PROOF AND INTUITION

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I. Two Types of Epistemic Systems

I would like to begin by describing two basic types of epistemic structures, one of which will be called "Intuition Intensive," the other "Logic Intensive." For the sake of concreteness, systems of both types may be thought of as at least partially ordered sets of beliefs; the ordering of the elements in a given system roughly reflecting a division between two kinds of elements: source elements, and derived elements. The source elements are the basic beliefs or "raw materials" of the system; that is to say, those epistemic holdings from which all the other holdings of the system are generated. The derived elements, therefore, are those beliefs which stem from other beliefs in the system and which ultimately are traceable back to the source elements.

There is, of course, nothing very new or interesting in this basic picture which describes what Intuition Intensive and Logic Intensive systems have in common. More interesting are their differences. These differences, obviously enough, reside in the different characters and positions of dominance assignable to the source and derived elements. Modem philosophy has given a great deal of attention to questions concerning the possible character of source knowledge; for example, whether it is given by the senses or by pure reason. We, on the other hand, are primarily concerned with questions regarding the character and role of derived knowledge, and shall have relatively little to say about source knowledge. It is in this general context of concern with derived knowledge that the distinction between what we are calling "Intuition Intensive" and "Logic Intensive" epistemic systems arises.

For our purposes, it is best to begin with a characterization of Logic Intensive systems and thence to define Intuition Intensive systems by way of contrast with them. Logic Intensive systems are so-called because of the heavy use they make of perceived connections of *content* among the propositions which form the content of the various beliefs making up a system. In a Logic Intensive system, knowledge of one proposition p may be extended to knowledge of another by establishing that the content of the former knowledge logically implies that of the latter. This extension of knowledge involves a *transfer* of warrants; that is to say, an extension to q of the selfsame warrant attaching to p. What is taken to justify this transfer of warrants is the knowledge that the content originally warranted subsumes the content to which it is transferred. The general principle, then, seems to be this: if w is a warrant for belief in p, then w is also a warrant for any proposition known to be subsumed by p. Purely contentive or logical analysis is thus granted the power necessary for producing an indefinitely large extension of the knowledge deposited by any given source. For us, this is the key feature of Logic Intensive systems.

Intuition Intensive systems, on the other hand, are marked by a general refusal to admit extension of knowledge based merely on considerations of subsumption of content. On this way of looking at things, epistemic warrant is a relatively "individualistic" affair. Each proposition demands a warrant tailored to it, and there is relatively scant opportunity for one proposition to borrow its warrant from another (or, equivalently, for one proposition to have its warrant transferred to another). It follows that comparison of content between propositions, or what might broadly be referred to as "logical manipulation" of propositions, offers relatively little opportunity for extension of knowledge. Knowledge, then, is a logically fragile or unstable affair, and establishing that the content of one warranted belief entails some other proposition cannot generally be expected to result in the production of a warrant for that proposition, since different beliefs generally require different warrants. Generally speaking, a warrant for p can serve as a warrant for q only if either (1) p and q are the *same* proposition (and not, say, merely logically equivalent), or (2) the warrant for p (though not itself a warrant for q) can be *transformed* into a warrant for q. Consequently, in an Intuition Intensive system, epistemically legitimate inference will generally trade not on the idea of a *transfer* of warrants, but rather on the idea of a *transformation* of warrants.

In brief, then, the picture is this. In Logic Intensive systems, warrants are detachable or separable from their contents and can be attached to other contents provided only that those contents are subsumed by the originally warranted content. One and the same warrant can thus serve as a warrant for many different propositions. In an Intuition Intensive system, on the other hand, warrants are generally bound more intimately to their contents so that they cannot be readily detached and transferred to another proposition. Thus, it is generally not the case that one and the same warrant can serve as a warrant for different beliefs. To get a warrant for one belief to provide a warrant for another will thus generally require that the warrant itself be changed or transformed.

Even this brief description of the differences separating Intuition Intensive and Logic Intensive systems of thought should make it clear that they differ radically both with respect to their views of what logic is, and with respect to the place granted to logic in the knowledge-gathering process. There are, however, at least two ways of conceiving of this difference; one which sees it as a disagreement concerning the ultimate nature of inference, the other which sees it as a disagreement over the role of inference in the knowledge gathering process. It is not our task here to give a detailed account of these two conceptions, but it may be helpful to give a brief outline of them.

On the former conception, a distinction is made between inference, as a type of practical knowledge, versus inference as a type of theoretical knowledge. According to the practical knowledge model, an inference from p to q represents the application of an *ability* or a knowledge of *how* to get from knowledge of p to knowledge of q. Moreover, there is no suggestion that this ability either consists in, or is mediated by, the grasping of any connection (in particular, any *logical* connection) between knowledge that p and knowledge that q. All that is assumed is that one has the ability to get from knowledge that p to knowledge that q.

Taking inference as theoretical knowledge, on the other hand, requires seeing an inference from p to q as representing the application of a piece of *theoretical* knowledge (or knowledge-that). More specifically, it takes inference from p to q to consist in a transfer of a knowledge producing warrant for p to a knowledge producing warrant for q based on the realization (i.e. theoretical knowledge) that the truth of p guarantees the truth of q. In other words, it sees inference as the application of a piece of logical knowledge; namely, knowledge that the truth of p entails the truth of q.

Now, such disagreement over the basic nature of inference might be used to underwrite a disagreement concerning the choice between Intuition Intensive versus Logic Intensive models of knowledge-gathering. After all, on the Logic Intensive model, it is precisely the logical knowledge that p entails q which is supposed to provide for the transfer of p's epistemic warrant to q. Indeed, the very mechanism of transfer is taken to consist in the realization that p's warrant counts as a warrant for q since p's truth guarantees q's truth. The epistemic essence of logical knowledge is thus identified with the provision for a transfer of warrants among the logically related beliefs. Given such a view of inference, a Logic Intensive model of knowledge-gathering is well nigh irresistible.

On the other hand, one who sees inference as a case of practical knowledge may well be impelled toward an Intuition Intensive view. For if knowledge of the entailment relations obtaining between propositions is neither necessary nor sufficient for one to make her way from one warranted belief to another, then logical knowledge will not be given a very prominent place in the knowledge-gathering enterprise. Indeed, it will be seen as largely irrelevant to this endeavor. One will look not for a means of *transferring* warrants from one belief to another, but rather for a means of *transforming* the warrant for one belief into a warrant for another; the idea being that different beliefs will typically require different warrants and, therefore, that what one wants from an inference is not a means of using a single warrant to cover more than one belief, but rather a means of changing one warrant into another so that, the class of warrants having been extended, the class of warranted beliefs might similarly be extended.

Disagreement over whether inference is to be ultimately conceived of as transference of warrant or as transformation of warrant is thus one basis upon which disagreement over whether a given body of knowledge should be seen as Intuition Intensive or Logic Intensive might be founded. But it is not the only one. For even if there were general agreement that inference essentially consists in some piece of theoretical knowledge whose function is to provide for the transference to the conclusion of a warrant attaching to the premise, it would not follow that even a logically connected body of knowledge (i.e. a body of known propositions known to be logically related in various ways - including such cases where it is known that the body is entailed by one of its subsets) would have to be seen as Logic Intensive. This is so because it is possible for a logically connected body of propositions, each of which is known, to be nothing more than an assemblage of relatively autonomous insights; the logical relations among its members being largely immaterial to the knowledge which *it* represents. In such a system, the knowledge of q which is accorded epistemic significance is not that which might result from knowledge that p and knowledge that p entails q, but rather that which results from some autonomous insight into q. In other words, new knowledge is obtained not by logically exploiting old knowledge, but rather by securing an intuition fitted directly to the new knowledge so that extension of knowledge is largely a matter of getting new intuitions.

This brief discussion of inference is only intended to indicate that there is more than one way to connect preference for an Intuition Intensive epistemology to a theory of inference. One way involves rejecting the usual conception of inference which sees it as proceeding from a transfer of warrant from one proposition to another through the establishment of a logical relationship linking their contents, and replacing it with a view of inference which sees the mechanism of inference as one which transforms one warrant into another. (Such transformation, of course, is assumed to change the *character* of the warrant and not just the content of what is warranted. Otherwise, there is no difference between a transference and a transformation model of inference.) Alternatively, one might leave the usual conception of inference intact but devalue its epistemic significance by maintaining that extension of knowledge generally requires procurement of new intuition, and that logical extrapolation generally fails to produce epistemic extension.

These alternatives will inform the discussion of the next two sections of this paper where we seek to apply the general framework sketched in this section to the question of whether mathematical epistemology should be conceived along Intuition Intensive or Logic Intensive lines. For our own part, we believe that there are reasons supporting an Intuition Intensive approach. However, the establishment of such a claim is not the chief aim of this paper. Rather, what we want to do is to show how the distinction between Intuition Intensive and Logic Intensive conceptions of mathematical epistemology is involved in large areas of the debate that has taken place in the philosophy of mathematics, especially in this century. We believe that the framework provided by this distinction yields a deeper insight into what is at stake in this debate. More specifically, we believe (1) that it offers a deeper understanding of what the crucial differences are that separate logicism, formalism, and intuitionism, (2) that it gives a more accurate identification of the chief problems facing logicism and, indeed, any philosophy which attributes much epistemological significance to the Axiomatic Method, and, finally, (3) that it locates an error in the usual way of understanding and formulating intuitionism (especially intuitionist logic); a formulation originates with Heyting's influential papers of 1930.

The plan of the paper is as follows. In the next two sections, we shall sketch the history of the Logic and Intuition Intensive approaches, noting as we go, the specific consequences for logicism, formalism, and intutionism alluded to above. In the final section we briefly discuss a different challenge to Logic Intensive conceptions of mathematical epistemology based on certain ideas of Poincare's, and point out some of its ramifications for the philosophy of logic.

II. The Logic Intensive Tradition

Logic Intensive approaches to mathematical epistemology go back at least to Aristotle. In the *Posterior Analytics* (cf. 71a 1-74a4) he characterized scientific knowledge of a truth as consisting (at least partially) in knowledge of which truths “cause” it. In his view, there is an objective ordering of truths, and true knowledge of a proposition consists in knowing its place in this objective hierarchy. The task of a science, therefore, is to provide a determination of that objective ordering which structures the truths of its subject matter, and the proofs or demonstrations of that science are intended for the purpose of making manifest the various portions of the ordering in question. Such a view of proper theorizing, of course, strongly suggests an *axiomatic* approach to theory construction; the axioms being identified with the first or primitive truths of a science (i.e. the truths of a science which do not rest on any others), and the proofs of the theory serving to “locate” its non-basic truths within the objective ordering which structures its truths. If perfect knowledge of a truth is just knowledge of the objective reasons for its truth (i.e. knowledge of its place in the objective ordering of truths plus knowledge of its “predecessors”), then an axiomatic system of proof, or something tantamount to it, would appear to be necessary for the perfection of knowledge.

This belief in an objective hierarchy of truths was also given a prominent place in the mathematical epistemology of Leibniz who spoke of a “natural ordering” of truths where the “reasons” or predecessors of a given truth p are not only what cause us to “give our assent to” p, but also, and pre-eminently, those things which cause p to be true (cf. New *Essays,* Bk. IV, ch.vii, para. 9. and ch. xvii. para. 3). Belief in this objective ordering of the mathematical truths in Leibniz can, of course, be seen as that which is expressed in his commitment to the Principle of Sufficient reason for mathematical as well as non-mathematical truths.

More recently, Frege, perhaps Leibniz’s most influential latter day disciple, wrote

> “The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? The further we pursue these inquiries, the fewer become the primitive truths to which we reduce everything; and this simplification is in itself a goal worth pursuing.”
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> Frege, GL: p. 2.1
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> “...we are concerned here not with the way in which [the laws of number] are discovered but with the kind of ground on which their proof rests; or in Leibniz’s words, ‘the question here is not one of the history of our discoveries, which is different in different men, but of the connexion and natural order of truths, which is always the same’.”
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> Frege, GL: p. 23 [brackets mine].

As such remarks suggest, it is natural for one who accepts such an objective hierarchical ordering of mathematical truths to embark upon a foundational program of a grand scale. For on such a conception, the ideal of knowledge becomes that of tracing truths all the way back to the absolute beginnings, and one’s understanding of a truth p is proportional to how far back she traces its origins in the objective hierarchy of truths. Consequently, any theory whose proofs trace out only a part of the "blood-line" of its theorems yields only a partial or truncated knowledge of its theorems, and thus does not produce the best possible knowledge of them.

It is in connection with this last, foundational impetus that both Leibniz and Frege imparted their distinctively "logicist" twist; that is, that all mathematical truths can ultimately be reduced to logical or analytical truths (the so-called "identities," for Leibniz, and the principles governing the relations between concepts and their extensions for Frege). Some, most notably Poincare[2], have on occasion described this logicist outlook as an attempt to drive intuition from mathematics entirely; to make mathematical knowledge a wholly intuition-free affair. But this distorts at least the position of Frege, who was content to accept intuition as the means of knowing the first truths while at the same time classifying them as logical or analytical in character.

Thus, the key question with regard to logicism of a Fregean variety is *not* whether mathematical knowledge is intuition-free. It is not, and both logicists and non-logicists may agree on that point. Furthermore - and I realize this sounds heretical - the central issue concerning logicism is not even whether the first principles are truly logical or analytical in character. The logicists, it seems, were committed to regarding mathematical truths as logical truths primarily because of (1) their deep commitment to the Aristotle-Leibniz conception of a hierarchy of mathematical truths, and (2) their belief, which had the status of a well-supported conjecture rather than a regulative principle, that whatever would be capable of lying at the base of this hierarchy would have to be possessed of such supreme elementariness as is likely to be found only among the most elementary truths of reason known to us; namely, the basic truths of logic. Acting in concert, these two beliefs suggest, of course, that all mathematical truths are logical truths.

But what harm would be done to logicism if it were discovered that the first truths are not logical or analytical in nature? It might be thought that Frege would then lose his argument with Kant. But this is not clearly so. For what is required of the first principles in the Fregean scheme is that they be "general" and not the property of "the sphere of some special science," and that they "neither need nor admit of proof"[3]. But what else would first principles be (at least the first principles of any very extensive hierarchy)?[4] Since they would have to lie at the base of a deductive hierarchy under which all the special sciences are subsumed, there is a relatively clear sense in which they could not fail to be, "general" and could not possibly be circumscribed by (i.e. be the exclusive property of) any "special science." The very deductive power of a scheme of foundational principles would thus seem to qualify them as logical truths using Frege's criteria of generality and non-specificity.

Thus it may be best to view any narrower standard of logicality for the first principles (as, for instance, Leibniz' contention that they all be logical identities - that is, truths of the form 'A is A') as offering a *conjecture* concerning what specific kind(s) of principles *will* in *fact* be found to lie at the base of the mathematical hierarchy when once that hierarchy is discovered. The truth of such a conjecture is, however, a matter of secondary importance to the logicist, and the thing that matters, most is finding a suitably simple set of truths having the deductive power requisite to serve as the foundation of the hierarchy of mathematical truths. Had Frege succeeded in locating such a set of truths, controversy concerning whether they were truly logical or analytical in character would have been relegated to the status of sheer hair-splitting. Their sheer power would have been evidence enough that they were "general" and not the truths of any "special science." Their basicness and elementariness would have shown that they "neither need nor admit of proof."

If what I have just said is right, then neither the question of whether mathematics is intuition-free, nor the question of whether the first principles of the objective ordering of the mathematical truths are analytic is the central question of logicism. What then *is* the central question? To answer this, we need to recall the most basic feature of the Aristotelian epistemology which underlies the logicist viewpoint; a feature that I would describe as its Logic Intensiveness. Recall that in the Aristotelian view, knowledge of a mathematical truth consists in knowledge of its place in the objective ordering of those truths; the fuller the knowledge of a truth's hierarchical location, the fuller the knowledge of that truth. Note also that the objective ordering spoken of here is supposed to be a logical ordering, so that the position of a truth can properly be taken as given by its proof from the first principles. From these two premises it follows that perfect mathematical knowledge of a given truth p consists in knowing a proof of p from the first principles.[5] In other words, one who knows foundational proofs (i.e. proofs from the first principles) for a set of propositions has as full and perfect a mathematical knowledge of those propositions as is possible. This, it seems, is about as full-blooded a Logic Intensive view of mathematical epistemology as can reasonably be imagined.

We suggest, therefore, that the central epistemological question with respect to logicism is neither whether mathematical knowledge is intuition-free nor whether the first principles of the foundational scheme are truly logical or analytical in character. Rather, it is the question whether mathematical knowledge is Logic Intensive in the way that logicism is committed to saying it is. Also at issue here is the role of the Axiomatic Method; specifically, its role in mathematical epistemology. For if that method is to be viewed as more than a metatheoretic convenience (i.e. a device which allows us to discuss metamathematical questions with precision), then it seems that some epistemological significance must be attached to the ordering which it induces on the theorems of a theory. Of course, such significance need not be seen as residing in a supposed objective ordering of truths. But it would seemingly have to imply at least that there is nothing significantly deficient in the mathematical knowledge of one whose knowledge consists in knowing an axiomatic proof for each truth of a given body of truths. As we shall see, such a view is open to serious challenge.

III. The Intuition Intensive Tradition

Both the Logic Intensive tradition and its Intuition Intensive counterpart agree (as over against a more broadly "empirical" mathematical epistemology) that mathematical knowledge is more than just knowledge of a mathematical truth. There is considerable difference, however, in the ways according to which the two approaches arrive at this conclusion. On the Logic Intensive side, the distinction arises because of the supposed objective logical hierarchy of mathematical truths, knowledge of which is essential to epistemic mastery of mathematical truth. The Intuition Intensive approach, however, comes to the distinction by way of a distinction between different types of epistemic extension or expansion. In particular, Intuition Intensive epistemology makes room for a possible extension, by sheer logical inference, of the class of mathematical truths known, but it denies that genuine mathematical knowledge is generally the result of such an extension. That is, it grants that in some sense *knowledge of mathematical truths* may possibly be extended by means of purely logical inference, but it does not allow that *genuine mathematical knowledge* is generally extendible in this way.

One finds elements of the Intuition Intensive tradition in the philosophy of Kant who, as is well known, argued against Leibniz'view of mathematics as being composed wholly of analytic truths. But Kant's chief concern seems not to have been that of establishing the Intuition Intensiveness of mathematical knowledge. Rather, it was to refute the Leibnizian position that mathematical truths, being analytic, are not truths about any particular objects of any specific kind. This ambivalence in Kant to the question of Intuition Intensiveness is reflected in the fact that his three most devoted and influential followers in this century (viz. Hilbert, Brouwer, and Poincare) were not all advocates (at least not equally) of an Intuition Intensive view. Brouwer and Poincare (whom Brouwer labeled a "pre-intuitionist") both adopted fairly robust Intuition Intensive approaches to mathematical epistemology, but Hilbert did not. Indeed, Hilbert's approach is in some ways even more radically Logic Intensive than Frege's.[6]

In the next section we will take up the views of Poincare. For the remainder of this section, however, we shall concern ourselves with the particular strain of Intuition Intensiveness found in Brouwer's thought and consider how it leads to a revision of the commonly accepted view of intuitionist logic which was first set forth in the writings of Heyting, and which dominates the contemporary literature on intuitionism.[7]

Though it is not commonly recognized, Brouwer had not one, but rather two critiques of logic. The better known of these (which we shall call *The Special Critique*) is a critique of certain specific principles of classical logic; most notably, the Principle of Excluded Middle. But though Brouwer rejects some of the principles of classical logic, there is a sense in which he viewed others of them as acceptable; namely, as *instruments* for locating propositions that admit of intuitionist proof. On Brouwer's view there is, of course, a big difference between something's being an instrument for the location of truth, and its being a giver of mathematical knowledge. Thus Brouwer insisted that use of even reliable logical instruments cannot be expected to produce genuine mathematical knowledge since that occurs only when the truth of a proposition has actually been experienced (cf. Brouwer [1954], p. 3(524); Brouwer [1955], p. 114 (552); here as elsewhere in this paper, page numbers in parentheses refer to those in Brouwer's Collected Works).

Nonetheless, within certain limits, the principles of classical logic can be used to *indicate* propositions to which an intuitionist proof or justification *corresponds*. Thus, in his Cambridge Lectures Brouwer makes the following comment.

"Suppose that, in mathematical language, trying to deal with an intuitionist mathematical operation, the figure of an application of one of the principles of classical logic is, for once, blindly formulated. Does this figure of language then accompany an actual languageless mathematical procedure in the actual mathematical system concerned?

A careful examination reveals that, briefly expressed, the answer is in the affirmative, as far as the principle of contradiction and syllogism are concerned, if one allows for the inevitable inadequacy of language as a mode of description and communication. But with regard to the principle of the excluded third, except in special cases, the answer is in the negative, so that this principle cannot in general serve as an instrument for discovering new mathematical truths."

Brouwer [1951], p. 5.[8]

Brouwer, therefore, does not see all principles of classical logic as being equally bad (even though this is, for him, just a way of saying that some are worse than others). Some (e.g. the principle of contradiction) are reliable indicators of truth even if others (e.g. the principle of excluded middle) are not. The reliable ones have some limited epistemic utility since, given an experience of one truth p, they can tell us that another truth q is experienceable. As we said above, this does not mean that they either can or do give us something that is epistemically as valuable as the experience of q's truth itself (which is what constitutes mathematical knowledge on Brouwer's view). Nor does it mean that they show us *how* to transform an experience of p's truth into an experience of q's, or even show us *that* this is the case (since q's truth might, at least conceivably, experienceable without the experience of its truth being derivable from that of p''). It means only that our knowledge of p can be extended to knowledge of the knowability of other propositions, given a proof of the reliability (i.e. intuitionistic acceptability) of the principles of logic used to affect such an extension. Some principles of logic, however, lack even this weak instrumental sort of utility, and those are the ones which are the object of Brouwer's Special Critique and which he feels must be rejected.

Brouwer, however, was generally careful to distinguish this Special Critique of classical logic, which urges the unreliability, even as mere instruments for locating the truth, of certain principles of classical logic from a more sweeping, but curiously less well acknowledged, *General Critique* of logic. Thus, in presenting the contrasts between classical and intuitionistic mathematic, he customarily proceeded by delineating what he took to be the three central elements of classical thought which the intuitionist rejects. Briefly, these three were. in something like their conceptual order, the following: (1) belief in the existence of unknown truths, (2) belief in the possibility of generating new theorems of mathematics sheerly by applying logic to previously proven propositions, and (3) free use of the principle of excluded middle. Brouwer sets these out in the following passage.

"Classical algebra or logic, founded by Boole, developed by De Morgan, Jevons, Peirce, and perfected by Schröder, furnishes a formal image of the laws of common-sensical thought. This common-sensical thought is based on the following, conscious or subconscious, threefold belief: First, in a truth existing independently of human thought and expressible by means of sentences called 'true assertions,' mainly assigning certain properties to certain objects or stating that objects possessing certain properties exist or that certain phenomena behave according to certain laws. Furthermore in the possibility of extending one's knowledge of truth by the mental process of thinking, in particular thinking accompanied by linguistic operations independent of experience called 'logical reasoning', which to a limited stock of 'evidently' true assertions mainly founded on experience and sometimes called *axioms* contrives to add an abundance of further truths. Finally, using the term 'false' for the converse of true, in the so-called 'principle of excluded third,' saying that each assertion is either true or false, independently of human beings knowing about this falsehood or truth."

Brouwer [1955]. p. 113(551) [9]

This remark makes clear the two-fold critique of logic which we wish to emphasize; namely, that there are two different critiques of logic present in Brouwer's thinking. One of these (namely, that which denies the third of the classical principles set out in the above remark) is the Special Critique. The other, which constitutes the rejection of the second of the three classical tenets mentioned, is the "General Critique". It proposes a far more sweeping restriction of logic in the pursuit of mathematical knowledge than the Special Critique. In essence, we take it as constituting an Intuition Intensive conception of mathematical knowledge.

The objection found in the General Critique is not, therefore, concerned with the use of some specific principle or principles of logical reasoning. Rather, it places a strict limit on the epistemic utility of any sort of purely logical reasoning whatsoever. On this view, it is wrong to think that one's mathematical knowledge can generally be extended by sheer *contentive* analysis of any sort.[10] That is, it is wrong to think that, generally speaking, mathematical knowledge can be extended from one proposition to another just by showing the former to logically subsume or entail the latter. Contentive analysis of a given piece of mathematical knowledge may reveal other contents or propositions of which mathematical knowledge is possible, but it will not itself give mathematical knowledge of them.

The only way that one piece of mathematical knowledge can give rise to another, is through the transformation of the experience which constitutes it into an experience that is constitutive of the other. This involves more than a mere derivation of the content of the one from the content of the other; it requires as well some sort of transformation of the one experience (qua mathematical experience) into the other. Presumably, however, it is also different from merely having two mathematical experiences in temporal sequence, since one might have two such experiences without the one being in any way derived from the other. The experiences in some sequences will be dependent; the experiences in others not. The articulation of such a dependency relation among epistemic experiences would thus yield a *logic* of mathematical knowledge or truth (on the intuitionist's conception of truth), and this then raises the question of how such a conception of logic might compare with the standard conception of intuitionist logic.

Heyting, who was the originator of the standard conception, has described the basic contrast between classical and intuitionist logic as that which separates a logic of mind-independent truth or existence, on the one hand, from a logic of knowledge, on the other (cf. Heyting [1958], p. l07). The description of intuitionism found in Brouwer, however, does not jibe with this conception of intuitionist logic. It does, to be sure, see intuitionist logic as a logic of mathematical knowledge. But it also sees mathematical knowledge as something whose character and/or value is intimately tied to experience or intuition. Because of this, it does not take the logic of mathematical knowledge to be a logic of knowledge per se, but rather a logic of some sort of *direct experience* or, perhaps better, a logic of knowledge by *direct experience.*

To get a better understanding of the differences separating these two approaches to intuitionist logic it is perhaps best to look at an example. So, to take a very simple case, let us consider the logical operation of conjunction. In Heyting's logic, conjunction obeys the familiar introduction and elimination rules. Thus, one is counted as having knowledge (*or* proof) of P&Q if one has knowledge (proof) of P and knowledge (proof) of Q. And similarly, one is counted as having knowledge of P and knowledge of Q if one has knowledge of P&Q. For convenience, let us refer to the former property as the *synthetic conjunctive closure* of knowledge, and the latter as the *analytic conjunctive closure* of knowledge.

The satisfaction of such properties as analytic and synthetic conjunctive closure depends critically on viewing Heyting's logic as the logic of a rather general type of knowledge, and not as the logic of a type of knowledge that is tied at all closely to anything like direct experience. For though it might be reasonable to believe, at least modulo certain idealizations, that some kind of knowledge of P&Q can be synthesized out of a given type of knowledge of P and a given type of knowledge of Q, there is no similar reason to believe that such a synthesis should preserve the particular *type* of knowledge with which one starts; especially if the type in question is knowledge by direct experience or intuition.

To see that this is so, it may be instructive to consider the analogous case of direct sensory experience. In ordinary sensory experience, we are limited in our ability to selectively focus our direct sensory attention. I am able to experience that the grass outside my office window is green when I look out my window, and I am able to experience that the book lying on the table behind me is blue when I turn away from the window and focus my attention on that table. But I cannot synthesize the two experiences into one because of my inability to focus my attention on the contents of the table and the surface of the mall outside my window at one and the same time. This is but one illustration of the fact that, generally speaking, there is a competition for my sensory attention that often makes it impossible to find a way of focusing that attention on diverse parts of my sensory environment within the confines of a single experience.

The direct mathematical experience of which Brouwer speaks would seem to be the same as sensory experience in this respect, and so I see no reason to suppose it possible that two mental mathematical constructions might always be merged into one (i.e. be combined and experienced as a single construction in the relevant sense). This being so, even so standard a principle as synthetic conjunctive closure would not qualify as a principle of intuitionistic logic, and it thus becomes clear that viewing intuitionist epistemology as Intuition Intensive leads to dramatic departures from Heyting's conception of intuitionist logic.

Perhaps a similar point should be made with respect to analytic conjunctive closure, though I am not sure about this. In thought I can surely separate my experience of the blueness of the book lying on the desk before me from my experience of its rectangularity. But can I separate these facets of my experience *in experience*? That is, can I transform my experience of this blue rectangular surface into an experience just of its blueness (or just of its rectangularity)? It is not obvious to me that this can be done. Nor do I see that the mathematical experiences with which Brouwer is concerned are any different from sensory experiences in this regard. Consequently, I see no very good reason to think that analytic conjunctive closure should be counted as a principle of intuitionist logic.

Such general observations concerning knowledge by intuition or direct experience as those just mentioned can only lead one to question the accuracy of the usual (i.e. Heyting's) account of intuitionist logic. For if intuitionist logic is to be conceived of not as the logic of knowledge in general but rather as the logic (if such there be) of knowledge by intuition or direct experience, then there would seem to be little reason to expect it to abide by the principles set forth in the standard logic for intuitionism.[11] A conception like the one we are suggesting, of course, leaves but a minor role for logic to play in the enterprise of mathematical knowledge, but such a devaluation of logic seems to be in keeping both with Brouwer's general epistemological views as well as his much advertised animus toward logic.

In addition, we feel that an account like the one we are offering here is needed to make sense of the distinction marked earlier between Brouwer's General and Special Critiques of classical logic. In our opinion the standard version of intuitionist logic is the result of focusing one's attention too narrowly on the Special Critique. That critique, as we have already noted, identifies certain specific principles of classical logic (e.g. the principle of excluded middle) which are to be rejected on account of their unreliability as instruments for determining which propositions are and which are not possible contents of intuitionist proofs. If one eliminates these instrumentally unreliable principles of classical logic from it, what results is precisely Heyting's logic. Hence, Heyting's logic can be seen as the logic resulting from the Special Critique.

However, as we have argued, Brouwer's critique of logic does not stop with the Special Critique. Indeed, the Special Critique is the more a matter of detail; the real heart of Brouwer's logical doctrines being the General Critique, which calls for a far more radical restriction of logic than that called for in the Special Critique, and implemented in Heyting's logic. To reiterate, in Brouwer's view, the chief epistemological problem concerning logic is not its unreliability, but rather the fact that mathematical knowledge consists essentially in experience; moreover, an experience that (like ordinary sensory experience) is inextricably bound to its particular content, and which therefore does not readily admit of extension by analysis or synthesis of contents. Logic may extend the *content* of a piece of mathematical knowledge by showing that the extended content is a possible content of mathematical knowledge, but it does not thereby truly extend *mathematical knowledge* since extension of content does not bring with it a corresponding extension of the experience necessary for a genuine extension of mathematical knowledge. In our view, then, Heyting's logic is really not a logic of mathematical knowledge, where such knowledge is conceived of according to the Intuition Intensive manner of Brouwer. Rather, it is a logic for a conception of knowledge that is considerably more stable under analysis and synthesis of contents; a conception of knowledge which is basically insensitive to the differences separating knowledge by direct experience from knowledge of other sorts, and which therefore does not see genuine inference as consisting in the transformation of experiential warrants. Because of this, we believe that the standard view concerning the character and identity of intuitionist *logic* bears rethinking.

IV. Poincare's Objection to a Logic Intensive Conception of Mathematical Knowledge

So far, we have been chiefly concerned with contrasting Logic Intensive approaches to mathematical epistemology with approaches (i.e. the so-called Intuition Intensive approaches) that assign a greater role to experience in mathematical knowledge. In this section we want to examine a different sort of reason for taking exception to Logic Intensive epistemologies. And we shall begin by noting an observation that played an important part in the philosophical writings of Poincare, but which seems to have been largely ignored or overlooked since that time. Put simply, Poincare's observation was this: there is seemingly a big difference between genuine mathematical insight or understanding, and an ability, perhaps even a superb ability, to logically manipulate mathematical truths. In Poincare's view, one might reach a state of utter perfection in terms of her ability to logically analyze and synthesize a given body of mathematical truths without thereby having reached a state of perfect, or even very appreciable, mathematical knowledge with regard to those truths. Poincare himself put the point in this way.

> "The logician cuts up, so to speak, each demonstration into a very great number of elementary operations; when we have examined these operations one after the other and ascertained that each is correct, are we to think we have grasped the real meaning or the demonstration? Shall we have understood it even when, by an effort of memory, we have become able to reproduce all these elementary operations in just the order in which the inventor had arranged them? Evidently not; we shall not yet possess the entire reality; that I know not what, which makes the unity of the demonstration, will completely elude us."
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> Poincare [1905]. pp. 217-18.

> "If you are present at a game of chess, it will not suffice, for the understanding of the game, to know the rules of moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book on mathematics would do if he were a logician only."
>
> Poincare [1905]. p. 218.

Poincare himself sometimes muddied the waters by putting his point by way of a contrast between one who can invent or create mathematics versus one who can only verify or confirm what someone else has created (cf. his contrast in VS pp. 217 f. between the creator of an axiomatic system and one who merely uses an already created axiomatic system to construct proofs). The true mathematician is, of course, a being of the former sort, while a being of the latter sort is what Poincare often derogatorily referred to as a "logician" (examples of which were Peano, Frege, Russell and Couturat). Put in this way, however, Poincare's point loses much of its philosophical appeal since it begins to look like a failure to properly distinguish discovery from justification, and thus to confuse mathematical epistemology with psychology.

But there may be another way to understand Poincare; a way that sees him as making an epistemological rather than a psychological point. Perhaps the strongest indication of the line of thought that I have in mind comes from Poincare's persistent use of mathematical induction as a piece of "mathematical reasoning *par excellence*" (SH, p. 37). Poincare's poetic description of induction is that it contains in "a single formula, an infinity of syllogisms" (SH, p. 37) arranged in a "cascade": the theorem is true of 1; if the theorem is true of 1, then it is true of 2; therefore, the theorem is true of 2; if the theorem is true of 2, then it is true of 3; therefore, the theorem is true of 3, and so on. Reasoning by mathematical induction is said by Poincare to "reduce" this infinity of syllogisms to two elements: namely, knowledge of the minor premise of the first syllogism (the so-called "basis clause" of the induction), and knowledge of the general principle which subsumes all the major premises as instances (the so-called "induction clause"). But these two elements of knowledge are, in a certain important sense, greater than the simple juxtaposition of the infinitely many syllogisms which fall under it. For what they do is to order the syllogisms. It is "the feeling, the intuition" of this ordering which enables one to see the infinite collection of syllogisms as a single unit of reasoning (cf. SM, p. 385).

It is hard to know exactly what Poincare meant by all this. Certainly part of what he wanted to do was to claim the superiority (from the standpoint of mathematical knowledge) of reasoning by mathematical induction over the purely "syllogistic" reasoning covering the same ground. But in what is such superiority to be taken to consist? Let us approach this question by briefly reviewing the reasons the logicists had for "logicizing" mathematical reasoning.

The logicist, it will be recalled, is interested in cutting up every piece of reasoning into small pieces so that he may be in an optimal position to tell what that section of the hierarchy looks like. He wants to force every piece of reasoning through as fine a sieve as possible so as to not overlook any element of the mathematical hierarchy that would not be brought to light if reasoning were allowed to proceed in terms of bigger, less thoroughly analyzed steps. The logicist's quest for rigor is thus motivated by his desire to get a picture of the hierarchy that none could claim is incomplete.

The very idea of such a hierarchy is, however, beset by some troubling questions. For, paradoxically, there comes a point where rigor and inference are actually at war with one another. The ideal of rigor is to make the steps of reasoning as small as possible - the perspicuity of a step of inference being inversely proportional to the "distance" separating its conclusion from its premises. However, as the "distance" separating premises from conclusion becomes smaller and smaller so does the movement involved in going from the premises to the conclusion; and as this movement decreases, so does the legitimacy of thinking of the transition as a genuine inference. At the limit lies the case where the premises and the conclusion are the very same proposition. Here there is the smallest step possible separating premises from conclusion; and so, there is perfect rigor. Equally, however, there is nothing like genuine inference. Rather there is only something like repetition or reiteration of the premises.

To have genuine inference, then, the very least that is required is that the conclusion be a distinct proposition from the premises. But satisfaction of this basic condition may not be enough to guarantee genuineness of inference. For the conclusion of a would-be inference may be a different proposition from each of its premises and there still not be enough "distance" between its premises and its conclusion to constitute it as a genuine inference. A general sort of case illustrating this point is the following. Let A and B be two propositions, and consider the "argument" whose premises are A and B and whose conclusion is A. According to what was noted above, this "argument" cannot be considered a genuine inference, because its conclusion is the very same proposition as one of its premises. Consider now the "argument" whose only premise is 'A and B' and whose conclusion is A. The conclusion of this "argument" is not the same proposition as any of its premises, so it passes the minimal condition placed on genuine inference. Still, this latter "argument" seems but a trivial rewrite of the former; one which gathers in a trivial way the two premises of the former argument into a single premise. It is therefore hard to think of it as any more genuine an inference than the former, despite the fact that it satisfies the minimal condition.

These considerations and others like them thus raise a serious question for the philosopher of logic and mathematics. Explicit containment of the conclusion of an argument in its premise-set frustrates genuine inference because it makes the "distance" between the premises and the conclusion too small. However, certain forms of implicit containment do this too. Therefore, what is needed is some account of implicit containment which gives a principled explanation of which sorts of implicit containment foster (or at least permit) genuine inference, and which do not.

I believe that something like this may be part of what motivated Poincare's opposition to the Logic Intensive approach of the logicists: He felt that chopping up a piece of non-trivial mathematical reasoning into logic-sized inferences threatened to distort the true character of that reasoning by substituting a series of relatively trivial moves for it. In the extreme case, this "logicizing" of mathematical thought might go so far as to replace a piece of non-trivial mathematical reasoning by a series of logical maneuvers so close to one another as to represent no significant movement at all. If such a thing proved to be possible, and there seems to be no guarantee against it, then there is something questionable about a proposal to represent mathematical reasoning as a series of logical maneuvers; for such a scheme can end up representing genuine inferences as non-inferences.

However, even if "logicizing" mathematical thought should never lead to such an extreme distortion of non-trivial mathematical reasoning, one must still be concerned about a possible general tendency for logicization to misrepresent the true character of mathematical reasoning. I think that this may have been Poincare's central concern. The elementary logical principles which the logicists used to connect the various truths of mathematics represents a "global" conception of logic; that is, a conception of logic which sees it as being independent of any given subject matter to which it might be applied. Indeed, the very elementariness that the logicists require of their principles of inference tends to drive them in the direction of such a conception; reasonings which pertain only to a particular subject matter are exactly the sorts of things that disguise material assumptions about the subject matter - material which the logicists believe should be explicitly declared in the axioms of that subject.

Poincare, on the other hand, seemed to favor a more "local" conception of logic according to which different subject matters are to be seen as characterized by and legitimating different forms of reasoning. The study of the whole numbers, for example, thrives on reasoning by mathematical induction whereas other areas of mathematics do not, but may have different forms of inference which serve to distinguish them. More generally, the point is this: part of what is involved in gaining epistemic mastery over a subject is gaining a mastery of its distinctive forms of reasoning; that is to say, gaining a mastery of its peculiar "local" logic. The logicists thus err in trying to present all subjects as operating according to the same "global" logic. And this, then, becomes one of the main fault of Logic Intensive approaches to mathematical epistemology, or those of the sort advanced by the logicists, at any rate.

Here, then, we have a rejection of Logic Intensive epistemology that is not based on assigning a special place to experience. Rather, it turns on considerations of how inference and logic are to be conceived; specifically, on how much "movement" is required for genuine inference, and on whether logic is best conceived of in global or local terms. These, I believe, are questions of great moment for the philosophy of logic. As the discussion in this section would indicate, I think they can also serve to deepen our understanding of what is at stake in the debate surrounding some of the major movements in the philosophy of mathematics in this century.

NOTES

1. Bolzano also believed in an objective order of mathematical truths, and that the chief task of the mathematician was to study this hierarchy and to make manifest its arrangement in his proofs (cf. Wissenschaftslehre, sections 394, 525).

2. Cf. Poincare [1908], pp.483-485.

3. In Frege's view, a truth is analytic if it is general and not within the sphere of any particular science.

> "...these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgment but the justification for making the judgment... The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts. i.e., to truths which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori."
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> Frege, GL: pp. 3-4.

Frege's characterization of the analytic/synthetic distinction is, of course, rather different from that found in Kant. Also, it is clear that the crucial issues here are what makes a truth a "general" truth, and what makes a science a "particular" science (e.g. Is it having a domain of particular objects as its subject-matter? Or might there be a domain of particular objects capable of simulating the mathematical behaviour of any set of objects?) The answers to such questions bear implications for our understanding of the place of intuition in Frege's thought and that of his adversaries, Brouwer and Poincare. Thus, in order for the account that I am sketching in this section to be fully developed, more would have to be said about these matters than I am saying here.

4. If the hierarchy were not very extensive, the first principles might look more like the postulates of some special science and so they might fail to meet Frege's criteria even though they were the first principles of the hierarchy in question.

5. Of course, things are not quite so simple. For since there may be more than one logically proper proof of a given theorem from a given set of axioms, some account must

be given of how to tell which proof reveals the proper ordering or, alternatively, why the objective hierarchy is not totally unique. This is thus another respect in which the account sketched in this paper is incomplete. Also worth noting at this juncture is a point that we shall return to in the final section; namely, that the logicist requires basic inferences of a highly elementary sort so as to be assured that she has left nothing out of her articulation of the objective ordering with which she is concerned. Such a concern not to "jump over" any elements of the objective ordering of mathematical truths, in our opinion, is that which motivated Frege's ardent quest for rigor.

6. The respect in which Hilbert's position represents an even more extremely Logic Intensive view than that of Frege's is this: Hilbert did not even require contensive or semantical analysis for the extension of knowledge; rather, he believed that such an extension could be affected through sheer symbolic manipulation. He did, of course, require a proof that such extension by manipulation be sound or reliable, but for him this did not necessitate a semantical interpretation of the symbolism thus manipulated. Thus, both Hilbert and Brouwer agreed that classical logic was instrumental in character. Where they differed was in their views of which parts of the instrument were reliable, and in whether genuine mathematical knowledge could result from the proper (i.e. provably sound) use of the instrument. It is Hilbert's affirmation of this latter possibility that made him a logicist in Brouwer's eyes. [N.B. It might be historically more accurate to say that Peano, Frege, Russell, and the other logicists were formalists in Brouwer's eyes, since his term for logicism was 'old formalism,' and his term for Hilbert's position was 'new formalism' (cf. Brouwer [1955], pp. 2-4). But the point is the same; to Brouwer, both positions represent an objectionably Logic Intensive approach to mathematical epistemology.]

7. The reader should be warned that the interpretation of Brouwer's intuitionism set forth here is unorthodox, especially as regards its view of the nature and role of logic. The standard view, as presented in the writings of Heyting, conceives of intuitionistic logic as a general logic of knowledge (cf. Heyting [1958], p. 107), whereas we view it as the logic (if indeed there is such) of a particular kind of knowledge; namely, knowledge by intuition or direct experience.

But though the view developed here is unorthodox, we believe it makes better sense of the Brouwerian texts than the standard view. At any rate, it offers an alternative way of reading those texts, and our current understanding of Brouwer's views does not seem to be so well established that it would not benefit from the consideration of such alternatives.

8. Similar remarks occur in Brouwer [1923] p. 336. Though there is some leeway in the traditional terminology employed by Brouwer, the principle of syllogism to which Brouwer refers is probably modus ponens, and the principle of contradiction that from A it follows that not (not A).

9. Cf. Brouwer [1948], p. 1243(488). Brouwer sometimes adds a fourth element of disagreement; namely, classical mathematics' restriction to predetermined infinite sequences.

10. I think that it is this sort of contensive analysis that Brouwer had in mind when (cf. the passage from Brouwer [1955] quoted earlier) he characterized "logical reasoning "as thinking "independent of experience."

11. Is there anything left of Heyting's logic once intuitionism is understood as operating with an Intuition Intensive conception of mathematical knowledge? One might look to the theory of the conditional for an affirmative answer. But even here there is a problem. For the usual intuitionist semantics for the conditional calls for an operation that takes *any* proof of the antecedent to a proof of the consequent. Yet if an operation is so versatile as to be able to turn *any* proof of the antecedent into a proof of the consequent, how can it attend to any of the traits of these proofs other than their *contents*? Would not such versatility require (or at least suggest) insensitivity to all aspects of these proofs, other than their contents? And if content is the only aspect of the proof that matters, can the operation in question really possess enough Intuition Intensiveness to qualify as a genuinely intuitionistic proof?