axioms of Power Set, Choice, and Replacement, which shall be defined presently. Suffice it to note here that these axioms are self-evident for finite sets but rather contentious when applied to infinite sets. These axioms require the Axiom of Infinity plus the notion that certain operations on finite sets can be extended to infinite sets. Such extensions can be shown to be plausible given an infinite, omniscient and omnipotent being.

5.5 The Power Set and Transfinite Numbers

First we consider the Axiom of Power Set. Georg Cantor (1845-1918), the founder of modern set theory, showed that from any infinite set A we can, in principle, generate a set of larger cardinality (i.e., size) by constructing the set of all subsets of A, called the power set of A. This is called the Power Set Axiom. By this means one can generate an ever-increasing series of infinite sets, their sizes corresponding to ever-increasing transfinite cardinal numbers.

Cantor justified his belief in infinite sets by his belief in an infinite God. He thought of sets in terms of what God could do with them. An infinite God would have no difficulty forming the power set of any given infinite set. Further, Cantor's belief in God as the absolute infinite led him to believe that not every collection of objects is itself eligible to be a member of a collection. He believed that the collection consisting of "everything" is divine, beyond human comprehension, and in such a way that it is over-qualified to be a member of some higher collection. The absolute infinite is God: there is no mathematical set of all sets.

Cantor defined a set as a combinatorial collection (i.e., consisting of terms enumerated in an arbitrary way). The alternative definition of sets as a logical collection (i.e., formed in accordance with a specific rule, like "the set of all sets") led to various paradoxes. Cantor's combinatorial definition was free from contradictions and avoided such paradoxes, as described by Lavine (76ff). In fact, the only sets necessary in mathematics are sets of integers, rationals, reals, and functions of reals. These mathematical sets are all combinatorial. Operations on such sets have never led to contradictions.

Even today, almost every attempt to motivate the principles of combinatorial set theory relies on some notion of idealized manipulative capacities of the Omnipotent Mathematician. Constructionists reject Cantor’s transfinite cardinal numbers, as well as the power set axiom, since the power set of an infinite set can't be constructed by finite methods. Mathematicians are unable to specify any sort of general procedure to list every possible subset of a given infinite set. Christopher Menzel comments:

> "In this sense, it is the Platonic axiom par excellence, declaring sets to exist even though humans lack the capacity to grasp or "construct" them." (Menzel: 218)

Plantinga notes that the theist has a distinct advantage when it comes to explaining sets and their properties. The existence of sets depends upon a certain sort of intellectual activity - a collecting or "thinking together". According to Plantinga,

> "If the collecting or thinking together had to be done by human thinkers, or any finite thinkers, there wouldn't be nearly enough sets - not nearly as many as we think in fact there are. From a theistic point of view, the natural conclusion is that sets owe their existence to God's thinking things together...Christians, theists, ought to understand sets from a Christian and theistic point of view. What they believe as theists affords a resource for understanding sets not available to the non-theist..." (Plantinga 1990:35)

5.6. The Axiom of Choice

The Axiom of Choice states that, from every family of sets, it is always possible to form a new set containing exactly one element from each set. This axiom plays a key role in modern mathematics and many theorems depend on it.

The axiom is uncontentious if the number of sets in the family is finite. For an infinite number of sets this is no longer so, since one can't specify exactly how a particular element is to be chosen from each set. Nor can one construct the new set in a finite number of steps. For these reasons constructivist mathematicians reject this axiom.

Again, for theists this poses no problem. An omniscient, omnipotent God could surely always form such a set. Cantor adopted the Well-Ordering Principle, which asserts that any set can be re-arranged so that it has a definite first element, because he believed in a God who could, for any given set, define a suitable relation so that the elements of that set could be well-ordered. The Well-Ordering Principle is equivalent to the Axiom of Choice since, once all the sets well ordered, one simply stipulates that the first elements of all the sets be chosen to form the new set.

The Axiom of Choice is sometimes objected to on grounds that it leads to counter-intuitive results. For example, the Banach-Tarski Theorem asserts that a solid ball can be decomposed into a finite number of pieces, which can then be re-assembled into two solid balls the same size as the first. This certainly seems strange. However, it is not a logical *contradiction*. Moreover, this theorem requires *unmeasurable* sets, which often lead to curious situations that contradict physical reality. The mathematics used on physical problems is almost always mathematics restricted to *measurable* sets (see Wagon: 218). Certainly, the decomposition doesn’t work on an actual physical object containing a finite number of discrete molecules. Finally, the required decomposition is so complicated that only God could actually carry it out.

5.7 The Axiom of Replacement

The Axiom of Replacement is a bit more difficult to state. Informally, if A is a set and if F is a function, then the Axiom of Replacement assumes that a new set can be formed by replacing each element x in A by its image F(x). Thus, from a given infinite set, new infinite sets can be formed following a replacement formula. Although the new infinite set can't be constructed in a finite number of steps, this axiom is self-evident for combinatorial sets and presents no difficulties for an omniscient and omnipotent being.

5.8. Justifying Mathematics as a Whole

Thus far we have argued for the theistic justification of classical deductive logic, the natural numbers, and the Axioms of Infinity, Power Set, Choice, and Replacement. These latter axioms form the basis for ZFC set theory (i.e., Zermelo-Fraenkel set theory plus the Axiom of Choice). Remarkably, virtually all of modern mathematics can be derived from ZFC set theory. Most mathematicians never use axioms beyond ZFC.

One problem remains. Even if the above concepts and axioms were sufficient to derive contemporary mathematics, how can we be assured that this basis is consistent? Can we prove that this system can never give rise to any contradictions? How much of mathematics is guaranteed to be consistent?

Some parts of mathematics have definitely been proven to be consistent. For example, Gödel himself proved the consistency and completeness of propositional calculus; first-order logic has also been shown to be complete and consistent. Euclidean geometry was shown to be consistent and complete in 1949 by the Polish logician Alfred Tarski (Monk: 234). The consistency of Euclidean geometry implies the consistency also of non-Euclidean geometry. Hence these portions of mathematics are indisputably sound.

For larger systems, such as number theory (i.e., Peano arithmetic) the situation is more difficult. Gödel showed that, for any system large enough to include Peano Arithmetic, the consistency of that system couldn't be proven within that system. To prove consistency for such systems one must necessarily appeal to axioms beyond that system.

Thus, for example, Gerhard Gentzen (1936) proved the consistency of Peano Arithmetic by using transfinite induction up to the transfinite ordinal epsilon-0. Transfinite induction is equivalent to the Axiom of Choice. Hence, if one can accept the Axiom of Choice, number theory can be considered consistent. The above theistic justification for the Axiom of Choice thus validates the consistency of number theory.

From the natural number system, we can introduce further definitions from which algebraic, real, and complex numbers can be produced, as well as analysis. Gentzen believed that the consistency of analysis could be proven by using transfinite induction up to some ordinal greater than epsilon-0. Extensions of Gentzen's method by G. Takeuti (1953) and others indicate the consistency of at least major portions of analysis.

Alternatively, we could secure mathematics via ZFC set theory. Its most contentious axioms - the Axioms of Infinity, Power Set, Choice and Replacement - have all been validated by our theistic approach. Can we do better, and prove that ZFC is consistent? Gödel proved in 1938 that, if ZF (i.e., without Choice) is consistent, then so is ZFC. Hence the addition of Choice to the ZF axioms can't do any harm even though, as we saw, it can lead to some bizarre results.

One method of proving consistency is to consider subsystems of ZF. Almost all of ordinary mathematics can be derived from ZF without the Axiom of Replacement. This system can be proven to be consistent by applying the Axiom of Replacement (see Cohen). Hence almost all of contemporary mathematics can be proven consistent if one accepts the Axiom of Replacement, which we argued was justified by theism.

A detailed theistic justification of set theory has been developed by Christopher Menzel (1987). He avoids possible numerical paradoxes by using an iterative concept of sets, coupled to a theory of types. Sets are to be regarded as formed in a transfinite sequence of stages. At each stage every possible set is formed from members that were formed at earlier stages. The number of stages is "absolutely infinite". Menzel presents a formal model proving the consistency of ZFC set theory via his scheme. Menzel's iterative approach to sets seems similar to that of Gödel (1964), who asserted that the iterative concept of sets implies the ZFC axioms.

Ultimately, then, at seems that the consistency and certainty of mathematics can be grounded upon the multi-faceted nature of God Himself. Trust in God generates confidence in mathematics.

6. Conclusions

In summary, mathematical realism offers the best account of mathematics. Realism, since it posits eternal, non-material entities, strongly supports theism. The physical actualization of one world, out of many possible choices, makes plausible the free choice of a creative, omnipotent Mind. Theism readily explains the intricate relations between mathematics, matter and mind.

Theism in turn supports mathematical realism, since God Himself possesses logical and numerical attributes. Further, the infinite, omniscient, and omnipotent God ensures the validity of two-valued logic, proof by contradiction, actual infinite sets and the Axioms of Infinity, Power Set, Choice, and Replacement. These, in turn, provide a solid basis for contemporary mathematics.

A few last points. It must be stressed that, although theism may provide a solid foundation for mathematics, the nature of mathematics is such that *human* mathematics will of necessity always be incomplete. Moreover, since human mathematicians are fallible, some published proofs may well turn out to be erroneous.

Finally, although in this paper we have concentrated on God's relation to mathematics, God's greatness does, of course, extend far beyond mathematics. Indeed, we praise God for His awesome might, holiness and love. We particularly thank Him for His undeserved grace in saving us from our sins through the merciful work of His Son, Jesus Christ.

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