# RECENT PROBLEMS IN THE FOUNDATIONS OF MATHEMATICS

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The navigator of a jumbo jet making a transatlantic crossing addressed his passengers over the intercom: "Folks, I have some, good news and some bad news. On the negative side, I'm sorry to say that we're lost. However, and more positively, we're making excellent time!"

To some observers, if goals have ever directed mathematical inquiry, then during the last half century they have been formulated and implemented by that same navigator. To understand the foundational crises that have haunted twentieth-century mathematics it is necessary to briefly review the effects generated by Gauss, Lobachevsky, and Bolyai who each developed non-Euclidean parallel axiom. Though of mathematical interest in their own right, the significance of the new geometries was greatly magnified when it was discerned that they could be used to adequately model physical space, even to the extent that Einstein's theory of relativity later employed as its model a non-Euclidean geometry developed by Riemann. The question that obviously presented itself was how could any given geometry be called true when it and others contradictory to it all could be interpreted so as to fit physical space?

Immediately it had to be admitted that by nature mathematics engaged in the construction of logical systems which were not necessarily actualized in reality. This not only enforced the unique features of mathematical inquiry as opposed to scientific research, but also decharged the potential for controversy, tension, or integration between religion and mathematics and in fact between mathematics and philosophy, culture, etc. Thus, more than at any time since Euclid the axiomatic method asserted itself as the soundest approach for mathematical thought while consistency and meaning were seen to be fundamentally more important than truth which had been lost.

Precisely, an axiomatic system employs certain undefined symbols as primitive terms and a collection of defined symbols whose meanings are expressed solely in terms of the primitives. Properties of the primitives and defined terms are supplied by axioms which accordingly govern their use in the deductive process. The symbols and rules of logic are also assumed, but once applied to the chosen axioms, theorems result which are distinguished from axioms in that they are deductive consequences of axiomatic presuppositions. This rather coldly abstract and highly formalized approach to mathematics divorces meanings from terms and views underlying structures as fundamentally more important than any possible interpretations. However, when meanings are assigned to the primitives so that every axiom and, by their derivable relationship to the axioms, each theorem becomes a true statement in the context of the assigned interpretation then a model has been formed.

It must be emphasized that truth as used in this setting is one that is entirely "context dependent" upon the meanings assigned to the primitives. For example, if the term “plane” is given its usual popular meaning and the term “line” is interpreted to mean “straight line”, then the statement “On a plane a line is the shortest distance between two points” would likely be labeled true. However, if “line” is interpreted as before and “plane” as the surface of a globe, then the statement becomes meaningless since on a globe there are no straight lines. Finally, if “line” is assigned the meaning “circle” and “plane” is understood in the usual way, then the statement would likely be labeled as false. The point is that truth is entirely relative to the interpretation of the primitive terms. It does not have the absolute nature that mathematical statements were accorded prior to the work of Gauss, Bolyai, Lobachevsky, and Riemann.

The existence of a model distinguishes an axiom system as being satisfiable, a property that is of particular importance when an axiom is to be tested for independence, since a given axiom A in axiom system S is said to be independent if both S and (S-A) + ~A are satisfiable. Furthermore, satisfiability implies consistency, for, if a system S generated contradictory theorems P and ~P, then once interpreted and a model formed both P and ~P would have to be true. But a statement and its contradiction cannot both be true.

Now, returning to the problem of consistency, a number of efforts were undertaken to thoroughly axiomatize the various branches of mathematics with a view to then testing them for this very desirable property. For any given axiom system it is not difficult to list some possible approaches to demonstrating freedom from contradiction. For example, one might write out all possible theorems that are derivable from the axiom set A and scan them for contradictory results. However, the impracticality of this approach is evident excepting its possible utility in the most limited and simplistic systems. Alternately, one may be tempted to rely on the proposition that satisfiability implies consistency. The problem with this approach often is that each attempted interpretation merely generates a model in some other branch of mathematics that is itself based upon some axiom system B. This kind of relative consistency reduces to an assertion that A is consistent if B is. In fact, this situation developed during the early part of this present century by which time progress had been made to the extent that many branches could be modeled in number theory and were therefore consistent if number theory was.

Going a step further Bertrand Russell and Alfred North Whitehead revived the logistic thesis of Frege which essentially asserted that all of mathematics, and specifically number theory, was derivable from or reducible to logic alone, and in exploring the implications of this thesis their Principia Mathematica supplied a system of symbols which permitted the standard codification of all statements in pure mathematics. Meanwhile, David Hilbert had developed a theory of proof by using a highly formalized logic (that essentially emptied mathematical statements of meaning) that could be used to prove the internal consistency of mathematical systems.

The flavor of Hilbert's program may be had by considering two simple results of the logic system commonly referred to as the propositional calculus. In what follows the variables p, q, r,...will stand for propositions. Formulas will be constructed by joining variables with the connectives ,□ , and ~, while |□ may be interpreted as “it is an axiom or derivable from axioms that:”. The axioms supplied on the screen are the same as those proposed by Russell and Whitehead: (See the Appendix.)

As a first result we prove the Law of Excluded Middle which interpreted states that either a proposition or its negation must be true.

<u>Theorem A</u>: |□ : p    ~p

The importance of the second result will become apparent in the discussion that follows its proof.

<u>Theorem B</u>: |□ :p□ (~p□ q)

The simplicity of Theorem B does not prevent it from generating the somewhat surprising consequences that the propositional calculus is demonstrably consistent. If it were not consistent it would be possible to derive from the axioms using the rules of deduction both a proposition r and its negation ~r. But the assumption of the derivability of both r and ~r would allow the application of Theorem B to derive any formula whatsoever. Explicitly:

| | |
|---|---|
| \|□ :r□ (~r□ q) | Substitution into Theorem B |
| \|□ :~r□ q | \|□ :r by assumption; Modus Ponens |
| \|□ :q | \|□ :~r by assumption; Modus Ponens |

Thus, if r and ~r are both derivable, then any q is derivable, and, to show consistency all that is needed is some method of identifying some nonderivable formula. In fact, for the propositional calculus, such a method exists.

By defining the truth values of ~p, p□ q, and p   q as tabulated in the table, it is possible to show as illustrated in the projected examples that 1) every axiom is a tautology, in the sense that it is true irrespective of the truth values of the component variables, and 2) the rules of deduction preserve the property of being a tautology. Accordingly, to obtain a nonderivable formula in the propositional calculus it is only necessary to write down a formula that is not a tautology. Such formulas a p   q and ~(~p   ~q) fail to be true for all values of p and q, thus are not tautologies, and consistency of the propositional calculus is proven. It may further be observed that, since truth tables provide a method of determining whether or not any formula formed from the variables p,q,... and the connectives ~, □ ,     is a tautology and thus derivable from the axioms, the propositional calculus is also complete. By way of definition, this means that the propositional calculus captures as axioms or theorems all statements that may be written using the definitions and primitives of the system and which are consistent with the supplied axioms. (See the Appendix.)

| p | ~p | q | p□ q | p q |
|---|---|---|---|---|
| T | F | T | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | T | F | T | F |

Unfortunately, the propositional calculus embraces only a small part of formal logic, and it is not sufficient to generate all the theorems of number theory. A question arises as to whether or not there exists a set of axioms that are demonstrably consistent and which generate every theorem of number theory.

Just as gains were being made towards the goal of establishing the internal consistency of large branches of mathematics as well as imbedding mathematical foundations in logic, Kurt Godel published a result that demonstrated serious limitations upon these attempts. It was shown that provable consistency and completeness are not both possible for an axiom set that attempts to embrace number theory. More specifically, if the axioms are provably consistent from within the system (such as was the case with the propositional calculus) then there exists a theorem of number theory not derivable from the axioms i.e. the system is not complete. On the other hand, if the axioms are sufficiently comprehensive that they generate all the theorems of number theory, then they are not demonstrably consistent from within the system.

This does not deny that classical mathematics is consistent, but rather it can never be established by mathematical proof that a contradiction will not arise in classical mathematics at some future time. On the other hand, the careful choice of axioms and of proof techniques may help to guarantee that such contradictions will not likely appear, but these very choices must be made within and be affected by reference to a larger context or viewpoint.

A similar system dependency has already been suggested for foundational areas of mathematics: the selection of axioms, proof theories, etc. within the context of some larger viewpoint.

Indeed, the context dependency of truth values as applied to axiom systems has already been illustrated. In fact, part of the problem with ascribing truth or falsity to mathematical statements derives from the very meanings assigned to such primitive ideas as "point", "line", "number", or any other primitive terms contained in the development of an axiomatic system. In particular, it seems entirely reasonable to expect that there must be some significant interpretation that may be assigned to the primitives of number theory so as to make it true. After all, number theory and arithmetic have enjoyed extensive and practical use not to mention the particularly strategic role number theory plays as a system in which many other branches of mathematics may be modeled. Yet, the nature of numbers eludes precise definition to the extent that formulations of the meaning of number range from formalistic assertions that numbers are merely symbols devoid of meaning to the conceptualistic view that numbers are creations of the mind to the opinion of the realist that numbers are abstract entities, existing in and of themselves.

Although there may be disagreement over the exact description of the nature of number, the structure of the number system is rather well known as it is generated from Peano's famous system of axioms. Despite his achievement in forming an axiomatic basis for the structure of our number system, Peano only characterized the naturals up to isomorphism. That is to say that .while Peano's axioms are categorical, the possibility of more than one interpretation of the primitives remains. Indeed, by definition an axiomatic system S is categorical if all models of S differ only in the meanings assigned to the primitives. Models of a categorical system differ only in terminology and not in structure. From a structural point of view there is only one model for a categorical system. Thus, if S is categorical all supposedly different models of S are in actuality isomorphic in the sense that there exists a 1-1 correspondence between the "points" of each model and for each true statement in one model there is a true corresponding statement in the other. The axioms by themselves do not identify anyone interpretation that satisfies them as being the right one to the exclusion of the other possibilities. Thus, while the naturals model the system, so does any arithmetic sequence. If some model is to be chosen as preferred above the others, then that choice would have to be made by appealing to some framework more comprehensive than just that provided by the axiomatic basis for the system. Historically, metaphysical and epistemological assumptions have exerted great influence in the making of such choices and recent history is no exception. Logicism, formalism, intuitionism, and other schools of mathematics all involve extra-mathematical philosophical commitments. Intuitionism is selected from among these as an example due to its penetrating criticism of ontological problems in mathematics and the rather drastic measures that were subsequently proposed for reform.

In the first third of this century a group of mathematicians led by L.E.J. Brouwer and committed to philosophical idealism (the objects of knowledge have reality only within the mind) derived a philosophy of mathematics from the theories of Immanuel Kant. The tenets of this resulting philosophy included a rejection of some of the axioms and methods of classical mathematics. For example, to assert the existence of a particular kind of mathematical object requires the construction of an algorithm that would produce the proposed object using only a finite number of steps. By this requirement the institutionist rejects the whole of Cantor's theory of transfinite numbers, a theory that had a1ready gained wide acceptance. The point is that the problem of existence was attacked by working out the implications of a general metaphysical position within the specific context of mathematics. That the results of this particular attempt were not acceptable to most mathematicians does not diminish the very salutary effect intuitionism had and which it continues to have after certain modifications were made in the original theory.

Thus, an excursion into the views of the intuitionist might appropriately begin with a consideration of infinite set theory, for, in alliance with other factors, Cantor's work acted as one of the effective catalysts that motivated Brouwer to adopt the very restrictive tenets that characterize the intuitionistic movement. For our purposes the work of Cantor organizes itself around two questions:

1. How may infinite sets of the same cardinality but of different ordina1ity be labeled so as to distinguish - between them?

In the process of solving this problem Cantor generated his famous Continuum Hypothesis which states that it is false that a cardinal number exists between the cardinal of the natural numbers and the cardinal of the continuum.

2. Does every set have an ordinal number? For example, do the reals have an identifiable ordinal number?

A response to this second question was given rather early in the development of infinite set theory by Zermelo in 1904 when he proved by a heavy dependence upon the Axiom of Choice that for any set there exists a well ordering. Unfortunately, this result also introduced a host of paradoxes and problems that have resisted solution. For example, by Zermelo's Theorem it is possible to well order the reals, but to date a method to actually accomplish such a well ordering has proven elusive. A more troublesome consequence of Zermelo's Theorem is expressed in the Burali-Forti Paradox wherein an ordinal number is generated which is larger than the greatest ordinal number.

Since Zermelo obtained his result as a consequence of the Axiom of Choice it has been demonstrated that the two (Zermelo's Theorem and the Choice Axiom) are actually equivalent. Furthermore it may be noted that transfinite induction is based upon Zermelo's Theorem. Accordingly, if the accumulated problems that have been introduced to the body of mathematics by Zermelo's Theorem are regarded as so severe as to require the rejection of the Theorem together with all of its concommitants, then the Axiom of Choice, transfinite induction, and large branches of modern mathematics based squarely upon these principles would crumble.

In fact, recent results have demonstrated that the two problems (i.e. 1. the problem that led to the Continuum Hypothesis, and 2. Zermelo's Theorem) are inextricably related. In 1940 Godel proved that the Continuum Hypothesis is consistent with the axioms of set theory together with the Axiom of Choice, while in 1963 Cohen obtained the same result for the negation of the Continuum Hypothesis. Meanwhile, Sierpinski, during 1947, demonstrated that assumption of the Continuum Hypothesis together with the axioms of set theory permits derivation of the Axiom of Choice as a consequence. Thus, assumption of the Continuum Hypothesis also yields the Axiom of Choice, Zermelo's Theorem, and the problems of finding some well ordering for the reals as well as solving the Burali-Forti Paradox. On the other hand, if the negation of the Continuum Hypothesis is assumed, then there exist cardinal numbers between the cardinal of the naturals and the cardinal of the continuum but to date no such cardinals have been found. Taken together, Godel's and Cohen's results elevate the Continuum Hypothesis to that of an independent axiom, and one can only wonder if perhaps it will perform as strategic a role in the development of mathematics as was played by Euclid's parallel postulate.

Historically, and prior to the work of Godel, Sierpinski, and Cohen, the intuitionists discharged the problems related to the Continuum Hypothesis by rejecting the very mathematical assumptions that generated them. While Brouwer became the main protagonist of the movement, others who in whole or in part reflect intuitionistic biases include Kronecker, Poincare, Borel, Weyl, and Heyting. To such as these, mathematical objects were regarded solely as creations of the mind. However, since not every imagined object exists in reality, the creative powers of the mind are bound by strict limitations. More specifically:

1. Formal logical consistency is required. Self-contradictions may not be brought into existence and any proposition that implies contradiction must be false. Accordingly, the Law of non-Contradiction is adhered to which states that a statement and its contradiction are not both true.

2. Anything that is admitted to existence must be constructible in finitely many steps. By this the Axiom of Choice is rejected for it brings into "existence" a non-constructible set.

3. Impredicative definitions are rejected since such a definition presupposes the existence of that which is supposedly being generated. For example, consider the set A of all sets. In order for a set to be well defined, each of its elements must be known. The set A contains itself as one of the members making its definition circular.

4. The Law of Excluded Middle (that every statement is either true or false) is limited except in obvious cases. Otherwise the possibility that statements are neither true nor false is entertained. Also the consequence of the Law of Excluded Middle that ~(~p)□ p is rejected. Both the intuitionist and the non-intuitionist would agree that ~p is false, and furthermore that ~(~p) must be true. However, the intuitionist would not make the next step and assert that ~(~p) is p. Rather, ~(~p) might be some third possibility distinct from both p and ~p.

The last two examples suggest that Brouwer and the intuitionists have replaced the common two valued logic system that asserts that any statement must be either true or false with a three valued system that admits truth, falsity, and undecidability. However, such a logic presents serious problems at the very outset to any proponent of it.

A more telling weakness of intuitionism than the implied 3-valued logic is the fact that both set theory and classical analysis are denied under the intuitionistic postulates. Yet classical analysis has successfully been applied again and again to physical and technical problems. The indictment brought against analysis by the intuitionist is based upon the repeated use of principles and proof techniques that are unacceptable to intuitionistic thought.

Despite the mentioned deficiencies, intuitionism possesses certain definite strengths that commend it. In particular, paradoxes are easily treated and even welcomed. For example, the Russell paradox, which is built on the set S of all sets not containing themselves as members, is discharged by the intuitionist by regarding the existence of S and the resulting self-contradiction as the product of a nonconstructive definition.

Secondly, problems of decidability were responded to by intuitionists before such problems were even raised. Certain well known conjectures have neither been proven nor disproven. Among these are Goldbach's conjecture that every even number greater than 2 is the sum of two primes.

It may be that these problems and others have the same solution as was obtained by Godel and Cohen relative to the Continuum Hypothesis, that is that it may be taken as an independent axiom. In fact, in 1931 Godel had demonstrated that any logical system that was consistent and sufficiently large so as to embrace number theory would fail to capture all the theorems that could be formulated in the system. Propositions could be formulated that would not be decidable within the system. Since the assumption that the Law of Excluded Middle is universally valid is equivalent to assuming that every problem is solvable, the intuitionist, in denying the universal validity of the mentioned Law, also implies as a corollary that not every mathematical problem is solvable. Thus, the existence of long unsolved but easily stated problems hardly surprise him.

Finally, the constructive methods of the intuitionist do not lead to contradictions, and furthermore, they reduce questions of existence to questions of constructibi1ity. The loss of large branches of classical mathematics is surely a high price to pay for these two very desirable properties. The question to be addressed is whether or not there exists some alternative to the drastic and stringent program of intuitionism that would lend consistency to mathematics without robbing it of some of its most productive branches.

The contemporary problem of mathematical existence and meaning is not a new one. The nature of reality was discussed by Plato and Aristotle in their theories of transcendent universals and immanent universals respectively. But, no matter what forms or; conditions were proposed for the existence of mathematical entities, there was, and there persists, an underlying faith that, mathematical systems somehow embodied or reflected reality with fidelity. In the present age due to the extent to which mathematics has become internalized by its own character and by that forced upon it by mathematicians with "tunnel vision", the meaning of mathematical existence and the relationship of mathematics to reality are less easily defined.

Is it not irresponsible to dismiss the possibility that assumptions contained within the Christian world view might be worked into a philosophical system that might profitably address the ontological problems that are within mathematics? Our discipline embraces much more than just the deduction of theorems from axioms. It involves a proof theory, a theory of meaning and truth, an historical/cu1tura1 process and identity, and each of these areas involves recognition of and commitment to assumptions. Christianity itself is delimited if it is not fleshed out into an encompassing world view, a view that 1) is informed by the various spheres of knowledge and 2) which in turn is informative to these same spheres. It is conceivable that mathematics has reached a point in its development that this latter possibility is not a viable one. But if so, then the proof that Christianity does not speak to mathematics should not come from a failure to make the attempt.

What is to be worked out is the proposition that the Christian viewpoint affects the content and methodology of mathematical foundations and then ultimately all of mathematics.

## APPENDIX

The axioms of the propositional calculus

1. |□ : (p  p) □  p
2. |□ : q□  (p  q)
3. |□ : (p  q) □  (q  p)
4. |□ : p  (q  r) □ q  (p  r)
5. |□ : (q□ r) □  [(p  q) □  (p  r)]

Rules of deduction

a) A formula F may be substituted for proposition p everywhere p occurs in a given formula.
b) Modus ponens: |□ : q is a direct consequence of |□ :p and |□ : p□ q

Definition

(p□ q) □  ~p  q

Theorem A: |□ : p   ~p  (The law of the excluded middle – either a proposition A or its negation must be true.)

Proof:

| | |
|---|---|
| \|□ : (q□ r) □  [(~p  q) □  (~p  r)] | Rule a: ~p in Axiom 5 |
| \|□ : (q□ r) □  [(p□ q) □  (p□ r)] | Def. of p□ q |
| \|□ : ((p  p)□ p) □  [(p□ (p  p)) □  (p□ p)] | Subst. p  p for q, p for r |
| \|□ : [(p□ (p  p)) □  (p□ p)] | Axiom 1 and modus ponens |
| \|□ : p□ p | \|□ *: p□ (p  p) by subst. of p for q in Axiom 2; modus ponens |
| \|□ : ~p  p | Def. of p□ p |
| \|□ : p   ~p | Subst. of ~p for p and p for q in Axiom 3; modus ponens |

Theorem B: |□ : p□ (~p□ q)

Proof:

| | |
|---|---|
| \|□ : [(q p) □ (p q)] □ {[~p (q p)] □ [~p (p q)]} | Subst. in Axiom 5 ~p for p, q p for q, p q for r |
| \|□ : [~p (q p)] □ [~p (p q)] | Subst. in Axiom 3 p for q, q for p; modus ponens |
| \|□ : [p□ (q p)] □ [~p (p q)] | Definition of □ |
| \|□ : ~p (p q) | Subst. in Axiom 2 p for q, q for p; modus ponens |
| \|□ : p□ (~p□ q) | Definition of □ |

Definition

| p | ~p | q | p□ q | p q |
|---|---|---|---|---|
| T | F | T | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | T | F | T | F |

Examples

1) Axiom 3

| p | q | p q | q p | (p q) □ (q p) |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | T | T |
| F | T | T | T | T |
| F | F | F | F | T |

2) Theorem B

| p | q | ~p | ~p□ q | p□ (~p□ q) |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | T |
| F | T | T | T | T |
| F | F | T | F | T |