One, Two, Three
Absolutely Elementary Mathematics
By David Berlinski
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“One, Two, Three” is an amazing journey into a “land of the numbers” that plays a foundational role for several major mathematical areas. Rooted in the simple *act* of counting (reflected in the fundamental idea of succession), the “Absolutely Elementary Mathematics” (AEM) invites the human person to participate in a reality (of the number) that can be “better understood”, “better described”, but never “bettered”. In surveying the diverse geography of this arithmetical land, David Berlinski (now living in Paris, with a Ph.D. in Philosophy from Princeton, several postdoctoral research appointments at major universities, and author of several bestsellers) engages the readers through meaningful historical snapshots and well chosen fragments from the interesting life stories of persons that were creators/discoverers of most of modern mathematics.

The selection of highlights in the remarkable saga of the mathematical experience goes back to what Alain Connes calls “archaic” (undifferentiated) mathematics at the beginning of history (as illustrated, for example, by the pragmatic Sumerian scribal art and the Neolithic hunter’s slash marks chipped on his ax handle). From the ancient Sumer to the life of the “mathematical skeptic” Kronecker (the “man of a single *yes*” - sure, "God made the integers" he said, however “all else is the work of man”) and the groundbreaking ideas of Georg Cantor, the pioneer of Set Theory (who considered freedom as the essence of the mathematical experience, which is a very interesting statement in the context set up by the book) it’s a long way. However, the ancient and the modern ways of experiencing mathematics turn out to be in a surprising consonance, illuminating each other.

The “transcendental fraternity” of the natural numbers, unfolding from 0 through the continuing action of the “successor” functional engine, is the *raison d'être* for the book. It’s the fraternity that inspired Dedekind’s arithmetical meditations and Gottfried Leibniz’ speculations on the 0/1 epiphanies of nothingness/ being in the numerical realm. The readers are invited to reflect on the subtleties of natural numbers (“elusive” according to Bertrand Russell) and the related formalism of Peano axioms. Clearly, the principle of mathematical induction (a "leap of faith" because it "never goes beyond what is finite") is a central element in this reflection process.

Addressing the interface with logic, the book leads the readers from the tricks of the mythical Delphic oracle up to modern axiomatic considerations, through the adventurous life of the medieval philosopher Pierre Abélard and the genius of Euclid (whose geometry demonstrations caused Lincoln to spend long nights in order to master them). The geometry fans would be happy to see that the mathematical contributions of Ptolemy (author of the theorem on inscribed quadrilaterals that is currently proved in serious high-school geometry courses) are noticed. The author introduces the recursive definitions for addition and multiplication, together with glimpses into the syntactical game of mathematical signs and the polyvalence of its interpretations. This is completed with an overview of the first steps in the development of the central axiomatic theories of the Abstract Algebra (groups, rings and fields) together with related historical developments featuring (among others) Évariste Galois, Sonya Kovalevsky, David Hilbert, Emmy Noether, and the "supreme mathematical theoretician" Alexander Grothendieck. Then back in time to ancient Egyptian practical problems, to Brahmagupta's quadratic formula, from there to the modern Abel-Ruffini theorem on the non-existence of solution in radicals to equations of degree at least five, and so on. The time arrangement follows a nonlinear pattern, consistent with the unpredictable emergence of the great ideas.

According to the author, AEM has a holistic nature (it stands "as a whole"). The historical development of AEM and its related structures and formal language is mysteriously entangled with the trans-historical (platonic, otherworldly) aspects of the integers: indeed, the mathematical treasure that keeps being uncovered throughout the history "does not reflect any contingencies at all". In this sense, one might see the process of uncovering itself as an archeology of the transcendental, illuminating the initial motto, "*We read to find out what we already know*", and suggesting that for the person engaging it, mathematics is essentially a *journey of self-discovery*. This suggests an interesting way to look at the "deep field" of mathematics: that's a whole lot of treasure stuff out there (actually more abundant than the one in the physical world, as the author notices), enough for the unfolding, in complete Cantorian freedom, of the process of self-discovery of every participating human person. Along these lines, a reading of the mathematical experience in a phenomenological key may prove to be beneficial.

The unfolding drama of the mathematical experience may be seen as the paradoxical birth of infinity in the manger of finitude. That may apply for the "deep field" of the arts and, why not, for that of physical/cosmological reality itself. This, together with the fact that the person immersed in the mathematical experience exhibits what the author calls "a willingness to see *in* the axioms something created entirely *by* the axioms" echoes the problem of "naming infinity", that inspired the Moscow School of Mathematics at the beginning of the 20^{th} century (in one out of the painfully many gulag horrors, Dmitri Egorov, its President between 1923 and 1930, was denounced as "*a reactionary and a churchman*", was thrown in prison, and died in 1931).

Turning to some constructive observations and suggestions, which do not diminish at all the obvious value of the book and could be easily addressed in a future edition, there are some obvious misprints, such as $(-x)(-y)$ instead of "–x–y" on p. 153 (third row from the bottom) and on the right hand sides of the three formulas on p. 154; on p.179, the second inequality, $a/b<b+md/a+mc$, should appear as $a/b<(a+mc)/(b+md)$, with $(a+mc)/(b+md)$ two paragraphs further down. As a suggestion, since the author introduced the concept of order towards the end, where the fields are also discussed, I think it might help to have a small section regarding "ordered fields", i.e., fields together with a relation of total order that is compatible with the field operations: in this context, somebody may ask which fields can be ordered by a suitable, compatible, order structure. There are a couple of equivalent conditions for this to happen, one of them being that the field element –1 cannot be written as a sum of squares in the field - in which case the corresponding object is called a "formally real field" (for example, the field of complex numbers is not formally real, because –1 is itself a perfect square there).

To conclude, this is a book that manages to address the depths of the mathematical *experience*, touching sophisticated issues pertaining foundations, philosophy, and the process of discovery itself. The exuberant style of the author is easily recognizable to be that of a talented teacher and speaker, always ready to support the abstract ideas with surprising analogies as well as with proofs combined with great stories that stimulate the reader's interest. The book is a good start for students (all of them, not only for mathematics majors) willing to take an in-depth look, with philosophical touches, at the immediateness of the mathematical experience. It seems to me that there is a well founded hope that many undergraduates who are not math majors will consider signing up for one after reading the book.