

Beyond Practicality: George Berkeley and the Need For Philosophical Integration in Mathematics

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As a teacher of mathematics, the number one question that I receive from students is undoubtedly “When am I ever going to use this?” This paper is ultimately a response to that question, though the response may go most unappreciated by those students who most need it. My basic response to this question is this: it is the wrong question, or at least it isn’t the question the student truly intended to ask. It is my belief that modern society has conditioned students (and people in general) to value things (including knowledge) for their practicality. The question my students are really asking is “Why should I value this?” and they expect a response in terms of how math will get them ahead in life, earn them more money, and in general fix all their problems. But why should practicality be considered such a virtue? Should mathematical study be pursued because it is useful? Or should it be pursued simply for its own sake? It is my contention that the latter option, valuing mathematical inquiry for its own sake in the general pursuit of truth, is a better mindset (or worldview) in which to approach the practice of mathematics. This paper will demonstrate one reason to support such a view: it actually leads to *more* practical applications of mathematical endeavors than would otherwise be discovered. In other words, someone who approaches mathematical inquiry for its own sake is in a better position to find practical applications of the material than the person who comes to the table with applicability as their sole (or main) driving motivation.¹ Historical support for this theory may be found in the life and philosophy of George Berkeley, an 18th century Irish philosopher and Bishop of Cloyne.

This paper will demonstrate how Berkeley’s philosophical approach to mathematics led to an increase in practical mathematical advances. Namely this paper will examine how Berkeley’s critique of the calculus as developed by Newton and Leibniz brought about the refinement of real analysis, through the likes of Cauchy and Weierstrass, and also the development of nonstandard analysis by Abraham Robinson – events that may not have transpired (at least perhaps not yet) had the wonderful practicality of the calculus simply been accepted. This paper will first examine Berkeley’s objections to the pervading mathematical philosophy of his day, particularly the outworking of these objections in his criticisms of calculus as mentioned above. Secondly it will fit Berkeley’s philosophical convictions on mathematics into his broader philosophical pursuits, demonstrating that Berkeley’s motivation behind his critiques went beyond examining the practicality of the calculus. His views of mathematics were firmly rooted in his philosophical and theological convictions. Finally, this paper will examine the historical implications of Berkeley’s work, with a particular focus on the development of nonstandard analysis. In conclusion it will be clear how Berkeley’s philosophical approach to mathematics set the stage for further practical advancements around the calculus and thus portray him as a prime historical example of elevating the pursuit of mathematical truth over the pursuit of mathematical utility.

The reader should note up front that Berkeley’s philosophical pursuits were explicitly marked by Christian belief (as will become clear in the section of this paper titled “Berkeley,

Mathematics, and the Christian Faith”). The main argument of this paper is that a worldview which integrates philosophy and mathematics is needed to best pursue mathematical applications, so the immediate follow up question is “how does one go about integrating philosophy and mathematics correctly?” From my own perspective I believe we can still learn much from Berkeley here, rooting our philosophy and mathematics in Christian belief.

Berkeley’s Objections to Abstractionism in Mathematics and his Criticisms of the Calculus

Berkeley was concerned with mathematics and its philosophical interpretation from the earliest stages of his intellectual life (Jesseph, 2005). To be sure, Berkeley held no qualms about the practicality of mathematics. In fact the situation was quite the opposite. Mathematics was on Berkeley’s theory an essentially practical science. He saw no benefit in contemplating numbers in and of themselves apart from their subservience to practice, to measuring and counting actual things, and to the promotion of the benefit of human life (Johnston, 1923). Berkeley did not hold a position that claims the practicality of mathematics is unimportant. Rather, he believed it was *so* important that practical theories must be scrutinized under a careful philosophical lens before they are universally approved. Berkeley held so strongly to this conviction (being heavily influenced by other areas of his philosophy as we will discuss in the next section) that he was prepared to challenge the received views of his predecessors in the philosophy of mathematics. Just as he approached the metaphysical, epistemological, or scientific doctrines of Descartes, Leibniz, or Newton with a critical attitude, Berkeley was prepared to challenge their accounts of mathematics, even if this meant rejecting the most widely received principles and successful mathematical theories of his day (Jesseph, 2005). If anything can be said about George Berkeley it is that his convictions (which as we will see were deeply rooted in Christian faith) continually led him against the grain of the accepted views in his contemporary academic culture.

The object of Berkeley’s attack was a philosophy of mathematics that characterized the discipline as a science of abstractions. Abstractionism regards mathematical objects as the result of a mental process that begins with perception but creates a special kind of non-perceptual object (Jesseph, 1993). The claim here is that the mind can “pare away” irrelevant features of perceived objects and thereby produce an object appropriate for the science of pure mathematics. As an example, Euclid’s definition of a line as “length without breadth” would be understood as an abstraction in which the mind mentally separates length from breadth, thereby forming an abstract object appropriate for the science of geometry.

Berkeley’s attacks on the abstractionist philosophy of mathematics surface in great detail in his writings on geometry and arithmetic (and then by extension on algebra). However, for the purposes of this paper we will look exclusively at his critique of calculus as this is perhaps his most well known attack on the accepted mathematical principles of the early seventeenth century, as seen in his treatise, *The Analyst* (though it should be noted that his criticisms of calculus derive from his misgivings on geometry). *The Analyst* demonstrates most clearly the mathematical theories Berkeley is rejecting and his reasons for rejecting them.

From the time of Apollonius (circa 250 BC) there was a great interest in the problem of tangents: how to draw a tangent to a given curve. There were two people instrumental in the development of this problem (1900 years after Apollonius): Gottfried Leibniz and Isaac Newton, who independently attacked the problem using coordinate geometry. Their work became known as the (differential) calculus. Though each developed arguments in a unique way, both had a common underlying assumption of the existence of infinitesimals; quantities so incredibly small

that they are said to be between nothing and something.² Berkeley's work was done primarily in response to Newton, but his criticisms apply equally to both methods. Berkeley's strongest argument comes in opposition to the most fundamental premise of calculus, what Newton termed the fluxion, or infinitesimal change; a term that corresponds roughly with what today is referred to as the derivative.

The derivative gives the slope of the tangent at any point of a curve (also referred to as the instantaneous rate of change). It is basically determined by starting with the geometric definition for the slope of a line between any two points, and considering two points that are infinitely close to each other, that is, two points that are separated by an infinitesimal distance. The basics of a Newtonian proof of a derivative supposing y to be equal to the curve x^n proceeded as follows:

$$y = x^n \tag{1}$$

Taking the positive increment denoted by $y + dy$ and $x + dx$ gives³

$$y + dy = (x + dx)^n \tag{2}$$

Expanding the right side by the binomial theorem gives

$$y + dy = x^n + nx^{n-1}dx + \frac{(n^2-n)x^{n-2}d^2x}{2} + \dots \tag{3}$$

Subtracting (1) from (3) gives

$$dy = nx^{n-1}dx + \frac{(n^2-n)x^{n-2}d^2x}{2} + \dots \tag{4}$$

Dividing both sides by dx results in an equation expressing the ratio between dy and dx at any point on the curve:

$$\frac{dy}{dx} = nx^{n-1} + \frac{(n^2-n)x^{n-2}dx}{2} + \dots \tag{5}$$

Because dx is infinitely small in comparison with x , the terms containing it can be disregarded and:

$$\frac{dy}{dx} = nx^{n-1} \tag{6}$$

Notice in this derivation, infinitesimals are both "something" and "nothing": *something* because you can divide by them; *nothing* because you can disregard them in the final solution.

Newton referred to this slope of the tangent as "the fluxion of the fluent x^n " (Stewart, 1996). The power of this result is that it allows what once would have to be determined by complicated and cumbersome geometric analysis to be determined by a simple algorithm and formula. The usefulness of this result does nothing however to impede Berkeley's criticisms of Newton's argumentation.

Berkeley's reason for rejecting this procedure is that Newton takes the quantity dx first to be nonzero, and then zero, while nevertheless maintaining results that can only be obtained under the supposition that the dx is nonzero. Specifically the move from (4) to (5) is impossible without dx being nonzero because otherwise there would be division by zero which results in an

undefined expression. Yet this supposition is contradicted in the move from (5) to (6), which can only be permitted if dx equals zero. Berkeley objected that either dx is not exactly zero – in which case the answers are wrong, albeit not by much – or it is zero, in which case you can't divide by it, and the calculation doesn't make sense. Berkeley was objecting to the calculation of a fluxion as a ratio between two quantities which both vanish. As he famously states in *The Analyst*: “What are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?” (Berkeley, 2002).

Berkeley is operating under the undeniable premise:

If, with a view to demonstrate any proposition, a certain point is supposed, by virtue of which other certain points are attained; and such supposed point be itself afterward destroyed or rejected by a contrary supposition; in that case, all the other points attained thereby, and consequent thereupon, must also be destroyed and rejected, so as from thenceforward to be no more supposed or applied in the demonstration. (Cajorie, 1919).

Newton appears to violate Berkeley's premise by treating dx as both nonzero and zero. This use of such an abstract dx is unintelligible to Berkeley.

Traditional concepts of rigor require that mathematics deal with objects that are comprehended clearly – the use of infinitesimals then requires some further justification if it is to be reconciled with the relevant canons of rigor and intelligibility. A major concern in Berkeley's philosophy of mathematics is to show that the infinitesimal dx is both inadmissible and unnecessary in a properly developed mathematical theory (Jesseph, 2005). Berkeley's rejection of abstraction entails a repudiation of the dominant philosophy of mathematics of his day, and it is no exaggeration to say that his entire philosophy of mathematics is founded upon his critique of abstraction (Jesseph, 2005). Berkeley viewed mathematics through a philosophical lens, desiring veracity over mere utility.

One final point to note is the response of Berkeley to the flaws he saw in the logic of calculus. Berkeley did not deny the utility of the new devices nor the validity of the results obtained (Boyer, 1959). He doesn't merely criticize the logic of Newton and Leibniz; rather, he recognizes that their results are indeed useful and practical (what Berkeley refers to as instrumental), so he sets himself to the task of explaining why something logically inconsistent is able to produce apparent solutions to problems; why something unintelligible and inconceivable works when applied to conceivable and intelligible things. Berkeley's answer is that, in taking infinitesimally small quantities from a given point ($x + dx$ and $x - dx$), Newton's calculus both simultaneously underestimates and overestimates, and therefore these mistakes cancel each other. This argument of the compensation of errors is widely contested, but what is important is not the proof itself, but rather the *means* that Berkeley employs to undertake the proof: using accepted methods of mathematical inquiry, deduction, and rigor.

Berkeley's Primary Philosophical Endeavors, their Underlying Motivation, and their Synthesis with Mathematics

Berkeley is known for his immaterialism; his view that there is no matter in the universe, and the only things that exist are minds (spirits). Matter, in his view, is by definition stuff external to any mind. Berkeley did not deny that there are objects such as books and trees, but he held that such physical objects exist only in minds, wholly constituted of ideas (Pereboom,

2006). This concept of immaterialism may seem a radical perspective to take but its radicalism might be tempered somewhat by understanding the deeply theological influence behind this notion. To Berkeley, all that exists, exists in the mind of God. Colossians 1:17 seems to support this view at least inasmuch as a sound Christian understanding is that reality is continuously and solely being held together by the divine will of God. Even if one doesn't hold to the strict immaterialism of Berkeley, one can at least appreciate his theological convictions and the resiliency with which he applied those convictions to his views of the nature of mathematics.

Berkeley's thought arose largely in response to the theory of knowledge put forward by John Locke.⁴ In Locke's account of knowledge, there exist objects in the material world with certain properties (or qualities). Primary qualities are those that are actually in the thing such as motion, shape, solidity, and extension. Secondary qualities are not in the thing itself but are the power it has by virtue of its primary qualities to create in us certain sensations, for example, color, texture, smell, taste, and sound. Tertiary qualities are the qualities a thing has to affect the primary qualities of other things (for example how fire can alter the properties of a piece of wood). All of these qualities impinge on us to create an idea. The idea of the object is itself made up of ideas of primary and secondary qualities. What people are immediately aware of on Locke's model is the idea of the object, not the object itself. This clearly shows there is a gap between the world external to us and the world of our mind.

Berkeley's strongest denial of the existence of matter involves his attack on the supposed mind-independence of primary qualities (Pereboom, 2006). Berkeley argues that perception of primary qualities varies in the same way as the perception of secondary qualities and that, consequently, primary qualities are as mind-dependent as secondary qualities. Berkeley argues that primary qualities cannot be conceived apart from secondary qualities. What it means for a thing to exist is that it is perceived; that there is an idea of the thing. Berkeley concludes that there is *nothing more to the thing* than our collective ideas. We find this notion applied to his philosophy of mathematics in his *Philosophical Commentaries*: "Axiom. No reasoning about things whereof we have no idea. Therefore no reasoning about Infinitesimals" (354).

Berkeley argues that the idea in the mind *is* the thing itself. There is no external world, all things are mind-dependent and specifically, in Berkeley's personal philosophy, they are dependent on the mind of God (as will be explained in greater detail below). Berkeley was not simply trying to bridge the gap presented by Locke, he was trying to do away with it completely. Berkeley saw Locke's views, and the skepticism it breeds, ultimately leading to atheism and he tried desperately to curb that path of western philosophy. "Berkeley's gaze was always fixed on his polemical and apologetic goals; and it was only insofar as the prevailing philosophical currents could be put at the service of these goals that he appropriated and adapted them in his struggle with the enlightened philosophy which attempted to sever all links with the classical and Christian roots of European culture and civilization" (Hanratty, 1995). Similarly it can be said that Berkeley regarded his criticism of calculus as part of his broader campaign against the religious implications of Newtonian mechanics which bred a deism that tended to distance God from His worshippers. This point will be analyzed further in the next section.

Berkeley's philosophical endeavors on accounts of knowledge were driven by his theological beliefs, and so in turn was his philosophy of mathematics. For Berkeley there is a direct relationship between the theory of knowledge put forward by Locke and the pervading philosophy of his day: mathematical abstractionism. "From the beginning Berkeley argued that

the skeptical, materialistic, and atheistic implications of the new philosophical movement were closely linked with the inflated status accorded to the recently expanded mathematical and scientific methods” (Hanratty, 1995). What the prevailing mathematicians of the time were employing, particularly Newton and Leibniz in the development of calculus, were abstractions that could not be conceived. It is impossible to have an idea of a quantity that is both zero and nonzero, to propose that such a quantity exists is utter nonsense.

It would be a mistake to see *The Analyst* as an isolated foray into mathematical terrain or as a work disconnected from Berkeley’s broader philosophical enterprise (Jesseph, 2005). “His attack on infinite divisibility found in mathematics...exhibits a strategy employed throughout Berkeley’s philosophical writings, that of showing us that we do not understand something we think we understand since the words we use refer to nothing intelligible” (Fogelin, 2001). Berkeley repeatedly insists that the capacity of the human mind does not extend to the comprehension of the infinite (Jesseph, 2005).

Berkeley, Mathematics, and the Christian Faith

“All Berkeley’s endeavors were directed to the defense of what he saw as the most important truth – that we are in a constant and immediate relationship of dependence on God. Berkeley’s philosophy, including his provocative denial of the independent existence of material substances in his early writings, was a provisional instrument in the service of his overriding apologetic aim” (Brykman, 1984). George Berkeley’s apologetic aims influenced every aspect of his philosophical work, even areas of his philosophy that on the surface would seem disjoint from theology; namely the philosophy of mathematics. Berkeley’s theological convictions clearly influenced his philosophy of mathematics. This is perhaps made most obvious by noticing how Berkeley’s philosophical convictions on mathematics fit into his broader philosophical pursuits, and how these pursuits were clearly marked by his Christian beliefs. Through this process it becomes clear how Berkeley’s apologetic motivation manifested itself not merely in his metaphysics, but in his philosophy of mathematics as well.

Understanding Berkeley’s Christian apologetics and his philosophy of mathematics and how the two tie together is the key to understanding *The Analyst*. I devote a section of this paper to this point because I believe that, as mentioned in the introduction, there is much to learn from Berkeley on *how* one integrates philosophy with mathematics (especially from a Christian perspective). I also believe that many have slighted the significance of Berkeley’s work because they misunderstand and confuse his motivations. It has been argued that Berkeley undertakes the challenge in *The Analyst* to demonstrate the reasonableness of Christian faith. According to Boyer: “The motive prompting his criticisms in *The Analyst* was as largely that of supplying an apology for theology as it was of inflicting upon the proponents of the new calculus a rebuke for the weak foundations of the subject” (Boyer, 1959). I believe this characterization is not quite correct. Perhaps Cajori phrases it better: “Mathematicians complain of the incomprehensibility of religion, argues Berkeley, but they do so unreasonably, since their own science is incomprehensible” (Cajori, 1919). The distinction here is that Berkeley is not setting out to make a defense of the Christian faith by setting it over and above the methodology of the calculus, rather he is setting out to demonstrate the unreasonableness of Newton’s technique and makes use of his platform for a very informal apologetic of Christianity. The distinction may be slight, but I believe it is vastly important; as we shall soon see in the following critique of Berkeley’s argument put forward by Douglas Jesseph.

As the full title of *The Analyst* informs us, this work was intended to address the question of “whether the Objects, Principles, and Inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.” Berkeley famously concluded that the methods of the modern analysis compare unfavorably with those of revealed theology, so that an “Infidel Mathematician” cannot consistently reject religion for its incomprehensible mysteries while holding up the calculus as a model of correct and convincing reasoning (Jesseph, 2007). For Berkeley the degree of faith in calculus is the same as in Christianity. Berkeley had admitted that the calculus is useful for solving problems but at the same time recognized that its basis was an inconceivable notion. This seems like a strikingly similar starting point as Berkeley’s reasons for Christian faith. “Berkeley’s grounds for belief in God take the form of an inference to the best explanation that begins with facts about our sense experience (namely, that we do not cause our own sensations, that unperceived objects persist, and that the sense of vision has the structural characteristics of a language). From these facts, he concludes that we are justified in believing that a supremely powerful benevolent spirit (i.e., God) is the cause of our sensations, because the order, coherence, and reliability of our perceptions cannot reasonably be explained any other way” (Jesseph, 2007). Why would Berkeley undertake the argument such as the compensation of errors when it would appear he has a different strategy for justifying calculus at his disposal within his own philosophy? As Douglas Jesseph goes on to state the question:

We might imagine a devout mathematician reasoning thus: “Whenever I use the calculus, I find I get correct results. The method itself presupposes entities very unlike those I typically encounter in mathematics, and I can’t actually claim to have a clear notion of just what they are. Nevertheless, the best explanation of the order, coherence, and reliability of my mathematical results is that there really are such things as fluxions, evanescent increments, or infinitesimals.” It is worth pointing out that the incomprehensibility of fluxions should pose no great barrier to this strategy. It is, after all, a fundamental point of orthodox theology that God is incomprehensible, and Berkeley’s grounds for believing in God certainly don’t imply that we have a full comprehension of the divine essence. Why not have faith in fluxions, or believe in the ghosts of departed quantities? Compared to the failure that is Berkeley’s thesis of compensating errors, this looks pretty attractive (Jesseph, 2007).

Ultimately the answer is that in Berkeley’s view, mathematics is in a different compartment of knowledge from first (theological) philosophy (Johnston, 1923). The result is a fundamental difference between mathematics and theology: theology deals with mysteries beyond (but not contrary to) human reason, while mathematics deals only with things clearly conceived and evident to reason (Jesseph, 2007). Though for Berkeley there may be as much to take on faith in calculus as there is in Christianity, he does not grant them both the same esteem. Though Berkeley uses *The Analyst* to discuss both faith and mathematics, his goal is not to emphasize them both equally.

Perhaps it will be instructive to make an analogy between Berkeley’s approach in *The Analyst* and the approach taken by another great Christian thinker, the Apostle Paul in his first letter to the church at Corinth. Paul became all things to all men, so that he may by all means

save some (1 Cor. 9:22). The position he takes may not be one he personally espouses so long as it is not contradictory to the Gospel and it can be used in spreading the message. A key argumentative technique of Paul's epistle to the Corinthians is his use of what have been termed "Corinthian Slogans." Using this technique, Paul concedes a slogan common to the Corinthians (such as "all things are lawful for me..." in 1 Cor. 6:12) that he does not believe is true. He then proceeds to demonstrate that if it were true, it would lead to problems ("...but not all things are profitable"). Paul is essentially saying that even if what the Corinthians believed was true, they should be able to see that it would lead to a contradiction. In a similar fashion Berkeley uses *The Analyst* to point out that if a mathematician believed in fluxions (even though he shouldn't) then it would be contradictory to deride religious belief for its lack of reasonableness. And again in the vein of the Apostle Paul, who was not satisfied at pointing out contradictions but spent considerable effort to demonstrate how the Corinthians needed to correct their beliefs, Berkeley demonstrates how the view of fluxions needed greater rigor in order to meet the standards of mathematics. Though his offering of the compensation of errors argument was clearly not divinely inspired, his methodology seems to be taking a page out the writings of Paul: truth must be established prior to practicality.

He doesn't use an argument that calculus should be taken on faith because his theology and apologetic concerns do not allow him to make such an argument. His goal is not to come to the conclusion that fluxions work and therefore should be taken on faith like a religion. His goal is to point out that fluxions work, so the lack of rigorous argumentation put forward in their defense needs to be remedied, a goal eventually accomplished by Robinson (as detailed below). In this process he throws in for consideration the fact that if one believed the argumentation for fluxions was already rigorous enough, then they should not belittle religion as being devoid of sound defense, since this would not be correct thinking by their standard of 'sound.' Berkeley's goal is not to demonstrate how religion is logically superior to mathematics, but to point out the logical flaws in the calculus. Hence his quote in *The Analyst* of Luke 6:41-42:

Why do you look at the speck that is in your brother's eye, but do not notice the log that is in your own eye? Or how can you say to your brother, 'Brother, let me take out the speck that is in your eye,' when you yourself do not see the log that is in your own eye? You hypocrite, first take the log out of your own eye, and then you will see clearly to take out the speck that is in your brother's eye.

In the end, both people in Jesus' illustration have something in their eye. The point is that one person has the bigger problem and doesn't realize it. Mathematics in Berkeley's view is meant to deal with things evident to reason while theology goes beyond reason (not to say it is unreasonable). If a mathematical theory has a logical (truthful) flaw, this is the greater problem.

Also, it might be said that if Berkeley were to argue that mathematical principles are to be taken on faith, would this not greatly reduce the uniqueness of the Christian faith? Would Christian faith then appear as something undertaken solely because it is useful? By avoiding a claim to faith in mathematics, Berkeley is demonstrating his recognition of the primacy of theology which is the foundation for his worldview and life work.

Historical Implications of Berkeley's Critique and the Development of Nonstandard Analysis

George Berkeley's publication of *The Analyst* has been considered the most spectacular event in the history of 18th century mathematics. At the very least it must be acknowledged as a turning point in the history of mathematical thought in Great Britain (Cajori, 1919). Berkeley's objections to Newton's conception of an infinitesimal as self-contradictory were well received from a mathematical point of view (Boyer, 1959). Although Berkeley's arguments based upon the inconceivability of infinitesimals lose their force in the modern view of the nature of mathematics, they made obvious the need for a logical clarification of many of Newton's terms. Berkeley's critiques were successful in making this fact appreciated. Within seven years of the publication of *The Analyst* in 1734 there appeared some 30 pamphlets and articles which attempted to remedy the situation (Boyer, 1959).

Berkeley's objection is based on understanding the work of Newton and Leibniz in a static fashion with dx a small but fixed quantity. To overcome Berkeley's argument, dx needed to be considered as a variable, concentrating not on the given function but rather on the process of approximation that arises when dx approaches zero. A rigorous mathematical theory of approximation processes was needed to defend against Berkeley's objections. Neither Newton nor Leibniz addressed this issue in their respective original works. It was not until Cauchy developed the key idea of the limit, and a few years later when Weierstrass provided a formal definition of this notion, that the calculus was placed on sound foundations (Devlin, 1998).

It can be argued that this process of refinement, that took nearly 100 years after Berkeley's critique was published, originated in Berkeley's work. Later in the 18th century several mathematicians tried to address the questions of foundations that had been raised by Berkeley. Four main schools of thought developed, three of which included: infinitesimals, limits, and the formal algebra of series. However, the largest school of thought on the foundations of calculus was in fact a pragmatic school – calculus worked so well that there was no real incentive to worry much about its foundations (Bradley and Sandifer, 2009) – an approach to mathematics that this paper is arguing against.

In 1742, Maclaurin published his *Treatise of Fluxions* in an attempt to demonstrate that Newton's calculus was rigorous because it could be reduced to the methods of Greek geometry. Maclaurin states in the preface that he undertook the work to answer Berkeley's attack. Maclaurin advocates a limit approach to the problem. In 1797 Lagrange addressed the foundations of calculus in his *Théorie des Fonctions Analytiques (Theory of Analytic Functions)*, the subtitle of which states: containing the principles of differential calculus, without any consideration of infinitesimal or vanishing quantities, of limits or of fluxions, and reduced to the algebraic analysis of finite quantities. The book was based on his analysis lectures at the École Polytechnique. Just two years after Lagrange died, Cauchy joined the faculty as a professor of analysis and started to teach the same course previously taught by Lagrange. He inherited Lagrange's commitment to establish the foundations of calculus, but he followed Maclaurin rather than Lagrange and sought those foundations in the formality of limits (Bradley and Sandifer, 2009). In 1821, Cauchy published *the Cours d'analyse*, to accompany his course in analysis. Not only did Cauchy provide a workable definition of limits and a means to make them the basis of a rigorous theory of calculus, but also he revitalized the idea that all mathematics could be set on such rigorous foundations. Cauchy's approach to infinitesimal calculus combined

of infinitesimals and the notion of the limit. Weierstrass and his (ϵ, δ) approach eliminated infinitesimals altogether. The evolution of thought on the foundations of calculus can be clearly seen to flow from Berkeley, to Maclaurin, to Lagrange, to Cauchy, and finally to Weierstrass. Thus it was Berkeley's pursuit of (mathematical) truth and departure from the school of pragmatism that led to our contemporary notions of calculus with which we are familiar.

More recently, Abraham Robinson restored infinitesimal methods in his 1966 book *Non-standard Analysis* by showing that they can be rigorously defined. The motivation for Robinson's work was to show how infinitesimals "appeal naturally to our intuition" (Dauben, 1995). Notice that Robinson was not trying to maneuver around Berkeley's criticism to obtain a more rigorous method. Rather, he addressed directly the quantities that Berkeley labeled as "inconceivable" to demonstrate how they can indeed be conceived and then applied. Robinson believed that if you have an idea that something exists, provide a model. Do not talk about infinitesimals vaguely in a meta-mathematical sense – define them explicitly. Robinson was a firm believer that a philosophical lens, specifically in mathematical logic, could benefit mathematics proper (Dauben, 1995).

Robinson's work is addressed more specifically to the shortcomings of the work of Leibniz rather than Newton, the subject of Berkeley's critique. However, his criticisms could apply to Newton just as Berkeley's criticisms could apply to Leibniz. Perhaps the more important connection to Berkeley can be seen in that Robinson's motivation to address the topic was inspired by the work of Berkeley. Robinson felt compelled to examine the history of mathematics (in part) to help justify his own attempts to comprehend the infinite. He appreciated philosophical objections to the infinite and to infinitesimals (Dauben, 1995). He considered at great length the opposition to the infinite in the writings of Locke and Berkeley. He wrote of Berkeley: "it is in fact not surprising that a philosopher in whose system perception plays the central role, should have been unwilling to accept infinitary entities" (Robinson, 1966).

Though Robinson's own philosophies toward mathematics varied depending on the subject and the time period of his life, it is interesting to note that his key work, *Non-standard Analysis*, was published in the series "Studies in Logic and the Foundations of Mathematics," whose editors included L. E. J. Brouwer and Arend Heyting. Brouwer and Heyting are the prominent faces of a philosophy of mathematics known as intuitionism (Shapiro, 2000). Intuitionism, like many philosophies, can vary in its individual statements, but the main underlying assumption of the movement is that mathematics is primarily a mental activity. The mathematician's main tool is not pen or paper, but her mind. This lines up very well with Berkeley's way of envisioning mathematics specifically and knowledge in general. It is clear that Robinson's work was influenced by the work, and philosophical approach, of Berkeley.

Robinson characterized Leibniz' belief in infinitesimals as: a useful fiction in order to shorten the argument and facilitate mathematical invention (or discovery). The reason for the eventual failure of the theory of infinitesimals is to be found in the fact that neither Leibniz nor his successors were able to state with sufficient precision just what *rules* were suppose to govern their system of infinitely small and infinitely large numbers (Robinson, 1966). What was lacking at the time was a formal language which would have made it possible to give a precise expression of, and delimitation to, the laws which were supposed to apply equally to the finite numbers and to infinitely small and large numbers as well (Robinson, 1966). What Robinson's

nonstandard analysis neatly avoided was the problem for which Berkeley had condemned Newton – namely that his infinitesimals were the ghosts of departed quantities (Dauben, 1995).

Robinson tackled infinitesimals by dealing with equivalence rather than equality. Nonstandard analysis is concerned only with the equivalence of two elements ($a \approx b$) which does not require $a = b$, but only that a is infinitely close to b , i.e. that $a - b$ is infinitesimal. Consequently nonstandard analysis only asserted that $dy/dx \approx f'(x)$.

The following are some definitions of nonstandard analysis (Stewart, 1996). Much like the natural numbers were expanded to include reals, and then reals were further expanded to include complex numbers, Robinson now defines the hyperreal. A hyperreal is infinitesimal if it is smaller than all positive standard reals. Anything not finite is infinite and anything not in \mathbf{R} (the set of real numbers) is nonstandard. Every finite hyperreal x has a unique standard part, denoted by $\text{std}(x)$. This again is similar to every complex number having both a real and imaginary component. This $\text{std}(x)$ is infinitely close to x ; that is the expression $x - \text{std}(x)$ is infinitesimal. Each finite hyperreal has a unique expression as standard real plus an infinitesimal. In nonstandard analysis there are actual infinities and actual infinitesimals. They are constants and not Cauchy-style variables. It's as if each standard real is surrounded by a fuzzy cloud of infinitely close hyperreals, its halo, and each such halo surrounds a single real, its shadow.

Robinson justified his method from the perspective that once irrational numbers were accepted, there was no reason not to admit other consistent concepts of numbers that drew upon infinitary processes in a similar way (Dauben, 1995). Although this might seem cumbersome notationally, the more complex formalism of nonstandard analysis was but a small price to pay for the removal of inconsistency.

To see this removal more clearly, we can apply Robinson's notation to Newton's calculation of the fluxion (derivative) as described above (Stewart, 1996). Specifically we can draw our attention to steps (5) and (6) on page 3.

$$\frac{dy}{dx} = nx^{n-1} + \frac{(n^2-n)x^{n-2}dx}{2} + \dots \quad (5)$$

Because dx is infinitely small in comparison with x , the terms containing it can be disregarded:

$$\frac{dy}{dx} = nx^{n-1} \quad (6)$$

Using Robinson's notation, (5) now becomes:

$$\frac{dy}{dx} = \text{std}\left\{nx^{n-1} + \frac{(n^2-n)x^{n-2}dx}{2} + \dots\right\} \quad (5^*)$$

where x is a standard real and dx is any infinitesimal.

By Robinson's definitions, (5*) leads directly to (6) without dx having to be disregarded. In other words the derivative is defined as the *standard part* of the ratio expressed in (5). The idea is perfectly rigorous, because $\text{std}(x)$ is uniquely defined. And it doesn't just disregard or forget about the "extra dx "; it is removed it altogether. Nonstandard analysis does not, in principle, lead to conclusions about \mathbf{R} that differ from those in standard analysis, rather it is the

method and the setting in which it operates that are nonstandard (Stewart, 1996). Robinson viewed nonstandard analysis from a formalist point of view syntactically (formalism being the philosophy that views mathematics as nothing more than a game or language with rules to be followed): he had not introduced new entities but “new deductive procedures” (Dauben, 1995).

The question then becomes what is the point of nonstandard analysis if it does not lead to new conclusions? How can it be said that Berkeley’s influence has led to *more* mathematical applications? Stewart summarizes the impact of nonstandard analysis and makes a comparison to the original work of Newton:

Any theorem proved by nonstandard methods is a true theorem of standard analysis (and therefore must have a standard proof). But that doesn’t tell us how to find the standard proof, nor does it give any idea whether the nonstandard proof will be shorter or longer, more natural or more contrived, easier to understand or harder. As Newton showed in his Principia, anything that can be proved with calculus can also be proved by classical geometry. In no way does this imply that calculus is worthless. In the same way, the correct question is whether nonstandard analysis is a more powerful piece of machinery. That’s not a matter for mathematical argument: it has to be resolved by experience. Experience suggests quite strongly that proofs via nonstandard analysis tend to be shorter and more direct (Stewart, 1996).

Nonstandard analysts have since made advancements in perturbation theory, computer graphics, polynomial equations, fast-slow flows, geodesics, boundary value problems, boundary layer flow, stochastic differential equations, and spin systems in mathematical physics. From Stewart’s analysis, perhaps another (and promisingly fruitful) application of nonstandard analysis is in pedagogical practice. It is clear that Berkeley’s critique of the calculus has produced more mathematical applications than (most likely) would have been developed if Newton’s fluxions had been accepted solely for utilitarian reasons. To argue otherwise would be akin to arguing that calculus was no new mathematical application compared to geometry (at least according to Stewart’s analogy).

Conclusion

This paper has demonstrated that George Berkeley’s philosophy of mathematics was deeply intertwined with his practice of mathematics. These philosophical convictions are what drove him to attack the methods of Newton in *The Analyst*, in the face of the widespread utility of the calculus. By not giving in to pressure from his contemporaries to accept the calculus because of its practical applications, Berkeley set the stage for the refinement of the calculus and then, surprisingly, even *more* practical applications of the theory. Berkeley’s denial that utility implies truth ultimately led to the limit definition of the derivative which sidestepped the problem of infinitesimals. It also led to Robinson’s *Non-standard Analysis* which gave a rigorous definition of infinitesimals and presented them as something that can be intuited and used constructively in mathematical applications. We have seen how Robinson’s work was heavily influenced by Berkeley’s thought and how it can ultimately be categorized in a similar vein: both Berkeley and Robinson recognized the need for the integration of philosophy and mathematics. Making philosophy subservient to the practice of mathematics and believing that philosophy is most useful as a retroactive tool for describing mathematical practice, will (ironically) never lead to rich and fruitful discoveries. Only by making use of philosophy to interpret and illuminate the intellectual enterprise of mathematics can we truly realize the fullest applications of truth to life.

In addition, this paper has demonstrated that George Berkeley's philosophy of mathematics was influenced by his theological convictions. This can be seen in the clear relationship between his opposition to abstractionism in mathematics and his response to Locke's account of knowledge that was undeniably driven by religious concerns. The interconnectedness of Berkeley's mathematical philosophy and his theology can also be seen in the apologetic opportunity he takes in *The Analyst*, and in his clear demarcation between theological pursuits and his secondary interest in mathematics. In the philosophy of George Berkeley we find a man of faith who is in the world but not of it (John 15:19); his theological convictions inform his philosophy while remaining distinct. For Berkeley, all Christian endeavors, including mathematical ones, should be driven by an apologetic for the faith.

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¹ I liken this to Stewart Shapiro's argument for seeing an interrelatedness between philosophy and mathematics over and against the view that he terms philosophy-last-if-at-all in *Thinking About Mathematics*. (New York: Oxford University Press, 2000), 14-15.

² The title of "infinitesimals" is referred to only by Leibniz. Though Newton and his proponents claim to have a much stronger proof of analytic calculus because they avoid the terminology of infinitesimals, Newton's proofs nevertheless employ the same essence; a quantity so small that it can be disregarded as equaling zero.

³ The notation for the increment "dx" and "dy" (or a small change in x and small change in y) was represented instead by "o" in Newton's derivations. These notations however were used by Leibniz and are perhaps more recognizable by modern readers. These terms describe the same quantities to which Newton referred.

⁴ The following description of Locke's account of knowledge is taken from Douglas Blount (unpublished course notes in ST625 History of Philosophy, Dallas Theological Seminary, Fall 2008).