Constrained Flexibility

A Review of Alberto Martínez's *Negative Math: How the Rules of Algebra can be Positively Bent*

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A poignant moment arises in *Stand and Deliver*, a film about a group of students at Garfield High School in East Los Angeles whose lives are changed by their mathematics teacher, Jaime Escalante. He has finally begun to connect with them. With wide smiles they are following his lead in repeating a standard algebraic rule: "A negative times a negative equals a positive." After several rounds of this chant there is a pregnant pause, after which Mr. Escalante—with an equally wide smile—asks, "Why?"

The camera then cuts to another scene, leaving the question unanswered. Some in the viewing audience may very well wish that the scene had continued, having remembered the rule but forgotten the explanation. Indeed, why does a negative times a negative equal a positive? Most people have accepted this maxim even if they never heard a good explanation for it. After all, the laws of algebra have the same status as the laws that came down on tablets from Mount Sinai, don't they?

Well, no, at least not according to Alberto Martínez, who has produced a book that is at once scholarly and readable. He argues that many of the rules of algebra could have been otherwise. The adopted ones came about by a variety of factors, and don't necessarily offer the best resource for dealing with every kind of scientific inquiry.

This attitude may seem odd: many thinkers view numerical theorems as independent of human construction. In the words of Martin Gardner, "If two dinosaurs met two others in a forest clearing, there would have been four dinosaurs there—even though the beasts were too stupid to count and there were no humans around to watch."

Interestingly, this opinion of mathematical independence may not be so widely held with respect to the theorems of geometry. The Kantian view that our minds are equipped with a category of (Euclidean) space—a category that exists inside us as a condition of knowledge—likely dissipated with the advent of non-Euclidean geometries in the mid 1800s. But a general conviction that our minds have a category of quantity—necessary for the perception of real things—seems to hold sway with the general public.

When the absolute truth of geometry fell from grace mathematicians became bolder in thinking that deviant algebraic constructs—hitherto deemed as nonsensical because they did not correspond with geometric ideas—could seriously be considered. In this spirit Martínez, although steering clear of abstract philosophical discussions, wants to show that even well-established algebraic axioms might be altered, especially those that do not fully match our experience

The first part of Martínez's book contains a sweeping survey of the historical development of algebraic systems, concentrating on the problems of dealing with negative quantities. Some readers

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may be surprised to learn that serious discussion of negative numbers spanned the centuries—from at least Girolamo Cardano's publication of *Ars Magna* (*Great Art*) in 1545, to Augustus De Morgan's *On the Study and Difficulties of Mathematics*, published in 1831. Even into the 1880s complex numbers had dubious status in the educational system at Cambridge University.

Martínez writes in an informal style: "If later tonight you go to sleep and tomorrow somehow forget everything else in these pages, then at least please remember this:" (page 192) He treats his readers to quotations from Ph.D. theses, textbooks, journal articles, and letters, including remarks from many central mathematical and philosophical figures (such as Berkeley, Cauchy, Comte, d'Alembert, Dedekind, Einstein, Euler, Gauss, Hamilton, Kant, Leibniz, Newton, Russell, and Wallis), all of whom had something to say about negative numbers.

So what is it that we are supposed to remember? Basically, that there is a conceptual difference between the signed numbers $^{+}4$ and $^{-}4$, and the signless number 4, just as a displacement of 4 steps to the right ($^{+}4$), or to the left ($^{-}4$) is not the same as a distance of 4 (signless) units. For Martínez, ordinary numerical methods do not capture this physical distinction. We may glean two other lessons: first, negative numbers, at least in part, grew out of abstractions from concrete numerical operations, such as $5 - 2 = 3$, to symbolic representations, such as $a - b = c$; second, in some very real ways it may make sense to stipulate that a negative times a negative equals a *negative.*

Indeed, even though most teachers can give plausible physical analogies in support of the "Escalante" rule, Martínez argues that there is an intrinsic asymmetry in it, one that would be erased if a negative times a negative equaled a negative. How is this so? Picture the standard number line. Multiplying together two numbers to the *right* of zero, say $^{+}2$ by $^{+}3$, gives $^{+}6$, effectively moving $^{+}2$ (the original number) a total of 4 (signless) units to the *right.* Why is it, Martínez asks, that numbers to the *left* of zero, when multiplied together, don't get displaced further to the *left*? Instead, multiplying together two numbers to the *left* of zero, say $^{-}2$ by $^{-}3$, also gives $^{+}6$, displacing $^{-}2$ (the original number) a distance of 8 (signless) units to the *right.* If, instead, $^{-}2$ times $^{-}3$ equaled $^{-}6$, the original number $^{-}2$ would be displaced 4 (signless) units to the *left*, giving perfect (bilateral) symmetry.

One can "prove" the Escalante rule of course, but only relative to certain assumptions, which Martínez says can be freely called into question. One such assumption, for example, is left-sided distributive law, which is not valid in Martínez's system, although the right-sided distributive law still holds. To make the book more accessible he gives some "proofs" by example. If his ideas were to be used in conjunction with a course for mathematics students, a good assignment would be to ask for more general proofs. His basic multiplication rule is that the product of two numbers gets the sign of the first number. Thus, the product of two positives remains positive, but the product of two negatives is now negative.

Obviously, this change means that the commutative law no longer holds: a negative times a positive is negative, but a positive times a negative is a positive. Martínez confesses that the commutative law is a good one for products positive numbers, and also for products of negative numbers. But why, he asks, must we think it necessary that the law should hold with (mixed) products of positive and negative numbers? After all, many operations we experience do not have this commutative property: $6 \div 3 = 2$, but $3 \div 6 = \frac{1}{2}$.

Martínez's point is weak here because the property of commutativity (or lack thereof) in the examples he cites depends on the operation itself (multiplication or division), and not on the numbers that are being manipulated by the operation. He might strengthen his argument with a physical analogy like the following. Take a book and place it flat on a table, so that you would open it in the usual way. Now consider two types of orientation changes you might make to the book: rotations and flips. The commutative law holds if rotations are followed by rotations, or flips are followed by flips. As confirmation, rotate the book by 90 degrees, then by 180 degrees. The same result occurs if you first rotate the book by 180 degrees, then by 90 degrees. Commutativity does not hold, however, if rotations and flips are combined: first, rotate the book 90 degrees counter clockwise (so that the binding is facing you); next, flip the book 90 degrees away from you (so that the binding is pointing towards the ceiling). Return the book to its original starting position and reverse these moves (i.e., perform a flip followed by a rotation). You'll discover that the binding is pointing towards you, a completely different result.

The second part of Martínez's book details case studies of what might result by tinkering with the standard rules of mathematics. What are the consequences of the Martínez system? As was just mentioned, on the one hand it would be necessary to give up the commutative law and the left-sided distributive law. On the other hand the asymmetry mentioned earlier would vanish. So also would the need for imaginary numbers: $\sqrt{^{-}1} = {^{-}1}$, because we would have $({^{-}1})^2 = {^{-}1}$. And there's more.

According to Martínez, these new rules give a more natural explanation for squaring differences. He illustrates with concrete examples for his readers, but we can make his ideas more general. Take the (traditional) formula $(a-b)^2 = a^2 - ab - ab + b^2$. Figure 1 illustrates the geometry for the standard interpretation.

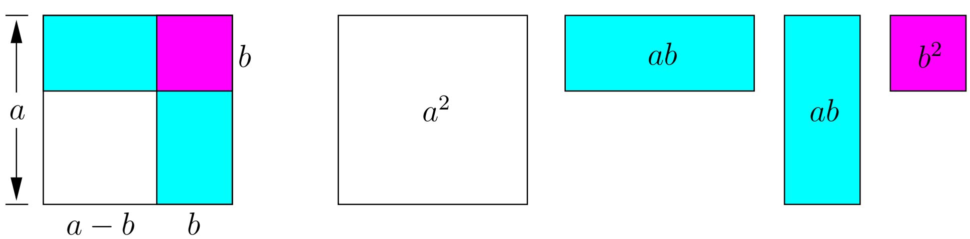

Figure 1: The traditional geometry of squared differences

From the original square of side length a, subtract b units from each side, leaving a square of side length $a - b$. The expression $(a-b)^2 = a^2 - ab - ab + b^2$ depicts two rectangles, each of whose area is ab, subtracted from the original square. As the figure shows, however, this procedure subtracts too much: the upper right square with area b^2 gets counted twice. To compensate, it is added back as $+b^2$ at the end. As Martínez observes, "Thus, we can understand why early algebraists concluded that 'minus times minus is plus.' "

There is, Martínez instructs us, another interpretation, one where no excess is subtracted at all. Figure 2 illustrates how it can be expressed, using concrete numbers this time, as

$$(5 - 2)^2 = 25 - 6 - 6 - 4 = 9.$$

Martínez states, referring to the square in the upper right corner, "Here we have a *negative* square: −4. The squares on the right side of the equation, +25 and −4, have opposite signs, just as the numbers on the left side, +5 and −2, have opposite signs." (page 159) On one level, this

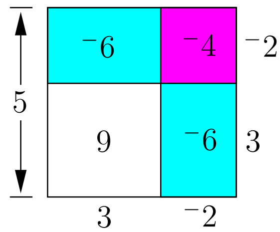

Figure 2: An alternate geometry of squared differences

construct looks quite neat, but how does the middle part of the expression $25 - 6 - 6 - 4$ work out? Using signed numbers, it comes from $(+5 +^{-} 2)^2 = (+5)^2 + (^{-}2)(^{+}3)(2) + (^{-}2)^2$. In abstract terms, Martínez states the rule $(a+b)^2 = a^2 + b(a+b)2 + b^2$, where it is understood that a is positive and b is negative.

What is the main point Martínez wants to make in producing these examples? It is to reinforce a general theme that permeates his book: the rules of mathematics are flexible.

Flexible, yes; arbitrary, no, and one might criticize Martínez for leaving us with the latter impression. In fact, the drawbacks he sees in the standard rules of multiplication are overstated, and he understates the disadvantages that would result if his proposed system were adopted.

Consider, for example, Martiníz's negative square. There is possibly more geometric simplicity in his approach, but at what price? The computations he proposes are more daunting, and the rule he gives at the end seems to come from nowhere. Note also the need to write $b(a+b)2$ instead of $2b(a+b)$, because $2b(a+b)$ would produce a positive number (remember, b is negative, and a positive times a negative yields a positive in Martínez's scheme). Furthermore, how do we get, geometrically, an area of $^{-}6$ for *both* of the above rectangles? If the rule is "multiply height by length" then the upper left rectangle gives an area of $^{-}6$, but the bottom one yields $^{+}6$. The reverse happens if we multiply length by height. But maybe the rule is, "If you have a rectangle with a negative value for one if its sides, the area is always negative, so calculate its area by using that side first in the product you form." Well, that seems a bit ad hoc, or at least no simpler than the addition of the duplicate area that occurs in the traditional approach.

This example suggests a more serious problem: the inability to simplify abstract algebraic expressions in a meaningful way. Let's illustrate this point by deriving the rule Martínez stated for the squared difference of two numbers. Using signed notation the expansion of the quantity $(^{+}a +^{-} b)^2$ begins in the usual way, with the right-sided distributive law:

$$(^{+}a +^{-} b)(^{+}a +^{-} b) = (^{+}a)(^{+}a +^{-} b) + (^{-}b)(^{+}a +^{-} b).$$

But now we're stuck. We cannot multiply out further using the left-sided distributive law, as it is not necessarily valid. Nor can we write $(^{+}a)(^{+}a +^{-} b)$ as $(^{+}a +^{-} b)(^{+}a)$, followed by an application of the right-sided distributive law, because the commutative law might not hold. If we knew, somehow, that a were greater than b, so that $(^{+}a +^{-} b)$ were positive, we could manipulate things to get Martínez's result, but only after quite an ordeal:

$$
\begin{aligned}
&({}^+a)({}^+a +{}^- b) + ({}^-b)({}^+a +{}^- b)\\
&\qquad = ({}^+a +{}^- b)({}^+a) - ({}^+a +{}^- b)({}^-b)\\
&\qquad = ({}^+a^2) + ({}^-b)({}^+a) - ({}^+a +{}^- b)({}^-b) - ({}^+a +{}^- b)({}^-b) + ({}^+a +{}^- b)({}^-b)\\
&\qquad = ({}^+a^2) + ({}^-b)({}^+a) - ({}^+a +{}^- b)({}^-b +{}^- b) + ({}^+a +{}^- b)({}^-b)\\
&\qquad = ({}^+a^2) + ({}^-b)({}^+a) + ({}^-b)({}^+a +{}^- b)(2) + ({}^+a)({}^-b) + ({}^-b)({}^-b)\\
&\qquad = ({}^+a^2) + ({}^-b)({}^+a) + ({}^-b)({}^+a +{}^- b)(2) - ({}^-b)({}^+a) + ({}^-b)({}^-b)\\
&\qquad = ({}^+a^2) + ({}^-b)({}^+a +{}^- b)(2) + ({}^-b)^2, \text{ whew!}
\end{aligned}
$$

To the extent that mathematics is flexible, its flexibility is constrained by requirements to make it not only consistent, but also a *useful* computational tool. In private e-mail exchanges Martínez confessed that his system lacked the latter quality.

Additionally, Martínez's criticisms of deficiencies in the traditional rules appear inconsistent. Recall, for example, his claim that the Escalante rule lacks symmetry. It indeed lacks the bilateral symmetry that he describes, but it has, from the standpoint of complex analysis, an elegant rotational symmetry. Later in his book, however (page 180), he defends the lack of bilateral symmetry in a graph generated with his new system by claiming that, though it fails to have bilateral symmetry, it exhibits rotational symmetry.

The rotational property involved in multiplying complex numbers is well known, as is the fact that complex analysis represents numbers as points on the xy coordinate system. Except for the origin itself they are determined by a certain distance from the origin (known as the *modulus*), and an angular rotation measured from the positive x-axis (known as the *argument*). To multiply numbers in complex analysis one simply multiplies their moduli, and adds their arguments. Thus, two negative numbers, each with arguments of π radians, combine to give a number with an argument of 2π radians, coinciding with the positive x-axis. Two positive numbers, by contrast, each have an argument of zero radians. When multiplied, the result stays on the positive x-axis. From the standpoint of rotations, then, everything is perfectly consistent.

Martínez recognizes the rotational capacity of complex arithmetic, but seems to dismiss its utility by saying it could be achieved just as easily by manipulating ordered pairs of numbers in specified ways. He gives an example of multiplication by i. It has the effect of rotating a point 90 degrees counter clockwise, which can be accomplished simply by interchanging its x and y coordinates. But what about multiplication by something more complicated like $-1+i\sqrt{3}$? This has the effect of rotating a number by 120 degrees. A more complicated mechanism for manipulating ordered pairs of numbers, of course, exists for mimicking this multiplication. Indeed, complex arithmetic consists precisely of ordered pairs of numbers and rules describing how to operate on them. But one reason the symbol i has stuck around is because it greatly facilitates numerical computation, something Martínez applauds. Given the enormous usefulness of complex arithmetic in various fields of science, it is astonishing to see Martínez site, as an *advantage* to his system, the elimination of imaginary numbers. In fact, that complex arithmetic often makes calculations with standard arithmetic much easier is the subject of some current serious philosophical speculation.

At the risk of sounding picky, here is one more criticism of Martínez's enthusiasm. His book has a section that discusses how various graphs appear in his new system. He notes that one can get different graphs by simply rearranging terms. For example, the equation $y = x^3 + 2x^2 - 8x$, generates a different graph than does the equation given by $y = x^3 + 2x^2 - x8$, because with

Martínez's rules $8x$ is always positive, whereas $x8$ has the same sign as x. The switch effectively alters the equation. Martínez then states, "Even if this were the only useful by-product of our artificial game of symbols we would yet have to judge our symbolic experiments as having been remarkably fruitful." (page 186) But the equation $y = x^3 + 2x|x| - 8|x|$ under the traditional rules gives the same output as $y = x^3 + 2x^2 - 8x$ with Martínez's scheme. Thus, his claim that, "we are able to construct a variety of curves that otherwise have no algebraic representation" does not hold up.

These criticisms are somewhat unfair: Martínez is not seriously proposing a new algebraic system. Rather, he aims to encourage the development of mathematical systems that match physical reality, and to consider that alternate systems might, in certain contexts, be a better match than ones currently in use. Martínez's research focus is on the history of Einstein's theory of special relativity, where certainly non-Euclidean geometry gives a nice logical model that neatly matches the physically observed properties. Quantum theory, by contrast, seems to lack a model: it has only mathematical equations that describe observed data precisely. In the words of Richard Feynman, referencing the famous "double-slit" experiment, "We cannot make the mystery go away by 'explaining' how it works. We will just *tell* you how it works." To the extent that successful research in quantum mechanics involves manipulating mathematical formalisms rather than searching for tools of physical representation, Martínez's point is weakened.

So who should read this book? Certainly, anyone with an interest in intellectual history would benefit from the first part, although there is a danger of getting the impression from the narrative that mathematical rules are completely arbitrary. The second part would be enjoyed by those interested in exploring the logical implications of tinkering with familiar constructs. Those that don't enjoy this activity would find that part long and tedious. The book also might be profitably included in a standard mathematics curriculum. In addition to giving students an historical perspective, it would also encourage them to stretch their creativity by devising new systems they may have previously thought were off limits.

Martínez took on this project because of his awareness that, generally, textbooks do not suggest how algebra was put together, and students of physics need to realize that they can adapt mathematics to fit physics, rather than just apply it. Do his ideas really constitute legitimate mathematics? Is anyone apt to consider seriously his proposed multiplication rules? Probably not, but these are the wrong questions to ask; Martínez's new rules are mostly for illustrative purposes. Besides the contribution he makes to help us better understand intellectual history, we might take a remark by Ludwig Wittgenstein to help us see another benefit. He writes (in *Philosophical Investigations*, Aphorism 67) "Why do we call something a 'number'? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fiber on fiber. And the strength of the thread does not reside in the fact that some one fiber runs through its whole length, but in the overlapping of many fibers." Martínez's book has the potential to cause the generation of many golden fibers that can be used in weaving the fabric of mathematics.