EPISTEMOLOGY TO ONTOLOGY

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It would probably take a book to do justice to this title. Nevertheless, I do wish to make some comments on the foundations of mathematics and then indicate how these ideas have implications in consideration of the existence question.

I

The pivotal question that demands an answer has to do with the nature of the whole mathematical enterprise. What is the nature of mathematical truth? What are we saying when we claim a theorem has been proven? We must be clear what it is we are talking about. In what sense is it true that a formula for solving quintic polynomials does not exist? In what sense is it true that for each complex polynomial there exists a zero of that polynomial in the complex numbers? Are such statements indubitable, probablistic, tautalogical, relative, or what?

Inseparable from this is the question of the existence of mathematical objects. Are the objects of mathematical statements created or discovered by mathematicians? Should we think of a compact Hausdorff topological space as a mental ideal object, i.e., give it a Platonic existence, or is the topologist talking about a real object much like the natural scientist?

What is mathematics all about, anyway?

Although such questions were largely ignored in our mathematical education, I believe we cannot ignore them now and be true to our calling as Christian mathematicians and educators. Can Jesus Christ be Lord of our lives except at those times when we pick up Thomas, Herstein, Dugungi or a host of other textbooks? I think not. We must seek to know how our Christian commitment bears on the mathematical activity in which we engage. For our own sanity and personal integrity we must not live in two universes, a theistic one and a mathematical one.

Moreover, I believe we suffer further from our lack of discussion of foundations of mathematics. We all have decried the "new math" - largely on the basis of its algebraically inadept graduates. Yet, as Reuben Hersh has pointed out at the San Antonio AMS meeting last year, it is our acceptance of the formalist position that has led to the axiomatization of elementary and secondary mathematics. Insofar as we view mathematics as formal symbol manipulation in an axiomatic setting, we will seek to do and show this "true mathematics" to students at the earliest opportunity. Secondly, until we know what mathematics is we will not be in a position to distinguish consistently valuable mathematics from that which belongs with the junk which appears from time to time in the Notices' abstracts. Could it also be true that some of us may be fearful of delineating the nature of mathematics, supposing that we will then no longer be paid for our mental game playing and will be forced to teach cook-book applicable mathematics?

Our discipline has declared that discussion of these philosophical issues is not proper work for a mathematician since the answers must be sought outside mathematics. As a consequence we have philosophy of mathematics written by philosophers. This writing at times seems like a sea of jargon, unintelligible to non-philosophers, and more often than not betraying the author's only nodding acquaintance with the foundations of mathematics. Since the early part of this century, mathematicians have largely abandoned the philosophy of mathematics and so no one is writing on the subject from a deep knowledge of functional analysis, algebraic topology, or group theory.

Most mathematicians live with a philosophical dilemma. If pressed on the question of the nature of mathematics, they will express some sort of formalist position. However, they will labor on their research firmly convinced that they are studying real objects. Reuben Hersh has given a name to such mathematicians, one which we should find amusing. He has called them "Sunday formalists."

An adequate philosophy of mathematics must account for both the success of mathematics per se and for its successful application. It is precisely the attempt to answer these two points that has led to the present state of philosophical schizophrenia on the part of most mathematicians. The solution lies not in answering the two independently but in finding an adequate worldview to explain both the mental activity of man and its relation to the world outside.

As Christian mathematicians we ought to be involved on two fronts in the discussion of foundations of mathematics. First, we should be analyzing the present philosophical options and discarding those which are not consistent with our Christian worldview. Secondly, we ought to be devising new philosophical systems to account for mathematics. We should not do this with the expectation that there will be found one Christian philosophy of mathematics. Nor should we necessarily expect to find a philosophical position which is exclusively Christian. A position consistent with a theistic worldview might very well be consistent with radically different presuppositional contexts. It is my hope that this conference will be a catalyst for doing these tasks as a collective venture among Christian mathematicians.

II

When we mathematicians turn to the philosophy of our discipline, we find our options are basically the ones which were present early in this century. Intuitionism, logicism, and formalism (or minor variants thereof) were not satisfactory then and they haven't become so in the intervening years. Intuitionism gives away too much of what we think of as mathematics and neither logicism nor formalism has been successfully carried out on its own terms. In fact, the latter cannot be, as Godel has shown. All fail to meet the double criteria I stated earlier.

Intuitionism, while keeping a foothold in the real world, cannot account either for the success of pure mathematics or of applied mathematics. Under its strictures too much of the theory and application ceases to be true. Such activity remains curiously useful but not verifiable, i.e., not mathematics.

The logicists propose to build mathematics on the solid foundation of logic and set theory. In the attempt they have successfully avoided the Russell type paradoxes but the price has been high. The foundation stones (postulates) are no longer obvious truths. Certain sets which seem to be clearly defined are prohibited because they lead to paradoxical situations. Moreover, there is no guarantee that other paradoxes and prohibitions are yet to be discovered.

The formalists have ceased looking for a foundation upon which to construct all of mathematics and are content to build each room of the mathematical house on a set of formal axioms without regard for their truth content. In this approach mathematics has found its source in the rational activity of the human mind. Formalism then can in great measure account for the success of mathematics per se but is unable to account for its applicability. Thus we see most present day mathematicians, who have been trained in the formalist mold, shifting to a Platonist view when in any context other than a philosophical one. Then their subject matter becomes an idealization of the real world rather than a vast mental chess game.

III

The criticism thus far has not been explicitly from a Christian perspective. In mathematics, as in the natural sciences, such explicit criticism must arise from those parts of a Christian world-view which embrace the natural world and the ability of man to perceive it and exercise his reason in understanding it. James Sire has aptly summarized these ingredients of a theistic world-view in his book, The Universe Next Door. They are:

> 1) God created the cosmos *ex nihilo* to operate with a uniformity of cause and effect in an open system. (p. 26)
>
> 2) Man is created in the image of God and thus possesses personality, self-transcendence, intelligence, morality, gregariousness, and creativity. (p. 29)

The second of Sire's propositions has consequences for all of mathematics while the first is particularly important in understanding applied mathematics.

Criticism of the three options from a theistic worldview is an area in which I venture with a rather light step. My own thoughts are still gestating and what follows is not as clear as I would wish. As I have mentioned, the intuitionists severely restrict mathematics and I feel severely underestimate the capabilities of the human mind. At times this rejection of the infinite is valuable but to reject it *in toto* seems to me to be invalid. Our ability to conceive of and work with infinite sets, procedures, etc. is ultimately due to our being made in the image of God, the infinite one. Our minds in part reflect his omniscience. It is also the recognition of the omniscience of God that leads me to reject the notion that certain propositions are neither true nor false. I find no difficulty with the idea that a proposition can be independent of an axiom system, but I find a three-place logic untenable. For a number defined to be 0 if □ contains in its decimal expansion no sequence of ten consecutive 9s and to be 1 if such a sequence does exist, I would firmly assert that either the number is zero or it is not. To postulate a third case would be to limit God's omniscience.

My discomfort with the logicist and the formalist positions is harder to pinpoint. It seems to rest in what I perceive as an implicit assumption of the autonomy of man. In particular, the formalist's repeated references to mathematics as a purely mental creation of man hints at this assumption.

Fundamentally, the success of mathematics is based upon the theistic proposition that man is made imago dei. It is our image bearing which gives us the mental equipment to do mathematics. Even as marred reflections of the omniscient Jahweh, who is the Truth, we are enabled to reason correctly and to learn part of the truth.

As Christians we can and ought to see the application of mathematics in the natural sciences as confirmation that our mental facilities were not completely marred by the Fall. That part of man still retains some of the image of God. As branches of mathematics find application, our confidence in the mathematical enterprise should increase. We need not claim that the mathematical models are true. Yet we are able to analyze the physical world and use descriptive models which are faithful representations. The correspondence between our mental model building and the natural world is not at all surprising since both the mind of man and the universe carry the imprint of the Creator.

Well then, where do we find ourselves in the controversy over the foundations of mathematics? With which school should we align ourselves? Or should we discard them all and design a system of our own? A little of both, I should think. We are in an enviable position. On the basis of a consistent and all encompassing worldview we can take the best of each position and avoid the pitfalls of each. Mathematics is a mental activity and to that extent the formalists are correct. We do in fact set up an axiom system and then investigate its properties. But we are not caught in the formalist box.

Because of our worldview and ultimately because of our connection with the Creator, we can rightly assert that our mental activity should usually be adequate to describe the natural world. On the other hand, the logicists are right in the assertion that the basis of mathematics is the logical activity of our minds. We do not find ourselves in the bind that either the logicists or the formalists have found themselves. Neither can carry out their desired program. The problems which they have encountered are not so devastating for us. As finite creations ourselves we do not necessarily expect to discover systems that we then are able to show as complete or consistent. The mere fact that in either system we can devise at least temporary patches (solutions) to the difficulties is a reaffirmation of the power of our mental equipment. For example, consider the work done by the students of Russell/Whitehead in distinguishing sets from classes and classes of the second order, etc. Or consider the work in metamathematics via models to establish relative consistency for various formal systems. We also have at times freely added the Axiom of Choice, the Generalized Continuum Hypothesis, or their negations to our axioms in order to make the system less incomplete (less restricted). In particular, topologists made use of the Tychonoff Theorem (based on AC or an equivalent) long before it became apparent that it and the equivalent AC are independent of the axioms of set theory (ZF).

We should expect that as new problems arise in the foundations of mathematics, ways will be found to deal with them. It may be that as our discipline expands we will find some errors to be corrected. The mathematical house will continue to need its remodeling and tidying up and such work will continue, just as the systems devised will continue to find new and unexpected applications in the sciences. The intuitionist position should be considered one of the axiom systems within mathematics, a rather restrictive one at that. In an age increasingly influenced by high speed computation there is much merit in seeking constructive results in addition to the usual existence and uniqueness proofs.

We can accept the notion from logicism that logic is the key ingredient and we can accept from formalism the idea that the creation of models and formal axiom systems is the heart of pure mathematics. In most of the history of mathematics these models have arisen as idealizations of natural world phenomena. In this century the order has quite often been reversed. The formal systems of pure mathematics have only later been applied. (Compare the use of modular number theory in error correcting codes - a fact which surely would vex the pristine pure mathematician G. H. Hardy if he were alive.) If asked whether the model is true, our experience with geometry cautions us against claiming that our models are more than descriptive. The success of general relativity indicates that our universe is probably not Euclidean. The true geometry might be a geometry we have not yet postulated, mightn't it? Many geometries, or more generally, many systems may be appropriate for the real world phenomena depending on the setting and the purpose for which the model is sought.

In summary, I believe that we can have mathematics as a mental chess game of the formalist sort and yet still have the Platonic idea of mathematics as an idealization of real world objects and phenomena. We needn't be schizophrenic mathematicians but rather holistic ones, not Sunday formalists but every day of the week Christian mathematicians.

IV

As I indicated earlier, the ontological question cannot be dealt with in isolation. The nature of the existence of mathematical objects is integral to a philosophy of mathematics. My remarks thus far bear on this issue. The objects with which mathematics deals have existence external to the mind of man. They exist in the mind of God. For many of these objects (particularly the finite ones) we can point to an external referent as well. Finite groups can all be realized as subgroups of sets of permutations on finite sets for example. Even infinite groups can be realized as subgroups of the group of permutations on a suitably large set. There are however some objects which as yet we have found no referent or model. Nevertheless I believe such objects and systems have existence prior to our discovery or postulation of them. There need not be external referents for everything, for surely God did not exhaust his mathematical knowledge in the creation.

Here too, theism allows us to avoid the perils of a formalist position as well as the limitations of an intuitionist one. The objects with which we deal are both pieces in our mental game and objects with real existence.

It seems that we have not quite made it to ontology. It is hoped that these remarks might help to guide the way for Christian mathematicians to and through a philosophy of mathematics. Wrong turns may be expected here and elsewhere, but let the journey begin.

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