Nineteenth Conference Of
The Association Of Christians In The Mathematical Sciences

Proceedings of the Biennial Conference held at
Bethel University May 29 - June 1, 2013

Association of Christians in the Mathematical Sciences

Nineteenth Biennial Conference Proceedings, May 29 - June 1, 2013

Bethel University, St. Paul, MN

Assembled by Eric Gossett

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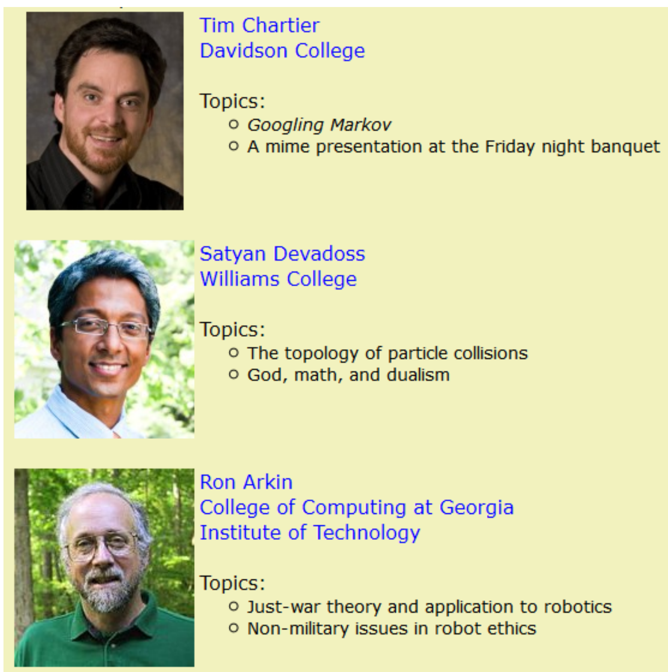
Introduction

The 19th Biennial Conference of the ACMS was held at Bethel University from May 29 through June 1, 2013. There were 111 registered participants, along with a number of family members at the conference.

The conference opened with Dr. Stephen Self of Bethel University presenting a brief concert on the 4,000-pipe Blackington Organ in Bethel's Benson Great Hall. The conference provided numerous opportunities for thoughtful presentations and discussions regarding many areas in mathematics and computer science, including consideration of faith-discipline interaction, service, and pedagogy. Of equal importance were the times of meeting colleagues from around the country (and several from other countries). These informal times offered opportunities for mutual encouragement and a chance to discuss matters of common interest. Some scheduled times of prayer were also introduced at this conference. The conference concluded with a formal worship service. Few academic conferences offer such a rich time for intellectual, emotional, and spiritual challenge and refreshment.

The following image from the conference announcement introduces the three invited speakers and their topics for the plenary sessions.



There were 48 15-minute presentations made during the parallel sessions. Of these, 25 have submitted more detailed versions for this proceedings. In addition, there were five panel sessions: a Service Learning Panel, one entitled "Does research matter?" (tailored especially for graduate students), a Faith Integration panel, one entitled "The opportunities and pitfalls of the Christian college scene", and a panel session to discuss the future of the computer science wing of the ACMS. The conference also provided a poster session, and a few late-night "birds of a feather" sessions.

The 2015 conference will be held at *Redeemer University College* in Ancaster, Ontario. Start getting your passport in order if you are not a Canadian. A conference announcement should appear in early 2015 at
`http://www.acmsonline.org/conferences/index.html`

An electronic version of this document (with color pages) is available at
`https://drive.google.com/folderview?id=0B0eg29iRQfp0dURzN2k2REIxTjA&usp=sharing`
That web page also contains materials submitted by a number of the parallel session speakers.

Eric Gossett
Conference Coordinator

2013 ACMS Biennial Conference

Wednesday, May 29

5:30 - 6:30 pm	dinner	dining center
7:00 - 7:45 pm	Opening session	Benson Great Hall
7:45 - 8:15 or 8:30	organ recital - Dr. Stephen Self (families welcomed)	Benson Great Hall
8:30 - 9:30 pm	reception for delegates/activity info session for families	Benson Great Hall

Thursday, May 30

7:15 - 7:45	Prayer time lead by Judy Palagallo	CC 331
7:45 - 8:45 am	breakfast	dining center
9:00 - 9:30 am	Welcome (Jeff Port Associate Dean of Natural Sciences); worship	CC 313
9:30 - 10:30 am	Tim Chartier: Googling Markov	CC 313
10:30 - 11:00 am	break	CC lounge
11:00 - 12:00 pm	Ron Arkin: Governing lethal behavior in autonomous robots	CC 313
11:45 - 12:45 pm	lunch	dining center
1:00 - 2:20 pm	parallel sessions	
2:35 - 5:45 pm	nature center/MIA (art museum)/Mall of America	
6:00 - 7:00 pm	dinner	dining center
7:10 - 9:00 pm	parallel sessions	
9:00 - 10:00 pm	Faith Integration Discussion	CC 331

Friday, May 31

7:15 - 7:45	Prayer time lead by Maria Zack	CC 331
7:45 - 8:45 am	breakfast	dining center
9:00 - 9:30 am	devotionals, announcements	CC 313
9:30 - 10:30 am	Ron Arkin: non-military issues in robot ethics	CC 313
10:30 - 11:00 am	break	CC lounge
11:00 - 12:00 pm	Satyan Devadoss: topology of particle collisions	CC 313
11:45 - 12:45 pm	lunch	dining center
1:00 - 2:30 pm	parallel sessions	
2:30 - 3:00 pm	break	CC lounge
3:00 - 4:30 pm	parallel sessions	
4:30 - 6:00 pm	poster session	CC lounge
4:30 - 5:15 pm	panel: The opportunities and pitfalls of the Christian college scene	CC 430
5:15 - 6:00 pm	panel: CS and ACMS	CC 325
6:30 - 7:30 pm	banquet	dining center
7:30 - 8:30	Tim Chartier: Mime-matics	CC 313 or CC 312
9:00 - 10:00 pm	birds of a feather sessions	

Saturday, June 1

7:45 - 8:45 am	breakfast	3900 Grill
9:00 - 10:00 am	Satyan Devadoss: God, math, and dualism	CC 313
10:00 - 10:50 am	ACMS business meeting	CC 313
11:00 - 12:00 pm	worship service	CC 312
12:00 - 12:30	box lunches	CC lounge

Abstracts for all parallel session presentations

A Bayesian/information theoretic approach to friendship Adam Johnson

Bayesian inference forms an expanding foundation for cognitive science. It has long been used to explain perceptual inference processes and more recently been used to learning hierarchical representations within cognitive processes. However, most research on Bayesian inference within the cognitive sciences has focused on offline (post-hoc) analysis of extant observational data rather than online sampling procedures that inform active learning. We combine hierarchical Bayesian approaches to structural learning from cognitive science with normative statistical sampling to develop a model of active learning. This statistically informed approach suggests that some observations are more meaningful than others. We then use this approach to model the interactions between multiple inferential agents that are capable of (1) selectively acquiring meaningful observations from others and (2) selectively offering meaningful observations to others. We hypothesize that these actions form the foundation of friendship.

A Different Approach Catherine L. Crockett

In this talk, I discuss an approach to getting science majors to rethink their study habits using two simple techniques. The techniques are showing students how to outline concepts in a manner that they understand and in-class quizzes with the intent to give a self-evaluation of where more study time is needed. After using both methods in two sections in a first semester calculus course, I surveyed the students to determine if these activities were successful. A majority of the students felt theses activities were helpful in the course and wanted to continue them.

A Mathematician's Reflections on James K. A. Smith's Desiring the Kingdom Bryant Mathews

The discipline of mathematics, perhaps more than any other, emphasizes the cognitive aspect of our being. In its attempt to reduce objects and statements to symbolic notation and formal logic, pure mathematics may appear to make little contact with the affective or sensory realms. Mathematicians even have a reputation of being so cognitively involved that their emotional and physical beings are left to atrophy. Yet mathematical discovery is itself a visceral experience for those who breathe its air and behold its vistas. In Desiring the Kingdom: Worship, Worldview, and Cultural Formation, James K. A. Smith argues that Christian education should be concerned not just with information but with formation, not just with the dissemination of ideas but with the shaping of desires. The communal practices in which we engage form our habits, whether virtues or vices, which in turn strengthen our desires for a particular version of "the good life." These desires then help to constitute our identity and give direction to our worship. What vision of 'the good life' do students absorb in our classes? Which desires motivate, and are reinforced by, their mathematical study? What behavioral and emotional habits do they develop through participation in our community? And finally, what pedagogical 'liturgies' can help our students to be conformed to the likeness, and not just the mind, of Christ? This talk will share some of my reflections, as a mathematician, on Smith's work, with the goal of sparking your desire to embody his holistic vision of Christian education.

Adventues In Flipping Roberto Bencivenga

Unlike many former students, I never had the experience of flipping burgers to support my education, but recently I have had plenty of exciting and successful experiences in flipping classrooms. The concept behind the "flipped classroom" model is gaining momentum and is briefly described in the document and video available at http://robertosmathnotes.weebly.com/2013-acms-conference.html. If you plan to attend this session, it is essential that you read the document and watch the video beforehand. Together they will take no more than 10-15 minutes, but will make the actual session much more meaningful. In my session I will address any issues and questions that you will bring in relation to flipping the classroom, as well as share some important lessons I have learned from my experience. Finally, the ideas we shall discuss there can be applied to academic settings as well as to any theological presentations, from individual evangelization opportunities to Sunday school and beyond.

Al-Khwarizmi: Father of Algebra? Calvin Jongsma

Adopting a historically defensible definition of "algebra," we will begin by exploring a few examples of algebra prior to al-Khwarizmi. We will then examine what algebra became through al-Khwarizmi's work. In conclusion, we will assess the historical importance of al-Khwarizmi's contributions for developments in European algebra.

An Investigation of Hi Ho! Cherry-O Using Markov Chains Nicholas Zoller

In the children's board game Hi Ho! Cherry-O, players attempt to move 10 cherries from their trees to a bucket in the center of the game board. A spinner determines whether a turn includes moving cherries from tree to bucket or bucket to tree. The winner of the game is the first player to move all of her cherries from her tree to the bucket. We model the game play using a Markov chain and calculate the expected number of turns needed to complete one game. Then we investigate what happens when the rules are changed. We discover that rules changes designed to either increase or decrease the length of the game have the desired effect. However, when rules changes are combined, we find that rules changes designed to decrease the length of a game can hide the effect of rules changes designed to increase the length of a game.

Analysis of Potentiation Bias Within Non-Random Data: Excitation Transfer Theory in Liturgy (Poster Session) Dallas F. Bell

Potentiation's enhancement of one agent by another agent causes the combined effect to be greater than accomplished separately. Over time a bias for repetition develops between those entities. Sequence data can be shown to be non-random. Beginning with mathematical sets, it is argued that there can be no true randomness. DNA sequences, biological and behavioral sequences are also demonstrated to be chronologically a priori of intellect to outcome. Excitation of emotions transfers an emotion to other emotions in cause and effect potentiation(s). The religious practices of liturgical standards culminate in the highest accretion sequence of collective behavioral bias for the human worship experience.

Calculus methods from the early 1600s Gordon Swain

The French mathematician Gilles Personne de Roberval lived from 1602 to 1675. He developed methods for finding areas and arclengths, as well as working on mechanics, and solving problems later attributed to Pascal and Torricelli. In this talk we will discuss ideas and methods that Roberval used to integrate $y = x^n$ in his Traite des indivisibles.

Case Study for Numerical Methods: Components in Audio Recordings Brian Turnquist

Linear Algebra students learn that the same vector can be expressed as a linear combination of different bases. If a transducer is sampled at regular intervals, then the resulting signal may be viewed as a vector in n-space. Examples would include audio recordings, force sensor recordings, and recordings of biological neural activity. Such vectors typically have a tonal component, an impulsive component made up of pops and crackle, and a noise component. A worthwhile case study in teaching Numerical Methods is to demonstrate that by expressing such vectors as a linear combination of a Fourier basis or alternately a wavelet basis, we can choose which features in the signal we wish to see.

Chaos and the Sea: Randomness and Purpose in the World. Troy Riggs

Is there any true wildness or chaos in the world? Some theological viewpoints suggest that there are not. But the Book of Revelation describes a day in which "there will be no sea." Does God interact with people outside of tightly ordered systems in purposeful ways? In this talk we examine how humans are able to work within stochastic systems or even intentionally use randomization methods to accomplish their purposes. Mathematicians, statisticians are able to gain knowledge and manage results in situations where direct control or precise prediction is extremely limited.

Chaos, Reality and Language Kyle Spyksma

Chaos Theory, the mathematical media darling of the '90s, has become less of a societal fad and research interest over the past couple of decades. However, from a mathematical physicists' perspective, issues surrounding Chaos Theory can be valuable aides in forming views on how mathematics, science and reality relate. In this talk, I will briefly explore how Chaos Theory can shape views of these relationships, with a focus on the language we use and (perhaps unintentionally) abuse when doing science and mathematics.

Code Ye into All the World: Leading a Successful Computer Science Missions Trip Tom and Darci Nurkkala

The global missions community goes wanting for skilled workers in almost every discipline. However, even students at a Christian institution that emphasizes global engagement remain largely clueless about the impact they could make in missions by leveraging their own academic specialty. Particu- larly in our on-line, cloud-based, mobile-enabled, global technology ecosystem, the need in missions for skilled workers in Computer Science (CS) and Information Systems (IS) has never been greater. At Taylor, our CS and IS students have ample opportunity to apply their skills to missions computing while on campus through both class and volunteer projects. But it's when students experience on-site work with full-time missions technologists that they develop an understanding and a vision for how they can contribute to missions by leveraging their own skill and passion. In this paper, we draw on our experience leading CS missions trips. We discuss the need for students to experience missions firsthand, and the student outcomes we have observed in intercultural awareness and spiritual formation. A key student outcome we explore is increased willingness to consider vocational missions service in both internships and full-time service after graduation. We also offer practical guidance for faculty or staff interested in leading discipline-specific missions trips with their students. Although our examples are drawn from our CS trips, much of the material here should be applicable across academic disciplines.

Computing Foundations for the Scientist Catherine Bareiss / Larry Vail

There is a need for a new style of supporting a computer course. Although it is widely recognized that computer technology provides essential tools for all current scientific work, few university curricula adequately ground science majors in the fundamentals that underlie this technology. Introducing science students to computational thinking in the areas of algorithms and data structures, data representation and accuracy, abstraction, performance issues, and database concepts can enable future scientists to become intelligent, creative and effective users of this technology. The intent of this course is not to turn scientists into computer scientists, but rather to enhance their ability to exploit computing tools to greatest scientific advantage.

Creation Care As A Focus For A General Mathematics Course John Roe

Issues of environmental sustainability are a focus of growing concern for many students, and this is a concern that unites profound worldview questions with extremely down-to-earth quantitative ones. A mathematics course built around an environmental theme is therefore a natural context for the integration of faith and learning. I will describe some of the structure for a course that I am developing, inspired by this idea, at a large public (secular) university.

Delaware, Dickeson, Assessment and How You Can Help Maria Zack / Greg Crow

How much release time should a chair receive? What is the cost per unit for a particular academic program? What is a student credit hour (SCH) anyway and why would anyone care? Why are so many boards enamored of Delaware, Dickeson and Assessment? The answer to these and many related questions will be presented in this talk. Analytics and various "efficiency measures" are becoming increasingly important in higher education and mathematicians and computer scientists are being regularly recruited to help university administrators make meaning from large volumes of data. Come and learn about this trend and how you can be of assistance to your institution.

Does research matter? (Panel) Judith Palagallo / Tim Chartier / Matt DeLong / Satyan Devadoss / John Roe / Francis Su

How do the requirements of teaching, research and service shape our career choices? Is the main purpose of a mathematician to produce research and to train students to continue research? Can our involvement in research influence our teaching in a positive way?

Euler and the Ongoing Search for Odd Perfect Numbers Brian D. Beasley

Leonhard Euler, after proving that every even perfect number has the form given by Euclid, turned his attention to finding odd perfect numbers. Euler established a basic factorization pattern that every odd perfect number must have, and mathematicians have expanded upon this Eulerian form ever since. This talk will present a brief summary of Euler's result and some recent generalizations. It will also note connections between odd perfect numbers and the abundancy index (the abundancy index of a positive integer is the ratio of the sum of its positive divisors to itself). In particular, finding a positive integer with an abundancy index of 5/3 would finally produce that elusive odd perfect number.

Explore Global Opportunities for Mathematics Scholarship, Teaching, and Service Ron Benbow

There are numerous overseas opportunities in which to apply your knowledge and interest in mathematics. These international experiences allow you to expand your scholarship, extend your teaching skills, to offer professional services to K-12 teachers or other university instructors, and provide much personal enrichment as well. Examples from recent professional experiences in Liberia, Haiti, Guatemala, and Ecuador will be shared to illustrate the connections to teaching, scholarship, and service. Information regarding MAA Study Tours, Fulbright Specialist grants, and other relevant organizations will be provided.

Factorizations in Strong Divisibility Sequences Stephen Lovett

A strong divisibility sequence is a sequence of integers a_n that satisfies $gcd(a_m, a_n) = a_{gcd(m,n)}$. Such sequences include many recursively defined sequences, including the Fibonacci sequence, and sequences created by repeatedly applying a polynomial to some initial condition. In this talk, we prove a factorization property about strong divisibility sequences in the more general context of a UFD, illustrating consequences for cyclotomic and dynatomic polynomials. Furthermore, a converse to this talk's main theorem provides a simple necessary and sufficient condition for a divisibility sequence to be a rigid divisibility sequence.

Faith Integration Panel Discussion Jamie K. Fugitt / Derek Schuurman

A panel session on the integration of faith and discipline in the undergraduate curriculum. As a part of the session, Russ Howell will present an opportunity coming from the Journal of the American Scientific Affiliation. Several submitted essays in response to a broader article will be selected by a committee of editors for publication.

Faith Integration Projects for Students Doug Phillippy

This talk will consider the use of projects to motivate students to think deeply about how their faith connects with mathematics. This talk will begin by describing what a faith integration project is, including the goals and objectives of such a project. The talk will briefly describe a number of projects written by the speaker, with a more detailed look at one of those projects. The talk will conclude by discussing how these projects are being used to assess how students are doing at articulating a maturing understanding of the connection between faith and mathematics.

Getting Freshmen Interested in the Infinite Nicholas Willis

How do you get freshmen interested in Mathematics? How do you integrate faith and Mathematics? How do you describe cardinality to non-majors? Solutions to these questions will be discussed in the context of a freshman seminar course at George Fox University.

Googol-part Fugue: Another Imagination of Game Theory and Divine Providence Gideon Lee

The problem of evil presents an intellectual hurdle for some to believe in a good and omnipotent God. The emergence of open theism could be seen as an attempt to make a stronger case for the free will defense. However, in denying divine foreknowledge as traditionally understood, open theism contradicts biblical revelation not only in its direct claims, but also when its logical implications for divine providence are worked out. The open theist Alan Rhoda has sought to explain through game theory how some degree of divine providence is possible under open theism. That explanation is astonishing since the open theist view of libertarian free will is intrinsically at odd with the rational actor model presupposed by game theory. In this essay, the free will defense of open theism and two other responses to the problem of evil are examined. Game theory and other mathematical theorems are employed in illustrating the theological claims. This essay seeks to show that the historic Christian doctrine of divine sovereignty can be reasonably explained given the presence of evil. The key is to recognize the biblical picture of the present age as a development ground and worthiness-demonstrating trial for a perfectible authentic humanity, chosen for a glorious leadership role in the new heavens and new earth, where everything will be knowable, optimal, and predictable.

Ideas For a Math Capstone Course Robert Brabenec

After teaching a majors capstone course on the history and foundations of mathematics for over forty years, I tried several variations this past semester. I will discuss these in my talk, along with an assessment of their effectiveness. Some of the ideas can be introduced into other mathematics courses for variety and enrichment.

Individualized Tests and Projects in Introductory Statistics Courses Clifford H. Wagner

I have written text processing software to enable instructors in introductory statistics courses to produce multiple versions of tests and projects. In the case of tests, having several similar but significantly different versions of a test helps reduce security concerns in large classes. In the case of projects, most students are more interested when they are asked to analyze their own personal data set. This software, Automatic Statistics Test Generator, is available for sharing and allows the instructor to add special code to a standard LaTeX file, and thereby produce multiple versions of a given document, each with its own answer sheet.

Insights on the Neyman-Pearson Lemma: Alternative critical regions, and their power. David E. Wetzell

The Neyman-Pearson Lemma is a powerful fundamental lemma in the area of hypothesis testing in Statistics. It gives the best test when testing simple vs. simple hypotheses. In this talk we would like to investigate testing a population mean $H_0 \quad \mu = \mu_0$ vs. $H_1 \quad \mu = \mu_1 > \mu_0$. As a result of the N-P Lemma, the best test is of the form, "Reject H_0 if $\overline{x} > c$" , where c is chosen so that the Type I error probability is a. Let n be small. What are some alternative decision rules of size a, what is their power in comparison to the best test? The talk should be of interest to a person who has had a first course in Probability and Statistics.

Integrating Faith through Writing in Calculus I Rebekah Yates

For many calculus students, there is no connection between mathematics and their faith beyond praying that God will help them survive the course. To create space for my students to explore how their learning in Calculus I and their Christan faith can inform and enrich each other, I give several short writing assignments over the course of the semester. I will share highlights from these assignments and student responses to them.

Inverted Classroom in Abstract Algebra: Daniel Kiteck

I was interested in trying something different than lecture in my abstract algebra class. I had heard of Inquiry-Based Learning, but I wasn't ready to make the full commitment of preparing a class where the students "found" proofs on their own to the main theorems, after being given minimal axioms. I decided, instead, on doing a type of "inverted classroom" where the students, in pairs, wrestled with the examples and proofs outside of class, so that they could present them during class-time. During class, I called on different students to present, where the students did not know who I would call on beforehand. Almost every class was almost entirely students leading the class. This resulted in rich class-times where the students were engaged, asking deep questions, and, getting more out of the material.

Invitation to the mathematical community: blog discussions in a transition course Kristin A. Camenga

Transition courses help students learn the basic logic and proof techniques they will need to be successful as a math major. However, these courses can also invite students more broadly into the discipline, preparing them to participate in the mathematical community. We share blog assignments and resulting discussions from a 7-week Introduction to Proof class which are intended to place the students' work with proof in a larger context. Topics include the interaction of math and faith, the nature of proof, and beauty in mathematics.

Lessons from "The Lesson of Grace in Teaching" Francis Su

At the Joint Winter Meetings in January 2013, I received the Haimo teaching award, and in my acceptance speech I chose to speak about how grace has shaped my teaching. In particular, I explained how giving and receiving grace can challenge the academic notion that you are defined by your accomplishments. I will share excerpts of this speech, as well as reactions to my speech, which subsequently went viral online and has been shared five thousand times on Facebook.

Life Lessons from Leibniz Andrew Simoson

The tri-centennial of Leibniz's death is nigh (2016). And 2013 is not too early to begin a special celebration of this man of mathematics. Besides being the co-discoverer of calculus and the implementer of binary numbers, formal logic, and formal languages, all of which foreshadowed the computer age, Leibniz is said to be one of the last to know almost everything that was known about almost anything. Professionally, his occupation was librarian in the princely court of Hanover in old Germany. Serving under three different princes, the last of whom became George I of England, Leibniz had to continually re-invent himself—somewhat like us older teachers and professors who have continually re-invented ourselves over the years as classroom technology changed from slide rule to hand-held calculators to computers to a profusion of computational schema and distance-learning on the web—-under changing administrations and expectations. Throughout his long life, he traveled extensively, maintained a vibrant, voluminous correspondence with a host of theologians, scientific savants, politicians, and friends. In fact, Leibniz is said to have "fine-tuned" the notion and practice of "the balance of power" among nations and pioneered the idea and practice of ecumenicalism within the fragmented church universal. He has much to teach us about math, life, and faith. In our time slot on the program—we give a short sketch with a few life lessons from this giant of a man.

Mapping Biblical Commandments to an Iterated Prisoner's Dilemma Framework Nathan Gossett / Adam Johnson

In his writings on Game Theory, an d the Iterated Prisoner's Dilemma in particular, Robert Axelrod outlined four properties that are predictors of a successful strategy: Niceness, Reciprocity, Forgiveness, and Understandability. On the topic of Reciprocity, Axelrod makes the claim that not only does The Golden Rule lead to a suboptimal strategy, but that one of the most successful strategies (Tit for Tat) shows that a command of "An eye for an eye" leads to a much more optimal strategy. In this paper, we will discuss the details of Axelrod's four properties, outline Biblical support for all four, and discuss how, within the framework of an Iterated Prisoner's Dilemma, neither "Do unto others..." nor "An eye for an eye" are the Biblical command that most closely matches the behavior of winning strategies in regards to the Reciprocity property.

Math Services (Birds of a Feather) Patrice Conrath

A gathering of those interested in math services (including Math Labs, testing, etc.).

Mathematical Affections: Assessing Values in the Math Classroom Josh Wilkerson

When am I ever going to use this? As a math teacher, this is the number one question that I hear from students. It is also a wrong question; it isn't the question the student truly intended to ask. The question they are really asking is "Why should I value this?" and they express their inquiry in terms of practicality because that is the language in which their culture has conditioned them to speak. While the utility of mathematical concepts are certainly important, we as educators need to utilize the mathematics classroom to address the more fundamental issue of fostering a proper sense of values. Learning has little meaning unless it produces a sustained and substantial influence on the way people think, act, feel, and ultimately worship. According to the NCTM standards it is through assessment that we most clearly communicate to students what aspects of mathematics are to be valued. This talk will address two essential questions:

1) *Why* is it necessary to develop assessments that equip students to not only know and practice but also love that which is true, good and beautiful?

2) *How* do we design worthwhile mathematical assessments that synthesize something seemingly non-objective like personal values with something seemingly non-subjective like mathematics?

The title of this talk is in homage to Jonathan Edwards' *Treatise on Religious Affections*. Edwards' goal was to discern the true nature of religion and in so doing dissuade his congregation from merely participating in a Christian culture (a mimicked outward expression) and motivate them to long for true Christian conversion (an inward reality of authentic Christian character). The purpose of this talk is to engage ACMS members in discerning the true nature of mathematical assessment and how we use it in the classroom: does it simply mimic the modern culture of utility by requiring outward demonstrations of knowledge retention and application, or does it aim deeper at analyzing true inward character formation? In closing, examples of affective mathematical assessments will be presented as resources for consideration and classroom use.

Modern Portfolio Theory Pays Dividends in Math Class (Poster Session) Ken Constantine

This poster indicates how simple linear regression, as employed in Modern Portfolio Theory, can be exploited as a finance-related application in a variety of Mathematics courses. The setting, with a linear relationship between a single investment's return and a market surrogate's return is first described. This setup can be a springboard to classroom examples or student projects including:

- Linear relationships – slope and intercepts – as in a PreCalculus course
- Simple linear regression – least squares and $\mathbb{R}^2$ – as in Introductory Statistics
- Calibration intervals – as in an upper level Statistics course

In addition to the technical content, discussion topics are indicated including:

- Implications of the fitted model for investment strategy
- Limitations of mathematical models and portfolio theory in actual implementation
- Financial stewardship – for personal finance, for charitable giving, for church investments

Normal Mode Analysis and Gaussian Network Model for protein structure fluctuations Jun-Koo Park

Functions of bio-structures are related to the dynamics, especially various kinds of large-amplitude motions. With some assumptions, those motions can be investigated by Normal Mode Analysis (NMA) and Gaussian Network Model (GNM). In this work, I review the NMA and GNM and evaluate GNM, based on how well it predicts the structural fluctuations, compared to experimental data. Then, I propose more refined GNM to reflect more physical interaction between atoms and compare the refined GNM with GNM.

Open Source Software: What it is, and why should we care? Karl-Dieter Crisman

We are all familiar with the difference between software on the desktop and the cloud, as well as software with varying price tags. But the distinction between proprietary and open source software is far less familiar, though I will argue in this talk it is also crucial. After giving an overview of what the question is really about (with an emphasis on mathematics instruction and research), I will summarize four main intersection points between Christian thought and the nature of open source software, at its best.

Pedagogical Enhancements to the DeSymbol Logic Translator Darren F. Provine / Nancy Lynn Tinkham

DeSymbol is a program that translates first-order predicate logic expressions into English. It is intended to be a practice tool for students who are learning logic for the first time or who are trying to refresh their memories if they need to use symbolic logic for an upper-level course. Students start with an English sentence and translate it by hand into symbolic logic notation; then they can check their work by using DeSymbol to translate their notation back into English. If the English sentence produced by DeSymbol differs significantly from the original English sentence, this helps the student to see what error was made in the logic expression. The latest version of DeSymbol adds support for prepositions, so that the student can now test expressions such as as $\text{on}(a, b)$ and $\forall X \forall Y(\text{on}(X, Y) \rightarrow \text{under}(Y, X))$. It also now supports a wider variety of idiomatic translations, including improved translations of common student mistakes. For example, the student who begins with the English sentence "All cats are mammals" and writes the expression $\forall X(\text{cat}(X) \wedge \text{mammal}(X))$ will see DeSymbol re-translate the expression as "Everything is a cat and a mammal", which helps the student to see why the expression is incorrect.

Philosophy Motivates Undergraduates in Mathematics Dusty Wilson

Is mathematics discovered or invented? After a sabbatical and years of study, I am not much nearer an answer, but I have watched this enigma transform students. The history and mystery of mathematics helps connect students from across the curriculum. Additionally, I have used the philosophy of mathematics as the core of small student seminars. These seminars have been foundational to prospective mathematics majors and drastically altered the academic pursuits of others. While appropriate within any small college setting where resources are limited and freedom great, I will present within the context most familiar to me. Given that over 40% of students receiving a bachelor's degree in mathematics earn credit at a community college, these simple steps at a public two year college may have a lasting and resounding influence on mathematicians and math education for years to come.

Philosophy of "spinning wheels" Loredana Ciurdariu

In this material I will speak about some well-known mechanisms studied by students and engineers emphasizing the impact which "spinning wheels" had and have in development of the society, on Christians and the church. Also the discovery of these machineries determines major changes in the people's outlook and leads to new trends in philosophy and Christianity. Then, I will give some examples from the Bible where "spinning wheels" it seems to appear: Judge 16:21, Ezekiel 1 and Revelation. In addition, an avi file where we can notice the movement of the "spinning wheels" (a crank-slider mechanism for example) will be attached.

Planning a Calculus II Class Dali Luo

A calculus II course typically includes these two parts: integration techniques and series. In this presentation, the following chart will be introduced which not only assists me in planning for the course but also helps the students to see the big picture of each of the two parts and, as a result, put together the numerous formulas.

	One-Mile Run	
Warm-up		review
Lap 1	Determination	preview
Lap 2	Persistence	new formulas
Lap 3	Endurance	more formulas
Lap 4	sprint	integration
Finish line		test

Reading Assignments and Assessments: Are Your Students Reading Math Text Before Class, After Class, Both, or Neither? Dave Klanderman / Mandi Maxwell / Sharon Robbert / Bill Boerman-Cornell

In his recent book What the Best College Students Do, Ken Bain defines a number of different types of students including "surface learners," "strategic learners," "routine experts," and finally, "deep learners." In our mathematics courses at Trinity, we have found examples of all of these student types, and a major determinant of their preferred approach to learning appears to be the ways and degrees to which mathematical texts and other written materials are read prior to class sessions. Each full-time member of the department both assigns and assesses the reading of mathematical materials prior to class sessions. Assessment methods vary significantly as well as the corresponding pedagogical choices. During this session, we discuss the results of a survey of over 100 Trinity undergraduates enrolled in a mathematics course during fall 2012. The courses included those at the introductory level such as statistics, calculus, and math for teachers, and those at the advanced level such as geometry, history of mathematics, and senior capstone seminar. Assigned reading materials ranged from textbooks, supplementary readings (e.g. Dunham's Journey Through Genius), and readings engaging a Christian worldview (e.g. Mathematics Through the Eyes of Faith). A summary of student evaluations of the assigned readings and related assessments will provide participants with issues for further reflection and potential future implementation in their own mathematics courses.

Service Learning in College Algebra Tedd Szeto

Azusa Pacific University, located in an extremely diverse part of Southern California, partners with many local agencies to provide its students with experiences that intentionally integrate academic learning with community service. For the past eight semesters, multiple sections of APU's College Algebra course have participated in this type of service-learning project by teaching prepared lessons to local fifth graders. This talk will detail the designing this program, discuss mutually beneficial aspects for all students involved, examine some of the challenges, and report the results from statistical data generated from the elementary students' pre- and post-tests.

Service Learning Panel Karl-Dieter Crisman / Josh Wilkerson / Dave Klanderman / Maria Zack

Many of us have wanted to incorporate service experiences in courses, or are being asked by our institutions to do so. Service-learning is a way of looking at service as being a partner with and leading to learning for our students. But in math, there are not a lot of resources to use! Our panelists will present classroom-tested ideas from several different levels of course, and we will end with a short time for more brainstorming among all participants.

Shaping a Digital World Derek Schuurman

The talk will present a book project about faith and computing that I have been working on - the book is called "Shaping a Digital World" and will be published by Intervarsity Press in June 2013. I hope to share my journey in writing the book along with an outline of the book.

Simulation Projects in an Operations Research Course Patrice Conrath

Bethel's junior/senior level course, Operations Research, involves a simulation project that utilizes the whole class as a project team. Five of the last eight course projects have helped to optimize various systems at Bethel. The remaining three projects include the local McDonald's drive through, a missions air base network, and most recently, a local food packing model for an agency called Feed My Starving Children. We will explore the advantages of each type of project and some of the impacts on students beyond mathematics/computer science knowledge (such as considering using their talents to help a non-profit). I will also share materials related to the project management structure, portfolio requirements, and other project management issues.

Successes and Challenges in Interdisciplinary Teaching Lori Carter

In the fall of 2012, after 2 years of research and planning, we launched a Computational Science minor at Point Loma Nazarene University. This meant that we were now inviting Biology, Chemistry, and Physics students to be primary players in mathematics and computer science courses. In some cases, we re-tooled existing computation-based classes to emphasize science-related examples and applications. In other cases we created brand new courses that were designed to blend the introduction of science-related problems with the introduction of computational tools to help solve them. In these courses, science students and computational students were on equal footing. Sometimes the science students knew more about the subject and were helping to teach the computational students, and sometimes it was the other way around. Most participants would agree that it has been a wonderful and fruitful adventure so far. But, it has not been without its surprises. The goal of this presentation is to share what we've learned about interdisciplinary teaching, projects, expectations, learning styles, and recruitment.

Teaching Complex Analysis as a Lab-Type Course with a Focus on Geometric Interpretations using Mathematica
Bill Kinney

I taught Complex Analysis for the first time in my career during the spring of 2013. I decided to do something "radical" and teach it as a lab-type course with a focus on geometric interpretations using the computer program Mathematica. We met in a computer lab and, during most meetings, we spent a large portion of our time experimenting and exploring using Mathematica to visualize key concepts in Complex Analysis. Because of this, there was a heavy emphasis on viewing analytic functions as conformal mappings as well as considering associated vector fields and flows. Mathematica was used to make the concepts "come alive" through its animation capabilities. The demonstration of these animations will be the main focus of my talk. I will also briefly show how I helped students learn more basic content through the use of many 10-minute video lectures (I also taught basic Mathematica code during these lectures).

Teaching Virtuous Computer Programming Victor Norman

Teaching computer programming means teaching students a skill – how to program a computer in a certain language. An instructor must teach the language syntax, data types, problem decomposition, debugging, and testing. However, an instructor of computer programming can teach more than just the skill of programming, but also incorporate and emphasize certain virtues, including hospitality and humility. This paper argues that a student who learns these virtues as they relate to computer programming will learn to produce code that is cleaner, better documented, more reliable, more robust, simpler, and better tested.

The Centrality of Christ: The Bell Curve As A Biblical Type of Christ Jason Wilson

Over two hundred years after his death, an unfinished notebook of Jonathan Edwards' was published for the first time in 1993. Edwards was a father of the Evangelical movement, but because his work on typology was not published until recently, it has received almost no attention. In his notebook, Edwards makes an explicit argument for extending biblical typology to nature in a biblically grounded manner. This study is an attempt to extend that research program into mathematics/statistics. We will consider the following proposition, "The normal distribution (the graph of which is the bell curve) is a biblical type of Christ." The basic idea is that as the normal distribution is the center of the discipline of Statistics, so Christ is the center of the plan of God (Eph 1:9-10). Evidence for the proposition will be given from four different statistical phenomena regarding the normal distribution: the Central Limit Theorem, limiting distributions, its striking conditional and marginal distributions, and its unique name.

The Founding of the Analytical Society at Cambridge University Richard Stout

Mathematics played a prominent role at Cambridge University from the end of the 1700's on through the nineteenth century, however by the early 1800's Britain was far behind the rest of Europe in producing important mathematicians and significant mathematics. Changes were needed and many historians point to the founding of the Analytical Society in 1812 as an important turning point in reforming British mathematics. One of the remarkable features of this group is that it was not composed of faculty members, but was a student-led organization, whose leadership was a group that the historian Joan Richards claims "went on to become the core of English science for the first half of the nineteenth century." While there is little doubt about the influence of the Analytical Society on British mathematics, this presentation will focus on the mathematical and institutional factors that contributed to the founding of the Analytical Society, especially the student-led efforts to establish a Cambridge branch of the British and Foreign Bible Society.

The Structures of the Actual World Walter J. Schultz / Lisanne D'Andrea Winslow

Scripture teaches that God has a plan for the universe. In this paper we argue that in order for it to function as a plan, it must have a temporal structure, a representational structure, and a proto-causal structure. This paper presents a formal model of the these three structures. As it turns out, the structures of God's plan are best understood as the structures of a musical composition. We, then very briefly describe its implications. The first is that this model (based ultimately in the doctrine of God) grounds a metaphysics of science. Second, it grounds a structuralist philosophy of mathematics. Third, since the composite structure is a complex relational structure (some of whose parts are themselves mereological sums defined predicatively) global self-reference is eliminated. Therefore, if in the spirit of Leibniz we take God's plan to be the actual world for a metaphysics of modality, we are able to preclude the incoherence that plagues set-theoretic constructions of platonic entities such as Roberts Adams' maximal propositions and Alvin Plantinga's 'book' on the actual world. Yet, in one sense, our model is a logical extension of Plantinga's 'actual world' as a maximal, temporally-invariant state of affairs. So, the model also grounds a metaphysics of modality in the biblical doctrine of God. Fourth, the model provides the basis of a world-inclusion semantics for systems of formal logic.

The Unity of Knowledge and the Faithfulness of God-The Theology of Mathematical Physicist John Polkinghorne
Matt DeLong

The Rev. John Polkinghorne is arguably the greatest living Christian voice in the dialogue on science and religion. A well-decorated mathematical physicist, Polkinghorne resigned his academic post mid-career to study for the Anglican priesthood. He has since become an influential theologian and a prolific author. Polkinghorne is widely admired by Christian academics for his thoughtful and winsome defense of the harmony between science and faith, and yet his theological views are not without controversy. This talk will give a brief survey of Polkinghorne's theology, including his thoughts on the Trinity, the Bible, creation, prayer, providence, and eschatology, with a particular view for the ways in which his mathematical and scientific thought has influenced his theology.

Using Anime and Manga to Strengthen Content Retention Eric Gossett

Mathematics and Computer Science are subjects that many Japanese consider to be normal parts of life. Consequently, these subjects appear fairly often in popular culture. In particular, anime (Japanese animated cartoons) and manga (Japanese comic books and graphic novels) contain many examples of significant content in these disciplines. I have found many of these examples to be helpful as tools to reinforce idea retention. Of great interest is the series of Manga Guides to many subdisciplines in the STEM fields, including linear algebra, statistics, and database. In this session I will share some of my favorite examples.

Computing Foundations for the Scientist

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ABSTRACT

There is a need for a new style of supporting a computer course. Although it is widely recognized that computer technology provides essential tools for all current scientific work, few university curricula adequately ground science majors in the fundamentals that underlie this technology. Introducing science students to computational thinking in the areas of algorithms and data structures, data representation and accuracy, abstraction, performance issues, and database concepts can enable future scientists to become intelligent, creative and effective users of this technology. The intent of this course is not to turn scientists into computer scientists, but rather to enhance their ability to exploit computing tools to greatest scientific advantage.

1. INTRODUCTION

Aided by powerful software packages, scientists today build complex visual system models, manage scientific databases, perform simulations, consult expert systems, and render results. If science students can understand this software as more than a black box, they can be positioned to better understand its value and results, and make more intelligent decisions about how to analyze and improve their results, and even when to rely on the software and when not.

The purpose of this course is to make future scientists more intelligent users of computing technology in their practice of science. At ONU an introductory course on computing foundations and their specific application to areas of science has been developed. The material in this course been developed in at least 14 modules that can be combined into one course, used as part of existing science courses, and used by students for independent learning.

2. BACKGROUND

The need for understanding of computational thinking in today's society by all disciplines is well documented [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. According to Peter Denning [6], "many whose lives are touched by computing want to know how computers work and how dangerous or risky they are, . . . and most everyone asks for an uncomplicated framework for understanding this complex field."

It is essential to understand what computational thinking is. It is understanding the fundamental concepts of computer science and applying them to most every area of life. According to Denning [6], this includes computation, communication, coordination, automation, recollections (windows of computing mechanics) and simplicity, performance, reliability, evolvability, and security as design principles. Not only is computational thinking conceptualizing and not programming[6], Lu [9] explains that programming should not be a student's introduction to computational thinking, just as proof construction is not a person's introduction to mathematics (arithmetic is). "The emphasis should be on understanding

(and being able to manually perform) computational processes, not on their manifestations in particular languages." Courses based on this should address properties such as "convergence, efficiency and limits of computation."
Computational thinking is foundational to the scientist [5, 7, 8, 10, 11, 12]. Nobel Physics Laureate Ken Wilson stated that "computation had become a third leg of science, joining the traditions of theory and experiment [5]." Hambrusch [8] states "scientific research is now unthinkable without computing. The ubiquity of computerized instrumentation and detailed simulations generates scientific data in volumes that can no longer be understood without computing."

Many have understood the charge to promote more computational thinking in disciplines outside computer science to mean the introduction of programming in these non-computing curricula. At a SIGCSE 2009 panel on the present and future of computational thinking, three leaders in this area (Astrachan, Hambrusch, and Settle) reported the inclusion of programming in non-computing courses [2][8]. A multi-disciplinary approach at the University of Nebraska [11] also works with biology (and some non-science disciplines), but still incorporates programming.

One current approach [9] covers computational thinking with biology students without programming, but is aimed specifically to biology majors at the junior/senior level, which precludes the possibility of adding a computing minor or cross pollination of ideas across disciplines. At Carnegie Mellon University [4], there is a course on computational thinking that does not include programming, but this course is designed for the general student population and does not focus on the areas specific to the sciences.
This course is unique in its effort to more broadly define computer literacy and fluency for the science student, and to focus on coverage of a variety of non-programming computing concepts that will enrich the science student's appreciation of computer technology as a valuable tool to be used in the creation of science.

3. WHAT IS IN THIS COURSE?

The Computing Foundations for the Scientist course has been developed at Olivet Nazarene University as the product of an NSF CCLI grant as a possible supporting course for most science majors. It has been designed to teach fundamental concepts of computing that are essential to scientists as they do their work. Details about the course can be found at the following URL.
http://cs.olivet.edu/twiki/bin/view/ComputationalScience/WebHome

This course has three major goals:
1. To help science majors understand the benefits and limitations of the technology they use
2. To equip science majors to better optimize the use of software in future courses
3. To help science students see connections between the many different sciences that they might not experience as they focus on their own particular major

To help the students understand the benefits and limits of technology, these are explained and demonstrated through the use of different science examples. For example, when XML is discussed, an XML document of the periodic table is used. To help them optimize the use of software, different software is used to solve different types of science problems, including simulation software, GIS software,

discipline specific websites, and spreadsheets. By using these (and other tools) to solve specific science problems, students see new and better ways to look at different problems. By taking specific, real life topics from many different science courses, students start to make connections between different sciences. Biology students can see that simulating population growth is similar in principle to simulating chemical reactions. Engineers can see that studying fluid dynamics is similar to some of the things geologists study.
This course was developed in a modular format with four modules explaining the background computing concepts and ten modules taking those concepts and showing how they impact different areas of science and math. They are structured in such a way that the ordering is flexible (as long as the prerequisite knowledge is covered first) and easily expandable with new computing modules and new science modules

The areas of computing covered by the different modules include: algorithm understanding, data accuracy and the source of errors, performance issues (including Big-O), reliability, simulation, visualization, abstraction, databases, data structures, and storage issues. These topics (and others) were chosen based upon their impact on the different science topics that were chosen. Other topics might be necessary if different topics where used.

The modules developed are as follows:

CSIS 1	Introduction to Computational Science
CSIS 2	Data types: Representation, Abstraction, and Limitations
CSIS 3	Procedures: Algorithms and Abstraction
CSIS 4	Self -Defining Data: Compression, XML, and Databases
BIOL 1	Bioinfomatics
BIOL 2	Cladograms
CHEM 1	Chemical Kinetics
CHEM 2	Molecular Modeling
ENGN 1	Design and Analysis with Engineering Spreadsheets
GEOL 1	Geographic Information Systems and Spacial Analysis
GEOL 2	Flow Analysis
MATH 1	Solving Equations
MATH 2	Curve Fitting
NSCI 1	Scientific Data Acquisition

Details on each of these modules (and future modules) and the course in general can be found at: URL: http://cf4s.olivet.edu.

The computer science modules were written by computer science professors. The biology, chemistry, engineering, and geology modules were written by biology, chemistry, engineering, and geology professors (respectively). The remaining modules were written by teams from different disciplines.

This course has been taught by one of the computer science professors using active learning techniques. Minimal time was spent lecturing (unless there was a complex issue that everyone needed help understanding). Instead the students (working in pairs) spent much of the class time reading the modules and working through the examples and questions embedded in each module.

Each module starts with an overview of the concepts being studied. It then gives a short lesson on a concept in that module followed by an on-line activity and questions. This repeats until the module is finished. At the end of the course, students present projects that they have developed on their own using some of the skills learned in the course. The individual module assignments, project, and exams combine to form the grade for the course.

4. WHY CONSIDER TEACHING THIS COURSE?

4.1 The Benefits We Experienced

There are four major benefits we experience from this course (besides additional benefits that came from the grant itself). The biggest benefit was a course that is designed to better meet the need of most SEM (science, engineering, and mathematics) majors. Many existing supporting courses fall into one of two categories: learning to program or learning to use specific software (such as MATLAB). While there are good reasons for these courses, most scientists do not write programs any more. They either have complex software that they can use or will work with a computing professional to develop very specialized software. In addition, being able to use software is not good enough for the scientists. They need to be able to understand the results, know if the results can be trusted (why or why not), and realize what technology can and cannot do for them. This course is better designed to meet these goals.

The second benefit that came from the course was experienced by the professors and students alike. Each of us learned a lot of science that was outside our disciplines. In addition, the SEM professors received a much better understanding of the underlying computing concepts and have been able to incorporate this knowledge into some of their advanced courses.

The third benefit was experienced by the computer science students that took this course. In many computer science courses, students learn the concept removed from the applications that use it. While we may discuss these areas in class, they have limited personal exposure to these areas. Computing students taking this course get a strong introduction into different ways computer science in used in the sciences. This has expanded their horizons to see additional areas that they might want to study and/or work in.

The fourth major benefit this course has brought to ONU is increasing the awareness that most students need a better understanding of technology. Being capable to use the computer for personal needs (such as word processing, socializing, research, email, etc.) is not sufficient in today's society. Most disciplines use technology in a very advanced way (and this is only going to increase). It is important for the users of technology to not treat it as a "black box" and accept the results without question. It is also important for them to know what technology is capable of and what is not. This course demonstrates such things to the SEM community on campus and is being used as a model when talking with other disciplines.

4.2 Additional Reasons

There are a number of additional reasons why a department might consider adopting a similar course. The first two reasons deal with costs. All the software used in this course is either free/public domain or already on the campus (Microsoft Excel). There is no need to invest in software. In addition, the modules themselves are free (to both the department and students). A department may take the modules and teach the course without much additional work. The students don't even need to buy a textbook!

One thing that might discourage someone from teaching such a course is limited expertise in the SEM areas. Being strong in the sciences is not necessary to teach this course. The modules are written so that sophomores in different areas can understand the science. So an engineer who has yet to study any biology can understand the biological modules. The biology student can read and understand the geology areas. The math students can following the chemistry covered. What is required is an understanding of the scientific method and the ability to use mathematics. (While there is some calculus in some of the sciences, students aren't even required to be able to do the calculus.)
A third reason to consider adopting such a course is that it is easy to expand. If an institution wants to expand the topics into another discipline (such as physics), this can be done easily. The existing computer science modules can be used (and more written if necessary) and a physics module can replace one of the science modules. If the institution has a researcher very involved in another area of chemistry and wants to include a module in that area, that researcher needs to just follow one of the existing templates to write a new one. As more and more institutions adopt such a course, there will be a wide variety of modules to choose from and to even vary the course from year to year.

The last reason to consider adopting such a course is probably the most important one. As one thinks about the future of computer literacy, we need to adapt from the old models as technology changes and how we use it. Very few people (other than computing professionals) write any major code. But when they use programs, there are some concepts that they need to be aware of. These concepts can be learned via programming (such as performance found in Big-O understanding) but can also be learned other ways. By focusing on the details of computing that are essential, time can be spent on helping the students apply them in areas that they will realistically encounter in their professional life.

5. WHAT NEXT?

There are four areas of additional work for this course. This first area will be done at ONU over the next 18 months. As the course is further refined, it will be taught again and additional formal assessment will be done. This will help us as we add modules (three are already planned), better connect the modules to each other, and find other ways to improve the overall experience.

The next area is to work with other institutions that might want to adopt this course. Such institutions need to be identified and resources. In addition, as other institutions use these modules, additional improvements will be identified.

As this work progresses and expands, additional science disciplines will contribute more modules and additional topics for existing disciplines will be added. More computer science modules will be added as needed. All new modules will be added to the repository.

The last area of future work is once again probably the most important. The philosophy of this course can be expanded beyond the sciences. Just as there are fundamental concepts about computing that are necessary for the scientists to understand so that they can do their work well, there are fundamental concepts that are essential for those in the health care industry, the artist, the communicator, the historian, and the educator (just to name the few). Similar courses can be developed for many different areas. The computer science modules can be used (and adapted to use different examples) as needed and additional ones can be used. The goal will be for all students to be able to learn the computing concepts that are important for them to succeed in their discipline. This need will only become greater as each discipline demands more and more complex technology to support them in their areas!

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7. ENDNOTE

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Euler and the Ongoing Search for Odd Perfect Numbers

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Abstract

Leonhard Euler, after proving that every even perfect number has the form given by Euclid, turned his attention to finding odd perfect numbers. Euler established a basic factorization pattern that every odd perfect number must have, and mathematicians have expanded upon this Eulerian form ever since. This talk will present a brief summary of Euler's result and some recent generalizations. It will also note connections between odd perfect numbers and the abundancy index (the abundancy index of a positive integer is the ratio of the sum of its positive divisors to itself). In particular, finding a positive integer with an abundancy index of 5/3 would finally produce that elusive odd perfect number.

1 Before Euler

"*As for me, I judge that one can find real odd perfect numbers.*"

– René Descartes

1.1 Euclid and Subsequent Conjectures

As noted in a number of historical accounts (including [1], [2], and [4]), the study of perfect numbers dates back even before the time of Euclid (ca. 300 B.C.). Euclid provided the following definition: A *perfect* number is equal to its parts. That is, a number is perfect if it equals the sum of its proper (aliquot) positive divisors. In addition, the first recorded result concerning perfect numbers is due to Euclid [5] (Book IX, Proposition 36).

Theorem 1. *If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.*

Using $n = 2, 3, 5$, and 7 in Euclid's formula produces the first four perfect numbers, well known to the ancients:

6, 28, 496, 8128, ...

Based on this short list, two conjectures about perfect numbers proved irresistible. Nicomachus (ca. 60-120) believed that perfect numbers alternate ending in 6 and 8, while Iamblichus (ca. 283-330) claimed that for each positive integer k, there is exactly one perfect number with k decimal digits [2]. Alas, both conjectures are false, as the next two perfect numbers are 33,550,336 and 8,589,869,056. On the other hand, a more intriguing question, also from Nicomachus [2], remains open to this day.

Conjecture 1. *Euclid's rule gives* all *perfect numbers; in particular, no odd number is perfect.*

In addition to his conjecture about the nonexistence of odd perfect numbers, Nicomachus, along with Theon of Smyrna (ca. 70-135), distinguished between deficient and abundant numbers [2]. In modern notation, given a positive integer n, we let $s(n)$ be the sum of the proper positive divisors of n. Then Euclid's definition translates into: n is perfect if $s(n) = n$. Similarly, n is *deficient* if $s(n) < n$, while n is *abundant* if $s(n) > n$. In particular, we note that every prime power is deficient, since $1+p+p^2+\cdots+p^{k-1} < p^k$ for any prime p and any positive integer k; hence there are infinitely many deficient numbers. The smallest abundant numbers are 12 and 18, which happen to be nontrivial multiples of the first perfect number, although the next largest abundant number (20) is not a multiple of any perfect number.

Motivated by such examples, and the fact that every abundant number smaller than 900 is even, Jordanus Nemorarius (1225-1260) made the following claims [2].

Conjecture 2. *(a) Every (nontrivial) multiple of a perfect number is abundant.*

(b) Every (nontrivial) divisor of a perfect number is deficient.

(c) No odd number is abundant.

In fact, the first two of these conjectures are true, but the third is false, since 945 is abundant. Since every multiple of an abundant number is also abundant, we may conclude that there are infinitely many abundant numbers, including infinitely many odd ones. This observation at least provides us with hope that there are odd perfect numbers, but we still have no proof or disproof of the existence of such a number.

1.2 Setting the Stage for Euler: Descartes and Frenicle

Interest in studying perfect numbers increased in the 1600s, with a particular focus on disproving the conjecture of Nicomachus. For example, René Descartes (1596-1650) believed that he could prove the following claims [2].

Conjecture 3. *(a) Euclid's rule gives all even perfect numbers.*

(b) Every odd perfect number has the form ps^2 with p prime.

We continue to follow Dickson's account in [2], examining the exchange between Descartes and one of his frequent correspondents, Bernard Frenicle de Bessy (1605-1675). Frenicle agreed with Descartes' conjecture about odd perfect numbers and further claimed that this prime p must be congruent to 1 mod 4. However, neither Descartes nor Frenicle offered a proof of either claim. To demonstrate his conviction that an odd perfect number exists, Descartes noted that if $p = 22{,}021$ were prime, then taking $s = 3 \cdot 7 \cdot 11 \cdot 13$ would produce the odd perfect number ps^2; alas, as Descartes knew, we have the factorization $22{,}021 = 19^2 \cdot 61$. Undaunted, Descartes wrote to Frenicle to suggest that replacing 7 or 11 or 13 in s might eventually work instead. Indeed, both Descartes and Frenicle found several examples of *even* multiperfect numbers – but no *odd* perfect numbers ...

2 Euler

"*Whether ... there are any odd perfect numbers is a most difficult question.*"

– Leonhard Euler

2.1 Completing the Work of Euclid on Even Perfect Numbers

As Dunham notes in [4], a letter from Christian Goldbach in 1729 may have initiated Leonhard Euler's work in the field of number theory, inspiring Euler to tackle the following claim by Pierre de Fermat.

Conjecture 4. *For each nonnegative integer n, $F_n = 2^{2^n} + 1$ is prime.*

This sequence of so-called Fermat numbers begins 3, 5, 17, 257, 65537, etc. In particular, F_n is indeed prime for $0 \leq n \leq 4$. But Euler showed the claim is false when $n = 5$, as 641 divides $F_5 = 4{,}294{,}967{,}297$. In fact, the reality about Fermat numbers could ultimately prove to be the exact opposite of the original conjecture, as no other primes have been found among F_n for $n \geq 5$ [1].

It is indeed an understatement to say that Euler went on to make many contributions to number theory. One example was the search for amicable numbers, a pair of positive integers m and n with $s(m) = n$ and $s(n) = m$. Only three pairs of amicable numbers were known before Euler, yet he proceeded to discover 59 new pairs [4]. Another example was Euler's introduction of the function ϕ to enumerate the positive integers not exceeding a given natural number n which are relatively prime to n. Euler applied this new function to generalize Fermat's Little Theorem, which states that if p is prime and $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$. Following is Euler's generalization of Fermat's result [1].

Theorem 2. *Given the positive integer n, if a is any integer with $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.*

In keeping with his usual creativity in applying new approaches to old problems, Euler reached a breakthrough in studying perfect numbers, a seemingly simple observation that nevertheless produced profound results. Instead of using $s(n)$ to sum the proper positive divisors of n, Euler introduced the notation $\int n$ to represent the sum of *all* positive divisors of n, including n itself. The modern notation for this function uses σ to replace $\int$, so that we now write $\sigma(n) = s(n) + n$.

Euler was able to show that σ is a multiplicative number theoretic function [1]; that is, if $\gcd(m, n) = 1$, then $\sigma(mn) = \sigma(m)\sigma(n)$. This property proved quite valuable for both computational and analytical use. For example, we may calculate $\sigma(360) = \sigma(2^3 \cdot 3^2 \cdot 5)$ as follows:

$$\begin{aligned}
\sigma(360) &= \sigma(2^3)\sigma(3^2)\sigma(5) \\
&= (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5) \\
&= (15)(13)(6) \\
&= 1170.
\end{aligned}$$

We note that if the product of the three factors in the second line is expanded via distribution, then the resulting 24 terms are precisely the positive divisors of 360.

More importantly, Euler used the multiplicative property of σ to prove the long-awaited converse of Euclid's theorem on even perfect numbers.

Theorem 3. *Euclid's rule gives all even perfect numbers.*

In particular, Euler made use of the ratio $\sigma(n)/n$, which we now call the *abundancy index* $I(n)$ of n. We will return to the abundancy index later, but for now, we simply note that n is perfect if and only if $I(n) = 2$; similarly, n is deficient when $I(n) < 2$ and is abundant when $I(n) > 2$.

Proof. We follow Euler's proof as outlined in [1] and [4]. If N is an even perfect number, then write $N = 2^{k-1}b$ with b odd and $k > 1$. Then $2N = \sigma(N) = 2^k b$, while $\sigma(N) = \sigma(2^{k-1})\sigma(b) = (2^k - 1)\sigma(b)$. Thus

$$\frac{\sigma(b)}{b} = \frac{2^k}{2^k - 1}.$$

Since $2^k - 1$ and 2^k are relatively prime, we know that $2^k - 1$ divides b. (At first, Euler seems to have believed that this observation was enough to conclude $b = 2^k - 1$, but he later corrected the gap in his proof as follows; see [2] and [16].) Then $b = (2^k - 1)a$ for some positive integer a, which implies $\sigma(b) = 2^k a$. But both a and b are positive divisors of b, so $2^k a = \sigma(b) \geq a + b = 2^k a$ and thus $\sigma(b) = a + b$. This yields $a = 1$, so $\sigma(b) = 1 + b$ and hence b is prime. $\square$

2.2 The Eulerian Form of an Odd Perfect Number

In addition to establishing that Euclid's rule produces all even perfect numbers, Euler was able to prove that every odd perfect number must have the following form [1].

Theorem 4. *If an odd perfect number exists, then it has the form $p^k s^2$, where p is prime, $\gcd(p, s) = 1$, and $p \equiv k \equiv 1 \pmod 4$.*

Before proceeding with Euler's proof, we pause to note that his result was not quite what Descartes and Frenicle had conjectured, as they believed that $k = 1$, but it came very close. In fact, current research continues in an effort to prove $k = 1$. For example, Dris has made progress in this direction, although his paper refers to Descartes' and Frenicle's claim (that $k = 1$) as Sorli's conjecture [3]; Dickson has documented Descartes's conjecture as occurring in a letter to Marin Mersenne in 1638, with Frenicle's subsequent observation occurring in 1657 [2].

Proof. We outline the proof of Euler given in [1], noting that once again, Euler employed the σ function in his argument. If N is an odd perfect number, then write $N = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ with each p_i an odd prime and each $k_i > 0$. Thus

$$2N = \sigma(p_1^{k_1})\sigma(p_2^{k_2}) \cdots \sigma(p_r^{k_r}).$$

Since $2N \equiv 2 \pmod 4$, exactly one factor $\sigma(p_j^{k_j})$ is congruent to 2 modulo 4, with the other factors all being odd. Also, since $\sigma(p_i^{k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{k_i}$, we may consider cases for p_i modulo 4:

(a) If $p_i \equiv 1 \pmod 4$, then $\sigma(p_i^{k_i}) \equiv k_i + 1 \pmod 4$. Since $\sigma(p_j^{k_j}) \equiv 2 \pmod 4$, we conclude $k_j \equiv 1 \pmod 4$. But every other $\sigma(p_i^{k_i})$ is odd, so k_i must be even for $i \neq j$.

(b) If $p_i \equiv 3 \equiv -1 \pmod 4$, then $\sigma(p_i^{k_i})$ is 0 modulo 4 if k_i is odd and is 1 modulo 4 if k_i is even, which immediately implies that $p_j \equiv 1 \pmod 4$. But 4 cannot divide $\sigma(p_i^{k_i})$, so k_i must be even for any prime p_i which is congruent to 3 modulo 4. □

With this theorem concerning the necessary form of any odd perfect number, Euler prepared the way for future mathematicians to refine his result and to continue progress toward a proof or disproof of the existence of odd perfect numbers. Just over one hundred years after Euler's death, another mathematician would indeed contribute significantly to the list of conditions needed for an odd perfect number.

3 After Euler

" ... a prolonged meditation on the subject has satisfied me that the existence of any one such [odd perfect number] – its escape, so to say, from the complex web of conditions which hem it in on all sides – would be little short of a miracle."

– James Joseph Sylvester

3.1 Sylvester's Web

In 1888, James Joseph Sylvester picked up where Euler left off, using the Eulerian form of an odd perfect number to establish a number of important results [6]. As Dickson [2] noted, in that year alone, Sylvester was able to prove:

- No odd perfect number is divisible by 105.
- An odd perfect number must have at least four distinct prime divisors. (Sylvester proved later that year that an odd perfect number must have at least five distinct prime divisors, and he conjectured that at least six were required.)
- If an odd perfect number is not divisible by 3, then it must have at least eight distinct prime divisors.

In addition to establishing this "web" of requirements for odd perfect numbers, Sylvester emphasized the importance of resolving such a question that dated back to ancient times. He referred to the issue as being a "problem of the ages comparable in difficulty to that which previously to the labors of Hermite and Lindemann environed the subject of the quadrature of the circle" [2].

Inspired by Sylvester's work, mathematicians have endeavored ever since to extend the web of conditions for odd perfect numbers. We outline a small sample of such conditions. For example, it took 37 years after Sylvester's claim before Gradstein proved that an odd perfect number must have at least six distinct prime divisors; that lower bound was subsequently improved to seven by Robbins and Pomerance (independently) in 1972 and to eight by Hagis in 1980 [19]. (Chein also proved the result for eight in 1979, but his dissertation was not published [6].) The current best known lower bound is nine, due to Nielsen in 2007 [12]. Nielsen also established that if an odd perfect number is not divisible by 3, then it has at least twelve distinct prime divisors [12].

Next, we follow the progress listed in [9] on finding lower bounds on the largest prime factor of an odd perfect number:

60 – Kanold, 1944

11,200 – Hagis and McDaniel, 1973

10^5 – Hagis and McDaniel, 1975

3×10^5 – Condict, 1978

5×10^5 – Brandstein, 1982

10^6 – Cohen and Hagis, 1998

10^7 – Jenkins, 2003

Since the publication of [9], the lower bound on the largest prime divisor of an odd perfect number has been improved, to 10^8 by Goto and Ohno in 2008 [7]. Iannucci has also shown that the second largest prime factor of an odd perfect number must be at least 10^4, improving on the previous results of 139 (Pomerance, 1975) and 10^3 (Hagis, 1981) [9]. In addition, Iannucci has proved that the third largest prime factor of an odd perfect number must be at least 100 [10].

We also note some of the congruence conditions that apply to odd perfect numbers. We have already encountered Euler's result that every odd perfect number must be congruent to 1 modulo 4. In 1953, Touchard established that an odd perfect number must be congruent to either 1 modulo 12 or 9 modulo 36 [18]. In 2008, Roberts refined this result, proving that every odd perfect number must be congruent to either 1 modulo 12, 117 modulo 468, or 81 modulo 324 [14].

Since Sylvester's time, mathematicians have woven more and more strands in the "web" which surrounds odd perfect numbers. Indeed, it would be quite difficult to list all of the additional conditions now known. Instead, we conclude this section by simply noting the recent contributions of Ochem and Rao: In 2012, they showed that an odd perfect number must be greater than 10^{1500}, must be the product of at least 101 (not necessarily distinct) prime factors, and must have a prime power divisor greater than 10^{62} [13].

3.2 Connections with the Abundancy Index

We recall that Euler applied the ratio $\sigma(n)/n$ in his proof that Euclid's rule gives all even perfect numbers. Using the modern notation and terminology $I(n) = \sigma(n)/n$ for the *abundancy index* of a positive integer n, we summarize in this section some of the connections between abundancy results and the search for odd perfect numbers.

To generalize an earlier definition, given an integer $k \geq 2$, we call the positive integer n *multiperfect* or *k-perfect* if $I(n) = k$. Descartes [2] established a number of results concerning multiperfect numbers, including the following connection between 3-perfect and 4-perfect numbers.

Theorem 5. *If n is 3-perfect and 3 does not divide n, then $3n$ is 4-perfect.*

Proof. We observe that since σ is a multiplicative function, so is I. Consequently, since 3 does not divide n, we have

$$I(3n) = I(3)I(n) = \tfrac{4}{3} \cdot 3 = 4.$$

$\square$

In 2000, Weiner [20] showed a remarkable connection between a specific abundancy index and the existence of an odd perfect number. We give his theorem and proof here, noting the similarity to the method used by Descartes in the previous result.

Theorem 6. *If there is a positive integer n with $I(n) = 5/3$, then $5n$ is an odd perfect number.*

Proof. We outline the key steps in Weiner's proof [20]. First, since $3\sigma(n) = 5n$, 3 must divide n. But then 2 cannot divide n, as otherwise, 6 would also, yielding the contradiction that n must be perfect or abundant. Hence n is odd, which implies that $\sigma(n)$ is also odd. In particular, this means that each prime factor of n must occur to an even power [1], so 3^2 must divide n. But then 5 cannot divide n, as otherwise, 45 would also, yielding the contradiction that $I(n) \geq I(45) = 26/15 > 5/3$. (Here, we have used the fact that $I(n) = \sum_{d|n}(1/d)$ (for example, see [1]), which in turn implies that if d is a positive divisor of n, then $I(d) \leq I(n)$.) Thus

$$I(5n) = I(5)I(n) = \tfrac{6}{5} \cdot \tfrac{5}{3} = 2.$$

$\square$

In 2003, Ryan [15] provided the following generalization of Weiner's theorem.

Theorem 7. *If there exist positive integers m and n such that m is odd, $2m-1$ is prime, $2m-1$ does not divide n, and $I(n) = (2m-1)/m$, then $n(2m-1)$ is an odd perfect number.*

Ryan also showed in [15] that if m is even but not a power of 2, then $I(n) = (2m-1)/m$ has no solution. Three years later, Holdener [8] proved another generalization of both Weiner's and Ryan's theorems, giving the following necessary and sufficient condition for the existence of an odd perfect number.

Theorem 8. *There is an odd perfect number if and only if there are positive integers p, n, and k such that p is prime, p does not divide n, $p \equiv k \equiv 1 \pmod 4$, and*

$$I(n) = \frac{2p^k(p-1)}{p^{k+1}-1}.$$

We note the direct connection between Holdener's result and the Eulerian form of an odd perfect number: If such positive integers p, n, and k do exist, then $p^k n$ is an odd perfect number, since

$$I(p^k n) = I(p^k)I(n) = \frac{p^{k+1}-1}{p^k(p-1)} \cdot \frac{2p^k(p-1)}{p^{k+1}-1} = 2.$$

In particular, n would equal s^2 in Euler's result.

In the opposite direction, since the non-existence of such a value of $I(n)$ would prove that no odd perfect numbers exist, Holdener and Stanton [17] have studied these rational numbers outside the range of I and have given them a descriptive name: Given a rational number $r > 1$, r is an *abundancy outlaw* if $I(n) \neq r$ for every positive integer n. Applying Holdener's theorem, we note for example that if 5/3, 13/7, or 17/9 is *not* an abundancy outlaw, then an odd perfect number exists. The "outlaw" status of these three rational numbers, along with others of the form in Holdener's result, remains unknown [17].

Such observations raise the natural question of the distribution of abundancy indices vs. the distribution of abundancy outlaws in $[1, \infty)$. On the one hand, Laatsch [11] proved in 1986 that $\{I(n) : n \in \mathbb{N}\}$ is dense in $[1, \infty)$, which raises the hope that perhaps 5/3 is in fact an abundancy index. On the other hand, Weiner [20] showed in 2000 that the set of abundancy outlaws is also dense in $[1, \infty)$.

While the question of the existence of odd perfect numbers remains open, we may apply Laatsch's result about the density of abundancy indices to create some interesting examples. In particular, we can make $I(n)$ arbitrarily close to certain "famous" numbers:

- $I(3 \cdot 17 \cdot 577 \cdot 665857) = 1.414213562371\ldots$
- $I(2 \cdot 3 \cdot 5 \cdot 11 \cdot 29 \cdot 277 \cdot 67927 \cdot 204109349) = 2.71828182845903\ldots$
- $I(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 163 \cdot 23509 \cdot 690120229) = 3.141592653589792\ldots$

In addition, we are able to make $I(n)$ arbitrarily close to 2 for odd n:

- $I(3 \cdot 5 \cdot 7 \cdot 11) = 1.9948\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 389) = 1.99993\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 383) = 2.00001\ldots$
- $I(3 \cdot 5 \cdot 7 \cdot 11 \cdot 389 \cdot 29959) = 1.99999998\ldots$

But the main question still remains, handed down to us by Euclid, Descartes, Euler, Sylvester, and many others: Can we make $I(n)$ *equal* to 2 for odd n???

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4.2 Helpful Web Sites

1. MacTutor History of Mathematics, J. J. O'Connor and E. F. Robertson:

   http://www-history.mcs.st-and.ac.uk/HistTopics/Perfect_numbers.html

2. OddPerfect.org:

   http://www.oddperfect.org

3. Wolfram/MathWorld, C. Greathouse and E. Weisstein:

   http://mathworld.wolfram.com/OddPerfectNumber.html

Explore Global Opportunities for Mathematics Scholarship, Teaching, and Service

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Many members of ACMS have more extensive overseas experience than I have and are more qualified to address the potential of mathematical global involvements on both personal and professional levels. These people are really not my primary intended audience. For those of you who have little or no overseas professional experience, the goal of this paper is two-fold: 1) to stimulate your thinking about international opportunities and how they might pertain to your professional interests and expertise, and 2) to provide some ideas about how you might get started.

The realization that I could use my mathematical knowledge and experience in an overseas setting came relatively late in my professional career. Even from childhood, I have always enjoyed the rewards of travel. But, my journeys were for personal enjoyment and enrichment. It was only about six years ago that I began to connect my professional and personal development goals in a global context. This brief paper will describe some of my recent global experiences within the framework of the three primary domains of academic work: scholarship, teaching, and service. Although we often distinguish these areas of our work, I have learned that in many global experiences, there is often much overlap among the three. Because there is neither time nor space to provide a detailed description of each trip, I would like to provide a brief overview of each of the three types of global experiences in which I have recently participated.

Teacher Mentoring (Teaching and Service)

My first experience in overseas teaching and service occurred in the summer of 2007 when I joined a Teacher Mentoring Team to Liberia, West Africa through Hope Corp, a ministry arm of World Hope International. WHI describes itself as "a faith-based relief and development organization alleviating suffering and injustice through education, enterprise and community health." The organization provides a variety of services to developing countries. Such services include anti-trafficking programs, education, child sponsorship, HIV/AIDS prevention and treatment, water projects/well drilling, microfinance loans, and rural development. The Hope Corps Teacher Mentoring program provides professional development workshops for teachers, primarily those teaching in Christian schools. I first learned about the organization through my local church and denomination which has a working relationship with World Hope. I became even more interested when a friend participated, as a science teacher, in a Mentoring Team to Sierra Leone in 2005.

My team consisted of 5 teachers, four from the U.S. and one from Germany, and a college-aged photographer. In a week-long workshop, we provided instruction in teaching reading, math science, and in child psychology and in biblically-based classroom management strategies. Approximately 85 teachers, from PreK – 9th grade, attended the classes held in an elementary school and church near Monrovia. The instruction in math encompassed both content knowledge and teaching methods because both areas were of critical need to the teachers. Most of them had no or minimal teacher training and also lacked much depth of understanding in mathematics. They had been taught math via rote learning and memorization and that was the only model of instruction with which they were familiar.

I prepared booklets of instruction and activities for 4 levels of classes, based on the five NCTM content standards (*Principles and Standards for School Mathematics, 2000*): number and operations, algebra, geometry, measurement, data analysis/probability. During the workshops, we worked through many hands-on activities to illustrate the meaningfulness, relationships, and logical nature of mathematics.

Two years later, in 2009, I joined another Mentoring Team to Haiti. Approximately 115 Haitian teachers attend the workshops held on the grounds of a seminary near Port au Prince. These workshops were partially funded by a grant from USAID for HIV/AIDS prevention and treatment. The organization of this workshop was similar to the one in Liberia with teachers receiving instruction in math science, and language arts content and pedagogy.

My experiences in teaching math at K-12 levels, before joining higher education, were great assets in these two overseas teaching/service experiences. Even so, there were many challenges that probably apply to most developing countries. These include meager facilities, lack of desired teaching/learning resources, climate differences, culture and language issues (we did had translators for Creole in Haiti), and the wide range of teachers' backgrounds, experience, and knowledge. For example, some participants had many years of teaching experience while others were just beginning their teaching career. Those of you who have been on any kind of "missions trip" to a developing country will certainly understand many of these obstacles and probably be able to add your own list of "challenges" to mine.

Despite the challenges, these kinds of opportunities are tremendously enriching as well. These overseas experiences have expanded my creativity, challenged my flexibility, enlarged my perspectives, and strengthened my faith. When you don't have the teaching materials to which you are accustomed, you learn how to make do with what you do have and explain things in different ways than you've ever done before! When your schedule keeps being altered and your plans are always tentative, you learn to be more flexible! When you encounter new ways of looking at customs, resources, people, and life in general, your perspectives are forever enlarged! When you have the opportunity to join other believers in worshiping in diverse cultural settings and worship styles and to see God working in amazing ways around the world, your faith in the power and grace of God is confirmed and tremendously strengthened!

International teaching has provided a new "reference point" for many of my subsequent personal and professional experiences. I find that I have much more patience with the occasional inconveniences of everyday life in the United States. Things like the lack of air conditioning or modern facilities or resources, or the appearance of potholes in my street, or crowded conditions on a bus, or delayed and cancelled flights have lost much of their "power of annoyance" over me. I merely think back to more challenging circumstances I've encountered overseas and realize that "I can handle this."

I now believe that the process of mentoring teachers in a developing country can be thought of as "service learning for the mentors." Assisting in the training of inadequately prepared teachers/mentees is indeed a service with far-reaching effects for the teachers, students, and community development. But it also includes a lot of learning for the mentors as well. There is no way to describe or identify everything I have learned through these interactive relationships with teachers in developing countries. These are great opportunities to serve and encourage others with whom you share common goals. I believe it is a "win-win" outcome for everyone.

I have talked about one organization (World Hope) because that has been my experience. However, there are a number of similar humanitarian organizations (some with a Christian foundation) that you might want to investigate. Perhaps some are affiliated with your church or other ministries with which you are already acquainted. The last page of this paper provides a partial listing of some organizations you might want to investigate for overseas opportunities.

MAA Study Tour (Scholarship)

In the spring of 2011, I joined with about 24 others to participate in the MAA Study Tour that year. It was entitled "Mathematics among the Ancient and Modern Maya" and included visits to Guatemala and Honduras. This well-organized and event-packed tour included explorations of ancient Mayan city ruins, various museums, and nightly lectures and discussions led by our archaeologist guide to investigate the ancient Maya civilization. We learned how to interpret Mayan glyphs of numeration, calendars, and historical events. This is information about the history of mathematics that I can readily incorporate into some my current college courses.

Just as interesting to me were our encounters with the current Maya (indigenous) people of Central America. As we visited numerous scenic, historical, and cultural sites, we learned how the modern Maya have both preserved their subculture but also been influenced by the larger "Spanish" culture around it. This was a time of heightened interest in studying Maya history and culture since many people were wondering then if the world would come to an end in December 2012, the end of the Mayan long-count calendar (it didn't).

As our group engaged in myriad topics on our study tour, I began to notice how multi-disciplinary our study had become. So, I started writing down some of the areas that were being addressed: religion, astronomy, mathematics, geography, archaeology, cultural anthropology, history, sociology, philosophy, political science, economics, geology, biology, chemistry, ecology, physics, music, art, linguistics, archeoastrononomy…. It was definitely the most interdisciplinary ten days of my life!

Depending on the destination, some trips may be somewhat expensive. Because of the international and academic nature of the trip, I did receive significant financial assistance from my university. Previous MAA Study Tour destinations, beginning in 2003, have included Greece, England, Mexico, Euler (Germany/Russia), Peru and Galapagos Islands, Egypt, and Italy. For more information on past and future tours, see: www.maa.org/StudyTour

Fulbright Specialists Program (Service and Teaching)

In the summer of 2010, I applied to the Fulbright Specialist Program with hopes that I could include a short-term international experience as part of my sabbatical during the spring of 2011. Because of various timing issues and the number of steps necessary to finalize all arrangements, that did not work out. However, I was accepted to the Fulbright Specialist roster in January 2011 and completed a two-visit experience in Ecuador in 2012 and 2013.

The Fulbright Specialist program awards grants to support short-term (2-6 weeks) overseas experiences for qualified U.S. faculty and professionals in select disciplines iin over 100 countries. It promotes linkages between U. S. academics and professionals and their counterparts at host institutions overseas. Approved projects focus on strungthening and supporting the development needs of the host institutions abroad and is not intended for personal research purposes. Eligible activities include short-term lecturing, conducting seminars, teacher training, special conferences or workshops, collaborating on curriclum planning, institutional and/or faculty development.

Here is an overview of how the program works. U.S. faculty and professionals apply to join a Roster of Specialists for a five-year term. Roster candidates are reviewed by peers in the same discipline and by the J. William Fulbright Foreign Scholarship Board. Eligible institutions wanting to host a Fulbright Specialist submit project applications through the Fulbright Commissions or U.S. Embassies in their home country. Projects are reviewed and approved by the Fulbright office in their home country and the U.S. Department of State.

In essence, the Fulbright program serves as a clearinghouse or "matchmaker" between U.S. specialists and overseas project applications. However, in practice, it appears that many or most specialists secure their own placement by contacting the overseas institution first and coordinating with them to structure the host's project description to match the qualifications and interests of the specialist. That was the situation in my case.

My project involved working with the University of Azuay (UdA) in Cuenca Ecuador. I chose to work at this particular university because of prior relationships between UdA and some of my colleagues at Taylor. But if you already know someone associated with an overseas institution, that might be a good place for your initial contact.

My work at UdA including the following activities: presentations to math faculty on research-based instructional strategies, formation of a professional development model for mathematics instructors, consultation/advising on the creation of a mathematics department and a master's degree program, course curriculum evaluation, and conducting workshops for math teachers at an affiliated high school.

In addition to my academic work, Ecuador was a wonderful place to visit. I was able to live with a gracious host family, worship in several churches, fellowship with missionary friends, visit Cajas National Park and Ingapirca (Inca ruins), and participate in many other local cultural events. It was a great opportunity to build new personal and professional relationships that, I hope, will continue.

Another benefit from the Fulbright Specialist Program is that Fulbright pays your travel expenses to the host country and the host institution is responsible for your in-country expenses, including transportation, housing, and food. In addition, Fulbright provides an honorarium of $200 per day for each day of the trip (including travel days).

See this website address for more information on the program:
http://www.cies.org/specialists/

A Word of Advice in Getting Started

If you have not traveled abroad professionally, a missions-type trip (e.g. teacher mentoring or other work) can be a good place to begin, if you have the necessary qualifications. Study Tours are also great ways to begin because they are highly structured and organized, leaving little for the participant to worry about. No previous experience is necessary; just jump in and enjoy the trip! I highly recommend the Fulbright Specialist Program but you may want to wait to apply for it until you have some previous overseas experiences to strengthen your qualifications. The following page provides some links to organizations that might have international programs that would be a good fit for your interests.

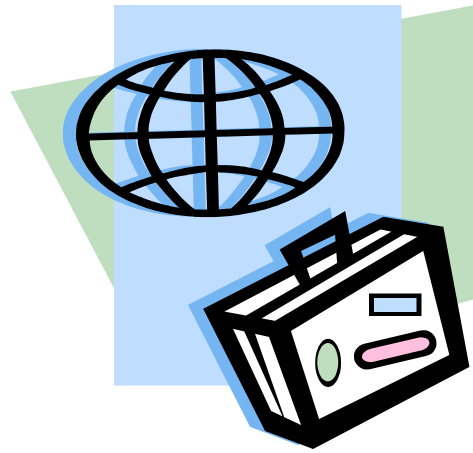
Conclusion

I want to close with two quotes that I love and that express important ideas that I have found to be true in my global experiences. The first comes from Miriam Beard, "*Travel is more than the seeing of sights; it is a change that goes on, deep and permanent, in the ideas of living*."

The second is from the late Rev. Bruce Larson, "*Every Christian is launched on a life of experiment, discovery, and faith from which he can report on new ways that God may be working in specific situations.*" I think that this perspective is especially true for those of us teaching in the sciences in Christian higher education. This life of "experiment, discovery, and faith" can be lived out anywhere, but I believe that international experiences multiply our opportunities to discover and participate in the many amazing things God is doing around the world.

Opportunities for Overseas Experiences in Mathematics

ACMS Biennial Conference: Bethel University
May 30, 2013



Beyond Borders
www.beyondborders.net

Cross Cultural Solutions
www.crossculturalsolutions.org/Choosing_your_program

Fulbright Specialists Program
http://www.cies.org/specialists/

Global Links—Global Educators Program
www.global-links.org

Helps International (Guatemala)
www.helpsintl.org/programs/education

International Teacher Training Organization—Teach English Overseas
www.tefl-tesl.com

Mathematical Association of America Study Tours
www.maa.org/StudyTour

Project Teach Haiti
www.project-teach-haiti.org

Study Abroad—Teach Abroad Programs
www.studyabroad.com/teach

Teach Abroad
www.teachabroad.com

Teachers Without Borders
www.teacherswithoutborders.org

United Planet
www.unitedplanet.org

World Hope International
www.worldhope.org/hopecorps/shortterm.htm

World Teach
www.worldteach.org

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PHILOSOPHY OF "SPINNING WHEELS"

LOREDANA CIURDARIU

Abstract. In this material I will speak about some well-known mechanisms studied by students and engineers emphasizing the impact which "spinning wheels" had and have in development of the society, on Christians and the church. Also the discovery of these machineries determines major changes in the people's outlook and leads to new trends in philosophy and Christianity. Then, I will give some examples from the Bible where "spinning wheels" it seems to appear: Judges 16:21, Ezekiel 1 and Revelation. It is also interesting to see 2 Kings 2:9-12, 2 Kings 6:13-18 and maybe Daniel 7:9.

In addition, an avi file where we can notice the movement of the "spinning wheel"(a crank-slider mechanism for example) will be attached.

1. About the history of the "Spinning wheels"

My name is the "spinning wheel" and after a lot of thinking I decided to tell you something about my travel in the world and the lived experiences. The wisdom of God was with Him when the universe was created and even since then the wisdom of God planned me to exist and to be used. I existed for a very long time in the imagination of people and it came a time when I was the means through which I fulfilled some dreams of them. However as almost all the discoveries which bring progress to the mankind, I knew that will come people who will search and shall find all the bad uses for these discoveries and like that I stayed to think and I let the wisdom of God to let me to light.

Even from the oldest times I had seen that people struggled hard in their endeavor to work the land. They started to think at me, especially those from Egypt who needed that the water during overflow of Nile to wet a large surface of land and to obtain better crops in order to be able to feed the families after which they paid their fees to the authority. Among the most ingenious devices designed to perform the most various operations can be listed the devices of throwing boulders used to the siege of citadels, different leveraging scales, different mill devices, different mining devices, lifting devices. Because the need was more and more astringent and God gave me the acceptance, I started to let them to discover me little by little after their power of understanding. Nevertheless some say that I was used first by those who built the suspended gardens. There was a man of science and Greek philosopher from Syracuse named Archimedes of whose research was focused on geometry, arithmetic and mechanics who began to think more and more to me and to the advantages that I could offer him. His originality consists precisely of the fact to consider the mathematics not only as a theoretical science as Pythagoreans, but also as a applicable practical science in the design of machines and installations. His inventions in the war engineering field and hydraulic engineering field remained famous such as the screw of Archimedes, the device hooking ships as well as other fight weapons used successfully in the war of Greeks against the Romans in a certain period.

The philosophy of the time, was inspired to a great extent from the Greek philosophy and mainly by the philosophy of Aristotle. Then appeared Christian thinkers such as Augustine, Thomas Aquinas and other great theologians of the church. "The wheels" could not develop, the mathematics was considered as a queen of sciences by the Greek philosophers and the applied sciences did not have time to develop. "Scientific knowledge, according to the Aristotelians, was concerned with establishing true and necessary cause of things", see [14].

And there passed times and times over people, their life was still very hard. There were periods of long wars, many devastating diseases, earthquakes, terrible winters and other natural calamities. Thousands of "spinning wheels" were used and destroyed in awful wars. And thus people became more bad and selfish. The "spinning wheel" was thinking about people but knew that they cannot think of it. Certainly these were the preparing conditions for the religious reform that would follow. And indeed it followed. After the period of crusades and of other conquests, followed the period of establishing the universities, first set up near churches and monasteries to study theology, then the period of Renaissance and religious reform and then the period of scientific revolution, all these forming the prerequisites necessary for trigging the industrial revolution.

Among the most older and popular inventions of people is numbered also the clock arose from their need to measure periods of time shorter than day, month and year. According to [6], between 1280 and 1320 there was an increasingly higher number of references to clocks and horologes in the registers of churches and this probably indicated the fact that a new type of mechanism for the clock was produces, the mechanical clock. "The former purpose of this device is administrative, the latter arises naturally given the scholarly interest in astronomy, science, astrology, and how these subjects integrated with the religious philosophy of the time", see [6]. Simple clocks would have announces intervals between set times of prayer. Also remained detailed descriptions of the clocks built by Richard of Wallingrad in St Albans by 1336 and by Giovanni de Dondi in Padua from 1348 to 1364, but also from other continents, in China there were concerns regarding the astronomic clock and also in the Muslim world. However The Salisbury Cathedral clock, built in 1386, is considered to be the world's oldest surviving mechanical clock that strikes the hours, according to [6], and one of the first pocket watches, called "Nuremberg Egg", made around 1510 and attributed to Peter Henlein, (Germanisches Nationalmuseum, Nuremberg). The theory of the mechanical clock had been translated into practical constructions very quickly, and the "spinning wheel" had seen as them in their desire to study the celestial phenomena and the sky of God, the astronomers were among the first who were interested by the improvement of the astronomic clocks. Clockmakers built smaller and smaller clocks and that became a technical challenge for them, as well as improving accuracy and reliability of these mechanisms. In the metalworking towns of Nuremberg and Augsburg, and in France, clockmaking flourished in the 15th and 16th centuries.

The peoples mentality changed, there were some fundamental changes in the church and in the society and there were created the premises necessary for the scientific revolution. One of the most important figure of scientific revolution , Galileo Galilei is the first who had successfully an independent academic course on Mechanics of Machinery at the University of Padua between 1597-1598 being aware by their importance. And thus the "spinning wheels" won the first battle, beginning to be popularized among the scholars gaining their independence towards the fundamental sciences as mathematics and physics. Some mathematicians thought at the "spinning wheels" in their way and appeared such an example, Leibniz wheel used in the arithomenter, the first mass-produced mechanical calculator. Another great mathematician who dealt with the study of mechanisms was L. Euler who was also a Christian, see his life. The next development in accuracy of the clock occurred after 1656 with the invention of the pendulum clock. According to [6], Galileo had the idea to use a swinging bob to regulate the motion of the a time-telling device earlier in the 17th century. Christian Huygens, however, is usually credited as the inventor, he had the first pendulum-driven clock made (to see the implications of the discovery of this mechanism in philosophy).

It was known for a long time that it can be built a steam car but its efficiency is due to the discovery of James Watt of 1775, who improved the steam car of Thomas Newcomen. There were some major discoveries in that period such as the discovery of some method to produce high-grade iron in a blast furnace fueled by coke rather than charcoal which was major step further in the production of iron as raw material for the Industrial Revolution. These created the conditions which led to the improvement of the steam car used in the textile industry. Because even if there were many who were interested by such mechanisms and related mechanisms, like sisters, brothers, uncles, aunts and other relatives of mine, none of them did not paid attention to what I really wanted to say, each of them used his invention in other directions than the one I wanted to suggest. So I had seen like James Watt who did not think too much at me "the spinning wheel" was the one who understood what I wanted to tell them, being also the most appropriate person for what he had to do having more features that I have. It was necessary to find by turn two persons to understand his inventions and to support financially his discoveries, the first being Matthew Boulton. James Watt, by his nature a practician not a theorist, was also the inventor and holder of patent of the steam locomotive and thus England became the first country in which began the Industrial Revolution. He was the one who introduced the name of horse power and after his name was called the kilowatt measurement unit. The steam machine required a so called "straight/line mechanism" that is a mechanism for the conversion of reciprocating motion (of the input piston) to rotary motion (of the output crank), according to

([1]). If until then we were a few families, a few relatives, since then we multiplied, we became colonies after the evolution, we specialized and appeared our classifications and eventually a whole nation was born and is born in the factories built by simple people for us. We started to work together, to coordinate our movements being more specialized in the performance of different operations it was established our hierarchy depending on the importance of the work performed, our sizes began to vary from very small to huge. It was a great step the world change after which understood the utility of these discoveries during the Industrial Revolution and so it began the reign and euphoria of "the spinning wheels". All the enthusiasm of that period resulted after the previous revolutions. Logically these had to succeed only in the order in which succeeded and mainly first were raised the universities which were set up and supported at the beginning by churches for the study of theology then came the period of Renaissance than the Scientific Reform and Revolution and at last the Industrial Revolution. However the evangelical awakening of the 8th century which began by J. Wesley and G. Whitefield took place in almost the same period when the Industrial Revolution began, England was still a country more isolated than the others from the European continent, and the English were the masters of seas. People must have a certain tranquility, the necessary motivation and favorable conditions to may think to such things, they had to be in a state of grace and to be peace. The industrial revolution started to have effect also in countries as France but especially then in Italy and Germany and later in countries which were English colonies. Besides the steam locomotive (1803), other great discoveries from the technique of the 9th century were also the steam ship (1807), the first electric engines appeared (1837), then the diesel engines, and the benefits of these discoveries began to reflect slowly on almost all the fields.

And "the spinning wheels" thought to another very popular invention, to a more fast means of transportation to maintain him in shape and to be within the reach of each man, the dandy horse, also called Draisienne or laufmaschine, which pertained more to entertainment for the beginning and proved its utility a little later. The discovery belongs to the German baron Karl von Drais and was the first human means of transport to use only two wheels in tandem. "It is regarded as the forerunner of the modern bicycle and was introduced by Drais to the public in Mannheim in summer 1817 and in Paris in 1818", according to [7]. There were many people who were concerned to improving this mechanism, one says that the first mechanically-propelled 2-wheel vehicle may have been built by Kirkpatrick MacMillan a Scottish blacksmith, in 1839 then in 1860s, Frenchmen Pierre Michaux and Pierre Lallement added a mechanical crank drive with pedals on an enlarged front wheel (the velocipede) and finally Starley's 1885 Rover, manufactured in Coventry, England, is usually described as the first recognizably modern bicycle.

Even the Russian mathematician Pafnuty Chebyshev, specialized in the theory of approximation was concerned with the study of mechanisms and used his theory for example "to find the optimal lengths of a four-bar linkage a (crank or rocker), b (coupler), c (rocker or crank), d (fixed link) where the centre point M of the coupler b executes an approximately linear path as long as possible," according to [1]. In their papers, the authors ([1]) showed that "Chebysev found a double rocker linkage (a=c) with rotating coupler and the relative dimensions a:b:c:d=5:2:5:4 and in this case the point M follows a straight line approximately with a relative deviation of 2/1000 from the exact line, length of this line being equal to the length of the fixed link." And it is known to a greater or smaller extent also the ideas of the personalities in their turn lead to shaping the philosophy ideas of the time.

It is time that "the people of spinning wheels new born" to cross all the earth, to cross more faster the oceans and to conquer the space, but then the people's curiosity did not stop here but "the spinning wheels" began their travel to other planets and stars, thus fulfilling the deep desire of man to know from what it is made the universe. The engine became the central part of any machinery, even the name of this job, "engineer" was formed from this word, "engine". "The people of the spinning wheels" held a council and a segment of it felt the call and decided to build and improve the locomotive, the car and all what is traveling on the land. Another part of it which loved the stretches of water felt attracted by the machineries which cross the seas and oceans. The last part of the people had another dream, that of flying and tried to build machineries which fly. As almost every time when it was talking about a new invention, "the spinning wheels" put in the heart of many people from more countries, from various social layers the idea to build a vehicle with engine which can be used by each person. Each put his footprint over the discovery made, these were improved in time but the most useful and easy to made, the one for which were found the necessary means of completion and the most profitable was the one which survived.

Nicolas-Joseph Cugnot is considered to have built a steam-powered tricycle that is the first full-scale, self-propelled mechanical vehicle or automobile in 1769, then in 1801, Richard Trevithick built and demonstrated his

Puffing Devil road locomotive, and then in 1807 Nicphore Nipce and his brother Claude probably created the world's first internal combustion engine which they called a Pyrolophore, installed in a boat on the river Saone in France. In this period, others, like Samuel Brown, Samuel Morey, and Etienne Lenoir with his hippomobile, each produced vehicles powered by clumsy internal combustion engines. There were several other German engineers like Gottlieb Daimler, Wilhelm Maybach, and Siegfried Marcus who working on the problem at about the same time, but Karl Benz generally is acknowledged as the inventor of the modern automobile, according to [8]. In 1879, he was granted a patent for his first engine, which was designed in 1878, and his first Motorwagen was built in 1885 in Mannheim, Germany. The first motor car in central Europe was produced by Czech company Nesselsdorfer Wagenbau in 1897. It is known that Daimler and Maybach founded Daimler Motoren Gesellschaft (DMG) in Cannstatt in 1890, but Benz, Maybach and the Daimler team seem to have been unaware of each other's early work. In 1890, Emile Levassor and Armand Peugeot of France began the automobile industry in France. In Britain, there had been several attempts to build steam cars had more or less success. In 1897, German engineer Rudolf Diesel, built the first Diesel Engine, according to [8]. Mass production began in 1914 in U.S., by Henry Ford's cars which came off the line in fifteen-minute intervals, much faster than previous methods, increasing productivity eightfold, while using less manpower, according to [8]. Combination of high wages and high efficiency used by H Ford was copied by most major industries. There were thousands of small manufacturers who competed to obtain the world's attention and this led to a very fast development of the vehicle industry. The improvement of the automobile did not end not even today, the competition between the car manufacturers being yet fierce. When "the spinning wheels" quarrel they produce a deafening noise (see intersections, highways) so it is preferable not to be provoked.

Between 1867 and 1896 the German pioneer of human aviation Otto Lilienthal developed heavier-than-air flight being the first person to make well-documented, repeated, successful gliding flights, according to [9]. The Wright brothers flights in 1903 was considered as the first sustained and controlled heavier-than-air powered flight. They said that Otto Lilienthal was a major source of inspiration for their decision to pursue manned flight and in 1906, Alberto Santos Dumont made what was claimed to be the first airplane flight unassisted by catapult. We had also two pioneers of human aviation in our country, T. Vuia and A. Vlaicu(1910).

Airplanes were tested as weapons in World War I and demonstrated their potential as mobile observation platforms, then proved themselves to be machines of war. Airplanes had a presence in all the major battles of World War II.

During the American Civil War, both sides successfully built working submarines. The second documented Confederate submarine was the American Diver, built in Mobile, Alabama. It was initially designed to be propelled by an electric motor but this proved too weak. A steam engine was installed next, but also was insufficient.

First air independent and combustion powered submarine was the Ictineo II, designed by Narcs Monturiol. Launched in Barcelona in 1864, it was originally human-powered, but in 1867 Monturiol invented an air independent engine to power it underwater. The first mass-produced submarine was a human-powered vessel designed by the Polish inventor Stefan Drzewiecki. In 1884 Drzewiecki built the first electric-powered submarine. Discussions between the English clergyman and inventor George Garrett and the Swedish industrialist Thorsten Nordenfelt led to a series of steam-powered submarines, the first being the Nordenfelt I. Peral was an all-electrical powered submarine, launched by the Spanish Navy. The other was the Gymnote, launched by the French Navy which was also an electrically powered and fully functional military submarine, according to [10].

In 1896, the Irish inventor John Philip Holland designed submarines that, for the first time, made use of internal combustion engines on the surface and electric battery power submerged. Holland VI was launched on May 17, 1897, at Navy Lt. Lewis Nixon's Crescent Shipyard of Elizabeth, New Jersey and then in 1900, the United States Navy purchased the revolutionary Holland VI and renamed it USS Holland (SS-1), America's first commissioned submarine. The submarines were also used in the Russo-Japanese War at the end of the 19th century, in World War I and in World War II. So it can be seen that many countries were interested by such an invention, countries such Russia, Turkey, Spain, Greece, Peru, France, America, Japan and were concerned over time to bring improvements, submarine being used a lot in wars and espionage and of sinking the enemy ships. Therefore this invention had a strategic importance for many countries showing the power "spinning wheels". Submarines can also be modified to perform more specialized functions such as search-and-rescue missions or undersea cable repair and can be used for undersea archaeology and can work at greater depths than are survivable or practical for human divers.

The term robot (from Czech robot) was used by Josef Capek and Karel Capek in their works of science fiction at the beginning of 20th century which described in his work since 1921 workers of human resemblance, which are grown in tanks. The image of humanoid robots took shape in literature, especially in the novels of Isaac Asimov in 1940's. Since 2000 the basic problems seem to be solved together with the apparition of ASIMO (Honda). So "the spinning wheels" seeing how far brought the imagination of people compared to what could be done at the respective moment and how concerned they are with the machineries, decided to appeal as many times in the past and to the help offered by other disciples and to construct them. The robots are made especially by the combination of the disciplines: mechanics, electrical engineering and informatics. In the meantime from their connection it was created a new discipline, mechatronics. For the achievement of autonomous systems is necessary the connection of more disciplines of robotics. Here is highlighted the relation of the concepts of artificial intelligence or neuroinformatics as well as their biological ideal biocybernetics. The actual industrial robots are not usually mobile, their operational field is limited. The industrial robots were used for the first time in Germany at welding works starting from 1970, according to [12]. The domestic robot works autonomous in the household. There are autonomous robots and the teleguided robots, the robots being divided in more categories like, mobile autonomous robot, humanoid robot, industrial robot, service robot, toy robot, explorer robot, walking robot and military robot, according to [12]. The explorer robots are robots which operate in places hard accessible and dangerous teleguided or partially autonomous. The robots are very useful because they can trace and defuse or destruct bombs or mines and there exists also robots which help for the search of people buried after earthquakes. These can work for example in a region being in a military conflict, on the Moon or Mars. Cryobots can be used in the research of polar capes on Mars and Europe.

And thus, "the spinning wheels" from the robots already reached the planet Mars before man. It is interesting to study how were reported and are reported all these inventors(see for example life of L. Euler and G. Galilei) to the local church how they were integrated in it, at the Bible, their philosophical conceptions, if they had, if they were believers or not, how faithful they were if they were afraid of God and which were their conceptions about divinity, world and life. Nowadays a dream of the "spinning wheels" is however to imitate as much its nature to copy it even if at their beginning among what has made them interesting were also some features which differentiated the mechanisms from the organisms (for example the sensitivity to pain). Today from the relation between biology and technique was developed the bionics.

2. Mechanistic philosophy (What say the wise men about "the spinning wheels")

The mechanism in general philosophical meaning is a conception which explained the reality understood almost univocally in material meaning, only by the movement between bodies. The mechanism consider also that the whole nature is a complicated machine. The two doctrine of mechanism in philosophy are both doctrines of metaphysics: the universal mechanism is a global doctrine about nature and the anthtropic mechanism is a local doctrine about humans and their minds, according to [13]. Universal mechanism is an antique philosophy which has close relations with materialism. As exponents of mechanism in the Greece philosophy were Democritus and Epicureans who considered the atoms and their movement the basic ontological principle.

Historians of the scientific revolution traditionally say that its most important changes were in the way in which scientific investigation was conducted, as well as the philosophy underlying scientific developments. The role of mathematics gradually increased and also new changes like mechanical philosophy, and empiricism appeared. The mechanical philosophers viewed natural substances, which had previously been understood organically, as machines. During the scientific revolution, under the influence of scientists and philosophers as Francis Bacon a sophisticated empirical tradition was developed in the 16th century, in contradiction with the previous approach of Aristotle by which the analysis of the known facts produces subsequent understanding of phenomena. The philosophy of Bacon of using an inductive approach to nature consists of to abandon assumption and to attempt to simply observe with an open mind.

By the mathematical-experimental methodology from the 18th century, by the Galilean founding of dynamics, by the laws of movement established by the Newtonian mechanics and then by other theories of the classic mechanics, the mechanist becomes programmatically the explanatory theory of the nature in its period of glory during the 18th and 19th centuries. Thus nature is going to be understood as objective order and assembly of ordered and comprehensible factual reactions only in determinist meaning, by assuming the causality principle as the unique capable to explain the reality rationally and properly at the level of the new epistemological statute of knowledge, according to [3]. This lacks both the appeal to the essences and finalities of Aristotle-scholastic physics, and the organicist conceptions of magic practices and of the animist visions of Renaissance, according to [3]. The finalism retake the prime place in the opposition towards the mechanist in the thinking of I. Kant and in the philosophy which followed him, especially in idealism. Then the nature and history began to be represented as being structured according to some well determined purposes or immanent either transcendent. In the contemporary era the electromagnetic physics, the theory of the fields and quantum physics demonstrated the impossibility of mechanist model to explain the reality. During the apogee or, when the power of "the spinning wheels" was very high, this mechanist vision influenced even the social sciences and especially the classic political economy. Actually many people from that time were so obsessed and excited by this mechanism, the clock that the mechanist from later started to think that, even more certain than the mechanism of a clock which indicated the twelve o'clock at an hour after which it indicated the one o'clock, the discoveries of the scientific revolution showed that any phenomenon can be explained in terms of mechanical laws and as any phenomenon must be completely determined regardless that it is talking about the past, present or future. The main representatives of mechanism in the 18th century were G. Galileo, R. Descartes, T. Hobbes and B. Spinoza. For example, R. Descartes interpreted the universe as a huge machine and the whole corporal reality in terms of extension and movement, and the philosophy of T. Hobbes (1651) are present the features of mechanist, anti-finalism and determinism. For him the knowledge does not consider goals or essences, but the material objects. As alternative to the vehement critics which he brought to the finalist vision, B. Spinoza sustains the idea that God orders geometrically the universe interpreting it as a great machinery consisted of parts, and in the soul of man would be only elements of thinking correlated by causal relations. However R. Descartes, argued that reality consists of two radically different types of substance, the extended matter and immaterial mind. The dualism of R. Descartes was motivated by the fact that it was improbable that mechanical dynamics to produce the mental experiences. The perception in anthropic mechanism it not that anything can be explained fully in terms of mechanics, but that anything about human beings can be explained in terms of mechanics the same as about the clock or engines. All the mechanical theories face with finding an explanation for the human mind. Like in the past, today also the main subjects of debate between anthropic mechanism and anti-mechanists are the mind and conscience and particularly the free will. The opponents of mechanist argues that anthropic mechanism is incompatible with intuition because they say that matter without conscience cannot explain completely the phenomenon of consciousness. "The spinning wheels" wanted to talk also about the philosophical theories from the last century, the hypothesis of K. Godel, but they wait to be more crystallized decanted all of them especially that there are still intense disputes between philosophers and they still have an opinion which cannot be changed. After 1960 some scholars like H. Putnam in his paper entitled "Minds and Machines", J. R. Lucas in "Mind , Machines and Godel", R. Penrose in his book "Shadows of the Mind"(1994) have debated about Godels incompleteness theorems and anthropic mechanism. They pointed out anti-mechanism arguments and H. Putnam also pointed out that the difference between what can be mechanically proven and what can be seen to be true by humans shows that human intelligence is not mechanical in nature.

All this philosophical trends influenced after a period of time (as always) the philosophy of people, see also Pascal's Pensees, maybe in the same way how the religious reform influenced the evangelical revival of the eighteenth century.

3. **A simple mechanism**

The relevance of the mechanisms models reached the maximum point during the industrial revolution in the European countries in the 19th century. The models of mechanisms are used today especially for didactic purposes in universities and can be considered an important part of the History of Mechanical Engineering. During the industrial revolution they were built for the market of specialized companies for learning but also as demonstrators for advanced solutions. Nowadays the focus is on the virtual models of mechanisms. An important Toolbox used in Matlab for engineer for virtual model is Simulink. The humble "spinning wheels" visited also the classrooms of students and noticed how the students learnt the knowledge taught by their professors. And they observed and were very happy because they saw that their name appear many times when professors explained the structure of the mechanisms. An example of classical mechanism, the crank-slider mechanism appears also in the construction of the steam locomotive. If "the spinning wheel" must explain the crank-slider mechanism under the understanding of all its explanation certainly would sound like this: If are known the lengths of elements AB, BC and BM and it is known that A is the fixed center (it is chosen A the origin of axes O and in this case the eccentricity e=0, the eccentricity being the distance from A to the right on which is moved the point C) of a circle of radius AB, B is moving on the circle, and C is moving linear on OX axis, its moving being determined by the movement of B, and M belongs to the line BC, to achieve the animation of the mechanism and to cross the trajectory of point M for a rotation of the crank, if there are requested to appear 400 frames on rotation, and the files to be saved on avi. file. AB element is called crank, and BC element slider.

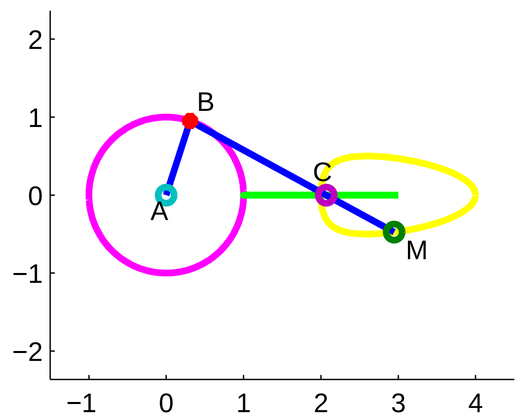

Figure 1: The crank-slider mechanism

In the second case are known the lengths of the segments AB=BC=CD=45 mm, EC=210 mm, eccentricity e=15 mm, the coordinates of the fix point A (0,0) and of the fix point D, xD=10 mm, yD=15 mm. When point B describes the circle of center A and of radius AB, point C shall move appropriately on the circle of center D and radius DC, the segment CD keeping its constant length, and point E shall move on a segment of line parallel with axis OX situated at the distance e=15 towards A. Here appear two cranks, ruling crank , driven crank and slider. In order to achieve the animation of the mechanism for a rotation of the crank if there are requested 400 frames on rotation, the files going to be saved as an avi. file.

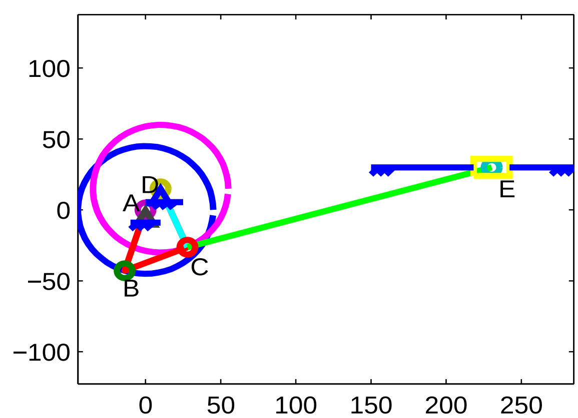

Figure 2: The double crank-slider mechanism

As projects, as laboratory themes, it is requested to draw and to animate using specific software mechanisms learned by the students within the educational subjects "Mechanisms and machinery", in order that they understand better how they are built and how they work. The students from the final years and from master currently use the program SolidWorks which is used also by engineers for the study of the kinematics and dynamics of mechanisms, but these drawings as well the performance of animation of the appropriate two mechanisms were made using the program Matlab because this software together with the software Mathematica or Maple are studied in the first year by the students within the laboratory hours at the educational subject "Mathematics for engineers". Examples of classical animated mechanisms can be also found in [15].

4. "The spinning wheels" in the Bible

The most happy is "the spinning wheel" when it thinks that "His wheels" as part of the throne of God, see Daniel 7:9, can stay in heavenly places continually in worship and prayer before Him an to praise Him, thus fulfilling its purpose as part of His creation. Then it feels an untold peace and reconciliation with itself. "The spinning wheel" remembers than how God told it to keep company to Samson in the dungeon of sufferance and to help him in the works given by the oppressive Philistines, Judges 16:21. And not only Samson was in a rough slavery but also the people of Israel and the state of mind of Samson reflects exactly the state of Israel people.

Another special mission full of joy of "the spinning wheels from the chariot of fire worn by some horses of fire" of the God was to take the Prophet Elijah, one of His two witnesses, to the sky in a whirlwind, separating him from his disciple Elisha. The request of Elisha to receive a bent measure from his spirit or was fulfilled because Elisha cried out "My father! My father! The chariots and horsemen of Israel!", see 2 Kings 2:12 and then struck the waters with the cloack threw by Elijah which divided in one part and another and Elisha crossed over, see 2 Kings 2:14. Actually in all the life of Elisha were repeated the miracles made by Elijah showing thus that he received what he asked, namely a bent measure from his spirit. Another wonder made by the prophet of Elisha in which were implied also the "spinning wheels" was the one from 2 Kings 6:8-23. Exactly when everyone considered their situation without escape, the citadel being surrounded by the army of Syrians with horses and chariots who were looking exactly for them, God gave them a solution. As much as it was the concern of the servant of Elisha when had seen the large enemy army and had not seen any solution, so easy and elegant God solved the problem showing them the mountain full of a greater army of heaven of horses and chariots of fire which surrounded Elisha. Thus while to those loved by God opened their eyes to see the spiritual reality, "Do not fear, for those who are with us are more than those who are with them" (2 Kings 6:16), to Syrians closed their eyes and Elisha lead them exactly to the emperor of Israel, whom he did not let to slaughter but put him to prepare a meal and then let them go. Then it was a time when "the heaven wheels" were put face to face to "the earth wheels" . The latter in their pride and conceit were not even aware of the presence of "the "heaven wheels" and by their power, while the heaven ones in their modesty looked at them with pity and prayed to God to avoid the fighting and not to be obliged to destroy them. There were two kinds of chariots, two kinds of eyes, earth and heaven, two kind of kingdoms and two kings. Another moment of the life of Israel people reminded by the Bible, in the Old Testament is in the Book of the prophet Ezekiel, in Ezekiel 1. The people were passing through states of great poverty, when Ezekiel was called in service by the prophet by the heavenly visions which he had. And "the spinning wheels" could heard about "the heaven wheels" in which it was the spirit of the living creatures who served to show the glory of God, see Ezekiel 1. The visions of the prophet Ezekiel; Then "the earth spinning wheels" understood that the latter, "the heaven wheels", take part of the secret and hidden things of God which cannot be yet discovered to people, see 1 Corinthians 13:12, because "For now we see in a mirror obscurely, but at that time face to face; now I know in part, but at that time I will fully know even as also I was fully known."

Maybe in future some engineers shall try to imagine something to resemble with what the prophet Ezekiel had seen in his visions from the Bible.

The industrial revolution plays an important role in the development of society, the machineries ("spinning wheels") changed the life of people, but not human being, they are very useful, but common and therefore the machineries will become more and more necessary in the life of people.

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Open Source Software: What is it, and why should we care?

(Presented May 30th, 2013 ACMS Conf., Bethel U.)

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1 Introduction

There is a certain distinction in talking about computer software that has implications for both mathematics and moral thought. To begin, let's look at two familiar distinctions.

One is that between local ("desktop") software and software as an online service (in "the cloud"). Here is a very non-exhaustive list.

Type	Desktop	Cloud
Productivity Tools	MS Office	Google Docs
Web Browser	FF, IE, Chrome	(None)
Learning MS	(None)	Blackboard
Math Tools	Matlab	Wolfram Alpha
Games	lots	lots

If we instead focus on the distinction between software you pay for and software you don't pay for, we get a similar list. (Note that some things might switch sides if you were a corporate customer.)

Type	Pay	No Pay
Productivity Tools	MS Office	Google Docs
Web Browser	(None)	FF, IE, Chrome
Learning MS	Blackboard	(None)
Math Tools	Matlab	Wolfram Alpha
Games	lots	lots

But there is another dichotomy you may *not* be as familiar with:

Type	*Proprietary*	*Open Source*
Productivity Tools	MS Office, Google Docs	Open Office
Web Browser	IE, Chrome	Firefox
Learning MS	Blackboard	Moodle
Math Tools	Matlab, Wolfram Alpha	Geogebra, Sage
Games	lots	lots

The second column is the subject of this talk.

2 What is Open Source Software?

We just saw examples of open source software in many genres. But what *is* open source software? First, we need a definition:

> The **source code** of a program is the original instructions to the computer, written by programmers.

This contrasts with the software itself, which usually is a binary file – one only the computer can really read and interpret. So we obtain two definitions.

> **Open Source Software** (OSS) is computer software whose source code may be freely modified and redistributed.

The opposite of this would be *proprietary* software, where the source code may not be modified or redistributed without express consent. (This will often be seen in end-user license agreements.)

This distinction may seem arcane, so it might help to see some (*important*!) related concepts that sound like OSS, but aren't.

- Many journals (such as the journal of the ACMS) and websites are open access resources. But it would violate copyright to modify many of these materials.
- Community wikis, like Wikipedia, can be modified. But they aren't source code.
- Many document formats, and the formats underlying the web, are open standards, or ground rules for the common good (similar to MLA style). They do not confer any rights, however.

You are probably already familiar with open source software. Some less familiar ones not mentioned above include the Linux operating system, the R statistics package, and the GIMP graphics software. The main places it does not have a foothold are social networking software in the cloud and programs to view copy-protected media.

3 Why Open Source?

There are many reasons why people and companies choose to use OSS. Naturally, there is one very popular reason; since one has the right to see the instructions to the computer, it nearly always is free to install and use! However, this talk takes the view that this is not *sufficient* grounds to use OSS, and in this section we introduce a larger set of rationales for (and against) its use.

In industry (including all big players), academia, and government, the following reasons usually are important.

- Much is extremely high quality for mission-critical positions.
- It is very customizable, since your engineers have the code.
- Paying for *support* instead of the software makes better business sense.
- Bug reports might be acted on more quickly (even by you).

The last point is directly related to political economist Steven Weber's notion of software as an *anti-rival* good. Essentially, software becomes more valuable when more people use it – even if they don't pay – since it gives more chances to see it work in real-life situations.

In addition, there can be good grounds not to use open source. In mathematical or pedagogical domains, the following considerations are important.

- Dedicated support staff is often absent. (This is a problem with using Moodle, for instance.)
- The software often lacks third-party or hardware support. (This is particularly important with Matlab replacements, since it is so ubiquitous in engineering.)
- For end users, the interface may have a high learning curve or not be up-to-date. (Anyone who has used early versions of WeBWorK, or ever used LaTeX, will know this.)
- There may be a 'sunk' curriculum investment in one program, and it can be hard to update materials. (This is true in mathematical and learning management contexts.)

However, it is important to note that all of these arguments often apply to *any* change of software! Think of the last calculus text decision your department made; it is likely that related considerations were all in play, even if the texts were not free.

Finally, there are some *mathematical sciences* reasons for favoring open-source software. First, and most importantly, it is vital that science be *reproducible*. In mathematics, examples are not proofs! A new algorithm for finding primes gives no guarantees without seeing the actual instructions to the computer.

On a similar level, it is important for pedagogical reasons for students to be able to understand algorithms (at least for majors). Being able to verify that the computer is using the algorithm in the book can be eye-opening –

and if it doesn't, can bring up a very fruitful discussion. In either situation, one might also want to know whether a correct algorithm has also been correctly *implemented*![1]

These are quite specific to the math and computer sciences; in design or psychology, it is evident that the software works or does not work. On the other hand, in linguistic fields it might be important to verify whether (say) automatic translation software really is doing something appropriate in its code *before* it makes a cultural faux-pas.

4 Theology and Open Source

However, for this audience it is most appropriate to look at the broader context. Many programmers in the open source movement – and it is a *movement* – see the question in broader terms than economic or technical ones, and this is a good starting point for theological reflection.

Many observers point to a change in the role of software and programmers over the decades, as software (not hardware) became the marketable product. Steven Weber again, in *The Success of Open Source*:

> The narrative of the programmer is ... of the craftsperson from whom control and autonomy were taken away.

That is, true craftsmen and -women wish to see their innovations built upon – not endlessly reinvented.

With that in view, open source licenses have a radically different way of thinking of intellectual property. One of the most pervasive open source licenses does not even allow modifications to be distributed without allowing subsequent modification ('copyleft'). This means that not only can one modify the code (open source), but it can *never be directly used in a proprietary product*. Regardless of one's views on this particular license (and it is controversial), it makes it clear that open source users and creators have a long history of explicit moral and political value judgments in their *self*-perception and motivation.

Thus it is no surprise that there are explicit connections to Christian thought. The following four rubrics distill most such connections; I would argue that these are all natural connections to our non-Christian colleagues as well.

Open Source Software, at its best:

- Is good stewardship
- Builds community
- Helps the underprivileged
- Promotes creativity

None of these things are guaranteed, or exclusive to OSS. (That is another talk.) But I *do* claim that, in general, open source has more potential to live up to these virtues than proprietary software, and elaborate briefly on this in the remainder of the talk.

4.1 Stewardship

First, the relationship to stewardship. Many of us purchase 'standard' proprietary software without really thinking about it. But what is the opportunity cost? House church leader Donald Parris suggests that, with Linux as the default operating system or by using OpenOffice[2], "You might be able to. . . boost a missionary's efforts."

This argument seems almost ridiculously straightforward, but one needs to watch for pound-foolishness. Though one blog post goes so far as to suggest that only churches with a lot of tech-savvy members need use OSS, it is true that with *any* technology one needs to be mindful of usability – even with things as simple as updating a church calendar. However, this is not a specifically OSS versus proprietary argument.

[1] I am indebted to my colleague Russ Bjork for this point.

[2] In personal communication, a colleague confirms that the Mormon (LDS) faith provides computers with OpenOffice for standard congregational use.

4.2 Community

Somewhat more surprising might be the very explicit connection to *community* that open source can foster. OSS user and developer communities are *very* explicit that they are communities; one extremely popular system (Drupal) advertises itself with the tagline, “Come for the software, stay for the community.” Given the long tradition of viewing Jesus as starting a new way of being human community (from the Johannine corpus to Bonhoeffer), there might be a connection – but what?

Larry Wall is the founder of one of the most popular programming languages, Perl. Perl is open source, and Wall (an evangelical Christian, by all accounts) has spoken often explicitly about this connection: “[I modeled] the Perl movement on another movement . . . the founder [said] ‘He who wishes to be greatest among you must become the servant of all’.” Indeed, because use and developing of OSS is completely voluntary, there is a real subversion of traditional hierarchies. Moreover, the best open source communities are very clear about wanting to develop healthy community, and I have personally seen the desire to meet in person someone long known only from an email forum.

This comes with a caveat; some OSS – and proprietary – communities are poisonous. Beware any discussion list which uses the term ‘n00b’ often! But what this means is that there is real opportunity for Christians modeling true community here. Especially because there doesn’t have to be as much of a marketing focus, frank discussion can flow more.

4.3 Helps Others

There has been a lot of discussion about whether software of any kind can be good for helping others. The argument may be similar to that over other technologies, juxtaposing the questions of whether a new technology is good because it promotes efficiency, or bad because it creates dependence.

This is where OSS *may* have a distinct advantage. Can we help the underprivileged with software? ‘Free software’ guru Richard Stallman certainly believes so: “. . . these words, attributed to Hillel: ‘If I am not for myself, who will be for me? If I am only for myself, what am I?’ ” Easy sharing and low cost certainly would seem to connect to broad Judeo-Christian ethic of helping the other. Still, although there are numerous examples of projects such as One Laptop Per Child and nations deciding whether to use Linux or open document processors, there are also examples of countries explicitly rejecting this in favor of the ‘mainstream’ products such as Microsoft Office. This is really directly connected to the notion of ‘open standards’ raised earlier, which we will not address.

Nonetheless, properly organized, OSS has three important advantages toward granting opportunity toward those not yet in the digital elite. Naturally, the cost argument is very significant here. In some circumstances, the only marginal cost for the end user and provider is that of bandwidth to download. More importantly, there are two things easier to do with open source software than proprietary software.

First, many situations in the developing world (and also in this country) have very low-end hardware, or at least somewhat outdated hardware. Typically, it is not worth the effort for a normal software provider to continue providing versions of their software for such situations; many web pages cannot be viewed without the latest upgrade to Flash or other video software, for instance, which is not available on many older computers. This is true for any software, but with open source the *potential* exists to keep things operating far longer than typically viable. Sage has kept up a version for decade-old Macs for some time now, for instance.

Similarly, the issue of localization is important; how do we provide the software in the language and with visual cues appropriate for different cultures? Although some companies with truly global ambition tend to provide more of this, with a lot of software it is difficult to use without English proficiency. This is far easier to deal with when all the hooks are available to provide translations.

These come with a large caveat that someone has to be found to actually *provide* this functionality! Open source is not a panacea. Still, the potential is greater here with a similar project, all else being equal, when the project is open source.

4.4 Creativity

In my view, the most important connection is perhaps the most surprising. Math and programming polymath Donald Knuth has this quote to prime our pump:

> I think people who write programs do have at least a glimmer of extra insight into the nature of God.

Christians believe that, as humans, we are created *imago dei*. But what does this mean? Many programmers (and others) argue that the creative impulse is at least a part of this[3]. In this case, anything that promotes (healthy) creativity is helping humans achieve their God-given potential; Larry Wall is very explicit about this.

But what does open source have to do with this? Here the key is that *anyone* can engage in this creativity. This is most of all true for those who can help in programming – worth thinking of with ourselves or our students. Open source gives anyone the potential to be creative without having to start from scratch, but rather in increments.

Still, most of us will not be programming – and presumably far more so for other students! Still, I do not believe that this removes them from the opportunities for living out this mandate.

Think of a creative artist who needs a whole ecosystem of people to help, such as glass artist Dale Chihuly. He has been physically unable to work on his oeuvre for decades, yet shepherds a whole pool of workers to bring his ideas to fruition. Each of these workers is also participating in the creative impulse, brings their own point of view, though under more hierarchical structure than most OSS projects.

In the same way, open source communities encourage *anyone* to participate by submitting bug reports, confirming fixes, providing translations, and so forth. Calvin College education professor Ron Sjoerdsma provides the Biblical narrative of Bezalel and Oholiab (Exodus 35) as a paradigm for this. These artisans were not just gifted to craft the beautiful ornaments for the tabernacle of the Lord, but were inspired to teach as well. This analogy of the need for skilled ones in technology to be mentors at the same time, to teach others to contribute in a community effort, is evident. How can we empower others to use their God-given creative gifts to benefit the whole community – with even tiny contributions being worthwhile? Open source software is a good place to start.

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Service-Learning Panel

(Presented May 30th, 2013 ACMS Conf., Bethel U.)

Panelists: Dave Klanderman, Josh Wilkerson, Maria Zack

Moderator: Karl-Dieter Crisman

This panel session is an opportunity for faculty to learn a little bit about service-learning and hear from some examples of its use by colleagues.

1 Introduction

What is service-learning? A service-learning component (of a course) is an educational activity which[1]:

- Renders some kind of useful service to society (typically a non-profit), and
- Provides a useful learning opportunity for the students involved, in close conjunction with the curriculum.

This can happen in contexts ranging from optional regular service to a required group consulting project.

There are many benefits to having such experiences. Naturally, there may be an institutional directive! However, there is also some documentation of this enhancing higher-order learning, and establishing a good bond between the university and the community (including good press). For the present context, the imperative for service in Christian thinking is a typical rationale, as well as the connection to enhancing student self-perception of moral development.

However, there can be barriers. Charles Hadlock of Bentley University says, "Unfortunately, the mathematical sciences are sometimes perceived as having a more difficult task to incorporate service activities in the curriculum." This is substantiated by an anonymous professor in an online survey otherwise not about mathematics: "I can think of no service projects in the community that will enhance student learning of the abstract reasoning skills they should be learning in mathematics." Though the panelists might disagree with this, the fact remains that there are few places where multiple service-learning ideas for collegiate mathematics are put together in one place (though there are scattered references elsewhere). Hence the need for our panel talks.

- MAA book: "Mathematics in Service to the Community", ed. Hadlock.
- A website of talks from the 2011 Joint Meetings
  `http://www.math-cs.gordon.edu/ kcrisman/SLTalks/`
- There is a forthcoming PRIMUS issue on the topic
- The Campus Compact website has a (very) few ideas

2 Statistics in Service – Josh Wilkerson

The purposes of this project, done with high schoolers taking AP Statistics in Texas, was to help a non-profit service agency. This required survey research for program evaluation or client needs assessment, which was identified by the class.

Students participated in a group providing the following four services:

- Meeting with agency and developing a survey instrument
- Conducting a survey
- Compiling and coding the data
- Analyzing the data

[1] cf. Charles Hadlock

One important aspect of this project was that students were under the command of the service agency, but the teacher was *not* in the direct chain of command, serving more as an outside consulting facilitator.

The project was a great success. A member of one group wrote an article for a local newspaper; this group analyzed the differences between volunteer efforts and county-provided resources in responding to a devastating set of wildfires. The fact that this topic was chosen by students helped truly pique student interest. For the same reason, student professionalism and ability to engage the community was improved and stretched. The survey was a tool for real interaction.

However, at the same time survey development and analysis cemented the importance of statistics; this was a real-life scenario, allowing for deep connections to the course material that could not have been achieved without such a project.

3 Serving the Institution – Maria Zack

At Point Loma Nazarene, every student engages in some sort of large, year-long senior project as part of the graduation requirements. Many of them end up doing a significant service project, using real data analysis and mathematical organization skills. In this event, students are divided into teams.

These projects often serve the institution by taking institutional data and helping out where the institution does not have enough resources to adequately make use of the data they acquire for reporting and other purposes. Some typical projects include the following:

- Analysis of chapel attendance patterns
- Development of undergraduate curriculum for CS 'service' courses
- Analysis of student retention data
- Creation of a website showing laptop specs for incoming students
- Using GIS to help the Admissions Department refine recruiting efforts

As one can see, this is a wide variety of projects. The cooperation of faculty and staff from all over campus is crucial for implementing something like this. At best, one has full buy-in from both the constituencies on campus desiring help, *as well as* from the faculty in the department itself. Mentoring these projects is time-intensive and requires at least some load for an instructor for the course.

For the students, many skills are developed and required. They engage in research – asking questions, finding details – and turning these very same questions into problems. More specific skills may include managing data (perhaps via databases), analyzing data, and representing it graphically. Finally, the key skill of producing *reporting* material is also achieved; these may take many forms.

4 Trinity Math Triathlon – Dave Klanderman

Trinity Christian College has been offering a Math Triathlon as a community outreach event for fifteen years. It has turned out to be an excellent opportunity for service-learning in a number of different courses as well, providing a nice glimpse into the intersection of outreach and service.

For most of its time, the triathlon has been an event for late middle school (grades 7 and 8) – see `http://tcc.trnty.edu/mathtriathlon/`. It primarily focuses on local Christian high schools and homeschool groups, but this is not essential to the running of such an event. It consists of individual events, team events, and so-called 'relay' events for the various teams. In addition, to facilitate grading time for the organizers, there is a more open-ended halftime activity.

Note that the website above has information about *all* Triathlon materials (past year events, performance statistics, sample halftime activities, etc.)! More recently, the triathlon has been expanded to an event for students in grades 3-6.

There were many different service-learning components. The most immediate one was in a special January term course for mathematics and math education students; these students design and implement the entire event! Needless to say, this was a very useful learning experience, and had huge amounts of service.

However, there were many other opportunities for service. Elementary education majors were able to obtain field education hours, and created the halftime activities (such as ‘Human Connect 4’). Some of them also created some of the problems, as the grade levels were not so far off. In an upper-level probability/statistics course, students did more in-depth analysis of item difficulty and the performance of each participating school and group – useful both for the host and the participating schools.

One interesting aspect was with last-minute ‘volunteers’ from courses not as directly connected, such as Calculus II. These students provided reflections on the experience and connections, for extra credit. However, it was typically only the students *also* enrolled (not necessarily concurrently) in one of the other courses serving this event who did indeed see connections with the material; this is an important point to note in general for service-learning – that it can be hard to explicitly connect the service with the learning. But in general the students made clear connections and felt part of a team putting on the event for the community.

Catherine Crockett

TITLE: A Different Approach

ABSTRACT

This paper discusses an approach used to encourage science majors to rethink their attitudes and study habits in a first semester calculus course. Two activities were used to enhance study habits. They are outlining concepts and in-class quizzes designed for self-evaluation of skills. After using both methods in two sections of the calculus course, the students were surveyed to determine if these activities were successful. A majority of the students felt the activities were helpful and wanted to continue them.

INTRODUCTION

Recently, I taught two sections of a course called "Math 144: Calculus with Applications" at a small, liberal arts university. The goal of the course is to develop a conceptual understanding of topics in first semester calculus and their applications. Most of the students who enroll in this course are science majors. While I have taught a variety of levels of calculus, this was my first time teaching calculus filled with science majors. I was looking forward to working with this particular type of student. Unfortunately, after grading the first exam it was clear that these motivated students were having trouble learning the material. The purpose of this paper is to share the activities I used and the student response to these activities.

The majority of students who enroll in Math 144 are freshmen on the Pre-Med track (i.e. Double biology and chemistry majors). Most students take this course in the second semester of their first year of college. The usual class size for a section is between 30 to 35 students. Since 2001, 90% of our students who applied to medical, dental or veterinarian schools have been accepted. These students are familiar with academic success and are highly motivated to earn top grades. However, the students' mathematical backgrounds often vary greatly and some suffer from math anxiety.

During the first five weeks of the semester the students maintained high attendance rates, took notes, answered questions and flourished in group discussions. On the surface, the two classes looked like groups of engaged, successful students. Unfortunately, the first exam revealed a different picture. The scores for the first exam in both classes were bimodal. I asked the students to answer a few background questions. It became clear that there were three different groups of students: students who had a working knowledge of calculus, students exposed to calculus but did not possess a working knowledge and some students who have never taken calculus in their academic career.

Further investigation revealed that the students were studying by memorizing examples and facts exactly as they were written in the text. For some students, they could not recognize different question types. While it is common for freshman to experience lower grades than they may be accustomed to, I took their concern into consideration. At the time, I was reading several articles / reports on STEM education and talking with colleagues in the STEM disciplines.

MOTIVATION

Why should we be concerned with how science majors learn calculus?

We should be concerned about how science majors learn calculus for the sake of both areas: the sciences and mathematics. First semester calculus is often the only mathematics course (besides statistics) that these students are required to take. By helping these students be more successful in calculus, they may be encouraged to take more mathematics. More mathematics for these students sets them up for more success in their careers. In an article titled *Challenges, Connections, and Complexities: Educating for Collaboration* (Jungck, 2005), the author maintains that biology students who learn more math and computer science are often more successful than their peers who learned less math. Another article, *The "Gift" of Mathematics in the Era of Biology* (Steen, 2005), takes this a step further and suggests science research is hindered when the researchers are missing mathematical skills.

Recommendations for closer alliance between mathematics and the sciences have come from several different sources. One such source is the *BIO 2010* report published by the National Academy of Science (Committe on Undergraduate Biology Education to Prepare Research Scienctis for the 21 st Century, 2003). This report is a study on the needed reform in undergraduate education in the life sciences. It states that the "connections between the biological sciences and the physical sciences, mathematics, and computer science are rapidly becoming deeper and more extensive". The first two recommendations for change from this report is the need for more mathematics and critical thinking skills in the life-science course.

Another publication addressing the needs and connections in mathematics and biology is *Math and Bio 2010: Linking Undergraduate Disciplines*. This research was a joint project of the Mathematical Association of America (MAA), the American Association for the Advancement of Science and the American Society for Microbiology. The project addresses and discusses the need for developing quantitative skills in science majors since the life sciences are becoming more quantitative. If we can help these students find success in calculus- it pays off for everyone.

After reading these reports, I wondered if the approach to studying science different than the approach for studying mathematics? I believe the current answer is yes. I interviewed a colleague whose research is in mathematics and science education. Our conversation focused on study methods and attitudes of science majors. She commented that memorization can be an effective study tool in life science courses since there are many facts and vocabulary the students need to learn. However, she cautions that it is not always the best model for learning (Maskiewicz, 2011). In calculus, the emphasis is on the concepts and the application of these concepts. Since most life-science courses have to cover large amounts of facts and concepts, application of these concepts often doesn't occur. For example, in *Connecting Biology and Mathematics: First Prepare the Teachers* (Sorgo, Fall 2010), the author remarks "...science courses rarely develop the skills associated with approaching novel or unfamiliar problems…".

APPROACH- How to help students learn calculus?

There are many resources on undergraduate education in mathematics. I found several articles in various journals discussing different study techniques. For example, in Primus, I found an article titled *Teaching calculus students how to study* (Boelkins, M.R. and Pfaff, T.J., 1998). The authors describe their success in giving the students a daily homework and study plan. However, I felt before introducing any study techniques to my students, it was important to address the students' approach. So the question became, "How do I encourage my students to have a different approach to learning mathematics?"

A colleague of mine referred me to Carol Dweck's research on the effects of beliefs about intelligence and "Achievement goal theory" (Elliott, E.S. and Dweck, C.S., 1988) (Mangels, J.A, Butterfield, B., Lamb J., Good, C.D. and Dweck, C.S., 2006). Dweck's research suggests if students have performance goals instead of learning goals, they are more focused on the validation of ability (i.e. grades). According to Dweck, students with performance goals are more prone to see a new problem or challenge as a threat rather than an opportunity to learn. In contrast, students with learning goals tend to view new problems or challenges as opportunities to learn and grow.

After the first exam was graded and returned, the following was displayed on a screen for the class to read and discuss.

Research by Carol Dweck

> Students for whom performance is paramount want to look smart even if it means not learning a thing in the process. For them, each task is a challenge to their self-image, and each setback becomes a personal threat. So they pursue only activities at which they're sure to shine---and avoid the sorts of experiences necessary to grow and flourish in any endeavor. Students with learning goals, on the other hand, take necessary risks and don't worry about failure because each mistake becomes a chance to learn.
>
> Dweck's next question: what makes students focus on different goals in the first place? People with performance goals, she reasoned, think intelligence is fixed from birth. People with learning goals have a growth mind-set about intelligence, believing it can be developed. "Study skills and learning skills are inert until they're powered by an active ingredient," Dweck explains. Students may know how to study, but won't want to if they believe their efforts are futile. "If you target that belief, you can see more benefit than you have any reason to hope for."

After reading the screen, the students were given time to think and respond. The students were asked to discuss their opinions and thoughts on the following questions:

1. Are you taking this class to get a good grade only or to get a good grade and learn some calculus?

2. Do you like to work on challenging problems? Why or why not?

3. When you do homework, do you first learn the concept and then work on the assigned homework problems or just go straight to the problems?

4. Do you believe intelligence is fixed at birth? If it is not fixed at birth, what can you do to increase your intelligence?

As a class, we discussed Dweck's research on goals and beliefs and their effects on our actions. Several of the students were willing to be transparent and expressed concerns such as: grades overshadowing learning, not having the freedom to fail and having a great deal of value placed on the perception of being smart. While not all of these issues could be addressed, the main goal was to get them thinking about their philosophy of learning and how it shapes their actions.

To encourage the students to have learning goals, two simple activities were incorporated into the class: outlining a concept together and self-evaluation quizzes. Before doing these activities, I explained to the classes why we were doing these activities based on Dweck's research. The new class routine became outlining concepts once early in the week and taking one non-credit quiz at the end of the week. Both activities were designed to take approximately 10 minutes of class time.

Outlining Concepts:

After asking the students how they studied, it was clear many of them were trying to memorize theorems and examples not written in their own words. Most of the students were using 3 by 5 note cards to help them study for their science courses. Since the students were already using the note cards, I decided to use them for outlining. Each student was given a 3 by 5 note card and put into a group with three other students. The students were asked to work together to identify the key pieces of information required for the specified topic. Next, each student had to determine how to place the information on the note card. I encouraged the students to write the information in a format that suited their own learning style.

Here is an example:

"What are the key points for finding the local minimum or maximum of a function? How could you express this information visually?"

For most students, the picture they drew was "the top of a hill" for the maximum and "bottom of a valley" for the minimum. They described how to find the local extrema in terms of "uphill" and "downhill". This led to a discussion on how to find the extrema when we don't know the graph of the function. We were able to discuss techniques for locating the local minimum and maximum in terms of the first derivative and how the students' pictures were illustrating the behavior of the first derivative.

Another example of a concept we outlined was optimization. Most of the students wrote out a check list of items to complete. While I wasn't too sure about the checklist idea, it turned out that the students themselves determine (correctly) the needed steps to solve these problems. In the end, they were able to put the concepts together to work out the process.

The first time we outlined in class, I started the conversation and did an example of a note card with input from a student on the overhead projector. Within a few weeks, the students were the ones discussing key points and putting their examples on the overhead projector without needing any input from me. One of the unexpected benefits of this activity was the stronger students became more engaged in the class.

Self-Evaluation Quizzes:

Once a week, each class was given a self-evaluation quiz. These were "volunteer" quizzes since the grades were not recorded. The goal was to give students an opportunity to test their knowledge at no risk to their grade. Most of the quizzes asked basic questions. Each quiz covered a topic that was taught earlier in the week. The quiz had two parts; first, write an explanation on how to solve the problem and second, use math to solve the problem. I wanted the students get into the habit of evaluating their level of knowledge on a concept. By having the two steps, the students could identify where they needed to study more. After the class went over the solution, the students were given a list of examples/problems on the same topic for their own practice.

Example:

Question: Suppose you are asked to find the local minimums and maximums of

$$f(x) = x^3 + 5x^2 - 36x \,.$$

Part 1: Write out the steps you would need to do to solve this problem.

Part 2: Using mathematical tools discussed in this class, find the local minimum and maximum of the function.

RESULTS:

Anonymous Surveys

Since this was an experiment, I wanted feedback from the students. I wanted to know if the activities were beneficial and should be continued. A short anonymous survey was passed out to each class after the second exam was taken, but before it was graded and returned. The survey consisted of five questions with space for comments. The first two questions asked the students to rate their opinion on a scale of 1 (strong negative) to 4 (strong affirmative). The next three questions where “Yes” or “No” questions about if they wanted to continue these actives. Then there was a space for comments.

Survey questions:

1. The note cards we wrote in class helped with the material.
2. The quizzes we took in class helped.
3. I would like more quizzes (they don't affect your grade).
4. I felt more prepared for Exam #2 than I did for Exam #1.
5. I would like more note cards.

Summary of the responses:

Out of a total of 58 surveys, 24 of the students said the note cards helped them very much and 23 of the students said the note cards help some (totaling 81.03 % of the students). For question 2, did the quizzes help, 35 students said it helped a lot and 15 said it helped some (totaling 86.2% of the students). 93.1% of the students said "Yes" they would like more quizzes. 82.75% said they felt more prepared for the second exam. 81.03% said they would like more note cards.

Trends:

While this survey is quite limited, the results do indicate that the students want help in learning how to study. Almost all comments written on the survey were positive such as "The note cards are helpful", "Thank you", and "Worked great". One student came to my office to declare she finally believed that she "NOT MATH STUPID”. The one negative comment received was about the note cards and it was "I lose them".

The scores for the second exam were significantly higher and no longer bimodal for both sections. However, the most important change was the increase in the student's confidence level. There was a change in the type of questions being asked during my office hours. Students were coming in asking about specific questions and examples, often citing where they thought they were missing some information. For example, when solving an optimization problem, I had fewer general questions on the set up of the problem and more questions about what to do with a solution not in the feasible domain. The students were thinking and not afraid to suggest ideas.

CONCLUSION

Based on this experience and data, these activities are beneficial to the students. In the future, I would like to continue these activities with a few modifications. I think giving the students more time to digest the material and post/ discuss their thoughts on Dweck's research could be transformative for them. For the quizzes, I would like to try a new format. The students would first take the quiz and then correct the quiz with a peer. Peer feedback could be more effective than going over the solution as a whole class. The success my students found in self-evaluation quizzes and outlining concepts encourages me to continue these efforts in future classes.

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Delaware, Dickeson, Assessment and How You Can Help

by

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Abstract:

How much release time should a chair receive? What is the cost per unit for a particular academic program? What is a student credit hour (SCH) anyway and why would anyone care? Why are so many boards enamored of Delaware, Dickeson and Assessment? The answer to these and many related questions will be presented in this paper. Analytics and various "efficiency measures" are becoming increasingly important in higher education and mathematicians and computer scientists are being regularly recruited to help university administrators make meaning from large volumes of data. This paper describes the issue and provides some examples of how faculty can help.

Introduction: Assessment, Delaware and Dickeson

For the last ten years, the documentation and reporting of the assessment of student learning has been part of the annual work of all academic departments. Much of the formalization of this process has been driven by regional accrediting bodies who have demanded answers to two questions:

- What do you expect your students to learn?
- Can you demonstrate that they are learning these things?

The scrutiny of each university's answers to these questions has increased in the last few years. Since the 2008 downturn in the global economy, a third question has been added to this list:

- What is the value of a college degree? In particular, are your students prepared for a profession when they graduate?

Publications such as Arum and Roska's *Academically Adrift* (Arum, 2011), which claims to demonstrate that students learn very little in the areas of writing, complex reasoning and critical thinking in their first two years of college, have increased the concern about the value of a degree. Certainly criticism of the methods used and conclusions drawn by Arum and Roska abound, but their criticism of higher education is now part of the political debate about the future of higher education. While data consistently demonstrate that a college degree significantly improves an individual's earning potential, new studies have reinforced the notion that what you major in matters (Zaback, 2012 and Grusky, 2013). While assessment is nothing new, the volume of data gathered and analyzed is going to continue to increase as federal reporting requirements grow in the next several years.

Since the release of *A Test of Leadership: Charting the Future of U.S. Higher Education* (aka “The Spellings Report”) in 2006, several key words have entered into the administrative lexicon of higher education. These words are: access, affordability, learning/quality, accountability and innovation (Spellings, 2006). To some extent, existing assessment processes address issues related to tracking and improving quality, however, there are increasing expectations for greater transparency of learning outcomes data. That means public access to summary data about student achievement, which requires a fairly high level of skill in summarizing and presenting data in ways that are meaningful to students, their families, the general public and legislators.

Since 2008, the issue of the affordability of higher education and the transparency in reporting costs has been prominently discussed in the media. This has led academic leaders to ask the question:

- Are our programs and departments functioning as efficiently and cost-effectively as possible?

Most institutions are looking at the expenses related to each of their academic programs in an attempt to reduce costs in order to hold down tuition increases. The National Study of Instructional Costs and Productivity – University of Delaware (www.udel.edu/IR/cost/) or just the Delaware Study is an attempt to benchmark instructional costs by program using very careful definitions and methodology. The study does recognize that there are real differences in the costs of programs (for example nursing programs are more expensive than literature programs) and the data provided by the Delaware Study gives each institution an indication of how its costs program by program compare with a national pool as well as a self-selected pool of 10 or more comparator institutions. This process was first begun by Michael Middaugh when he was at the University of Delaware and some of his findings can be found in *Understanding Faculty Productivity: Standards and Benchmarks for Colleges and Universities* (Middaugh, 2000) as well as in his subsequent book *Planning and Assessment in Higher Education: Demonstrating Institutional Effectiveness* (Middaugh, 2009). Some examples of the work that is involved in computing the main statistic for the Delaware Study, the student credit hour (SCH), are given in the examples below.

While the Delaware Study produces some useful information about benchmarking, it does not provide benchmarks for non-curricular units in the university. Academic program and services prioritization, which in fact encompasses non-academic units as well as academic units, attempts to address three questions:

- How do we make judgments about the efficiency and effectiveness of non-curricular areas of the university?
- How do we make decisions based on the data from all university departments?
- How are resources allocated to strengthen the institution?

One process for addressing these questions is contained in the book *Prioritizing Academic Programs and Services: Reallocating Resources to Achieve Strategic Balance* (Dickeson, 2010). What Dickeson and Ikenberry suggest in the book is quite radical. They suggest assessing the strength of programs using a number of metrics and then deciding which programs to eliminate and which programs to retain based on those metrics. Dickenson and Ikenberry call for the bottom *one-third* of programs to be eliminated. Because of the extremely sensitive nature of this process, it is essential that individuals with good skills in pulling, combining and analyzing data are involved in defining the metrics and generating the values for those metrics.

In addition to trying to cut costs, universities are looking for ways to increase revenue. Much of that conversation has been centered on the word innovation and the book *The Innovative University: Changing the DNA of Higher Education from the Inside Out* (Christensen, 2011). It is not surprising that much of what is discussed in this book relates to the use of technology in education. It is essential that faculty with a knowledge of both the strengths and the limitations of technology engage fully in the campus conversation. Too often, the administrators making significant decisions about the use of technology have very limited experience with technology; they need honest open counsel from faculty experts.

Why Does Your Board Care about Assessment, Delaware, Dickeson and Innovation?

First and foremost, your Board is interested in these issues because they hold your institution in trust. That means that they must be concerned with the long-term viability of the school. A general consensus is emerging that higher education has entered a "new normal" and that even as the economy recovers, the higher educational context will not return to what it was before 2008. This is attributed to many factors including a slow recovery in the economy, a shift in employer expectations for new graduates and the increasing use of technology in higher education (with the conversation fueled by the emergence of MOOCs and the money funding them). All of this means that trustees have to ask more in-depth questions about:

- The cost of institutional programs and processes;
- The affordability of the institution with a particular focus on the size of annual tuition increases that can be sustained;
- Ways that technology can improve student learning and keep costs down; and
- Possible new initiatives that can diversify the university's revenue streams.

Many of these ideas come from the business world, and like it or not, the United States Department of Education is requiring ever more detailed reporting on business processes. Because all of our universities are financially dependent on our students' ability to receive federal and state financial aid, it is impossible to not do as the Department of Education and the associated regional accrediting bodies ask. These requirements are generally organized around the "Spellings Report" criteria of access, affordability, quality, accountability and (sometimes) innovation.

Your Board is reading much of the same material that you are reading about higher education. This means that they are also concerned about:

- The role of for-profit institutions and the ways that they have cut into market share for traditional higher education;
- The emergence of MOOCs (Massively Open Online Courses) backed by well-funded organizations such as Coursera, Udacity and edX and the uncertainty about whether or not MOOCs will have a significant impact on higher education; and
- Online degrees from major institutions which have the potential to reduce what our institutions can charge for degrees targeted at adult learners. For example, Georgia Tech just announced a

partnership with Udacity and AT&T to offer a $7,000 master's degree in Computer Science that will be a Georgia Tech degree.

Finally, there are significant demographic changes at work among the college aged population in the United States. Those attending university are much more diverse in both race/ethnicity and age than they have been in the past. This has a number of implications for admissions, financial aid, student support and curriculum. If universities are going to provide educational access for more than just traditional 18-22 year old students from middle class families, changes are going to have to be made. One of the most significant of these changes is trying to determine how to address the needs of adult learners. Additionally, data from the National Center for Educational Statistics (NCES) (http://nces.ed.gov/programs/digest/2012menu_tables.asp) shows that number of students attending university increased by 25% from 1997-2007 and is expected to increase by 25% in the 10 years ending in 2017. Someone will provide that education.

			1997-2007	
Undergraduates	**1997**	**2007**	**Change**	**%Change**
Full-Time	7,418,598	9,840,978	2,422,380	32.7%
Part-Time	5,031,989	5,762,793	730,804	14.5%
Total	12,450,587	15,603,771	3,153,184	25.3%

		Projected	**2007-2017**	
Undergraduates	**2007**	**2017**	**Change**	**%Change**
Full-Time	9,840,978	12,141,135	2,300,157	23.4%
Part-Time	5,762,793	7,374,008	1,611,215	28.0%
Total	15,603,771	19,515,143	3,911,372	25.1%

Boards know that in order for our institutions to remain financially healthy, it is necessary to find ways to attract and educate the growing number of individuals participating in higher education.

So Why Did (or Should) That Administrator Call You?

The problems of interest to university Boards and administrators involve complex data management and analysis. Computer scientists and mathematicians bring a critical set of skills to the table. You:

- Can pull and store data so that it is useful;
- Can synthesize information from multiple sources;
- Can build models and create algorithms;
- Can keep *n* definitions in your head at the same time;
- Know data analysis;
- Can detect and make sense of patterns and anomalies that others miss;
- Can chase down outliers; and
- Have experience with communicating technical results to a non-technical audience.

These are essential skills for the task at hand.

In addition, mathematics and computer science are considered "useful" degree options that lead to stable and high paying careers. Thus you can more easily navigate some of the tough political issues because it is highly unlikely that programs in your department would be cut. However, that does not mean that your department will escape the need to change and you can bring a faculty perspective to conversations about the implications of particular institutional changes.

What Can You Do to Make Yourself Useful?

First and foremost, you need to read. Become conversant in the material in the books and articles at the end of this paper. Follow the higher education media by reading the daily summaries of trends. Some of the best are:

- Inside Higher Ed (free): www.insidehighered.com. You can sign up for their email updates.
- The Chronicle of Higher Education (this has a modest subscription fee, but more than likely you can access an electronic copy through your university library): www.chronicle.com.
- National Association of Independent Colleges and Universities (free and good if you are in a private institution): www.naicu.edu. You can sign up for their email updates.

Second, offer to help. Spend some time talking to institutional leaders (Provost, Deans, President, CFO) in order to get a better understanding of the issues with which they are wrestling and how your skills may assist them. The administrators in your institution may not realize that you have both the skills and the willingness to serve.

Some Specific Examples of Data Projects

Below we list a few samples of the types of data analysis projects on which the authors have worked in the last few years.

Faculty Load Release Time: Our institution has just completed a review of load releases within each academic department. One would assume that the academic unit head (either department chair or school dean) would be the recipient of the majority of the load release within their department or school. This was in fact the case in only a minority of departments. The best explanation is historical accretion of small amounts of load release for immediate compelling needs. Left unexamined over decades, this led to the bloat that was discovered.

The new allocations assign a total load release budget to each academic unit with a requirement that a minimum number of units be spent on academic administration, usually in the form of a chair or school dean. There is a certain "yuck factor" to performing the role of chair or school dean and this was assigned a yearly allotment of 6 semester credit hour base release for undergraduates and a complexity dependent base number of units for graduate programs (par value 8). This "yuck factor" covered the first 2500 undergraduate SCH's and the first 750 graduate SCH's per semester. Thereafter, a sliding scale was implemented based on the number of SCH's generated. The sheer number of full-time faculty and adjuncts to be managed created another sliding scale factor (about 0.2 units per FTE faculty member). To take one example, the department of Mathematical, Information, and Computer Sciences will now receive 6 units of base (the "yuck factor"), 3 units for SCH production, and 1 unit for faculty management.

The formula was arrived at by setting up a spreadsheet where the values of base SCH cut-offs and the SCH values needed for each additional unit of departmental release could be changed. In addition, the ratio of release units to FTE faculty managed was also changeable. The resulting cost of each change was displayed as a dollar figure above or below the current total allocation. In the end, some tweaking beyond the model was done by the provost. The result of the process was that it helped the academic leaders get a sense of the implications of their individual preferences within the model. Ultimately, the normalization of release time saved the school about $200,000 annually.

The Computation of the Cost of a Student Credit Hour (SCH): Since the goal of this work is to compare the revenues and costs in the same discipline within a host of institutions, it is necessary to compute the relevant values in a consistent manner. For the Delaware Study, the key ratio that is computed for a department or program is the cost per student credit hour. We will focus first on the revenue side of the equation. Even though many colleges have a flat tuition rate for taking between 12 and 17 credits per semester, we will use the Student Credit hour (SCH) as a proxy for revenue.

Let us begin with a straight-forward example from a History program which is housed in the History and Political Science department. In this case the cost per student credit hour for this History program will be compared with that of other History programs. For a single course example, consider HIS 233 which is a three credit semester-long course. Suppose that section 1 meets from 11:00-11:55 MWF for the whole semester. If there are 60 students taking HIS 233 section 1, then there are 3 x 60 = 180 student credit hours generated for this section of this course. This sort of section is the norm on many campuses and hence is easy to keep track of. However, some of the other sections of the same course may only have 2 or 3 students enrolled. With such a low enrollment, this is probably an independent study section. When looking at institutional records, the giveaway is that there is no time or room associated with most of the independent study sections. Local knowledge and relationships with the folks in the History and Political Science department will be necessary to figure out which of the low enrollment courses with assigned rooms and meeting times are in fact independent study sections. The Delaware Study has different roles for standard class sections, independent studies, practicums and labs in the computations of SCH.

For a less straight forward example, let us turn to the Biology program which is housed in its own department. The difficulty comes from lab sections. Suppose that BIO 214 is a five credit semester-long course. Suppose that section 1 meets from 11:00-12:10 MWF for the whole semester and for this lecture, the students receive 4 units of credit. It also meets from 2:30-5:30 T for a lab and for the laboratory students receiver 1 unit of credit. This follows the pattern of the earlier example. For the lecture portion, the 24 students enrolled in the course generate 4 x 24 = 96 SCH. For the lab portion, they generate 1 x 24 = 24 SCH. For this section then there are 120 SCH's generated. The same procedure would be used for each of the other sections. The difficulty arises in the quirky nature of the discipline. At our institution, and some other institutions, this type of course is actually in the catalog as a four hour course. The rationale is that the lab counts for zero credit. Thus only one section of the 4 semester hour of credit lecture for BIO 214 is listed in the schedule. Each lab section is listed separately for zero semester hours of credit. It is clearly the case that the faculty members do convince most of the students that this makes sense. Surprisingly, the faculty members themselves who teach the labs do not view them as having zero load credit for their own schedules or wages. In addition, Delaware's methodology does not allow for zero unit labs so an institutional reporting methodology needed to be created. For BIO214 and all other courses with zero credit labs, ¾ of the units (in this case 3) are reported as lecture units and ¼ of the units (in this case 1) are reported as laboratory units.

We turn now to the expense side of the equation. That is, what does it cost to provide all of the SCH's generated in the program? All of the salaries, benefits, and taxes paid on behalf of each employee must be combined for all faculty (full and part-time) and for all support staff. In addition, direct program budget expenditures are recorded. The total cost is then divided by the total SCH's computed earlier to produce the cost per student credit hour. This ratio pertains to a single program and can be compared with the same program at other institutions.

The Delaware Study methodology uses faculty load records and contract information to determine full-time equivalent faculty members in the various employment categories (Tenure/Tenure Eligible, Other Regular faculty, Supplemental Faculty and Teaching Assistants). The fixed ratio is 12 units of faculty load counts as one FTE.

There are some interesting issues that come up when you go about opening program schedules and faculty load records for individual departments. The first question to ask is "Is this section cross-listed?" If it meets in the same room, at the same time, with the same instructor for some part of the course, then it is cross-listed. As we saw earlier, it can be rather difficult to determine if something is an independent study section of a course that typically meets in a regular manner. The rule of thumb is the Duck Standard: If it walks like a duck, and it sounds like a duck, it is a duck. If it sure seems like an independent study and clarity cannot be achieved from the available local knowledge, it is declared to be an independent study. If a class appears to be cross-listed, it is declared to be cross-listed.

Another set of interesting issues comes up when two different programs claim the same class. The Delaware Study methodology is to assign the course to the department from which the instructor's salary is drawn. A more difficult setting is where the faculty load records for multiple programs claim the same 12 students who meet at the same time in the same room. In this case, a keen political sense is needed. It is tricky to know when to point out that you have just opened a door and been hit with a falling skeleton. Sometimes, the right approach is to quietly put the bones back and then to save the knowledge until it can actually be acted on by the right person at the right time. In short, you have to anticipate who will be upset and why. Political skill matters!

Data Management: The examples above concern only a few of the variables that need to be calculated for the Delaware Study. To complete the study required a collection of spreadsheets. One contained salary and benefit data for all the professors and for the staff whose work directly supported the academic departments and schools. Another contained the faculty load records for each of the collections of academic departments which are organized as say "The College of Arts and Sciences." The actual fall schedules with individual course enrollments were stored in another spreadsheet. Some of this data was supplied by units such as HR, but the majority was simply pulled from the Informix database using ODBC connectivity and Microsoft's® Query functionality sitting on top of Excel®.

Beyond that we found it very helpful to have all of data for the programs in a single department stored and analyzed in the same spreadsheet. We kept versions of such spreadsheets to allow for surface level reasonableness checks and for more systematic auditing.

Communicating Results: When the technical results are in place, it is incumbent on the researcher to find the right language and pictures to convey the information to each of the audiences. The books by Edward R. Tufte come to mind as helpful (Tufte, 2001). Often the numerical results must be conveyed in narrative form. At other times, graphical presentation works best. Usually it is a combination of the two forms that will best communicate the ideas to the audience.

Below is an example of how the Delaware Study data was presented for all of the academic departments. What is shown below is fictitious data with the formatting that we developed.

White Columns: When providing the data for the national comparison group, the Delaware Study provides 25th, 50th and 75th percentile levels for the cost of each program in your Carnegie class in their data base. Those values are indicated in the white columns for Program 1 and Program 2 in Fictitious Department. Note that the costs are quite different between programs with Program 2 being much more expensive nationally.

Colored Columns: A linear approximation was used to identify the 37th, 62nd and 90th percentile levels. This information was simply used to give departments some sense of where their values were in relationship to the overall data.

The Asterisk: This indicates where the cost per student credit hour falls for Program 1 and Program 2 in Fictitious Department. As you can see Program 1's cost is in the 50th-62nd percentile range, as indicated by the green shading, within institutional expectations. But Program 2's cost is in the 75th-90th percentile range. That is higher than expected as is indicated by the orange shading.

The P: The Delaware Study allows each institution to select a set of at least 10 peer institutions to also use for benchmarking. We found that not all of the programs offered in our institution were offered by our peers, but if there is a peer comparison, it is marked with a P. So Program 1 had a peer comparator number and it looks like Program 1 aligns well with the costs of our selected peers for the same program.

	25%	25%-37%	38%-50%	50%	50%-62%	63%-75%	75%	75%-90%	90% +	
Fictitious	180			220	* **P**		260			Program 1
Department	220			270			400	*		Program 2

While it took multiple days to identify a way to present all of the relevant data in a single graphic, the net result was that academic leaders from all disciplines could make meaning from the data. There were many subsequent conversations that started with questions of the type "What factors are making the cost per credit hour for Program 2 so high?" This data gave us a starting point to have those conversations.

As you can see, the data management, analysis and presentation work that is needed by higher education institutions is as complex as it is important. The authors of this paper encourage you to get involved, you have skills that your institution needs.

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The Unity of Knowledge and the Faithfulness of God: The Theology of Mathematical Physicist John Polkinghorne

Matt DeLong*

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O God, because without you we are not able to please you,
mercifully grant that your Holy Spirit may in all things direct and rule our hearts,
through Jesus Christ our Lord, who lives and reigns with you
and the Holy Spirit, one God, now and forever. Amen.[1]

1 Introduction

The Rev. John Polkinghorne, KBE, FRS, is arguably the greatest living Christian voice in the dialogue on science and religion, having written on "the cosmos, the nature of the universe, relativity, chaos theory, string theory, critical realism, philosophy, nature, theology, the end of the world, the Trinity, the character of God, [and] divine action... [1, p. 181]," among other things. A well-decorated mathematical physicist, Polkinghorne resigned his academic post mid-career to study for the Anglican priesthood. He has since become an influential theologian and a prolific author. Polkinghorne is widely admired by Christian academics for his thoughtful and winsome defense of the harmony between science and faith, and yet his theological views are not without controversy.

In this paper we will give a brief introduction to Polkinghorne's life and work. We will give an introduction to Polkinghorne's approach to philosophy and theology. We will introduce the two most significant influences on Polkinghorne's development as a theologian and philosopher of science. We will then give a necessarily telegraphic review of some of the topics addressed in Polkinghorne's theology, including his thoughts on science and religion, natural theology, evil, providence, prayer, resurrection, the soul and eschatology. We will then conclude with a few short examples of Polkinghorne's thoughts on mathematics.

2 Biography

John Polkinghorne was born on October 16, 1930 Weston-super-Mare, England to George and Dorothy Polkinghorne. His dual loves of science and theology were nurtured by his schooling, in which his mathematical precociousness was recognized and encouraged, and by his church and home life, from which his lifelong Christian faith was birthed. In addition to these joys, the sadnesses of his youth helped to shape the mathematical physicist and theologian that he would become. His sister Anne died at age 6 in 1929, before John was even born. And his only other sibling, Peter, died in 1942 flying in World War II for the Royal Air Force. This close intimacy with natural and human evil, as well as the puzzlement of unanswered prayer, shaped John's

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[1] A traditional Anglican prayer from *The Book of Common Prayer* that John Polkinghorne often prays before public events [1, p. 14].

thinking on the nature of created and divine reality, as well as our understanding of it through science and theology.

After a promising start to his mathematical education in primary and secondary school, John served briefly in the military. This too shaped his future course, as one of his duties was to teach elementary mathematics to apprentices studying to be engineers. Thereafter, John was given a scholarship to Trinity College, Cambridge, from which he completed both an undergraduate degree in Mathematics and then a Ph. D. in Physics. While at Trinity John's life was significantly changed in two other ways. During one meeting of the Christian Union, John felt called by God to deepen his commitment to his faith in Christ, which he did. In addition, John met and began dating Ruth Martin, a statistics major at Girton, whom he eventually married.

After John completed his doctorate and they were married, John and Ruth moved to Pasadena, CA, where John took a Post-Doctoral position at The California Institute of Technology. There John was on a team that developed the mathematical models that predicted the eventual discovery of quarks. John then return to the UK to teach for a couple of years at Edinburgh, before returning to take a post in Mathematical Physics at Cambridge.

In 1977, having "done his bit for the subject [1, p. 54]" of physics, John did something that shocked the community of mathematical physicists. He resigned his faculty position in order to study for ordination in the Church of England. John very much enjoyed his biblical, theological and pastoral studies, approaching them as the scientist that he was.

After graduation and ordination, John entered parish ministry when he became the vicar in tiny Blean. After a few years serving the church and the people of Blean, John was called back to Cambridge once again to be the dean and chaplain at Trinity Hall. His ministerial career then rejoined his academic career when he was elected to the presidency of Queens' College, Cambridge. After administering this role for a little over a decade, John retired from any official academic positions in 1996. However, up to the present day, he has remained prolific as a writer, public speaker, consultant and advocate on issues relating and related to science and religion [1, 5, 10].

3 Publications and Honors

Since his first book, *The Analytic S-Matrix* (1966, with Eden, Landshoff and Olive), Polkinghorne has written or edited seven book on science and mathematics, including his personal best-seller, *The Quantum World* (1985). His research and writings in science were significant enough that Polkinghorne was elected as a Fellow of the Royal Society in 1974.

As impressive as it is, Polkinghorne's scientific output pales in comparison to his prolific theological output. His first theological work was *The Way the World Is* (1984), which was the beginning of a corpus that now includes over 30 theological volumes either written or edited by Polkinghorne.

Most of Polkinghorne's theological books are short, on the order of between 100 and 200 pages. Recently, some have undertaken the task of trying to gather and summarize the whole of Polkinghorne's theological writing in a way that is accessible. Oord (2010) edited for Polkinghorne selections of his writings chosen by Polkinghorne himself to represent the best of his work, and organized into twenty-two topical chapters [27]. More accessible is Giberson and Nelson's 2011 biography of Polkinghorne [1], which is both an engaging introduction to a winsome and thoughtful man, and also a succinct overview of his thought.

Polkinghorne's writings on theology and science have been so significant that he's been called by Ian Barbour, "...one of the pioneers in the field [of science and religion]." He's been described by Philip Clayton, as "...the most-read defender of the religion-science dialogue." He's also been called "...Britain's leading scientist theologian..." by Keith Ward [27]. These three are all highly significant praises, given that they are coming from some of the leading thinkers and writers in the dialogue between Christianity and science.

The recognition of Polkinghorne's work goes beyond the anecdotal. He has been invited to give several prestigious lectures, including the Gifford Lectures at the University of Edinburgh, the Terry Lectures at Yale University, and the Parchman Lectures at Baylor University. Polkinghorne was even knighted as a Knight Commander of the Order of the British Empire, although as is customary for priests in the Church of England, he is not called "Sir." Perhaps the most telling sign of Polkinghorne's significance as a leader in the dialogue between science and faith is his 2002 Templeton Prize for Progress on Science and Religion [1, 5, 10].

4 Philosophy and Method

4.1 Critical Realism

Both as a scientist and as a theologian, Polkinghorne looks through the philosophical lens of critical realism; he believes that our perceptions can constitute a reliable guide to what really is. In One World, Polkinghorne argues for the inadequacy of non-realist philosophies of science such as positivism, in which the aim of science is merely to arrive at theories that are intersubjectively agreed upon by scientists, and idealism, in which the perceptions of science are said to be merely the results of our observational procedures [22, pp. 21-31].

Polkinghorne instead insists that the terms of our scientific understanding "are dictated by the way things are [22, p. 2]." Polkinghorne is a realist in part because he sees the observed success of science as pointing to the fact that scientists are "gaining a tightening grasp of an actual reality [22, p. 2]."

However, Polkinghorne gives three reasons for his belief that the only defensible realism is a critical realism. The first is that science can only ever claim verisimilitude, i.e. it can only ever give adequate accounts of limited physical regimes, never a complete account of all physical reality. Second, our everyday notions of objectivity prove insufficient for regimes far removed from our familiar experience, such as the strange world of Quantum Mechanics. Finally, the judgment and discretion of scientists do play a role [22, p. 21].

For Polkinghorne, the philosophy of critical realism applies to theology as well as to science. Whereas in science he believes that what we know from nature is a trustworthy guide to what really is, in theology he believes that what we know from God's revelatory acts is a trustworthy guide to God's real nature [18, p. 101]. For science, this trust is underwritten by its fruitfulness. For theology, this trust is "directly underwritten by the faithfulness of the God so revealed, who will not be a deceiver [18, p. 101]."

4.2 Bottom-up Thinking

Derived from his critical realist philosophy, Polkinghorne uses a method of enquiry that he calls "bottom-up thinking." Instead of starting with presupposed answers, whether in science or theology, and then looking for evidence to support those answers, a bottom-up thinker looks at the evidence and then draws conclusions that the evidence suggests [24, p. 145]. Bottom up thinking, in other words, moves "from evidence to understanding [12, p. 127]."

In science, for example, Polkinghorne uses a bottom-up approach to his interpretation of Quantum Mechanics. While acknowledging that there are empirically adequate deterministic accounts of quantum uncertainty, he reads the situation "from below" as the observed unpredictability indicating real randomness [9, pp. 39-40]. Polkinghorne makes a similar move in interpreting chaotic dynamical systems. He interprets the epistemological indeterminism of a chaotic system as aligning with a true ontological indeterminacy [9, p. 41].

Polkinghorne also applies bottom-up thinking to theology. For example, in regards to the trinity, Polkinghorne is persuaded by the so-called "Rahner's Rule [28]." In this, the economic

Trinity, God as God is known through creation and salvation history, is identified with the immanent trinity, God as God in Godself [18, p. 101]. As a second example, Polkinghorne believes that our understanding of who Jesus was pivots on the critical event of his resurrection from the dead [10, p. 120]. Here he is of course following in the good company of the Apostle Paul (1 Cor. 15:14).

5 Influences

5.1 Polanyi

Perhaps the greatest influence on Polkinghorne's philosophy and method was Michael Polanyi, the Hungarian chemist and philosopher, and in particular his book, Personal Knowledge. Polanyi rejected the idea that scientists were purely objective, and instead believed that all knowledge is personal.

Scientists, like all other observers, cannot separate themselves from their experiences and backgrounds [1, p. 32]. Scientists attempt as a community to rise above this, and collectively, albeit imperfectly, move towards an objective description of reality. In this way, the scientific community is not altogether different from a religious community.

As Polkinghorne puts it, "All human knowing is from a particular point of view, which will offer opportunities for insight but be bounded by its inherent limits [1, p. 32]." Thus properly practiced and understood, both science and faith seek to understand ultimate realities, but recognize that they fall short of perfect understanding. Polkinghorne summarizes Polanyi's thinking into the maxim "To commit myself to what I believe to be true, knowing that it may be false [1, p. 32]."

5.2 Moltmann

While the philosophy of Polanyi had the greatest impact on Polkinghorne, the philosopher, Polkinghorne, the theologian, was most greatly influenced by the theology of Jurgann Moltmann. Moltmann, a German theologian, came to faith in Christ as a prisoner in an Allied Prison camp, and his theology is profoundly shaped by the horrors of the Holocaust.

Moltmann's theology has influenced Polkinghorne's thinking on a number of topics, including the Trinity, the incarnation, the resurrection, creation, and eschatology. However, Polkinghorne's thinking was most impacted by Moltmann's seminal work, The Crucified God, which shaped Polkinghorne's thinking on evil and suffering.

For Moltmann, the cross of Christ is where God participates in the suffering of the world. As Polkinghorne sees it, this is a critical insight for theology, particularly as it wrestles with its deepest challenges. "The concept of a suffering God affords theology some help as it wrestles with its most difficult problems... [21, p. 22]."

In fact, for Polkinghorne this insight is not just critical for theology, it is critical for faith. "The insight of the crucified God is at the heart of my own Christian belief...and indeed of the possibility of that belief [13, p. 44]." The towering intellectual, who now wrestles with the questions of evil and suffering on a world-wide and even cosmic scale, was once the sensitive 12-year-old boy wrestling with the question of why the God that he lifted his brother, Peter, to during WWII allowed him to perish in his plane over the Atlantic Ocean. Moltmann's insight allows him to see this God not as uncaring for Peter, and for John, but as suffering with them in their loss.

6 Examples of Theological Thought

Although one may assume, since Polkinghorne was a physicist before he was a priest, that his theological concerns have to do only with issues at the obvious intersections of science and faith.

Although he does address such questions as the origin of the universe or cosmological arguments for the existence of God, his theological thought is much more wide ranging than that. He also brings the insights of science to bear on such theological issues as the question of evil, providence, prayer and the soul. Moreover, on topics for which the insights of science have no obvious input to his theologizing, such as the scriptures, the resurrection of Jesus, and Trinitarian theology, Polkinghorne brings his methodology as a scientist of bottom-up thinking to bear. Below we give very brief summaries of just a few of the many theological issues addressed by Polkinghorne in his prodigious writings.

6.1 Science and Religion

There are today many people of varying perspectives, including both atheists and Christians, who see science and religion as being in fundamental conflict. Such people might view someone, like Polkinghorne, who is both a physicist and a priest much as they would a "vegetarian butcher [21, p. 1]." By contrast, Polkinghorne quite obviously sees no conflict between his roles as scientist and Christian believer. In fact, for him it seems "a natural and harmonious combination [21, p. 1]."

To Polkinghorne, science and theology share a "cousinly relationship in their search of truth [21, p. 15]." Their focus is on different dimensions of truth, and yet they both are built on the conviction that there is truth to be sought. Thus, they are complementary, not competing, pursuits. Polkinghorne pursues both, striving to look "with both the eye of science and the eye of religion" because this binocular vision enables him "to see more than would be possible with either eye on its own [27, p. 3]."

Moreover, for Polkinghorne both science and theology deal with the surprising and counter-intuitive character of reality, which although lending itself to verisimilitudinous description, can never be exhaustively grasped. In his Terry Lectures, Polkinghorne demonstrated the cousinly relationship between science and theology by finding five points of analogy between the struggle for understanding quantum theory and the struggle for understanding Christology. In both cases, there were moments of enforced revision, a period of unresolved confusion, new synthesis and understanding, continued wrestling with unsolved problems, and finally deeper implications of the new theory. [13, p. 29].

6.2 Natural Theology

Polkinghorne defines natural theology as "the search for the knowledge of God by the exercise of reason and the inspection of the world [25, p. 8]." Unlike those who would deny the possibility of such knowledge, such as Barth for example, Polkinghorne believes in the possibility of deriving some limited knowledge of God from natural theology. On the other hand, unlike those who would derive "proofs" of God's existence from reason or nature, such as Anselm or Aquinas for example, Polkinghorne sees the claims of natural theology as not being logically coercive demonstrations of the existence of God but rather as offering theistic belief as a credible and coherent possibility [13, p. 10].

Polkinghorne rightly eschews the possibility of theology providing answers to the questions of all the other disciplines, but rather affirms that theology must utilize the insights from the other disciplines in order to set them "within the most profound context available [25, p. 7]." Moreover, since theology views reality in its totality, Polkinghorne invites every form of human understanding to contribute to the theological task, saying somewhat cheekily that "Theology cannot just be left to the theologians... [25, p. 8]."

For Polkinghorne the two major contributions of natural theology are that the natural world, and our scientific understanding of it, point to a Divine Mind and a Divine Purpose. To Polkinghorne, the hints of such divine presence are most clearly seen in the vastness of the universe and in humanity, those in which the universe has become aware of itself.

The Divine Mind is to Polkinghorne a most reasonable explanation for why mathematical beauty is consistently a guide to scientific truth about the created order, and moreover why humans are endowed with the capacity to appreciate this beauty. "The 'unreasonable effectiveness of mathematics' in uncovering the structure of the physical world (to use Eugene Wigner's pregnant phrase) is a hint of the presence of the Creator, given to us creatures who are made in the divine image [13, p. 4]." To find hints of the Divine Purpose, Polkinghorne does not look towards the fruitfulness and diversity of life, because he believes that Darwin has abolished that form of the argument from design. Rather, fitting of a physicist, Polkinghorne sees this Divine Purpose in the anthropic fine-tuning of the universe. "What we have come to understand is that ... [the laws of nature must] take a very specific, carefully prescribed form. Any old world will not do [13, p. 6]."

6.3 Evil

Polkinghorne is charitable towards those who do not believe in God, saying that "Atheists aren't stupid... [1, p. 158]." Like many, he understands the greatest challenge to belief, for both the skeptical and the faithful, to be a perpetual one–the problem of evil and suffering. "Of all the difficulties that hold people back from religious belief, the question of the evil and suffering in the world is surely the greatest [19, p. 138]." Polkinghorne believes that this is both a logical and an existential challenge, and it is not just the existence of evil and suffering that frame the challenge, but also their scope, particularly in light of twentieth-century atrocities, such as the Holocaust [19, pp. 145-6].

Polkinghorne separates the problem into two parts: moral evil and natural evil. He acknowledges that there is not a clear-cut distinction between the two, for evil moral choices can exacerbate the suffering brought on by natural evil and vice-versa. However, the distinction is important in considering a theodicy. Whereas, the responsibility for moral evil lies with human beings, it seems that the responsibility for natural evil ultimately lies with God [19, p. 138].

Polkinghorne considers, and dismisses, two common approaches to theodicy. The first is to lay the blame for the natural evil in creation at the Fall of Adam and Eve. Polkinghorne believes that a literal reading of Genesis 3:14-19, as indicating that an original act of moral evil led to a curse upon creation that is the source of natural evil, is incompatible with the long history of life that includes death as a seemingly necessary component. Instead he interprets the Fall in mythical terms as a turning away and an alienation from God, the only source of hope for transcending mortality and our finite existence [19, p. 139-140].

The second approach that Polkinghorne finds lacking is to deny the absolute reality of evil. He rejects the view that evil is merely a deprivation, or absence, of the good. Here again he refers to the Holocaust as reason for not being "quick to be dismissive of the possibility that there are also non-human powers of evil loose in the world [19, p. 140]."

Polkinghorne's preferred approach to theodicy is a variant of the greater good theodicy. In regards to moral evil, he claims the common free will-defense: "a world with freely choosing beings, however bad some of their choices may prove to be, is a better world than one populated only by perfectly programmed automata [19, p. 141]." He does, however, note that in a post-Holocaust world, this cannot be claimed without a quivering voice. His approach to natural evil is similar, and he terms it "the free-process defense [19, p. 143]." In this view, creation is a place of evolving processes, where creatures, in fact all of creation, is allowed to participate in making itself. This necessarily includes the cost of death as a prerequisite to the possibility of new life. Moreover, some things such as cancerous tumors and earthquakes just happen, not for a locally specifiable purpose, but because of "the brute fact of occurrence [19, p. 144]."

6.4 Providence

Polkinghorne's view of God's actions in the world is derived party from scientific considerations and partly from theological considerations. Of primary significance is his view of creaturely freedom articulated in the free will defense and free process defense cited above. "I believe that God wills directly neither the act of a murderer nor the devastation wrought by an earthquake, but both are permitted to happen in a world that is more than a divine puppet theatre [24, p. 141]." However, Polkinghorne does believe that God interacts with creation in particular ways in particular circumstances, rejecting the deistic view of an uninvolved, clockmaker God [1, p. 72].

A key question for thinkers at the interface of science and theology is, how can we "speak with integrity about the notion of God's acts in the world, whilst at the same time accepting with necessary seriousness what science can say about the world's regular processes [13, p. 48]?" Towards an answer to this question, Polkinghorne assumes that human choice is real and not illusory [13, p. 49]. He also relies heavily on the recent scientific insights, both from Quantum Theory and Chaos Theory, that there are inherent unpredictabilities in the physical world [13, p. 50].

However, a key philosophical question is whether these unpredictabilities are merely epistemological or whether they are also ontological. Although Polkinghorne acknowledges that there is no logically forced decision on how closely epistemology should model ontology, he believes that the working experience of the vast majority of scientists and the cumulative success of science provide a rational defense of this position [13, pp. 52-3].

Polknghorne examines and rejects several common approaches to understanding divine agency. He rejects a minimalist approach, in which God's acts are confined to his faithful upholding of the universe, as being an inadequate account of history and personal experience. He rejects the idea that God acts only by influencing people, partly because that leaves God without causal activity throughout the vast majority of the history of the universe. He rejects process theology, with its account of God's actions as being limited to a divine lure in each unfolding event of the universe, because it relegates God too much to the margins. Instead he favors an analogical approach, where we understand God's actions in relation to creation analogously to the way that we understand our actions in relation to our bodies [13, pp. 54-6].

This does not mean that Polkinghorne sees creation as God's body, both because that view lacks the degree of coherence and interdependence of biological organisms and because it would leave God in thrall to the changes, and even long-run cosmic futility, of the universe [13, p. 57]. Instead, Polkinghorne views our actions in our bodies as coming from top-down causality, not reducible merely to the bottom-up interactions of its constituent parts. By analogy, he believes that God also acts in creation using top-down, holistic causality [13, pp. 57-8].. However, this begs the question of where we might find possibilities for understanding the "causal joint [3]," which connects God's top-down agency to creation.

Polkinghorne admits that he is in the land of conjecture, saying that "With the nature of human agency still mysterious, we can hardly dare to aspire to more than hopeful speculation when it comes to talk of divine agency [13, pp. 59].." However, as one trying to integrate scientific and theological views of the world he believes that such conjecture is necessary. He believes that the unpredictabilities identified by Quantum Theory and Chaos Theory provide possibilities for understanding the causal joint. For various reasons, including the difficultly of understanding the relationship between the micro world and the macro world, he does not believe that Quantum Theory is the most useful place in which to look for this causal joint. Instead, he favors finding it in the structured unpredictabilities of Chaos Theory, in which negligible changes in initial conditions can radically affect long-term outcomes, but only within the confines of a strange attractor of possibilities [13, pp. 59-62].. Polkinghorne believes that God might execute his willed intentions through "top-down causality at work through 'active information [13, pp. 63].'"

Polkinghorne derives several theological consequences from this view of God acting through top-down information input. These include the following four observations.

1. This view is a modern, scientifically dressed translation of the long-held discussion of the hidden work of the spirit.

2. There will be in inextricable entanglement among the actions of God, the actions of human, and the unfolding process of nature. Hence, save for the eyes of faith, the divine actions in the world are necessarily hidden from view.

3. God does indeed act in large measure in the world through faithfully maintaining predictable natural processes.

4. Although God acts consistently in comparable circumstances, unprecedented circumstances provide occasion for God to act in unique ways. This then is how we may understand divine miracles in the context of this present discussion.

5. Since the physical universe is one of true becoming, where the future being formed in temporality, God does not know the future in exhaustive detail. Omniscience is self-limited in the kenotic act of creation. [20, ch. 4].

On the last point, Polkinghorne would be thus be described as an Open Theist. "Even God does not yet know the unformed future, for it is not yet there to be knownthough undoubtedly God sees more clearly than any creature can the general way in which history is moving [24, p. 141]."

6.5 Prayer

"Can a scientist pray?" is the question that Polkinghorne asks as the theme of his second of three Parchman Lectures at Baylor University in 2002 [17]. Of course, this might depend upon what one means by the question. Prayer can involve many different aspects, such as worship, waiting in stillness and contemplation [20, p. 80]. And Polkinghorne believes that scientists pray prayers of adoration, even if they do not know it, when they marvel at the wonder of creation as revealed through science [1, p. 72].

The question, then, brought forth by Polkinghorne above is in regards to petitionary prayer. Scientists attempt to describe the orderly and predictable nature of the world. Can a scientist pray for that which seems to go against this order? "There is a great deal at stake for Christianity... if we can answer 'Yes' to the question, 'Can a scientist pray a petitionary prayer in a way that is positive, without impugning his or her scientific integrity?'" [14, p. 28]. As argued in the subsection on Providence, Polkinghorne is not a deist, rather he believes in a personal God who acts in particular as well as in general ways. Moreover, this God, through the words of Jesus, invites us in a shockingly direct way in Matthew 7:7 to ask things of him [20, p. 80].

However, Polkinghorne rejects the idea of prayer as akin to magic, in which we use spiritual or other powers to get what we want from God. [1, p. 65]. Of course, God already knows what we want and what we need. Thus, to Polkinghorne "prayer is neither the manipulation of God nor just the illumination of our perception, but it is the alignment of our wills with his.... That alignment is not just a passive acceptance of God's will by human resignation..., but it is also a resolute determination to share in the accomplishment of that will... [20, p. 80]." Polkinghorne notes that prayer is also where God asks us to commit ourselves to what we desire, citing Mark 10:51 [20, p. 83]. In other words, prayer is genuine encounter of persons; it is a "collaborative personal encounter between man and God, to which both contribute [20, p. 80]."

Polkinghorne believes that the future is genuinely open to both human and divine agency, hence the viability of believing in petitionary prayer. Moreover, he believes that God is a good Father who is perfectly good and who desires to give the best to each of his children [20, p. 83].

Polkinghorne also acknowledges, soberly no doubt when recalling the deaths of his brother Peter and his good friend John Robinson, among other times of seemingly unanswered prayer, that when talking about prayer we must admit to the "strangeness of individual human destiny [17]."

Nevertheless, even in the face of death, such as at the deathbed of his beloved wife Ruth, Polkinghorne believes that prayer brings us into the presence of the God "to whom all hearts are open, all desires known, and from whom no secrets are hidden [2]." And being in the presence of this loving God heals our hearts, even if God does not answer a petitionary prayer by granting physical healing, but rather allows nature to take its course [1, p. 72].

6.6 Resurrection

"It is absolutely clear that something happened between Good Friday and Pentecost [10, p. 109]." So begins Polkinghorne's argument for the historicity of the resurrection of Jesus of Nazareth within the larger context of his defense of the Nicene Creed. For Polkinghorne, as for Paul (Cf: 1 Cor. 15:14), this question is not merely one of intellectual interest. Rather, for Polkinghorne the resurrection "is the pivot point on which the Christian faith turns [10, p. 109]." Whereas Polkinghorne accepts much historical criticism of the Biblical text [26, p. 8], he finds plenty of warrant for a motivated belief in a risen Lord and Christ.

Like the bottom-up thinker that he is, Polkinghorne asserts that there is much "evidential motivation for believing that Jesus was indeed raised from the dead [26, p. 77]."' The list of evidences that he gave in his Gifford Lectures included

- the transformation in the disciples,
- the exaltation of a crucified man is too paradoxical a notion to have been formed merely from a process of reflection,
- a swoon would not convince the disciples that Jesus had conquered death,
- hallucinations would not account for the variety of times and places of the resurrection appearances,
- the antiquity of the written testimony of the resurrection in the letters of Paul,
- the appeal to the witness of those still living at the time,
- the common theme in the appearances of the difficulty of recognizing Jesus,
- the emphasis on corporeality in Luke's account,
- the empty tomb,
- the fact that women were the earliest witnesses,
- the lack of a cult associated with the burial place, and
- the lack of triumphalism–even the shock and doubt of the disciples,
- the fact that the early Christians were Jews who began to worship on the first day of the week rather than the seventh,
- the ongoing experience in the church of the living Christ, and
- the contrast between the early Christian assertion of Jesus's resurrection and the contemporary expectations of both Jews and Gentiles in the ancient world. [10, p. 109-119].

However, also being a critical realist, Polkinghorne acknowledges the “hermeneutical circle in which the significance of Jesus and the truth of his resurrection inextricably interact with each other [10, p. 120]. That is, he allows that those who would look at the evidence with a prior conviction of a completely closed and uniform universe would find in the resurrection accounts only legend, not historical fact. But such an assumption rules out *a priori* any openness to what the New Testament is trying to say, which is that there is something unique about Jesus. “If God was indeed present in Christ in a unique way, then his story may rationally be believed to contain unique elements [26, p. 78].”

Polkinghorne not only gives motivation for believing in the resurrection, he also discusses its significance. For Polkinghorne, the resurrection of Jesus is a three-fold vindication. First, it is a vindication of Jesus, and his uniquely holy life and godly character. Second, it is a vindication of God, who was not found in the end to have abandoned the one who had fully and faithfully trusted in the Father. Finally, it is a vindication of the deep-seated human intuition and hope that death and futility do not have the last word, but that in the end all will be well. Rather, referencing 1 Cor. 15:22 and Col. 1:18, Polkinghorne sees what happened to Jesus as a foretaste and a guarantee of the ultimate fulfillment of humanity and the cosmos [10, p. 121].

6.7 The Soul

What is the soul? For John Polkinghorne, the simple answer is, “My soul is the pattern that is me [18, p. 161].” In wanting to take seriously both that human beings have emerged through a long and continuous process and the qualitative difference of the novelty of their emergence, Polkinghorne believes that human beings are psychosomatic unities. He does not absolutely rule out the possibility that human beings have a spiritual soul that is separable from the body. However, he finds it more satisfying to posit that our material and our mental and spiritual aspects are complementary and inseparable. Moreover, he believes that this view accords well with Hebrew and early Christian thought [19, p. 46-7].

In this view, have we then lost the soul? Polkinghorne does not think so. He believes that his perspective is very similar to those of Aristotle and Aquinas, who taught that the soul was the form of the body, although Polkinghorne puts it in modern scientific dress. Polkinghorne locates the soul in “the almost infinitely complex *information-bearing pattern* (emphasis in the original) in which the matter of the body is at any one moment organized [19, p. 47].”’ This information-bearing pattern is dynamically changing and developing, and it reflects unique human capacities, experiences, and relationships [19, p. 48]. However, it carries its own unique signature, about which Polkinghorne remarks that “A mathematician would say that there were invariant characters, preserved in the course of unfolding transformations [16, p. 107].”

If indeed “human beings look more like animated bodies than like incarnated souls [16, p. 104],” then one implication is that souls do not have intrinsic immortality. If that is the case, then how are we to understand what happens to us at our physical death? To Polkinghorne’s understanding, death is a real end, it is not the ultimate end. Polkinghorne believes that the faithfulness of God provides the basis for a coherent hope in the preservation of the information bearing pattern in the divine memory after a person’s physical death. He notes the similarity of this notion to the Hebrew concept of Sheol. Moreover, he believes that the resurrection of Jesus Christ provides the basis for trusting “that God in the eschatological future will re-embody this multitude of preserved information-bearing patterns in some new environment of God’s choosing [16, p. 107].”

6.8 Eschatology

For Polkinghorne, scientific insights on the long-run futility of the universe, not to mention the obvious fact of the not-so-long-run futility of each mortal human life, provide a significant challenge to Christian theology. This challenge, he believes, is met by the Christian virtue of hope. And this hope is eschatological. [16, p. 93] Polkinghorne agrees with Moltmann, who writes "From first to last, and not merely in the epilogue, Christianity is eschatology, is hope, forward looking and forward moving, and therefore also revolutionary and transforming the present [7, p. 16]." Moreover, according to Polkinghorne there is only one possible basis for this hope, and that is "the eternal faithfulness of the God who is the Creator and Redeemer of history [16, p. 94]." Trust in this faithfulness comes from "the knowledge that God is the One who raised Jesus from the dead [16, p. 94]."

Writing like the mathematician that he is, Polkinghorne summarizes his approach to eschatology in terms of four propositions. Indeed, in his view *any* viable approach to eschatology must be based on the following.

1. "If the universe is a creation, it must make sense everlastingly, and so ultimately it must be redeemed from transience and decay.

2. If human beings are creatures loved by their Creator, they must have a destiny beyond their deaths. Every generation must participate equally in that destiny, in which it will receive the healing of its hurts and the restoration of its integrity, thereby participating for itself in the ultimate fulfillment of the diving purpose.

3. In so far as present human imagination can articulate eschatological expectation, it has to do so within the tension between continuity and discontinuity. There must be sufficient continuity to ensure that individuals truly share in the life to come as their resurrected selves and not as new beings simply given the old names. There must be sufficient discontinuity to ensure that the life to come is free from the suffering and mortality of the old creation.

4. The only ground for such a hope lies in the steadfast love and faithfulness of God that is testified to by the resurrection of Jesus Christ [16, p. 147-8]."

Polkinghorne sees the scope of redemption to be cosmic, citing Rom. 8:18-25 and Col. 1:15-20. Just as the body of the Lord was not merely resuscitated and reconstituted, but rather transmuted and glorified, the new creation will be of matter transmuted from the transient matter of this creation [16, p. 113]. "This implies that the new creation does not arise from a radically novel creative act *ex nihilo*, but as a redemptive act *ex vetere*, out of the old [16, p. 115]."

Thus, creation is a two-step process. This world, with its evolving history and transience, is one that, although sustained by a faithful Creator, is held at some metaphysical distance from the Creator, who in a kenotic act makes way for the creation. The world to come, however, will attain eschatological *theosis*, in which creation will be wholly sacramental and share fully in the divine life. In other words, although Polkinghorne rejects panentheism as the reality of the present world, he does believe in a form of eschatological panentheism (Cf: 1 Cor. 15:28) [16, p. 113-4].

In much of the Christian tradition, the Four Last Things–Death, Judgment, Heaven, and Hell–have been the main topics of eschatological thought [16, p. 124]. Some of Polkinghorne's reflections on death have already been offered above in the subsections on Evil and The Soul. Here we will merely add that because of Polkinghorne's unswerving conviction that the love of God is everlasting, he does not see death as fixing a person's eternal destiny. He believes that God, like the father in the parable of the Prodigal, will always stand ready to greet the wanderer who returns. In addition, he believes that because of the mercy of God, those who never had a chance to hear and respond to the gospel in this life will not be denied that opportunity port mortem [16, p. 127].

Moreover, Polkinghorne believes that the judgment to come is not a fearful thing, but a hopeful thing. Although it will be "a painful encounter with reality, in which all masks of illusion are swept away [16, p. 131]," it is a necessary process in completing the work of salvation. For

Polkinghorne, the idea of the cleansing fire of God (Cf: 1 Cor. 3:11-15) approaches a purgatorial concept, and so he believes in "some revalued and re-expressed concept of purgatory [16, p. 133]" that prepares us for life in the face of a holy God. Polkinghorne conceives of judgment as "the divine antidote to human sin, just as resurrection is the divine antidote to human mortality [16, p. 131]."

We have already addressed some of Polkinghorne's ideas about the new creation, and its life with God, above. Here we will merely add that Polkinghorne subverts the concern that somehow everlasting life will be boring. Rather, reminded of 1 Cor. 2:9 (derived from Isaiah 64:4), Polkinghorne looks forward to "the unending exploration of the inexhaustible riches of God, a pilgrim journey into deepest reality that will always be thrilling and life-enhancing [16, p. 135]."

Although Polkinghorne does not believe that God ever withdraws his offer of mercy and forgiveness, even post mortem, he also does not believe that anyone "will be carried into the kingdom of heaven against their will by an overpowering act of divine power [16, p. 136]." Finding his imaginative depiction of hell not in Dante but in Lewis [6], Polkinghorne sees hell as a place that is deliberately absent of divine life. He finds the notion of annihilationism (or conditional immortality) somewhat persuasive, but he thinks that it would signal a defeat of the purposes of God's love. Rather, he is a hopeful universalist. He is open to the idea "that in the end and in every life, God's love will always be victorious [16, p. 137]." However, he is reluctant to claim *certainty* of universal salvation, because of the danger of moral indifference and cheap grace [16, p. 137].

While bringing the full powers of his thought to bear on the scriptural, theological, philosophical and scientific issues surrounding questions of eschatology, at the end of the day he admits of the difficulty of reaching definite conclusions on much of what awaits us at The End. Rather he encourages a hopeful waiting, demurring that "in many cases, the appropriate answer must be 'Wait and see.' Yet we may so wait in confident hope, because that hope is grounded in the everlasting faithfulness of God, the One who raised our Lord Jesus Christ from the dead [16, p. 138]."

7 Some thoughts on Mathematics

Polkinghorne makes frequent reference to mathematics–the subject that he had found to be "entrancing" as a precocious youth [1, p. 38]–throughout his writings. Sometimes he draws theological implications from general observations about mathematics, such as its role as the language of science or the experience of mathematical beauty. At other times he relies on specific results from mathematics–such as Chaos Theory, Gödel's Theorems, or non-Euclidean geometry–to provide grist for his theological mill. A full examination of Polkinghorne's thinking on mathematics would be a good follow-up project to this brief overview of his theology. In fact, in one of his books his thoughts on mathematics get their own chapter [13, Ch. 6]. Here we will merely note three observations that Polkinghorne derives from mathematics.

First, Polkinghorne believes that mathematical truths provide evidence of the existence of a non-physical realm. He notes that most mathematicians, himself included, are intuitive Platonists, who believe that they are discovering, and not inventing, mathematics. Thus, most mathematicians "believe that the object of their study is an everlasting noetic world which contains the rationally beautiful structures that they investigate [19, p. 5]."

Second, Polkinghorne sees mathematics as pointing to a Cosmic Mind. In particular, as the language of science, mathematics is "unreasonably effective [29]" at describing the physical world. Moreover, mathematicians often have a sense for mathematical beauty. "There is a thrill in encountering a beautiful equation which I believe is a genuine, if rather specialized, form of aesthetic experience [1, p. 37]." In fact, this sense of mathematical beauty is regularly used by scientists as a guide to truth. Together these observations contribute to the mystery of the universe's intelligibility, which to Polkinghorne is best explained by the fact that humanity is created in the image of a rational and aesthetically-minded Creator [18, pp. 63-65].

Third, Polkinghorne finds in mathematics reasons for epistemic humility and some basis for contentment with less than a fully developed understanding of reality. Seeing the search for truth as an adventure not a procedure, Polkinghorne notes that circularity of thinking is involved even in mathematics. "We have too long been bewitched by Euclid. A linear view of knowledge, as if it arose from building upon an unchallengeable foundation, does not work even in mathematics, as the nineteenth-century discovery of alternative geometries, and the twentieth century recognition of the Gödelian incompleteness of axiomatized systems, make only too clear [10, p. 32]." Indeed, we should be circumspect about what we expect that we can prove about God. "If we can't prove the consistency of arithmetic, it seems a bit much to hope that God's existence is easier to deal with [10, p. 57]."

8 Conclusion

Throughout Polkinghorne's prolific writings, many themes emerge. These include

- the compatibility of science and faith,
- the unity of knowledge,
- the possibility of motivated belief,
- the importance of Trinitarian and incarnational theology,
- freedom and openness in creation,
- the everlasting faithfulness of God,
- God's love for all creation, and
- the supremacy and centrality of Christ.

Polkinghorne's efforts in thinking and writing could be characterized by the injunction from 1 Thessalonians 5:21, "Test everything: hold fast to what is good (ESV) [4]," which is one of his favorite Scripture verses [1, p. 180]. In so doing, he necessarily thinks like the scientist and mathematician that he is. However, he recognizes the limits of science in theology, writing that the "scientific avenue into theological thinking will seek to give due weight to science, but it would be fatal to allow it to become a scientific take-over bid, affording no more than a religious gloss on a basically naturalistic account [13, p. 86]." He emphatically affirms that the god of Spinoza and Einstein, which is little more than a cypher for the rationality and order of the universe–is not the God and Father of our Lord Jesus Christ, nor of John Polkinghorne [13, p. 86].

Polkinghorne holds some beliefs that may run afoul of much American, Protestant, Evangelical thought. For example, although he believes that Scripture uniquely bears witness to the incarnate Word, he calls the notion of an inerrant text "idolatrous [26]." Although he believes that God is actively and intimately involved in creation, he is an open theist who believes that God does not exhaustively know the future. Although he affirms that all human hope is founded in the particularity of the death and resurrection of Jesus, he believes that there is no question "that God is truly experienced in other faiths [23, p. 105-6]." And although he believes in judgment, he is hopeful that in the end love will win and hell will be found to eventually be empty.

Nevertheless, thinking Christians who may disagree with Polkinghorne on some points should still find his work helpful for testing everything and holding onto what is good. Polkinghorne may be particularly helpful with finding ways to take with due seriousness the historic, orthodox claims of the Christian faith and the more recent, scientific observations about the age of the universe and the evolving character of life. He may be helpful for those faithful who doubt–those who seek a motivated and hopeful belief without being able to claim absolute certainty. He is in many ways and on many topics a guide to a middle path–neither entirely conservative nor entirely liberal. Finally, and perhaps most importantly, he can help his readers maintain an unswerving focus on the love of God for all humanity and all creation, as demonstrated and vindicated through the death and resurrection of Jesus Christ. For through all his life as a mathematician, scientist, priest, college president, theologian and family man, John Polkinghorne has been guided by his "heart's desire, [which is] to know the love of God [11, p. 105]."

9 Books by Polkinghorne on Science and Mathematics

1. The Analytic S-Matrix (1966, jointly with RJ Eden, PV Landshoff and DI Olive)
2. The Particle Play (1979)
3. Models of High Energy Processes (1980)
4. The Quantum World (1985)
5. Rochester Roundabout: The Story of High Energy Physics (1989)
6. Quantum Theory: A Very Short Introduction (2002)
7. Meaning in Mathematics (2011, edited, with contributions from Timothy Gowers, Roger Penrose, Marcus du Sautoy and others)

10 Books by Polkinghorne on Theology

1. The Way the World is : The Christian Perspective of a Scientist (1984)
2. One World (1987)
3. Science and Creation (1989)
4. Science and Providence (1989)
5. Reason and Reality: The Relationship Between Science and Theology (1991)
6. Quarks, Chaos and Christianity (1994)
7. The Faith of a Physicist: Reflections of a Bottom-Up Thinker (1994)
8. Serious Talk: Science and Religion in Dialogue (1996)
9. Scientists as Theologians (1996)
10. Beyond Science: The wider human context (1996)
11. Searching for Truth: Lenten Meditations on Science and Faith (1996)
12. Belief in God in an Age of Science (1998)
13. Science and Theology: An Introduction (1998)
14. The End of the World and the Ends of God (2000, edited with Michael Welker)

15. Traffic in Truth: Exchanges Between Sciences and Theology (2000)

16. Faith, Science and Understanding (2000)

17. The Work of Love: Creation as Kenosis (2001, editor)

18. The God of Hope and the End of the World (2002)

19. The Archbishop's School of Christianity and Science (2003)

20. Living with Hope: A Scientist Looks at Advent, Christmas, and Epiphany (2003)

21. Science and the Trinity: The Christian Encounter With Reality (2004)

22. Exploring Reality: The Intertwining of Science and Religion (2005)

23. Science and Providence: God's Interaction with the World (2005)

24. Quantum Physics and Theology: An Unexpected Kinship (2007)

25. From Physicist to Priest, an Autobiography (2008)

26. Theology in the Context of Science (2008)

27. Questions of Truth: Fifty-one Responses to Questions about God, Science and Belief (2009, with Nicholas Beale)

28. Testing Scripture: A Scientist Explores the Bible (2010)

29. Reason and Reality: The Relationship Between Science and Theology (2011)

30. Science and Religion in Quest of Truth (2011)

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Mapping Biblical Commandments to an Iterated Prisoner's Dilemma Framework

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Abstract

In his writings on Game Theory, and the Iterated Prisoner's Dilemma in particular, Robert Axelrod outlined four properties that are predictors of a successful strategy: Niceness, Reciprocity, Forgiveness, and Understandability. On the topic of Reciprocity, Axelrod makes the claim that not only does The Golden Rule lead to a suboptimal strategy, but that one of the most successful strategies (Tit for Tat) shows that a command of An eye for an eye leads to a much more optimal strategy. In this paper, we will discuss the details of Axelrods four properties, outline Biblical support for all four, and discuss how, within the framework of an Iterated Prisoner's Dilemma, neither Do unto others... nor An eye for an eye are the Biblical command that most closely matches the behavior of winning strategies in regards to the Reciprocity property.

1 A Brief Summary of the Prisoner's Dilemma

The Prisoner's Dilemma, along with the Iterated Prisoner's Dilemma variation, is a standard game that has been used to understand strategic behavior across competitive interactions between animals in behavioral ecology, *homo economicus* in micro- and macro-economic systems, and even state-actors within global political systems. A detailed discussion of the problem can be found in a wide variety of sources (see [1] for a dizzying array of applications). Here we focus on a formal synthesis that emphasizes Axelrod's computational approach [2].

In the classic problem, two criminals are captured and interrogated separately. Each prisoner has the option of staying silent or betraying their partner. Neither prisoner knows which option the other will choose. If both prisoners stay silent (cooperate), both will be convicted of a lesser crime and serve a short jail sentence. If both betray each other (mutual defection), they will be convicted of the large crime and serve a longer jail sentence. However, if one prisoner stays silent and the other betrays (cooperate/defect), the silent prisoner will be convicted and serve the longest possible sentence, while the betrayer will walk free in exchange for his cooperation. The pay-off matrix can be seen below.

	A stays silent	A defects
B stays silent	Both serve X years	A serves W years, B serves Z years
B defects	A serves Z years, B serves W years	Both serve Y years

Formally, the prisoner's dilemma requires that $W < X < Y < Z$ where the goal is to get the lowest payoff (reverse the inequality if the goal is to receive the highest payoff). For a single round of the Prisoner's Dilemma, a prisoner who betrays will always end up with a higher payoff than staying silent, regardless of what the other prisoner does. In the Iterated Prisoner's Dilemma, the two parties play the game repeatedly. An additional constraint of $W + Z > 2Y$ is generally added in the iterated form of the problem. The move from the one-shot version to an iterated version produces a shift in the optimal behavioral strategy. A simple default or bias toward betrayal is optimal in the one-shot version, while strategies that incorporate the parties' previous interactions prove better in the iterated version.

It should be noted that the Prisoner's Dilemma is not a universal description of all interactions, but is instead a description of a specific (although common) scenario where two parties must chose whether to cooperate with each other or not, and where the payoff values match the inequalities listed above.

2 Axelrod's Four Properties of Optimal Strategies

The optimal strategy for an Iterated Prisoner's Dilemma problem depends on what strategies are used by the other players. However, [2] points out that four properties appear to emerge in any optimal strategy. We summarize Axelrod's explanation of these properties as follows:

1. **Be Nice,**
   An optimal strategy will never be the first player to choose the "defect" option.

2. **Retaliate,**
   An optimal strategy will retaliate against a player who has chosen to defect.
   If your opponent is not cooperating, you should also refuse to cooperate in future rounds.

3. **Forgive,**
   An optimal strategy will return to cooperation if the other party begins to cooperate.
   This property acts as a counterbalance to the previous property. Ongoing retaliation (via defection) is a sub-optimal strategy. Forgiveness incentivizes cooperation for a formerly uncooperative party – particularly if the other party has made amends or demonstrated good faith by shifting back to cooperation.

4. **Be understandable,**
   An optimal strategy will provide sufficient information that another party can, to a degree, predict the consequences of their choice.
   The optimal strategy for a party who believes the other party is acting randomly is to avoid cooperation. Random behavior, in effect, reduces the iterative version of the prisoner's dilemma to the one-shot version.

Several strategies match these criteria. We outline two well-known strategies below.

2.1 Example 1: Tit-For-Tat

One strategy that is often optimal, or at least nearly optimal is named Tit-For-Tat. It is a simple strategy that exhibits all four properties. The strategy is simple, and has only two rules.

1. In the first round, choose to cooperate.

2. In all subsequent rounds, do whatever your opponent did in the previous round.

We can see that it meets the first property. It will never defect until its opponent has already done so. If its opponent doesn't cooperate, Tit-For-Tat will refuse to cooperate in the next round, meeting the second property. If its opponent starts cooperating again, Tit-For-Tat will return to cooperation in the next round, meeting the third property. Finally, with only two rules, Tit-For-Tat is very easy to understand, meeting the fourth property.

2.2 Example 2: Win-Stay/Lose-Shift

Within the behavioral ecology literature, the iterated prisoner's dilemma has been used to explain the development of reciprocal altruism within competitive biological systems [2]. [5] propose Win-Stay/Lose-Shift as an alternative to Tit-For-Tat. Win-Stay/Lose-Shift (also referred to as "Pavlov" or "Simpleton") can be described with a single rule:

1. Cooperate if and only if both players chose the same option in the previous round.

Win-Stay/Lose-Shift is based on standard learning and foraging strategies in which an animal will return to a potential food source where it has found food and avoid a potential food source where it has found no food. Simulated competitive evolution of agents within the iterated prisoner's dilemma showed that a generous Tit-For-Tat [1] strategy and Win-Stay/Lose-Shift strategies were most adaptive (though each strategy showed distinct advantages in different

[1] see below section 4

reward domains). Furthermore, if viewed within the context of foraging, an initial bias toward stay and sample a foraging site (cooperation) seems reasonable given that an animal has already invested the resources in reaching the foraging site. The advantage that Win-Stay/Lose-Shift offers over Tit-For-Tat or even generous Tit-For-Tat, as we will see below, is the intuition it provides regarding social movements when an iterated prisoner's dilemma allows group formation.

In general, the Win-Stay/Lose-Shift strategy adheres to Axelrod's four properties for optimal strategies. Though the simplest form of Win-Stay/Lose-Shift doesn't meet the first property, it can be easily amended with no added complexity relative to the Tit-For-Tat strategy above. If the opponent doesn't cooperate, Win-Stay/Lose-Shift will refuse to cooperate in the next round, meeting the second property. The third property is partially met because a path back to cooperation exists – though the path is not initiated by an opponent cooperation. And like Tit-For-Tat, Win-Stay/Lose-Shift is very simple and meets the fourth property.

2.3 Forgiveness vs. Pre-Forgiveness

Tit-For-Tat and Win-Stay/Lose-Shift behave identically for rounds in which they previously cooperated: they cooperate if their opponent previously cooperated and defect if their opponent previously defected. However, differences between these strategies emerge once an opponent defects. Tit-For-Tat requires that the opponent cooperates prior to once again cooperating. In a sense, Tit-For-Tat forgives only after adequate repayment has been made since it requires the opponent to forfeit any advantage it may have gained through defection. In contrast, Win-Stay/Lose-Shift cooperates if both players defect. The Win-Stay/Lose-Shift strategy to forgive prior to repayment is a gambit that suggests the gains resulting from a fast return to mutual cooperation will offset the requirement of repayment.[2]

3 Axelrod's View of The Golden Rule

While properties 1 and 3 can easily be matched with directives from the Bible (see Section 6), and property 4 is unlikely to raise any concerns, property 2 does initially seem to be in contradiction with New Testament themes of "turning the other cheek". In his book (Chapter 7, section 4), Axelrod even makes the claim

> Perhaps the most widely accepted moral standard is the Golden Rule: Do unto others as you would have them do unto you. In the context of the Prisoner's dilemma, the Golden Rule would seem to imply that you should always cooperate, since cooperation is what you want from the other player. This suggests that the best strategy from the point of view of morality is the strategy of unconditional cooperation rather than Tit-For-Tat. [2]

However, Axelrod goes on to list the downsides of unconditional cooperation, which (at least in a Prisoner's Dilemma) inevitably results in a non-altruistic player taking advantage of the unconditionally cooperative player.

3.1 Comparing Unconditional Cooperation and Tit-For-Tat

A Tit-For-Tat strategy will show some of the same characteristics as an Unconditional Cooperation strategy. Both will default to cooperation, and both are ready to forgive. However, Tit-For-Tat is much more resistant to being taken advantage of by a defecting player. In the event that the other player defects, Tit-For-Tat will also defect in the future, thus discouraging the other player from continually defecting.

3.2 Comparing Unconditional Cooperation and Win-Stay/Lose-Shift

Win-Stay/Lose-Shift, on the other hand, exhibits two interesting behaviors. First, unlike Tit-For-Tat, Win-Stay/Lose-Shift is willing to in effect "pre-forgive" after a mutual defection round and be the first to cooperate. This is more similar to Unconditional Cooperation than Tit-For-Tat's insistence that the other player be the first to show repentance. However, Win-Stay/Lose-Shift also has the property of exploiting a "sucker" (in the words of [5]). If Win-Stay/Lose-Shift defects when the other player cooperates, Win-Stay/Lose-Shift will continue to defect as long as the other player fails to reciprocate the defection. In this way, Win-Stay/Lose-Shift is quite different from both Unconditional Cooperation and Tit-For-Tat.

[2]It should also be noted that Win-Stay/Lose-Shift exploits an opponent that continues to make amends. As a result, we might argue that for Win-Stay/Lose-Shift there is no payment that is sufficient to warrant forgiveness.

4 Response to Error

One wrinkle that must be addressed is response to errors. An error in the context of the Prisoner's Dilemma game can be understood as a player deviating from its own strategy, having its choice miscommunicated, or having the other player misinterpret its choice.

Both Tit-For-Tat and Win-Stay/Lose-Shift have a danger of falling into a prolonged back-biting pattern in the event that the other player defects. For instance, if two players, one using Tit-For-Tat and the other using Win-Stay/Lose-Shift were playing, they would both tend toward mutual cooperation. However, if one of the players were to make an error that leads it to defect, the two players would enter a repeating pattern of alternating cooperation and defection[3] rather than the more optimal pattern of mutual cooperation.

Under the assumption that error is possible, the solution becomes building grace (unwarranted cooperation) into a strategy. If either strategy had a random chance of cooperating even after being taken advantage of, that would be enough to break the cycle and return to mutual cooperation.

Under the standard statements of TFT and WSLS, their probabilities of cooperating after a round where they received a pay off of $[W, X, Y, Z]$ would be $[1, 1, 0, 0]$ and $[0, 1, 1, 0]$ respectively. In other words, if TFT received payoff W or X, that would indicate that the other player cooperated in the previous round, so TFT would cooperate in the next round. If WSLS receives a payoff of X or Y, that would indicate that both players picked the same option in the previous round, so WSLS would cooperate in the next round. Any other payoffs would result in defection in the next round. If we wish to introduce some error tollerance into the system, we could change the probabilities of cooperation for TFT and WSLS to something like $[1, 1, e, e]$ and $[e, 1, 1, e]$ respectively where e is the probability of cooperation even following a payoff that would normally trigger a defection and is bounded by $[0, 1]$. This would be enough to introduce the possibility of breaking the back-biting cycle described above, at least after a while. The amount of grace, e could be adjusted based on the assumed likelihood of of an error occurring.[4]

5 Optimal Regimes

In a perfect world, the best outcome for all players in the iterated prisoner's dilemma comes from an 'always cooperate' strategy that is uniform across the population of players. 'Always cooperate' is, however, an unstable equilibrium and catastrophically fails with the introduction of even a single defector.

The incentive to defect arises from the inequality $W < X$ – the difference in outcomes given an opponent's cooperation – and the inequality $Y < Z$ – the difference in outcomes given an opponent's defection. The incentive to defect increases as $\frac{W}{X} \to 0$ and $\frac{Z}{Y} \to 0$. It decreases as $\frac{W}{X} \to 1$ and $\frac{Z}{Y} \to 1$. The incentive to cooperate arises from the fact that defection increases the likelihood that an opponent will defect in the future which removes the possibility of the best outcomes W, X for the foreseeable future.

The central difference between Tit-For-Tat and Win-Stay/Lose-Shift strategies is the path they take to cooperation. Tit-For-Tat is highly risk averse and tends to be more optimal when $\frac{W}{X} \to 0$ and when the other player's strategy is highly uncertain or exploitative [5]. Win-Stay/Lose-Shift tends to be more optimal in intermediate regimes [5]. Nowak and Sigmund's findings were based on genetic algorithm-based simulations that constrained the level of variation across generations. This slow level of variation is quite different than the massive level of variation between competition and relatively little variation within each competition in Axelrod's approach. Indeed, [5] found that generous Tit-For-Tat usually evolved first in their simulations and was only later replaced by generous Win-Stay/Lose-Shift .

As a final note, the deterministic versions of Tit-For-Tat and Win-Stay/Lose-Shift given by their respective conditional probabilities $[1, 1, 0, 0]$ and $[0, 1, 1, 0]$ respectively are suboptimal. Introduction of stochasticity into all parts of the condition probability increases their performance.[5]

[3] Specifically, in round i, WSLS would defect while TFT would cooperate, in round $i+1$ WSLS would defect while TFT also defected, in round $i+2$ WSLS would cooperate while TFT defected, and the pattern would repeat from there.

[4] Note, that as long as $e < 1$ these error-tolerant strategies remain differentiated from a strategy of Unconditional Cooperation, which would have a probability vector of $[1, 1, 1, 1]$.

[5] Yes, even introducing a random chance of defection in response to prior cooperation.

6 An Examination of Biblical Instructions for Personal Interaction

We propose that the directives from the Bible (in particular, the New Testament), taken together as a whole, exhibit the four properties identified by Axelrod. Furthermore, we believe that Axelrod mis-characterized one of these properties given the context of a Prisoner's Dilemma situation.

6.1 Be Nice

> Do to others as you would have them do to you. —*Luke 6:31*

Property one is seen in any number of verses tracing back to Jesus' second commandment to love your neighbor.

6.2 Retaliate vs. Do Not Let Yourself Be Taken Advantage Of

If we use 'retaliate' or 'punish' as our second property, we will run contrary to several Biblical imperatives.

6.2.1 Turning the Other Cheek

Note that we are specifically instructed to refrain from seeking revenge.

> Do not take revenge, my dear friends, but leave room for God's wrath, for it is written: "It is mine to avenge; I will repay," says the Lord. –*Romans 12:19*

Instead, we are given an instruction that is quite difficult to follow:

> But I tell you, do not resist an evil person. If anyone slaps you on the right cheek, turn to them the other cheek also. –*Matthew 5:39*

However, even this second instruction does not necessarily run contrary to optimal strategies we have discussed so far. Indeed, the most robust strategies will have elements of grace, or assuming that the initial offense may have been committed or perceived in error, thus warranting continued cooperation. However, even if the offense was indeed intentional and on-going, must an optimal Prisoner's Dilemma strategy involve 'retaliation'?

6.2.2 Re-characterizing the Second Property

It is property two that we believe has been mis-characterized by Axelrod. Rather than labeling it as "Retaliation" or "Punishment" as Axelrod does, we believe that property two is more accurately described as "Don't let yourself be taken advantage of".

We note that in other parts of his book, Axelrod uses the word "Reciprocity", which is closer to our interpretation.

> I urge you, brothers and sisters, to watch out for those who cause divisions and put obstacles in your way that are contrary to the teaching you have learned. Keep away from them. —*Romans 16:17*

> If your brother or sister sins, go and point out their fault, just between the two of you. If they listen to you, you have won them over. But if they will not listen, take one or two others along, so that every matter may be established by the testimony of two or three witnesses. If they still refuse to listen, tell it to the church; and if they refuse to listen even to the church, treat them as you would a pagan or a tax collector. —*Matthew 18:15-17*

> Warn a divisive person once, and then warn them a second time. After that, have nothing to do with them. —*Titus 3:10*

Note that rather than recommending unconditional cooperation, we instead find directives to sever ties with known bad actors if they refuse to cooperate. Withdrawing, or excommunication is, in our opinion, a more appropriate description of Axelrod's second property than "retaliation". This intuition meshes well with the Win-Stay/Lose-Shift strategy. And it is consistent with the exhortation to move on from those who are unwelcoming.

> If anyone will not welcome you or listen to your words, leave that home or town and shake the dust off your feet. —*Matthew 10:14*

> If people do not welcome you, leave their town and shake the dust off your feet as a testimony against them. —*Luke 9:5*

6.3 Forgive

> If your brother or sister sins against you, rebuke them; and if they repent, forgive them. Even if they sin against you seven times in a day and seven times come back to you saying I repent, you must forgive them. —*Luke 17:3-4*

Here again we see a rather clear statement of Axelrod's third property. As difficult as it may be, we are instructed that if someone who previously acted against us repents (with the change in actions implied by the original language of this verse), we are to forgive them and return to a relationship with them.

But when should we forgive: prior to repentance or after repentance? Luke 17:3-4 suggests that repentance is necessary prior to forgiveness in much the same way that Tit-For-Tat suggests that forgiveness ought to proceed. However, a counterargument could be put forward that in much the same way that we were forgiven in the midst of our sin (Ephesians 4:32; Colossians 3:13) or loved first by God (1 John 4:19), we should be willing to 'pre-forgive' in a similar way to Win-Stay-Lose-Shift.

6.4 Be Understandable

The fourth property is, oddly enough, probably the hardest to explicitly align with specific Biblical passages.[6] However, if we view the Bible, particularly the passages on lawful behavior, as an openly available guide to acceptable behavior, there is a clear, albeit implicit prompt toward predictable behavior.

7 Conclusion

Our analysis focused on Axelrod's characterization of optimal behavior within the iterated prisoner's dilemma. Axelrod's original analysis highlighted four properties observed in most winning prisoner's dilemma strategies: Be nice, retaliate, forgive, and be understandable. We suggest that Axelrod's second property, what he calls retaliation, can be better understood as reciprocation or not letting others continually take advantage. Though this is a relatively subtle difference in the characterization of the second property, it has important ramifications for Axelrod's further critique that the golden rule, "Do unto others as you would have them to unto you," is suboptimal.

We argue against Axelrod's suggestion that the appropriate instantiation of the golden rule within the iterated prisoner's dilemma is an "always cooperate" strategy. Instead, we argue that a deep understanding of the golden rule within the context of the prisoner's dilemma is in much greater alignment with the four properties he highlights and the re-interpretation of the second property. Within this context, we read each property Axelrod outlines as part of the golden rule,

- *Property 1*: We wish to meet generosity or begin interactions with cooperation, so let us cooperate.
- *Property 2*: We wish to be able to remove ourselves from exploitation, so let us imagine that others will as well.
- *Property 3*: We wish to be forgiven when we have exploited others, so let us forgive.
- *Property 4*: We wish to understand others and to not be surprised by another's actions, so let us be understandable and predictable ourselves.

We suggest that each property Axelrod highlights is, in fact, how most people would like to be treated within strategic interactions. While our treatment doesn't necessarily fit the purely "rational," homo-economicus model of human agency and strategic decision making, it does capture standard behavioral findings on human risk aversion [4] and altruistic punishment [3].

Finally, Tit-For-Tat and Win-Stay/Lose-Shift strategies receive different biblical support. In some cases, a path to interpersonal forgiveness occurs through penance and complete repayment as in Tit-For-Tat. However, in other cases interpersonal forgiveness occurs prior to penance and without repayment as in Win-Stay/Lose-Shift. We believe that an examination of the contexts within which these different approaches to forgiveness arise will provide intriguing new grounds for understanding biblical conceptions of forgiveness, reciprocity, and community.

[6] At least without delving into some questionable interpretation, such as trying to take Matthew 5:37 or James 5:12 out of context or picking only the most favorable English translations.

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Al-Khwārizmī
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30 May 2013

1 OUTLINE

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2 ✠ What is Algebra?

2.1 ◇ Tacit Definitions

✠ What is Algebra? Some Tacit Definitions

- Popular Perceptions

  Algebra is a branch of mathematics that ...
  - calculates with letters as if they were numbers.
  - manipulates symbolic equations to solve them.
  - provides formulas for science; generalizes arithmetic.
  - causes confusion, frustration, and grief for beginning students.

- Mathematics Educators' Definitions

  Algebra is a branch of mathematics that ...
  - provides symbolic representations of problems, using letters.
  - develops procedures for transforming and solving equations.
  - symbolizes general relationships/laws for numbers.
  - studies abstract structures (groups, fields; lattices; ...).

2.2 ◇ Definitions by Math Histories, Ed Research

- Outlook Assumed by (Older) Histories of Mathematics
  - Conventional periodization: Rhetorical, Syncopated, Symbolic
    - Assumes symbolic form is central/definitive
    - Values developments for advancing toward representation via letters
    - Treats early algebra as proto-algebra at best
    - Ironically, denies full-algebra status for Arabic contributions

- Outlook Offered by Recent Educational Research
  - Symbolization perspective
    - Move beyond narrow syntactic concerns
    - Focus on symbolization process (representations, transformations)
    - Symbols permit mechanization and further abstraction.
  - Generalization focus
    - Algebra offers a way to generalize/make general statements.
    - Algebra reasons with general claims, within a system of symbols, according to syntactic rules.

2.3 ◇ Definition Informed by HoM

- Refined Definition Informed by History of Mathematics
  - Algebraic features transcend and contextualize symbolization
    - Algebra is the premier quantitative problem-solving instrument.
      - Arithmetic calculates outputs; algebra solves for unknown inputs.
      - Algebra treats unknown quantities the same as known quantities.
      - Algebra uses efficient, systematic problem-solving methods.
    - Algebra enables mathematical modeling. (not our focus)
    - Algebra studies abstract structures. (also not our focus)

- History of Algebra Can Serve Mathematics Education
  - It can suggest ideas for teaching and learning algebra.
  - It can offer study materials for exploring/enriching algebra.
  - It can help refine our concept of algebra for educational research.

3 ✠ Babylonian Algebra

3.1 ◇ Background

✠ Babylonian Algebra

- Historiographic Details
  - Sources: half a million cuneiform tablets to decipher/synthesize
  - Early 20^{th} century history of mathematics
    - Mathematics studied largely in isolation from culture
    - Algebra seen as having attained a moderate level of abstractness
  - Late 20^{th} century history of mathematics
    - Mathematical developments now related to cultural contexts
    - Algebraic developments interpreted using more careful analysis

- Cultural and Mathematical Context (1800 BC)
  - Mesopotamian scribes: administrators and teachers
  - Computational expertise: sexagesimal place-value arithmetic
  - Algebraic problems: arising out of a surveyors' riddle tradition?
  - Algebraic solutions: terms suggest geometric substratum

3.2 ◇ Babylonian Quadratic Algebra

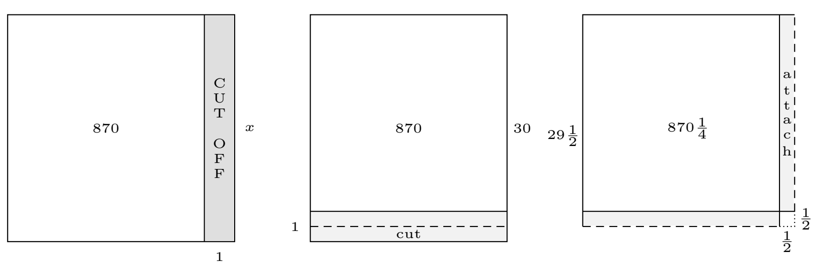
- Babylonian Quadratic Algebra: Problem and Solution

  - Babylonian quadratic problem BM 13901 #2
    I subtracted the side of a square from its area; it was 14,30 [870]. Find the square's side.

  - Babylonian solution: algorithmic calculation sequence
    (1) Halve: $1 \div 2 = \frac{1}{2}$.
    (2) Square: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
    (3) Add: $870 + \frac{1}{4} = 870\frac{1}{4}$.
    (4) Take the square-side: $\sqrt{870\frac{1}{4}} = 29\frac{1}{2}$.
    (5) Add the sides: $29\frac{1}{2} + \frac{1}{2} = 30$.

    The original square's side is 30.

- Babylonian solution's geometry: cut-and-paste



$x^2 - x = 870$

$x^2 - x + \frac{1}{4} = 870\frac{1}{4}$

$x - \frac{1}{2} = \sqrt{870\frac{1}{4}} = 29\frac{1}{2}$

$x = 29\frac{1}{2} + \frac{1}{2} = 30$

3.3 ◇ Assessment

- Are Babylonian quadratic solution methods algebraic? *YES!*
  - They solve for unknown inputs.
  - They calculate with unknown inputs like ordinary quantities.
  - They provide systematic methods of problem solving.

- Main features of Babylonian algebraic problem solving
  - Algorithmic step-by-step solution process; no formulas
  - Geometric medium, employing dynamic cut-and-paste methods
  - Geometric transformations match symbolic solutions
  - *Completing-the-Square* – versatile procedure – gave birth to algebra: recreational problem solving became the art of problem solving

4 ✠ Medieval Arabic Algebra

4.1 ◇ Background

✠ Arabic Algebra

- Cultural and Mathematical Context
  - Spread of the Arabic Empire (630 – 730; Spain to India)
  - Growing familiarity with other cultures' literature, philosophy, science, mathematics, etc.
  - House of Wisdom established (Baghdad, 825); intellectual center for translation, scholarship, and scientific research
  - Arabic algebra: probably indigenous origins; later Greek influence
  - al-Khwārizmī's mathematical texts
    - Arithmetic text introducing Indian numerals and reckoning (825)
    - Kitāb al-jabr w'al-muqābala (Calculation by al-Jabr and al-Muqābala): the founding Arabic text on algebra (830)
    - al-Khwārizmī's texts very important for European mathematics

4.2 ◇ Contours of Arabic Algebra

- Contents and Organization of al-Khwārizmī's Algebra
  - Doxological dedicatory preface (2 pages)
  - Systematic treatment of equations, all done in words (45 pages)
    - Six standard equation types identified (combinatorial criteria)
    - Standard equation types solved algorithmically
    - Standard solution procedures justified/illustrated geometrically
    - Computing with algebraic expressions and radicals
    - More complex equations (6 examples, 34 problems) reduced to standard types for solution by verbal transformations
  - Algebraic applications (no genuine quadratic solutions)
    - Commercial problems: Rule of Three (2 pages)
    - Measurement problems: area, volume calculations (15 pages)
    - Islamic inheritance problems (110 pages)

4.3 ◇ Canonical Quantities & Equations

- Types of Quantities Compared in al-Khwārizmī's Algebra
  - Numbers
  - Roots/Sides
  - Squares

- al-Khwārizmī's Standard Equation Types
  - Simple types

    (1) Squares equal to roots $[ax^2 = bx]$
    (2) Squares equal to numbers $[ax^2 = c]$
    (3) Roots equal to numbers $[bx = c]$
  - Compound types

    (4) Squares and roots equal to numbers $[ax^2 + bx = c]$
    (5) Squares and numbers equal to roots $[ax^2 + c = bx]$
    (6) Roots and numbers equal to squares $[bx + c = ax^2]$

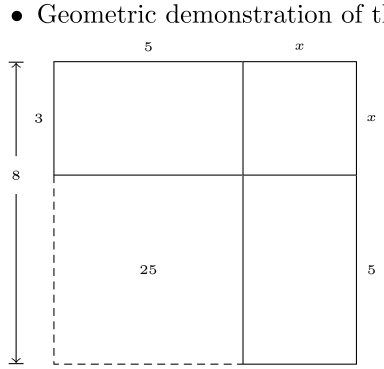
4.4 ◇ Arabic Quadratic Algebra

- al-Khwārizmī's Algebra: Some Problems and Solutions
  - Type 4 equation: squares and roots equal to numbers
    One square and ten of its roots equals thirty-nine. Find the root and the square.
  - al-Khwārizmī's algorithmic solution procedure (all in words):

    (1) Halve the number of roots: five.
    (2) Multiply this by itself: twenty-five.
    (3) Add this to thirty-nine: sixty-four.
    (4) Take this number's root: eight.
    (5) Subtract half the original roots: three.

    Three is the root of the square sought; the square is nine.

  - Geometric demonstration of the solution



Draw a square,
and add rectangles of length 5
to two sides of the square.
Total area, as given, is 39.
Complete the square:
25 is added to the area, giving 64.
The side of the large square is 8;
the small square's side is 5 less: 3.

$x^2 + 10x = 39$

$x^2 + 10x + 25 = 39 + 25 = 64$

$(x + 5)^2 = 8^2$

$x + 5 = 8$

$x = 8 - 5 = 3$

- Fully symbolic formulation of solution algorithm

(0) $x^2 + bx = c$

(1) $b \div 2 = \dfrac{b}{2}$

(2) $\left(\dfrac{b}{2}\right)^2 = \dfrac{b^2}{4}$

(3) $\dfrac{b^2}{4} + c$

(4) $\sqrt{\dfrac{b^2}{4} + c}$

(5) $x = \sqrt{\dfrac{b^2}{4} + c} - \dfrac{b}{2}$: a version of the Quadratic Formula.

- Type 5 equation: squares and numbers equal to roots
  One square and twenty-one equals ten roots. Find the square.

- al-Khwārizmī's algorithmic solution procedure:

(1) Halve the number of roots: five.
(2) Multiply this by itself: twenty-five.
(3) Subtract twenty-one from this: four.
(4) Extract the root: two.
(5) Subtract this from/add this to half the roots: three/seven.

Three/seven is a root of such a square, which is nine/forty-nine;
for each solution, one square plus twenty-one equals ten roots.

- al-Khwārizmī's geometric justifications for type 5 and 6 equations: more complex figures

- Sample problem 5 in al-Khwārizmī's Algebra

  I divided ten into two parts. Multiplying each part by itself and adding these products together, the sum was fifty-eight. Find the two parts.

al-Khwārizmī's Solution	Modern Symbolic Counterpart
Let one of the parts be a thing and the other ten minus that thing. Multiply ten minus a thing by itself: it is one hundred and a square minus twenty things. Also multiply a thing by a thing; it is a square.	Let x, $10 - x$ be the two parts. Then $(10-x)^2 = 100 + x^2 - 20x$.
The sum of these products is a hundred plus two squares minus twenty things, which equals fifty-eight.	
Now take the twenty negative things from the hundred and the two squares and add them to fifty-eight; then a hundred, plus two squares are equal to fifty-eight and twenty things [done by *al-jabr*].	So $(10-x)^2 + x^2 = 100 + 2x^2 - 20x = 58$.
Reduce this to a square by taking half of everything. It then becomes: fifty and a square are equal to twenty-nine and ten things.	Thus, $100 + 2x^2 = 58 + 20x$. Halving, $50 + x^2 = 29 + 10x$.
Reduce this [*al-muqābala*], by taking twenty-nine from fifty; there remains twenty-one and a square equal to ten things.	Subtracting/combining like terms, $21 + x^2 = 10x$.
[Now al-Khwārizmī starts the standard solution procedure for a Case 5 equation.]	
Halve the number of roots; it is five.	$10 \div 2 = 5$
Multiply this by itself (twenty-five) and subtract twenty-one; four remains.	$5^2 = 25$ $25 - 21 = 4$
Extract the root, it is two.	$\sqrt{4} = 2$
Subtract this from half the number of roots (from five); there remains three. This is one part.	$5 - 2 = \underline{3}$
The other is seven, the root added to half the number of roots.	$5 + 2 = \underline{7}$

4.5 ◇ Assessment

- Are al-Khwārizmī's solution methods algebraic? *YES!*
  - They solve for unknown inputs.
  - They calculate with unknown inputs as with numbers.
  - They systematically and efficiently solve equations.

- Algebraic features of Arabic solution methods
  - Equations are categorized, and canonical forms are identified.
  - Solutions are algorithmically found by operating on known and unknown quantities.
  - Solution procedures are geometrically demonstrated.
  - Algebraic expressions are computed, and equations are manipulated, to reduce equations to canonical form – albeit verbally.
  - Solution methods match symbolic algebra solution procedures. However, they are not yet fully uniform, nor do they give formulas.
  - Nevertheless, algebra is now systematically organized into a discipline, a science or theory of equations
  - Arabic algebra becomes the springboard for further developments in later European circles.

5 ✠ Summary

✠ Summary: Educational Lessons from HoAlg

- Norm of Concrete/Holistic Beginnings
  - Begin concretely, keeping the central problem-solving goal of algebra in mind
  - Model problems appropriately, using a variety of concrete approaches and effective procedures, including geometric ones

- Norm of Progressive Comprehension
  - Use more complex models and procedures as needed and as students are ready to use them
  - Introduce symbolic abstraction and operations gradually, in parallel with concrete representation and manipulations

- Norm of Efficient Uniform Procedures (future talk)
  - Reveal the limitations of a narrowly concrete approach (homogeneity, dimensionality, positivity)
  - Demonstrate the power and simplicity of an even more systematic/uniform abstract symbolic approach

- Value of History of Mathematics for Learning Algebra
  - Offers curricular and pedagogical insights to teachers
    - Suggests ways to highlight/connect/explain key ideas
    - Suggests ways to avoid difficulties
  - Provides enrichment and exploratory materials for students

- Comments or Questions?

Teaching Complex Analysis as a Lab-Type Course with a Focus on Geometric Interpretations using *Mathematica*

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Abstract

I taught complex analysis for the first time in my career during the spring of 2013. I decided to do something "radical" and teach it as a lab-type course with a focus on geometric interpretations using the computer program *Mathematica*. The students and I met in a computer lab and, during most meetings, we spent a large portion of our time experimenting and exploring using *Mathematica* to visualize key concepts in complex analysis. Because of this, there was a heavy emphasis on viewing analytic functions as conformal mappings as well as considering associated vector fields and flows. *Mathematica* was used to make the concepts "come alive" through its animation capabilities. A description of some of these animations will be the main focus of this paper. I will also briefly discuss how I helped students learn more basic content through the use of many 10-minute video lectures (I also taught basic *Mathematica* code in these video lectures).

1 Introduction

The idea of a "flipped" or "inverted" classroom has become trendy in recent years at all levels of education [1] (http://techonomy.com/2012/08/flipped-classrooms-turn-learning-on-its-head/). As of early June in 2013, *Wikipedia* [8] describes it as

> "Flip teaching (or flipped classroom) is a form of blended learning which encompasses any use of technology to leverage the learning in a classroom, so a teacher can spend more time interacting with students instead of lecturing. This is most commonly being done using teacher-created videos that students view outside of class time. It is also known as backwards classroom, reverse instruction, flipping the classroom, and reverse teaching." *http://en.wikipedia.org/wiki/Flip_teaching*

It is a long-term goal of mine to ultimately make all the classes I teach into some version of a flipped classroom, but it does take a lot of work to get there. Pedagogically-speaking, this is a vision I want to implement in my courses based on the opinions I have listed below.

1. Students are more engaged during class, especially if they are held personally accountable for the activities they do in class. Accountability for this engagement can take place during class or after class. If they are available, teaching assistants can be valuable resources to help keep students accountable through assessment of student products related to these activities.

2. The classroom should be, and can be, the best place to gain exploration skills, strengthen conceptual understanding, learn how to ask good questions, and improve problem-solving skills because of direct instructor guidance. It can also be the best place to gain experience with problems that are deeper and more "authentic". In fact, the classroom is an ideal place to collaboratively solve problems that might take a week or even a month to solve together. The instructor can guide this problem-solving teamwork by giving the students activities that will hopefully lead them to a solution. This also provides the instructor with opportunities to hone his or her own skills in identifying good problems and in working with the students to solve such problems.

3. Basic skills can be practiced in a flipped classroom, but it is a better use of class time for them to be practiced outside of the classroom. There are also many online resources to help students improve their basic skills. Students should still be assessed on their basic skills; perhaps through gateway exams and probably on regular exams as well.

4. Hopefully, the end result of all this will be better learning and better preparation for future learning and jobs. At the most basic level, there should be more learning that occurs just because of more classroom engagement happening. The students will hopefully develop a deeper desire to learn because of the confidence gained, because of the focus on exploration, and because of the connections made to real-life through significant and authentic problems.

5. As a Christian, I believe that the creation of a better learning environment in the mathematics classroom is a big part of God's call on my life. I want to glorify Him through the study and teaching of mathematics and to help people find their own calling, develop their own skills, and make the world a better place. I also think this environment will help me develop better personal mentoring relationships with more of my students because it will facilitate more time for one-on-one interaction.

6. The creation of online materials in both written and video format can help an instructor and an institution to broaden their audiences. For instance, instructors can create blogs or *YouTube* channels that, if done well, can become relatively popular around the world. I believe this should be considered a valid form of professional activity and development at many institutions and it also can be thought of as a calling from God to make the world a better place and to glorify Him.

7. All residential colleges and universities may be economically forced towards some form of this model for many of their courses. In order to differentiate ourselves from purely online instruction and provide value commensurate with our tuitions, we need to become more engaged with our students in the deeper issues and problems related to our subjects. Sufficiently motivated and disciplined students can learn basic skills and factual knowledge on their own online. Why should they go to college unless we offer them something deeper? Unmotivated and undisciplined students will benefit from a more "hands-on" and relationship-focused approach as well.

During the spring of 2013, I taught complex analysis at Bethel University for the first time. I had plans to implement this vision for the reasons mentioned and I had a small amount of success doing so the first couple weeks. I was, however, thwarted in fully realizing complex analysis as a flipped classroom for the whole semester because my computer broke down a couple weeks into the course. By the time it was fixed a week and a half later, I was too far behind to catch up, both in terms of creating *Mathematica*-based computer activities for the course and in terms of making instructional videos. In order to survive during the intense busyness of the semester, I reverted back to my standard mode of teaching. In this standard mode of teaching, I still made extensive use of *Mathematica* with my students, just not in an activity-oriented framework. In this rest of this paper, I will describe how I made use of *Mathematica*. The main focus will be on how I used *Mathematica* to illustrate fundamental concepts in complex analysis, but I will begin with a description of a few of its capabilities for more basic courses.

2 Using *Mathematica* as a Tool to Implement a Flipped Classroom in Basic Courses

Mathematica is a very powerful software application. It was known mostly for its symbolic computation capabilities during the first fifteen to twenty years of its development, though in the past five to ten years its visualization, numerical computation, data-access, and data-manipulation capabilities have been greatly improved. I have been making use of it extensively in my courses for about fifteen years. For the most part, I have used it as a calculation-checking and visualization tool for the purpose of illustrating concepts to my students. But I have also used it fruitfully in my own research in ordinary differential equations [2,3] and actuarial mathematics (done as a student/faculty summer research team with Jacob Smith) [4]. I have also required my students to use it to do real-world ("authentic") projects, especially in calculus-sequence courses.

One key tool for realizing *Mathematica*'s visualization capabilities is the command `Manipulate`. In fact, `Manipulate` can be used to make many kinds of interactive output, including interactive numerical output. For instance, the line of code `Manipulate[Table[Prime[n],{n,1,m}],{m,1,10,1}]` will make a slider-enabled output box (shown in Figure 1) that will show the first m primes for different integer values of m from 1 to 10 ("Prime[n]" returns the n^{th} prime and "Table" creates the list of primes from the first prime through the m^{th} prime. Figure 1 shows the animation "frozen" at $m = 8$.

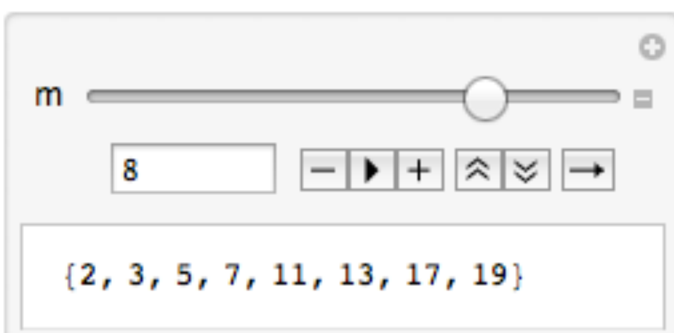


Figure 1: Snapshot of `Manipulate` animation to show the first m primes when $m = 8$.

As another example, with the help of Binomial[n,k], which returns the binomial coefficient $\binom{n}{k}$, we can quickly create an interactive version of Pascal's Triangle (see a snapshot in Figure 2 with $m = 9$) with the line:

```
Manipulate[Column[Table[Binomial[n,k],{n,0,m},{k,0,n}],Center],{m,0,10,1}]
```

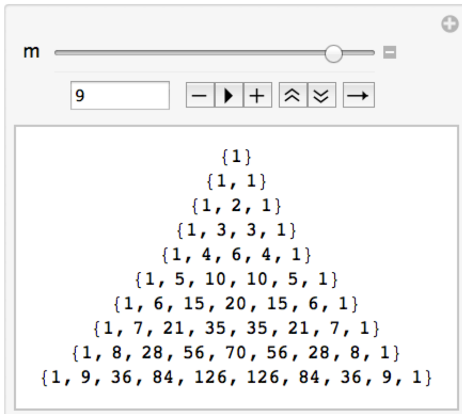


Figure 2: Snapshot of `Manipulate` animation to show Pascal's Triangle with $m + 1 = 10$ lines.

In a first-semester calculus class, one of the most fruitful uses of `Manipulate` is to help the students visualize the concept of local linearity. In Figure 3 we see a snapshot of an animation used to illustrate this concept as we zoom in on the graph of $f(x) = x^2$ near the point $(1,1)$. The animation parameter ϵ is used to determine the size of the "zoom box" (it's the "radius" of the box relative to the maximum norm $\|\cdot\|_\infty$).

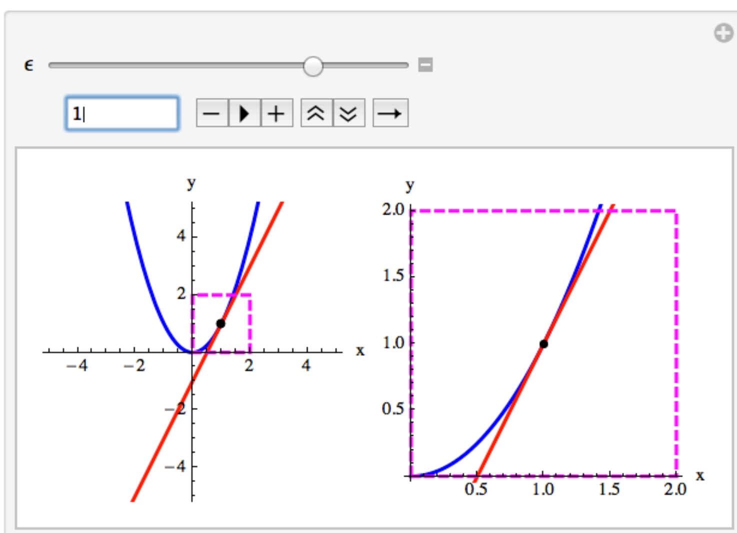


Figure 3: Snapshot of `Manipulate` animation to show the local linearity of $f(x) = x^2$ near $x = 1$.

```
x[t_] := Cos[3 t]; y[t_] := Sin[5 t]; r[t_] := {x[t], y[t]}; v[t_] := r'[t]; a[t_] := r''[t]; T[t_] := v[t]/Sqrt[v[t].v[t]];
Nunit[t_] := T'[t]/Sqrt[T'[t].T'[t]]; κ[t_] := Abs[x'[t] y''[t] - y'[t] x''[t]]/((x'[t])^2 + (y'[t])^2)^(3/2); R[t_] := 1/κ[t]; speed[t_] := Sqrt[v[t].v[t]];
dist[t_] := NIntegrate[speed[τ], {τ, 0, t}];
Manipulate[
 Grid[
  {{Show[ParametricPlot[r[t], {t, 0, b}, PlotStyle → {Thick, Blue}], ListPlot[{r[b]}, PlotStyle → {Black, PointSize[.04]}],
     Graphics[{Thick, Red, Arrow[{{0, 0}, r[b]}]}], Graphics[{Thick, Magenta, Arrow[{r[b], r[b] + r'[b] / 4}]}],
     Graphics[{Thick, Darker[Green], Arrow[{r[b], r[b] + r''[b] / 8}]}], Graphics[{Thick, Orange, Circle[r[b] + R[b] * Nunit[b], R[b]]}],
     PlotRange → {{-2, 2}, {-2, 2}}, AxesOrigin → {0, 0}, AxesLabel → {"x", "y"}, TicksStyle → 15, ImageSize → Medium],
    Plot[κ[t], {t, 0, b}, PlotStyle → {Thick, Orange}, PlotRange → {{0, 2 π}, {0, 25}}, AxesLabel → {"t", "curvature"}, ImageSize → Medium]},
   {Plot[{speed[t], dist[t]}, {t, 0, b}, PlotStyle → {{Thick, Magenta}, {Thick, Blue}}, PlotRange → {{0, 2 π}, {0, 25}},
     AxesLabel → {"t", "speed (magenta) & distance (blue)"}, ImageSize → Medium],
    Plot[R[t], {t, 0, b}, PlotStyle → {Thick, Orange}, PlotRange → {{0, 2 π}, {0, 25}}, AxesLabel → {"t", "radius curv"},
     ImageSize → Medium]}}], {b, .0001, 2 π}, LabelStyle → Large]
```

Figure 4: Code for Manipulate animation of the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ and associated quantities.

One of my favorite uses of *Mathematica* is to model motion using parametric curves and to animate the position, velocity, and acceleration vectors as well as the curvature, radius of curvature, and osculating circle. Figure 4 shows the (relatively small amount of) code used to create Figure 5. This animation uses b, the right endpoint of the interval $[0, b]$, as the animation parameter (the snapshot freezes the picture at $b = 3$). It plots the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ over the interval $[0, b]$ along with the position vector, velocity vector (scaled down), acceleration vector (scaled down), and osculating circle. We also see graphs of the curvature function and radius of curvature function along the right as well as the distance traveled and speed functions at the lower left.

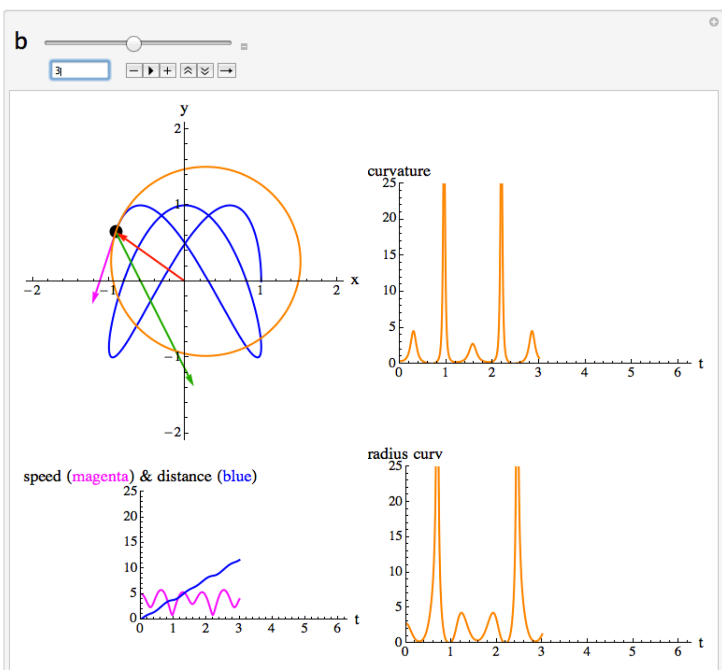


Figure 5: Snapshot of Manipulate animation of the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ and associated quantities.

Conversion of my calculus-sequence courses to flipped classrooms will involve helping the students learn how to create these *Mathematica* animations for two purposes: 1) to aid their understanding and 2) to help them solve, or at least model, more complicated and authentic problems and situations. Instruction about the relevant *Mathematica* code will occur both in videos I make (where basic topic-oriented instruction also occurs) and in class through activities, individual help, and whole-class discussion and illustration. I have many ideas for week-long (or month-long) in-class problems. For instance, in Multivariable Calculus ("Calculus 3"), I have a goal this fall (2013) to see whether or not we can be successful in using *Mathematica* to model some of the following kinds of motion (all of which will also require us to learn how to set-up and use *Mathematica*'s differential equation-solving capabilities): (1) the motion of a mass on a spring (harmonic motion), with and without friction, (and with and without external forcing) (2) the motion of a pendulum, with and without friction (and with and without external forcing), (3) projectile motion with and without air resistance in a fixed three-dimensional reference-frame, (4) accurately modeling the orbits of the first four planets around the sun, (5) projectile motion with and without air resistance in a rotating three-dimensional reference frame on the surface of a sphere (so the Coriolis effect must be taken into account), (6) launching a satellite into orbit, (7) devising a trip to the moon and back, (8) (piecewise) construction of a roller coaster and a car moving along the track in a physically-accurate way. I'm confident I can help the students be successful for (1), (2), (3), and (4). I'm less sure of my capabilities in helping them with the others. But we will try and see whether we succeed or not. I'm confident there will be joy in using the gifts God has given us in our attempts even if we do not achieve full success.

3 Using *Mathematica* as a Tool to Implement a Flipped Classroom in Complex Analysis

In complex analysis, one of the first fundamentally new concepts encountered is the geometric interpretation of complex number multiplication as resulting in the multiplication of moduli and the addition of arguments, modulo 2π. In Figure 6, we see a snapshot of an animation used to illustrate this concept. This time, however, the interactivity is not occurring through the use of a slider, but rather by clicking on each factor in the product (the red dots) and moving each of them around the plane with two degrees of freedom of movement. This is accomplished in *Mathematica* through the use of the `Locator` command.

Interactivity can occur through the use of sliders as well as cursors simultaneously. For instance, Figure 7 shows the 5^{th} roots of a complex number. The original number can be moved around using the cursor, while the order of the root m can be changed with the slider. The unit circle is shown for reference and to observe what happens to the moduli of the roots when $|z| > 1$ versus when $|z| < 1$.

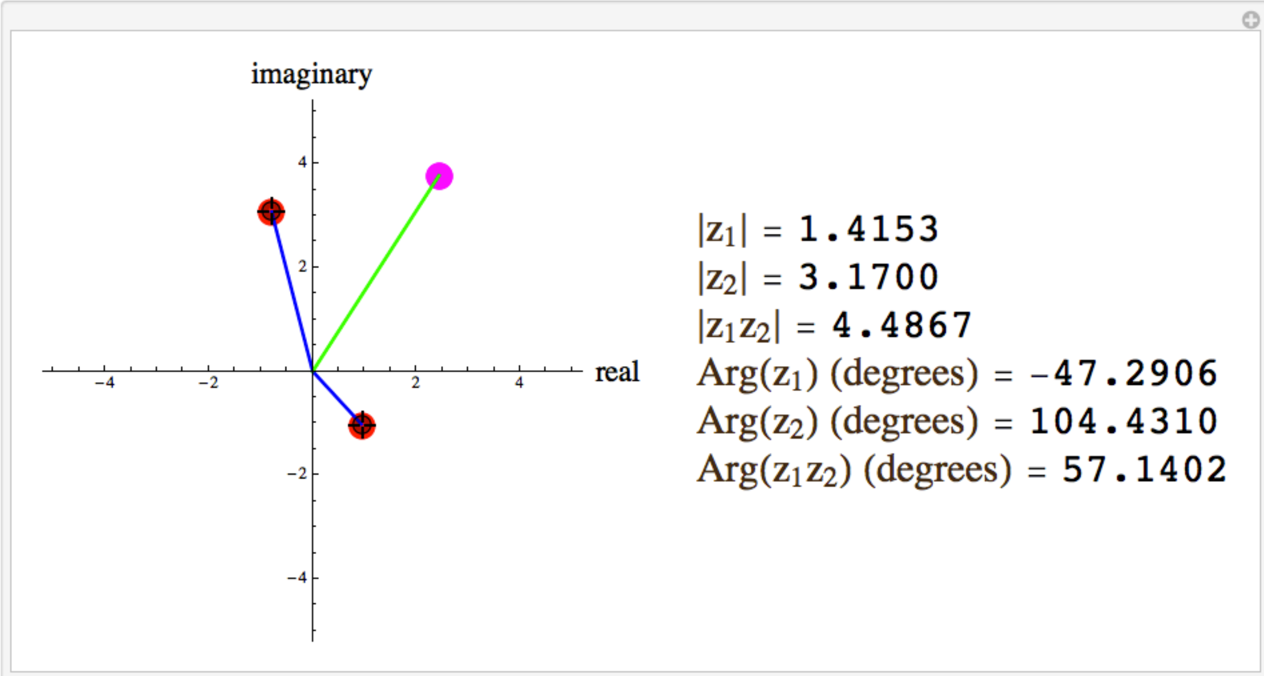


Figure 6: Snapshot of `Manipulate` animation to illustrate the geometric interpretation of complex multiplication. The interactivity occurs by moving the two cursors around in the plane.

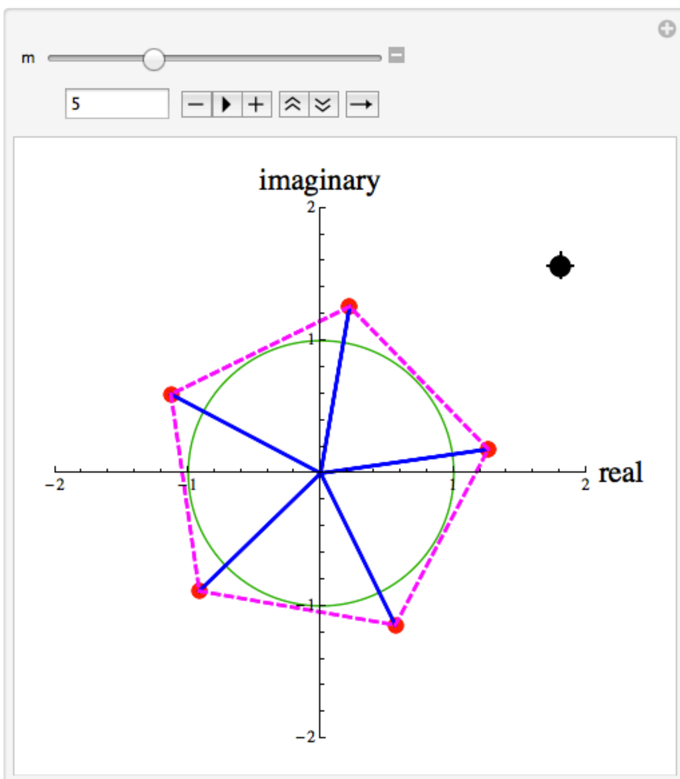


Figure 7: Snapshot of `Manipulate` animation to illustrate the m^{th} roots of a complex number. The interactivity occurs by moving the one cursor around in the plane as well as by changing the value of m.

From my perspective, the two most important uses of *Mathematica* in basic complex analysis are to illustrate how a complex-valued function of a complex variable can be thought of as a mapping (and how the derivative can give us information about that mapping) and how a complex-valued function of a complex variable has associated vector fields (and how the integral can give us information about those vector fields). For instance, we spent a lot of time exploring the function $w = f(z) = z^2$ as a mapping $f : \mathbb{C} \longrightarrow \mathbb{C}$ by using *Mathematica* to visualize the corresponding mapping $(u, v) = \Phi(x, y) = (x^2 - y^2, 2xy) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The command NestList can be used to quickly iterate a function and, combined with the commands Manipulate, Graphics, and ListPlot, create an animation of the orbit of a point under $w = f(z) = z^2$ (see Figure 8).

```
Φ[{x_, y_}] := {x^2 - y^2, 2 x * y};
Manipulate[Show[Graphics[{Thick, Blue, Circle[]}],
  ListPlot[NestList[Φ, pt, n], PlotStyle → {Red, PointSize[.05]}],
  PlotRange → 2, Axes → True, AxesOrigin → {0, 0}, TicksStyle → 15,
  AspectRatio → Automatic, AxesLabel → {"real", "imaginary"}],
 {n, 0, 10, 1}, {{pt, {3 / 5, 4 / 5}}, Locator}, LabelStyle → Large]
```

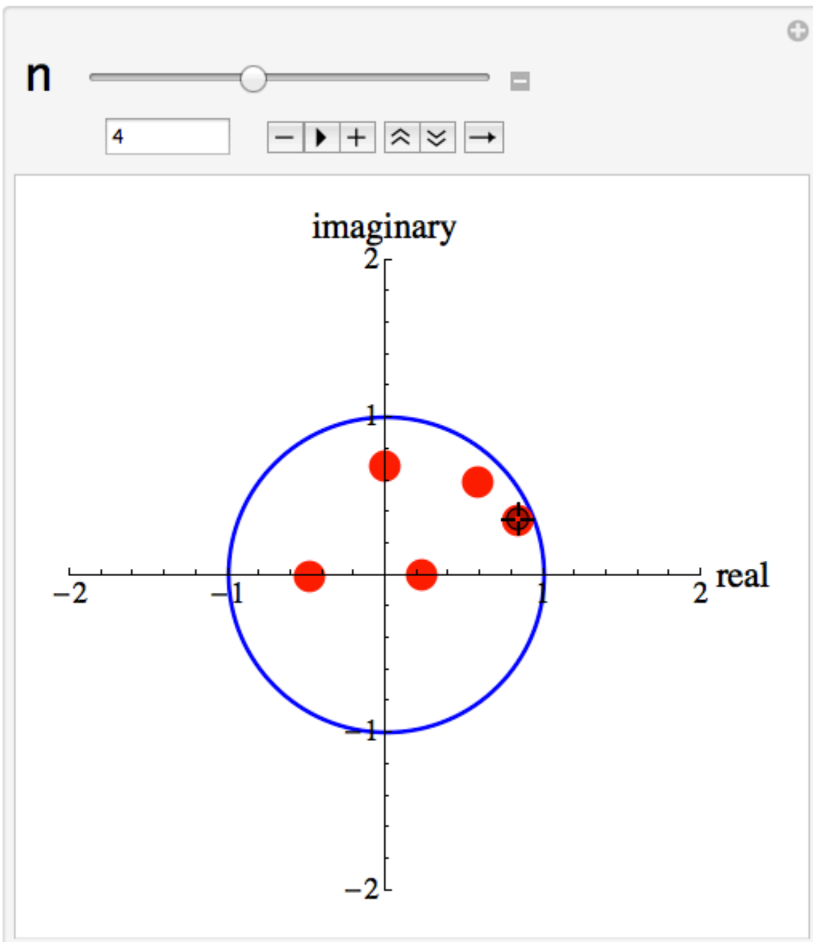


Figure 8: Snapshot of Manipulate animation to illustrate the orbit of a point under $w = f(z) = z^2$.

The Jacobian matrix for the mapping is $J(f) = J(\Phi) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and the Jacobian determinant is $\mathrm{Jac}(f) = \mathrm{Jac}(\Phi) = u_x v_y - u_y v_x$. When f is analytic, the Cauchy-Riemann equations imply that $\mathrm{Jac}(f) = \mathrm{Jac}(\Phi) = u_x^2 + v_x^2 = |u_x + iv_x|^2 = |f'(z)|^2$ on the domain. For $f(z) = z^2$, we then have $\mathrm{Jac}(f) = \mathrm{Jac}(\Phi) = 4|z|^2$ which means that f shrinks regions completely in the set $|z| < \frac{1}{2}$ and expands regions completely in the set $|z| > \frac{1}{2}$. Figure 9 shows how ParametricPlot and NestList can be combined to visualize images of a small disk inside $\frac{1}{2} < |z| < 1$ under successive application of $f(z) = z^2$. Note how the image initially gets larger but eventually gets smaller as the image points converge to zero.

```
Φ[{x_, y_}] := {x^2 - y^2, 2 x*y};
Manipulate[
 Show[ParametricPlot[NestList[Φ, {x, y}, n] /. {x → r*Cos[t] + c[[1]], y → r*Sin[t] + c[[2]]},
   {t, 0, 2 π}, PlotStyle → {{Thick, Red}}],
  ParametricPlot[{{.5*Cos[t], .5*Sin[t]}, {Cos[t], Sin[t]}}, {t, 0, 2 π},
   PlotStyle → {{Thick, Blue}, {Thick, Magenta}}], PlotRange → 2, AxesOrigin → {0, 0},
  AxesLabel → {"real", "imaginary"}], {n, 0, 5, 1}, {{r, .1}, .01, .2},
 {{c, {.6, .6}}, Locator}, LabelStyle → Large]
```

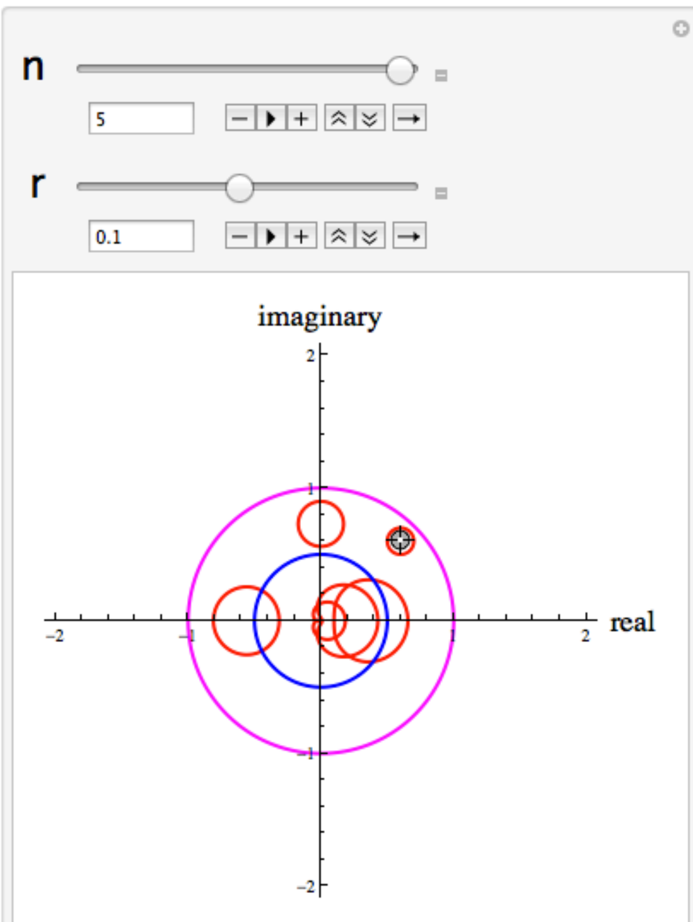


Figure 9: Snapshot of Manipulate animation to illustrate images of a small disk in the set $\frac{1}{2} < |z| < 1$ under iteration of $w = f(z) = z^2$.

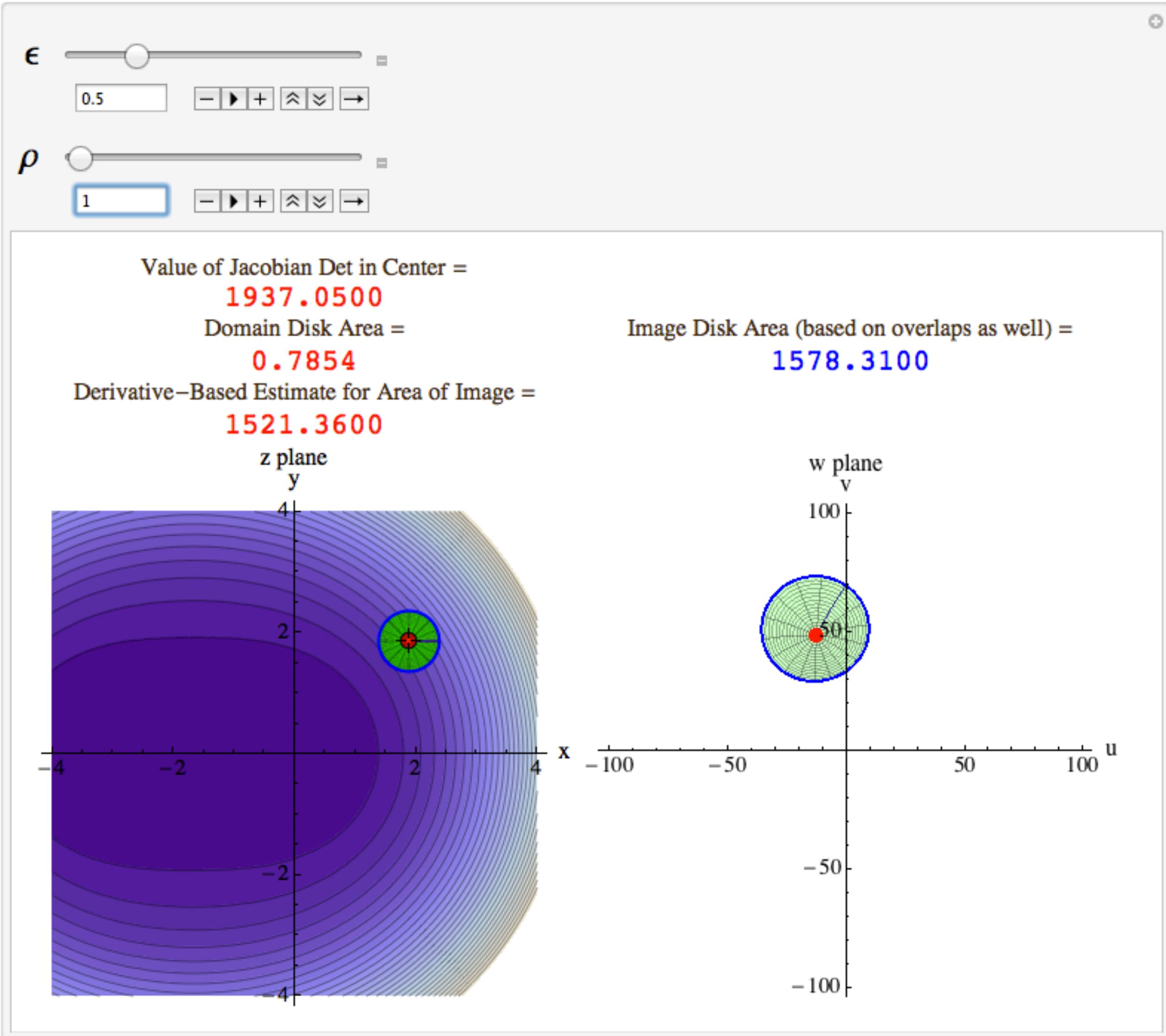


Figure 10: Snapshot of Manipulate animation to calculate the area of the image of a disk under a mapping.

Through the change-of-variables formula for double integrals, we can also program *Mathematica* to help us calculate the exact area of an image of a disk and compare it to the product of $\mathrm{Jac}(f)$ and the area of the domain disk when the disk is small. Figure 10 is the output of such a scheme that I made with my class this spring. The parameter ϵ is the radius of the domain disk and the parameter ρ gives the *relative* radius of the blue circle compared with the whole disk (here we just take $\rho = 1$ in order to focus on the area). The command `NIntegrate` was used to calculate the exact area of the image (including overlaps if the mapping is not one-to-one on the domain). The mapping in this example is $w = f(z) = z^3 + 5z^2$. Note that conformality of the mapping can also be inferred by focusing on how orthogonal families of curves get mapped to orthogonal families of curves. The contour map (level curves) of the function $\mathrm{Jac}(f) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is also shown on the left side of the picture. When moving over this contour map with the cursor (without clicking), the value of $\mathrm{Jac}(f)$ shows up in a box. Darker shading corresponds to lower values of $\mathrm{Jac}(f)$ and lighter shading corresponds to higher values.

The interpretation of the derivative as giving the local "amplitwist" effect of a mapping can also be illustrated (i.e. the interpretation of $dw = f'(z)\,dz$ as a linear transformation that involves a dilation and a rotation; see the text by Tristan Needham [5]). The output of *Mathematica* code to illustrate this concept is shown in Figure 11. The function is $f(z) = z^2$ so that $f'(z) = 2z$. The short vectors on the left are mapped to vectors on the right and are dilated by a factor of approximately $|f'(2+i)| = 2|2+i| = 2\sqrt{5} \approx 4.5$ and twisted by approximately $\text{Arg}(f'(2+i)) = \text{Arg}(4+2i) = \arctan\left(\frac{1}{2}\right) \approx 26.6°$. Once again, we can also see the conformality being illustrated by focusing on the orthogonal sets of curves in each picture.

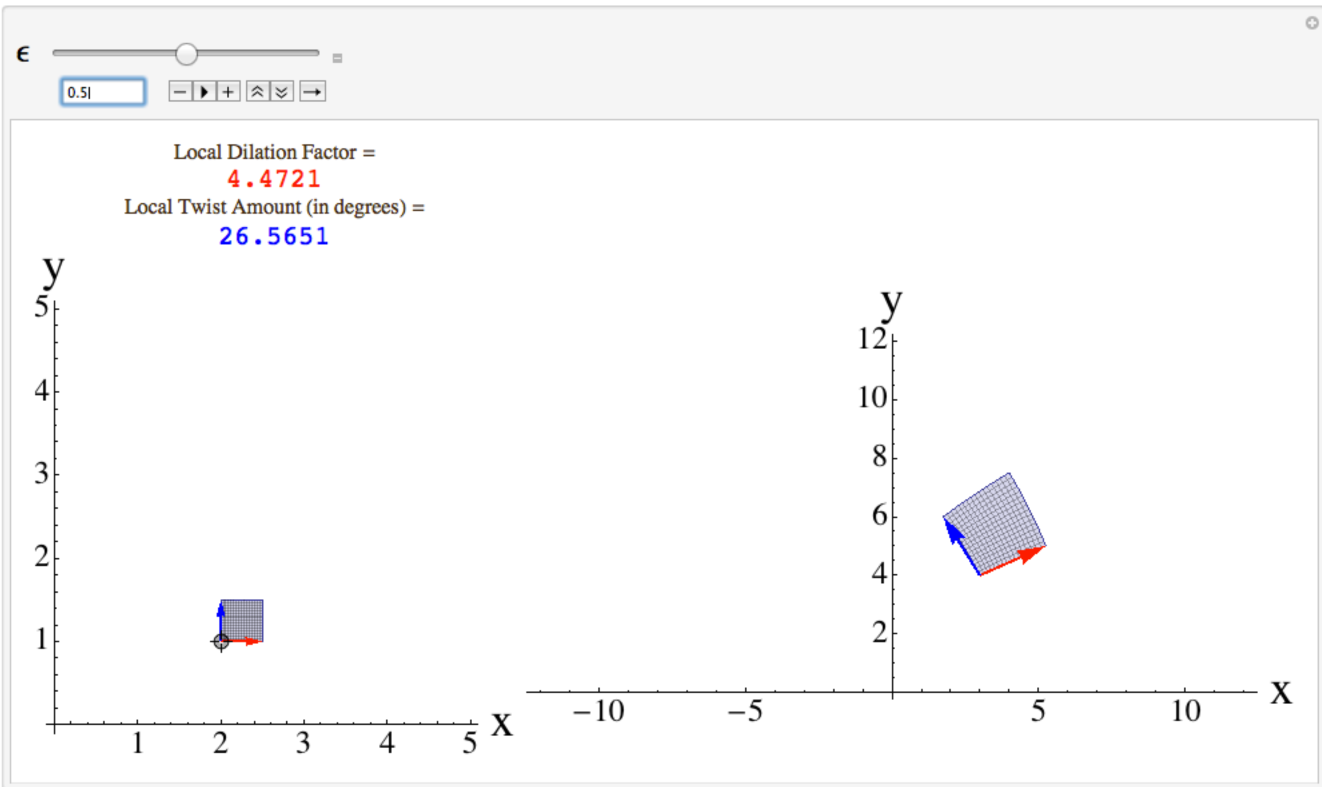


Figure 11: Snapshot of `Manipulate` animation to illustrate the derivative as representing the local "amplitwist" for $f(z) = z^2$ near $z = 2 + i$.

For complex integration, there are two fruitful perspectives that we discovered as a class that can be illustrated nicely with *Mathematica*. Let $w = f(z) = u(x, y) + iv(x, y)$ be a complex-valued function of a complex variable. If we think of $f(z)\,dz$ as $(u + iv)(dx + i\,dy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy)$, then the integral $\int_\Gamma f(z)\,dz$ can be interpreted as $\int_\Gamma \langle u, -v\rangle \cdot d\mathbf{s} + i \int_\Gamma \langle v, u\rangle \cdot d\mathbf{s}$, where $d\mathbf{s} = \langle dx, dy\rangle$ and the integrals $\int_\Gamma \langle u, -v\rangle \cdot d\mathbf{s}$ and $\int_\Gamma \langle v, u\rangle \cdot d\mathbf{s}$ are ordinary line integrals of the vector fields $\langle u, -v\rangle$ and $\langle v, u\rangle$. As such, they can be interpreted in terms of their circulation relative to the oriented contour Γ [6].

For instance, if f is the non-analytic function defined by $f(z) = f(x+iy) = (x+2y) + iy^2$ and Γ is the unit circle, oriented counterclockwise, then $\int_\Gamma f(z)\,dz = -2\pi + i\pi$. This means that $\int_\Gamma \langle u, -v\rangle \cdot d\mathbf{s} = \int_\Gamma \langle x+2y, -y^2\rangle \cdot d\mathbf{s} = -2\pi$ and the average value of $\langle u(z(t)), -v(z(t))\rangle \cdot z'(t)$ is -1, while $\int_\Gamma \langle v, u\rangle \cdot d\mathbf{s} = \int_\Gamma \langle y^2, x+2y\rangle \cdot d\mathbf{s} = \pi$ and the average value of $\langle u(z(t)), -v(z(t))\rangle \cdot z'(t)$ is $\frac{1}{2}$. The output of *Mathematica* code that illustrates this is shown in Figure 12. The background vector fields are very much scaled down, and the vector fields along the unit circle are scaled down as well, but not as much. The value of b gives the current value of t and is set at $b = \pi$ in the snapshot.

Of course, if $w = f(z)$ is analytic in a simply-connected domain, then these circulations will be zero when Γ is a closed contour in the domain because of Green's Theorem and the Cauchy-Riemann equations (both implying Cauchy's Theorem under the extra hypothesis of continuity of the derivatives of u and v). To be more specific, if Γ is a positively-oriented simple closed curve in the simply-connected domain of analyticity and $\mathcal{D}$ is the closure of its interior, then Green's Theorem, under the assumption of continuity of the derivatives of u and v, will produce the equations $\int_\Gamma \langle u, -v\rangle \cdot d\mathbf{s} = \iint_{\mathcal{D}} (-v_x - u_y)\, dA$ and $\int_\Gamma \langle v, u\rangle \cdot d\mathbf{s} = \iint_{\mathcal{D}} (u_x - v_y)\, dA$, from which the conclusion follows.

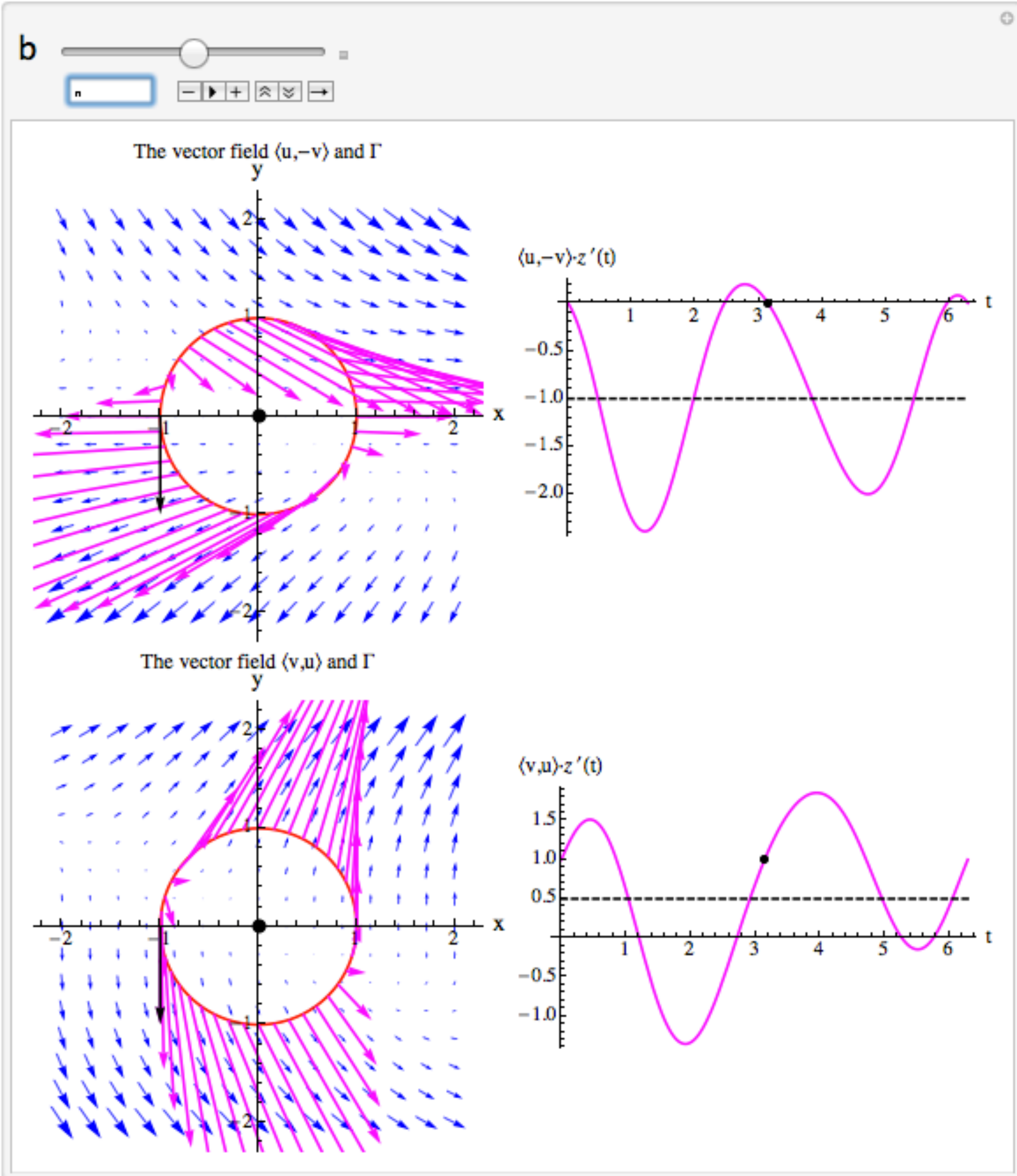


Figure 12: Snapshot of `Manipulate` animation to illustrate the integral as a combination of the circulation of the vector fields $\langle u, -v\rangle$ and $\langle v, u\rangle$.

Finally, when $w = f(z)$ has an antiderivative $F(z)$ in a domain (which will be true if $f(z)$ is analytic in the domain and if the domain is simply-connected), it is also fruitful to think of $\int_\Gamma f(z)\,dz$ as ΔF, where ΔF is the change in the value of F as z varies over the oriented contour Γ. When f is analytic and Γ is a closed contour in the domain of analyticity, then $\Delta F = 0$ and we confirm Cauchy's Theorem. If $f = u + iv$ and $F = U + iV$, then $\nabla U = \langle U_x, U_y\rangle = \langle U_x, -V_x\rangle = \langle u, -v\rangle$ and $\nabla V = \langle V_x, V_y\rangle = \langle V_x, U_x\rangle = \langle v, u\rangle$ and we can visualize $\Delta F = \Delta U + i\Delta V$ by focusing on the three-dimensional graphs of $U(x,y)$ and $V(x,y)$.

For example, if $f(z) = 3z^2 = (3x^2 - 3y^2) + i(6xy)$, then $F(z) = z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. If Γ is the oriented line segment from $z = 0$ to $z = 1 + 2i$, then $\Delta U = 1^3 - 3\cdot 1\cdot 2^2 = 1 - 12 = -11$ and $\Delta V = 3\cdot 1^2\cdot 2 - 2^3 = 6 - 8 = -2$ (and therefore $\int_\Gamma f(z)\,dz = -11 - 2i$). In Figure 13, we see by looking at the red curve that these values do seem to be correct.

```
F[z_] := z^3; z[t_] := (1 + 2 I) t;
Grid[
 {{Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -20, 20}, PlotStyle → Thick], Plot3D[Re[F[x + I * y]], {x, -2.1, 2.1}, {y, -2.1, 2.1}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Re[F[z[t]]]}, {t, 0, 2 π}, PlotStyle → {{Thickness[.02], Red}}], ImageSize → Medium,
    PlotRange → {{-2.1, 2.1}, {-2.1, 2.1}, {-11, 11}}, ViewPoint → {2, 1, 1}, AxesLabel → {"Re(z)", "Im(z)", "Re(z^3)"}, BoxRatios → {1, 1, 1}],
   Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -20, 20}, PlotStyle → Thick],
    Plot3D[Im[F[x + I * y]], {x, -2.1, 2.1}, {y, -2.1, 2.1}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Im[F[z[t]]]}, {t, 0, 2 π}, PlotStyle → {{Thickness[.02], Red}}], ImageSize → Medium,
    ViewPoint → {2, 1, 1}, PlotRange → {{-2.1, 2.1}, {-2.1, 2.1}, {-2, 2}}, AxesLabel → {"Re(z)", "Im(z)", "Re(z^3)"}, BoxRatios → {1, 1, 1}]}}]
```

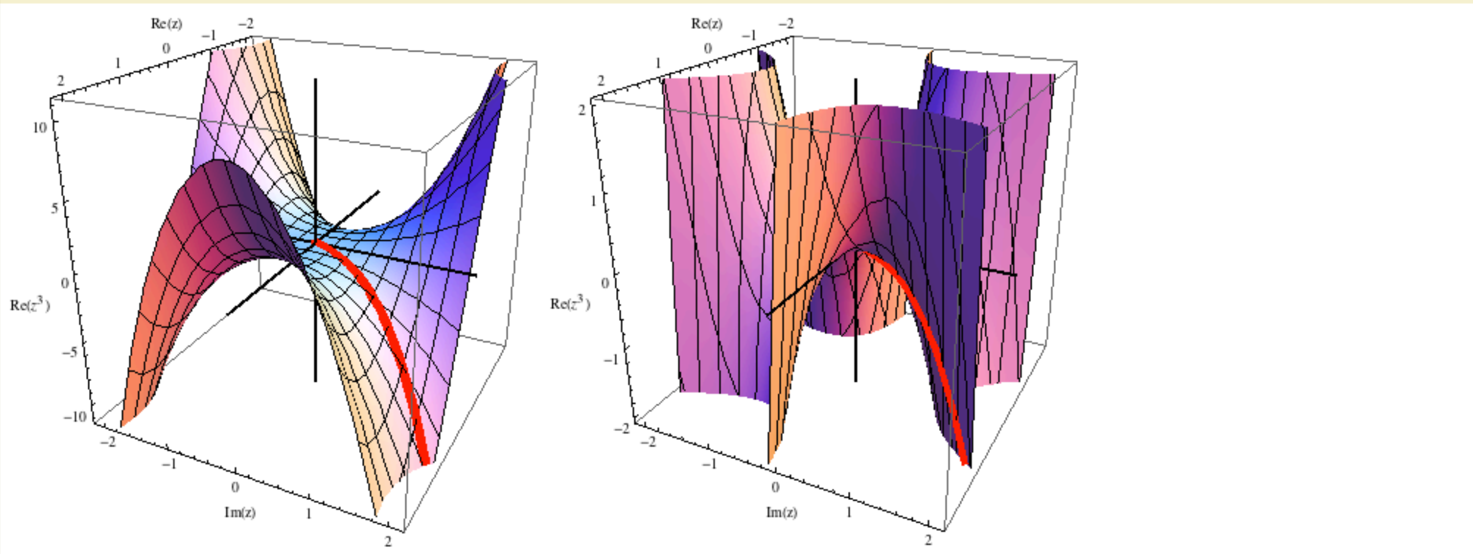


Figure 13: Illustating the value of the Integral $\int_\Gamma 3z^2\,dz$ in terms of ΔU and ΔV, where $\Delta F = \Delta U + i\Delta V$ and $F(z) = z^3$.

This mode of thinking can also help us make sense of integration of meromorphic functions around contours containing poles giving integrals that are nonzero and dependent on the residues of the function at the poles. As the most fundamental example of such a situation, if $f(z) = \frac{1}{z}$, then we know that f is analytic on $\mathbb{C} - \{0\}$ but that, for example, the principal value of the logarithm $F(z) = \mathrm{Log}(z) = U + iV = \ln|z| + i\mathrm{Arg}(z)$ is only an antiderivative of f on the slit domain $\mathbb{C} - \{x + iy : x \le 0, y = 0\}$. If Γ is, for instance, the unit circle, oriented counterclockwise, then we also know that $\int_\Gamma f(z)\,dz = 2\pi i$. This cannot be strictly interpreted in terms as $\Delta F = \Delta U + i\Delta V$. It can, however, *almost* be interpreted in this way. If we let w^+ be the limiting value of $F(z)$ as z approaches -1 from above the real axis in the complex plane and let w^- be the limiting value of $F(z)$ as z approaches -1 from below the real axis in the complex plane, then $w^+ = \pi i$ and $w^- = -\pi i$ and $w^+ - w^- = 2\pi i$. Furthermore, if we let v^+ be the limiting value of $V(z) = \mathrm{Arg}(z)$ as z approaches -1 from above the real axis in the complex plane and let v^- be the limiting value of $V(z) = \mathrm{Arg}(z)$ as z approaches -1 from below the real axis in the complex plane, then $v^+ = \pi$ and $v^- = -\pi$ and $v^+ - v^- = 2\pi$ and $w^+ - w^- = (v^+ - v^-)i$. In other words, if we think in terms of limiting values, we can still think about $\int_\Gamma f(z)\,dz$ as being "essentially" given by $\Delta F = \Delta U + i\Delta V$ and we can visualize it with the 3-dimensional plot in Figure 14.

```
F[z_] := Log[z]; z[t_] := E^(I * t);
Grid[
 {{Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5}, PlotStyle → Thick],
    Plot3D[Re[F[x + I * y]], {x, -1.2, 1.2}, {y, -1.2, 1.2}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Re[F[z[t]]]}, {t, 0, 2 π}, PlotStyle → {{Thickness[.02], Red}}],
    ImageSize → Medium, PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}, {-2, 1}}, ViewPoint → {2, 1, 1},
    AxesLabel → {"Re(z)", "Im(z)", "Re(Log(z))"}, BoxRatios → {1, 1, 1}],
   Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5}, PlotStyle → Thick],
    Plot3D[Im[F[x + I * y]], {x, -1.2, 1.2}, {y, -1.2, 1.2}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Im[F[z[t]]]}, {t, 0, 2 π}, PlotStyle → {{Thickness[.02], Red}}],
    ImageSize → Medium, ViewPoint → {2, 1, 1}, PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}, {-π, π}},
    AxesLabel → {"Re(z)", "Im(z)", "Im(Log(z)))"}, BoxRatios → {1, 1, 1}]}}]
```

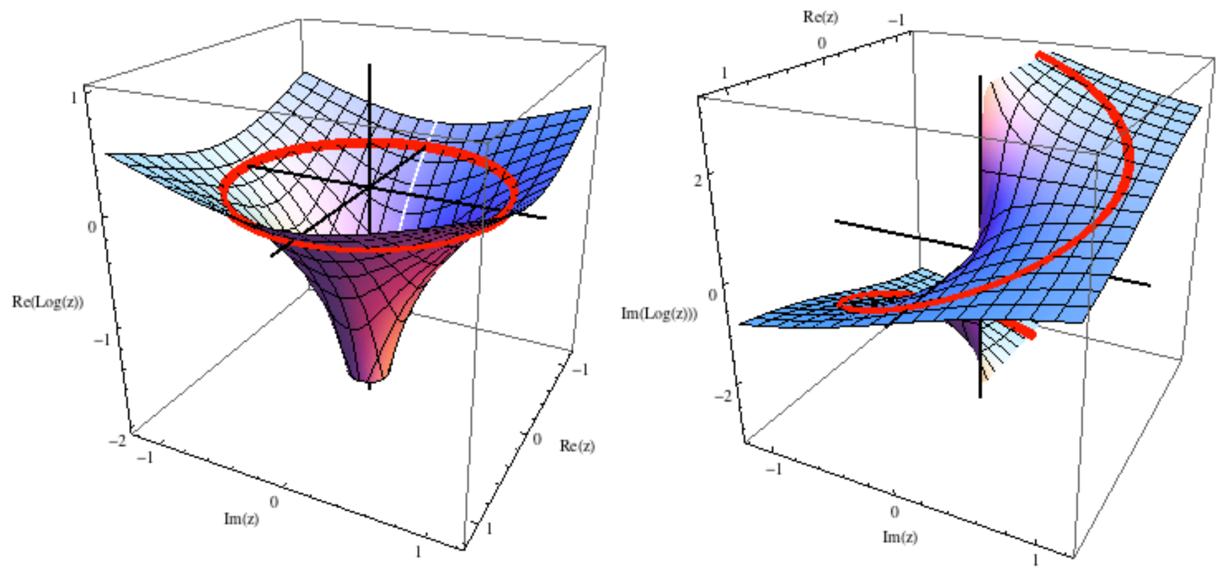
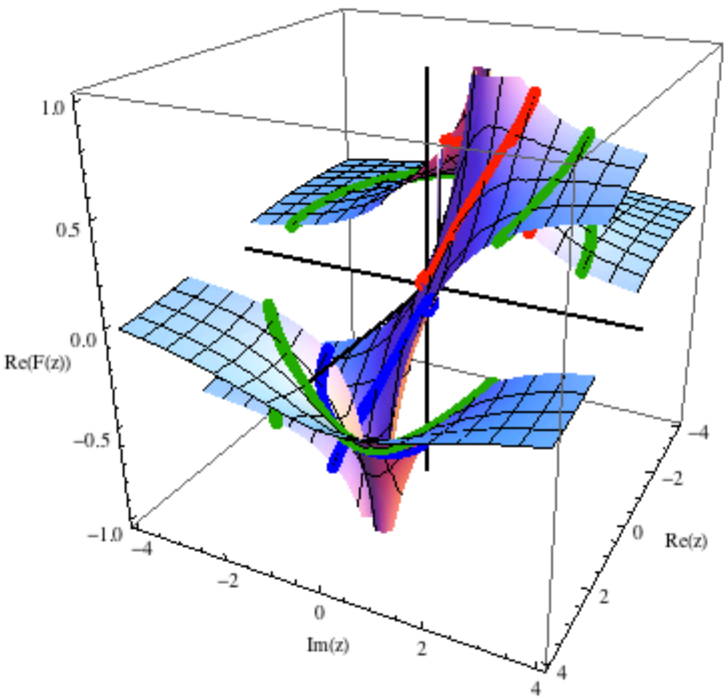


Figure 14: Illustating the value of the Integral $\int_\Gamma \frac{1}{z}\,dz$ in terms limiting values of of ΔU and ΔV, where $F(z) = U(z) + iV(z) = \mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$.

This is even more significant in more challenging examples, such as for the function $f(z) = \frac{1}{(z-i)(z-2)}$. The residue of this function at $z = i$ is $-\frac{2}{5} - \frac{i}{5}$ and the residue at $z = 2$ is $\frac{2}{5} + \frac{i}{5}$. If Γ_1 is $|z-i| = 1$ (positively oriented), Γ_2 is $|z-2| = 1$ (positively oriented), and Γ_3 is $|z| = 3$ (positively oriented), then the Residue Theorem gives $\int_{\Gamma_1} f(z)\,dz = \frac{2\pi}{5} - \frac{4\pi}{5}i \approx 1.25664 - 2.51327i$, $\int_{\Gamma_2} f(z)\,dz = -\frac{2\pi}{5} + \frac{4\pi}{5}i \approx -1.25664 + 2.51327i$, and $\int_{\Gamma_3} f(z)\,dz = 0$. If we let $U(z) + iV(z) = F(z) = \int f(z)\,dz = \left(\frac{1}{5} - \frac{2i}{5}\right)\tan^{-1}\left(\frac{z-2}{1+2z}\right) + \left(\frac{1}{5} + \frac{i}{10}\right)\mathrm{Log}\left((z-2)^2\right) - \left(\frac{1}{5} + \frac{i}{10}\right)\mathrm{Log}\left(1+z^2\right)$ (determined with *Mathematica*'s symbolic computational abilities; *Mathematica* can calculate residues and Laurent series expansions as well) over an appropriate domain with appropriate branch cuts, then Figure 15 shows that each integral can be thought of as "essentially" $\Delta U + i\Delta V$, where these quantities ignore the discontinuities and also are calculated using appropriate limiting values. It is difficult to "see" the correct changes in this example, but it can be confirmed more convincingly by using *Mathematica*'s ability to allow the user to rotate these pictures using the cursor.



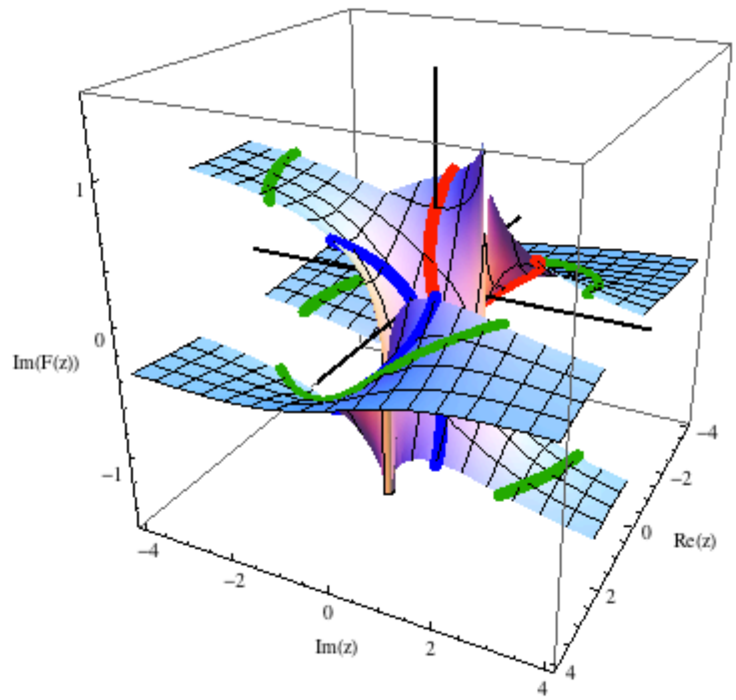


Figure 15: Illustrating the value of the integrals $\int_{\Gamma_j} \frac{1}{(z-i)(z-2)}, dz$ for $j = 1,2,3$ in terms of using appropriate limiting values to calculate ΔU and ΔV, where $F(z) = \int f(z)\,dz$ over an appropriate domain.

Other uses of *Mathematica* in my complex analysis course included: (1) visualizing how the plane can be mapped onto the Riemann sphere, (2) visualizing the definition of an open set (by considering simple examples where it is easy to calculate how the radius of a neighborhood of each point can be appropriately shrunk as the point approaches the boundary to keep the neighborhood inside the set), (3) illustrating the truth of the Cauchy-Riemann equations for analytic functions by focusing on how the local contour maps of u and v are approximately rotations of each other (up to an additive constant) near each point, (4) illustrating the truth the maximum principle for harmonic functions, (5) illustrating the non-conformality and non-injectiveness of a complex analytic function near a critical point, and (6) illustrating the Gauss-Lucas Theorem about how the critical points of a polynomial lie in the convex hull of the zeros of the original polynomial (it is also interesting to graph the level sets $u = 0$ and $v = 0$ simultaneously).

The Wolfram Demonstrations Project [9] (*http://demonstrations.wolfram.com/*) also has many resources for constructing models and illustrating concepts in many subjects, including complex analysis. As of early June in 2013, a search of "complex analysis" at that website produced 173 such "demonstrations" that can be downloaded and used in class (with appropriate credit given). I am particularly fond of code to construct the Mandelbrot set and associated Julia sets.

4 Use of Video Lectures

As stated in the introduction, my goal in all this is to make my classrooms "flipped" and to have the students ultimately be constructing these kinds of pictures and models for themselves. However, the vast majority of students still need basic instruction, as well as more help with the *Mathematica* code than I can give them during our class periods. Therefore, an important part of what I am working on to flip my classrooms is to make video lectures, each of which I typically try to make about ten minutes in length. The lectures focus on: (1) basic computational methods and concepts for each course as well as (2) constructing *Mathematica* code to illustrate these methods and concepts and to enable better use of *Mathematica* in the classroom.

Because of the difficulty I had with my computer this spring, I was only able to fully implement this vision of flipping the classroom for the first couple weeks, when we were in Chapter 1 of our textbook [7]. As of early June in 2013, I uploaded twenty-eight such lectures for Complex Analysis onto my channel on *YouTube* [10] (I plan to make more videos related to this subject the rest of the summer of 2013, as well as more videos about other subjects). The URL for my channel is *http://www.youtube.com/user/billkinneymath* and you can also find my channel by doing a Google search "Bill Kinney Math". If you watch these videos you will get a better sense of what I am trying to do than you'll get from my attempt to explain it here.

Would I ever want to use someone else's lectures? Yes, if I like what someone else is doing, I would make use their lectures (though I have not taken much time to look beyond some of the Khan Academy lectures). However, though it is very time-consuming, I very much enjoying making lectures in this way. Because of this, I think, for the most part, I will rely on my own lectures.

5 Conclusion

I hope that I have inspired you to consider flipping your classroom and to consider using *Mathematica* in your teaching, whether you flip your classroom or not. Please let me know if you have any questions or if you have ideas you would like to share with me. I am especially interested in calculus-related projects that can be modeled and analyzed with *Mathematica*.

6 References

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Googol-part Fugue: Another Imagination of Divine Providence and Game Theory

Presentation for Association of Christians in Mathematical Sciences (ACMS) Conference 2013

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Abstract

The problem of evil presents an intellectual hurdle for some to believe in a good and omnipotent God. The emergence of open theism could be seen as an attempt to make a stronger case for the free will defense. However, in denying divine foreknowledge as traditionally understood, open theism contradicts biblical revelation not only in its direct claims, but also when its logical implications for divine providence are worked out. The open theist Alan Rhoda has sought to explain through game theory how some degree of divine providence is possible under open theism. That explanation is astonishing since the open theist view of libertarian free will is intrinsically at odd with the rational actor model presupposed by game theory. In this essay, the free will defense of open theism and two other responses to the problem of evil are examined. Game theory and other mathematical theorems are employed in illustrating the theological claims. This essay seeks to show that the historic Christian doctrine of divine sovereignty can be reasonably explained given the presence of evil. The key is to recognize the biblical picture of the present age as a development ground and worthiness-demonstrating trial for a perfectible authentic humanity, chosen for a glorious leadership role in the new heavens and new earth, where everything will be knowable, optimal, and predictable.

It is a perfect Monday for running in Boston. Sunny sky and the temperature in the mid-50s is something to be thankful for, as most still remember the 90 degree temperature endured in last year's marathon. Over 24-thousand runners participate in this year's event. Many more come from all over the world to cheer and witness this oldest continuously running marathon tradition, now in its 117th year.[1] Notably, some run today in memory of the 26 victims of the Sandy Hook Elementary School shooting in Newtown, Connecticut. The race began with 26 seconds of silence in memory of the victims. The 26-mile marker features a Newtown city seal surrounded by 26 stars.[2]

But if anyone had hoped that this day, April 15, 2013, would bring any peaceful closure to the senseless violence that took place at Sandy Hook, they would be sorely disappointed. At 2:49 pm, two bombs exploded 13 seconds and 210 yards apart near the finish line. Three people were killed and 264 injured, with some losing their limbs, leaving a bloody and gruesome scene.[3]

"Who did this?" The coincidental symbolic significance of April 15 both as the Patriot Day and the Tax Day certainly didn't go unnoticed, not the least by those who always have a proverbial ax to grind either against radical Islam or the Tea Party movement.[4]

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[1]The Boston Marathon began in 1897, inspired by the first modern Summer Olympics held in Athens, Greece the preceding year. It is one the six major world marathons and attracts almost half a million visitors to Boston every year. Boston Athletic Association. "Boston Marathon History: Boston Marathon Facts" (http://www.baa.org/races/boston-marathon/boston-marathon-history/boston-marathon-facts.aspx)

[2]RT, 4/16/2013. "Compound tragedy: Boston Marathon blasts mar memorial for Sandy Hook victims" (http://rt.com/usa/newtown-families-boston-marathon-unharmed-917/)

[3]Wikipedia. "Boston Marathon bombings" (http://en.wikipedia.org/wiki/Boston_Marathon_bombings). The Boston Globe. "The Boston Marathon bombings" (http://www.bostonglobe.com/metro/specials/boston-marathon-explosions)

[4]For example, The Huffington Post, 4/16/23. "Tea Party Nation Blames Boston Bombing on Obama, 'Radical Islam'" (http://www.huffingtonpost.com/2013/04/16/tea-party-nation-boston-bombing_n_3094621.html)

The investigation reached a turning point when the bombing suspects were identified through CCTV captured videos. On Thursday, the FBI decided to "crowd-source" the investigation by releasing the videos, expressing hope that the public would "rat them out." Later the same night, a MIT policeman was gunned down in Cambridge and a public transit policeman was seriously wounded. The owner of a hijacked vehicle managed to escape, connecting the shootings to the bombing suspects. The authority chased down the vehicle in the neighboring Watertown. One suspect was killed and another escaped. Both the gunned down MIT policeman and the killed suspect were 26-year-old.[5]

Much of the metropolitan Boston was in an unprecedented lock-down on Friday, even though block-by-block search failed to capture the escaped suspect. Serendipitously, as the authority suspended the search leaving all Boston residents to brace for the uncertainty of the nightfall, a resident in Watertown reported spotting a bloody man hiding inside his boat in the backyard. The suspect was captured live after a round of gunshots. President Obama held a prime time news conference to bring the tragic week to a close.[6]

A few days later, the 26-year-old man who managed to escape from his hijacked SUV granted the Boston Globe an exclusive interview. "The story of that night unfolds like a Tarantino movie," the journalist observes, "bursts of harrowing action laced with dark humor and dialogue absurd for its ordinaries, reminders of just how young the men in the car were. Girls, credit limits for students, the marvels of the Mercedes ML 350 and the iPhone 5, whether anyone still listens to CDs ..."[7]

If Tarantino movies resemble real life stories, it is because one element always seems intentionally elusive. "*What is the point of all these?*" In real life, meaning is the one question that many may ask but few volunteer to answer. When the subject is gratuitous evil, is there ever a speakable *why* behind the who, when, where, and how? Yet, if not, how do we even begin to make sense of life?

Dark is the New Black

When I originally proposed this presentation, I had in mind a number of **game theory**[i] based arguments directed against **open theism**[ii] that demonstrate how mathematical concepts could be helpful in clarifying theological debates among Christians and illustrating biblical concepts to scientific-minded unbelievers. The recent tragedy in Boston inspired me to rearrange my materials. At the heart of open theism is the free will defense against the problem of evil, and that apologetic impulse must be considered. In this final form of the essay, I have repainted my arguments with the problem of evil as the background. In this section, I will state the problem of evil and survey its contemporary relevance. Then, in the following sections:-

- The Free Will Argument of Open Theism: I outline the reasoning of open theism by placing the free will defense as its principal premise.
- Divine Providence and Game Theory: I argue against an open theist proposal that by appealing

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Siorta, David. Salon.com, 4/16/2013. "Let's hope the Boston Marathon bomber is a white American" (http://www.salon.com/2013/04/16/lets_hope_the_boston_marathon_bomber_is_a_white_american/)

[5]Boston.com, 4/19/2013. "Slain MIT Officer Sean Collier remembered as brave, devoted policeman" (http://www.boston.com/metrodesk/2013/04/19/mit-police-officer-sean-collier-killed-the-line-duty-during-confrontation-with-marathon-bombing-suspects/okOsk0WUnFyGB1yQ6CxuBI/story.html)

[6]PBS News Hour, 4/19/2013. "President Obama's Statement Following Arrest of Bombing Suspect" (http://www.youtube.com/watch?v=R6c6aKX9fu4)

[7]Boston.com, 4/25/2013. "Carjack victim recounts his harrowing night" (http://www.boston.com/metrodesk/2013/04/25/carjack-victim-recounts-his-harrowing-night/BhQWGzarWee8MZ6KtMHJNN/story.html)
The Boston Globle, 4/26/2013. "Interviewing 'Danny'" (http://www.bostonglobe.com/2013/04/26/BJQU6L0PZwaKJYdL7VOnIJ/story.html)

to game theory, divine providence becomes possible under open theism. I contend that the **rational actor**[iii] in game theory and the **libertarian free will**[iv] actor in open theism are irreconcilable.

- Humanity 2.0 in Googol-part Fugue: I submit that the compatibility of divine foreknowledge and human freedom is quite conceivable if humanity is endowed with perfect knowledge and wisdom in the **new heavens and new earth**.[v] Basing my imagination in part on **optimization theory**[vi], I contend that the perfect humanity will choose rationally, optimally, and predictably. Hence, the future would be fully foreseeable even for human beings, not to mention God. The conceivability of such a future serves as a counter-example to the claim that divine foreknowledge and human freedom are logically contradictory.

- Soul Making: I examine the soul making theodicy which sees the adversity of the present age as a necessary condition for the development of the human soul. I suggest that the **incompleteness theorems**[vii] may help illustrate why certain experiences such as humility, faith, hope, compassion, and forgiveness are possibly obtainable in the state of imperfect knowledge, which makes the present age necessary. The ensemble methods in **statistical learning theory**[viii] may serve as a model for the deliberative process by which life perspectives are integrated.

- Humanity on Trial: I suggest that the notion of "theodicy" is ultimately misleading because humanity is the one on trial. The worthiness of humanity to serve as the ruler of the creation is being tested and God sees fit to permit evil as part of the trial. The concept of **control samples**[ix] in experimental design helps explain the presence of the inauthentic and the unredeemed. The idea of **double blind**[x] helps explain the limited divine intervention.

Due to space limitation, I will focus on articulating my perspective in the main text and leave the brief introductions of the mathematical and theological concepts (marked in **bold**) to the end notes. Suggestions for further readings are found in the footnotes. Whether my conclusions prove persuasive or not, I hope to succeed in showing that mathematical concepts have an informative role to play in theological reflections. My intention is not to rehash the vast amount of literature devoted to the problem of evil, but rather, to highlight a few common sense arguments that could be intuitively persuasive to scientific minded and mathematically versed non-believers. There are big questions in life that call for the integration of the entire spectrum of human knowledge and wisdom. The problem of evil is one of them.

Taken as an argument against the existence of God, the **problem of evil**[xi] in its deductive form may be stated as follows:

1. There are evil things in the world.
2. God is supposed to be omniscient, omnipotent, and good.
3. If God is unaware of the evil things, he is not omniscient. But if God is omniscient, he is either unable to eliminate the evil things, which implies that he is not omnipotent, or he is unwilling to eliminate the evil things, which implies that he is not good.
4. Therefore, the supposedly omniscient, omnipotent, and good God does not exist.

I echo many who have observed that the problem of evil counts among the biggest intellectual obstacles for people to come to faith, with the qualification that the obstacle seems bigger for people brought up in a monotheistic culture.

If you come across an unbeliever from Europe, chances are, the problem of evil could come up fairly soon in any discussion of religions. Hans Küng called it the rock of atheism.[8] Writings of many so-

[8]Küng, Hans. On Being a Christian, trans. Edward Quinn. (Garden City, New York: Doubleday, 1976), p. 432.

called new atheists testify to that claim.[9] Leonard Mlodinow recalled movingly the holocaust survival story of his mother in his book "War of the Worldviews: Science vs. Spirituality" co-authored with Depak Chopra.[10] To him and many secular Jews, the memory of holocaust renders the concept of a good and omnipotent God empirically unbelievable because nothing that the Jewish people ever did seems to deserve that savage horror.[11]

Chinese also suffered tremendously during the second world war. But half a century of communism on top of a traditional culture permeated with Buddhism positioned the average Chinese further down the scale of atheism.[12] I have come to know many Chinese intellectuals who have never in their lives seriously thought about the problem of evil. Sin is a foreign concept for them to begin with. But evil as meaningless chaos? "That's just the way it is!"[13] Entropy increases monotonically in this universe until it reaches the inevitable fate of heat death.[14] Closer to the present time, an asteroid probably hit the earth 66 million years ago and wiped out the dinosaurs.[15] It could well happen again. Given what we know about complexity theory and butterfly effects, why should anyone be surprised when all the sudden things inexplicably fall apart?[16] All it takes is a neural misfire!

At the other end of the worldview spectrum, I also have south Asian friends who see life's ultimate reality as a perfectly balanced justice maintained by an impersonal transcendental being. They believe that when you do evil, you carry bad karma with you into your next reincarnation.[17] Whatever suffering you have in this life, you earned it in your previous life. As the libertarian in the west would say, "life is what you make it!" You have nobody to blame but yourself. You are your own avatar!

When former atheists and pantheists come to the Christian faith, they bring along certain solutions to the problem of evil. Biblical or not, the solutions are there. In contrast, those raised in a monotheistic culture were often brought up with an image of God resembling a loving grandfather who sits on his armchair to spoil his grandchildren. It is that picture of God which seems most at odd with the existence of evil.

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[9]Brown, Neil. "New Atheism and the problem of evil" Compass, Summer 2013 Volume 47 Number 1, p. 29-32.

[10]Chopra, Deepak and Mlodinow, Leonard. War of the Worldviews: Science vs. Spirituality. (New York: Harmony Books, 2011) p. 62-63, 234-235.

[11]For example, see the blog post of Rabbi Alan Lurie, "How could God have allowed the Holocaust" (http://www.huffingtonpost.com/rabbi-alan-lurie/how-could-god-have-allowe_b_1207672.html)

[12]Chan, C.W. "Good and Evil in Chinese Philosophy" The Philosopher, Volume LXXXIV, 1996 (http://www.the-philosopher.co.uk/good&evil.htm) Feng, Youlan. History of Chinese Philosophy, Volume II. The Period of Classical Learning (from the Second Century B.C. to the Twentieth Century A.D). Trans. Derk Bodde. Ch. XIV Liu Chiu-Yuan, Wang Shou-jen, and Ming Idealism. Part 6, Origin of Evil.

[13]Phil Collins and David Crosby popularized the saying with their 1990 single from the album "… But Seriously," which features a CD cover of a boy riding a bike away from a bomb explosion, referring to the violence the UK experienced during the the Irish Republican Army conflict.

[14]Adams, Fred and Laughlin, Greg. The Five Ages of the Universe: Inside the Physics of Eternity. (New York: The Free Press, 1999) p.153-182 describes the last of five ages as the "dark era" where there is heat death and "never-ending annihilation."

[15]Renne, Paul R. et. al. "Time Scales of Critical Events Around the Cretaceous-Paleogene Boundary," Science, 8 February 2013, Vol 339, no. 6100 pp. 684-687

[16]Mitchell, Melaine. Complexity: A Gudied Tour (New York: Oxford University Press, 2009), ch. 2 "Dynamics, Chaos, and Prediction," p. 15-39 identifies three consensus opinions in complex system theory. First, "seemingly random behavior can emerge from deterministic systems, with no external source of randomness". Second, "the behavior of some simple, deterministic systems can be impossible, even in principle, to predict in the long term, due to sensitive dependence on initial conditions." Third, "although the detailed behavior of a chaotic system cannot be predicted, … there are some higher-level aspects of chaotic systems that are indeed predictable."

[17]Reichenbach, Bruce R. (April 1989). "Karma, causation, and divine intervention" Philosophy East and West (Hawaii: University of Hawaii Press) 39 (2): 135–149 [145]. doi:10.2307/1399374. Retrieved 2009-12-29. [http://ccbs.ntu.edu.tw/FULLTEXT/JR-PHIL/reiche2.htm]

Agnostics with a shopping mall view of religions ask questions about evil in their own ways, too. One victim who died on April 15 was a Chinese graduate student from Boston University. The only child in her family and an outstanding student, she happened to be a statistician and was involved with the international Christian fellowship at the historic Park street church. Among her friends who have no religious affiliations, some would probably ask: why didn't *her* God save her?

The English word “evil” stands apart from words like sin and pain because of its dark and inexplicable quality. Even before 9/11, the popular culture in the USA has been signaling a rising sense of uncertainty about the world in the collective consciousness. I recently did some searches on the IMDB web site for the feature films containing the word “dark” in their titles. From 1980 to 1984, there were 10. From 2010 to 2014, there are 134 including those in production. Counting also the word “darkness” adds 41. The ratio relative to all feature films grew from 0.07% to 0.51%. Counting only those grossing over 1 million dollars, the comparison is 0% and 1.5%.

<u>Feature Films with the the word “Dark” and “Darkness” in the Title</u>

	1980-84	1985-89	1990-94	1995-99	2000-04	2005-09	2010-14
“Dark”+“Darkness”	10 + 1	17 + 5	32 + 5	35 + 12	29 + 12	71 + 23	134 + 41
Total Features	14707	15949	16365	16534	18517	26179	34605
Ratio	0.07%	0.14%	0.22%	0.28%	0.22%	0.35%	0.51%
Grossing over $1M	0 in 220	3 in 746	5 in 915	3 in 964	4 in 954	4 in 1047	9 in 591

In finance, institutional investors trade among themselves via opaque exchanges known as dark pools using dark liquidity, with volume surpassing open exchanges. Some have suggested that dark pools and high frequency trading have led to much more frequent emergence of “financial black swans driven by ultrafast machine ecology.”[18]

Even scientists are picking up this language fashion. A 2009 Nature article declares that “Dark is the New Black” in cosmology.[19] Cosmologists now estimated that only 4.9% of the universe is made up of ordinary matter, while dark matter and dark energy add up to 95%.

I suppose most people would like to see darkness and evil eliminated, regardless of how they understand those words. Nevertheless, the bible foretells ever escalating magnitude of natural and human caused disasters in the last days (Matt. 24, 2 Thes. 2:9-12, 2 Tim. 3:1-5). Christians must be prepared to give an account for our hope in the midst of evil and sufferings (1 Pet. 3:15).

The Free Will Argument of Open Theism

The continued interest in open theism is a sign of the time. The “dark is the new black” mood of our time strengthens the case of some atheistis and that reality calls for a rational Christian response. To understand open theism, it is helpful to place the free will defense against the problem of evil as its principal premise. The main ideas of open theism can then be worked out as follows:

1. Even though God is omnipotent, God cannot do the logically impossible. Once God gave people free will, God cannot stop people from making bad choices. Hence, God cannot be held responsible for any evil human deeds if people have free will.
2. Because the utmost desire of God is to have people loving him by free choice, God sees a greater good in giving people free will.
3. If God knows the future, the future is objectively determined and free will is a mere illusion.

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[18] Johnson, Neil et al. “Financial black swans driven by ultrafast machine ecology” arXiv:1202.1448 [physics.soc-ph] (http://arxiv.org/ftp/arxiv/papers/1202/1202.1448.pdf)

[19] Massey, Richard. “Cosmology: Dark is the New Black.” Nature 461, p. 740-741 (8 October 2009) doi:10.1038/461740a

4. If free will is an illusion, God is responsible for evil.
5. But God is good.
6. Hence, free will cannot possibly be an illusion. Hence, the future cannot possibly be objectively determined. Hence, God cannot possibly know the future.

The first point is a version of the free will defense. Augustine was often recognized as the first major proponent of the free will argument.[20] In the 20th century, C. S. Lewis[21] and Alvin Plantinga[22] famously used the free will argument to defend Christianity against atheism.

The second point is often called the greater good argument for free will. There are also other greater good arguments that explain the existence of evil without free will. One only has to think about popular adages like "no pain, no gain" or "pain is the bitter medicine for the soul." In a later section, I describe another greater good argument known as the soul making defense. Open theists do not necessarily disagree with other greater good arguments. However, they are persuaded that unless the free will argument stands, God is culpable for the evil in the world.[23]

The free will defense addresses primarily the how and not the why of evils. Augustine thinks that our mind is created to be rational. Our rationality prevents us from understanding what is irrational. But evil is irrational.[24] Trying to understand evil is like trying to see darkness, there is nothing to be seen. Augustine concludes that it is impossible to ask *why* evil happens. We must be content with understanding *how* evil happens. In other words, we can ask "*who did that*" but "*why the person did that*" will remain a mystery in the ultimate sense.

It is worth noting that Augustine modified his view on free will over the years. When he wrote "On Free Choice," his view is mildly libertarian: God only observes human free choices from eternity and does not control them.[25] But only a few years later, Augustine realized that his view of free will leaves a hole even in the how question: if free will is an unknowable cause, does it really describe how things happen? In the end, while he maintains that God cannot be held responsible for evil, Augustine is convinced that no human free choice could be made independent of the sovereignty of God.[26]

Open theists reckon any determinism as irreconcilable with libertarian free will. Unlike the latter Augustine, most open theists do not further dissect free will beyond a self-originating cause. Their departure from the traditional Arminian understanding of divine foreknowledge could be seen as working out the Arminian understanding of free will more consistently.[27] The latter Augustine would probably ask the open theists how they ontologically *ground* the human free decision. In computational terms, wouldn't their libertarian free will necessitate either a kind of impersonal "oracle

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[20] Larrimore, Mark ed. The Problem of Evil. (Malden, MA: Blackwell, 2001) ch. 10 "Augustine, City of God" p. 54-61.
[21] Lewis, C. S. The Problem of Pain. (New York: Touchstone, 1996)
[22] Plantinga, Alvin. God, Freedom, and Evil. (Grand Rapids, MI: Eerdmans, 1974)
[23] Boyd, Gregory. Is God to Blame: Beyond Pat Answer to the Problem of Suffering. (Downer Groves: IVP, 2003), p. 186: "God sometimes predestines events, but he doesn't predestine individuals. He sometimes uses the evil intentions of people to fulfill his predestined plans, but he doesn't predestine people to have these evil intentions."
[24] Augustine, "On Free Will," 2.20.54: "... We cannot doubt that the movement of the will, that turning away from the Lord God, is sin; but surely we cannot say that God is the author of sin? God, then, will not be the cause of that movement; but what will be its cause? If you ask this, and I answer that I do not know, probably you will be saddened. And yet that would be a true answer. That which is nothing cannot be known."
[25] Scott, T. Kermit. Augustine: His Thoughts in Context. (Mahwah, NJ: Paulist Press, 1995), p. 158-168.
[26] Ibid, p.169-211.
[27] Hasker, William. "Foreknowledge and Necessity," Faith and Philosophy 2, no. 2 (April 1985), p.121-157 suggests that "those who seek to maintain the compatibility of free will and foreknowledge are in the end forced to abandon, implicitly if not explicitly, the libertarian conception of freedom," p. 154. See also: Frame, John. No Other God (Phillipsburg, NJ: P&R Publishing, 2001), p. 39-40. Ware, Bruce. God's Lesser Glory. (Wheaton, IL: Crossway Books, 2000), p. 31-42.

machine" which is higher than God, or a kind of "kernel functions" for Fate() that is opaque even to God, or both? Following most humanist libertarians, open theists nevertheless seem either unwilling to answer the grounding question or see that as unnecessary.

The strident stand open theism stakes out over free will is remarkable, considering how inconclusive the debate over free will has been among the academic philosophers.[28] Even traditional Arminian theologians sympathetic to open theism seem to see wisdom in downplaying the issue of free will by stressing instead the goodness of God.[29] But if there is one premise the open theists seem unwilling to give up, it is free will. Hence, a better circumstantial interpretation of their motivation is found in the tactical success they see in Alvin Plantinga – how Plantinga employed the free will defense to defeat the deductive form problem of evil and helped usher in a renaissance of Christian philosophical scholarship.[30] Plantinga perceptively identified the concept of free will as a foundational premise that most of his atheist colleagues in the philosophy department would be quite unwilling to surrender. I suppose from the view of an atheist philosopher, an argument against theism cannot really be worth quite as much as the notion of free will, without which personal moral responsibility is difficult to defend.

Ronald Nash recalled a meeting in which Plantinga made the suggestion that natural disasters could conceivably be explained as the works of demons exercising their free wills.[31] A Jewish rabbi protested at the back of the room in disbelief and questioned if Plantinga expected the audience to believe in a personal devil. Puffing his cigar, Plantinga explained that it does not really matter what one personally believes or even if the suggestion turns out to be true. All he was saying is that such a suggestion is *conceivable* and that is sufficient to defeat the deductive problem of evil. Now, that is what I meant by *tactical!*

However, the success of tactical arguments rest on understanding the assumptions of the people one is talking to. As the sons of this age often say on Wall Street, the nature of fads is that there is always a piling on phase that extends into a long tail, way after a stock attained its fair price. More than a quarter century after the deductive problem of evil declared dead,[32] it is at least fair to ask: is the sacred

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[28]Fischer, J. M., Kane, Robert, Pereboom, Derk, Vargas, Manuel. Four Views on Free Will. (Malden MA: Blackwell Publishing, 2007), p.1 introduces the subject of free will by pointing out the diversity of opinions: "There are many different ways of thinking about the nature of free will, and there are serious disagreements about what would constitute an adequate theory of free will. Much of the tradition has taken 'free will' to be a kind of power or ability to make decisions of the sort for which one can be morally responsible, but philosophers have also sometimes thought that free will might be required for a range of other things, including moral value, originality, and self-governance. Two other claims often made about free will are hotly disputed among philosophers; and authors of this volume will take different sides on these claims. One is the claim that free will requires 'alternative possibilities' or the power to do otherwise, and the other is the claim that free will requires that we are the 'ultimate sources' of our free actions or the ultimate sources of our wills to perform free actions."

[29]Olson, Roger E. "Is Open Theism a Type of Arminianism?" [http://www.patheos.com/blogs/rogereolson/2012/11/is-open-theism-a-type-of-arminianism/]

[30]Clark, Kelly James. "Introduction: The Literature of Confession" in Philosophers who Believe: The Spiritual Journeys of 11 Leading Thinkers (Downer Groves, IL: IVP, 1993), p. 10-13 observes that a major factor "in the revival of Christian philosophy was the presentation, publication and subsequent discussion of Plantinga's 'Advice to Christian Philosophers' … The philosophical and Christian boldness of Plantinga's address engendered an immense flowering of Christian philosophy in the subsequent decade. … Alvin Plantinga is perhaps best known among philosophers for his penetrating analyses of the problem of evil and the rationality of religious belief. From the time of the Ancients it has been alleged that there is an incompatibility between an omniscient, omnipotent and wholly good God and the fact of evil. Plantinga's free will defense demonstrates that this contradiction is only apparent and that the existence of evil does not logically disprove the existence of God."

[31]Nash, Ronald. Near the end of lecture record on "The Problem of Evil" in the course of Christian Apologetics from Reformed Theological Seminary at Apple's iTunes U.

[32]Mackie, J. L. The Miracle of Theism. (Oxford: Clarendon Press, 1982), p.154.

cow of some atheist philosophers of the last century still worth a bigger shrine today?

Divine Providence and Game Theory

Open theists deserve recognitions for their willingness to address the problem of evil. But their denial of divine foreknowledge is beyond the pale of biblical (Num. 23:19, Ps. 33:11, Ps. 139:1-4, Isa. 14:27, Isa. 40:13-14, Isa. 46:10, Rom. 11:33, Heb. 4:13, Heb. 6:17). Their refusal to consider other alternative model of human free agency is disappointing.[33] Many published studies have pointed out all sorts of obvious difficulties when the implications of open theism are logically worked out.[34] For example:

- If God does not know the future certainly, how do we understand prophecies that involve the faithful response of God's people? (Heb. 11) Were they compelled? For instance, did John the Baptist respond willingly to become the second Elijah?
- Given libertarian free will, what prevents human beings from sinning again in the new heavens and new earth? (Dan. 12:3, Rom. 8:19-21, 1 Cor. 8:12, 2 Cor. 4:16-18, Rev. 22:3-5)
- Conversely, what is the basis for claiming in the first place that all will sin? (Ps. 51:5, Ps. 58:3, Rom. 3:23, Rom. 5:12, 18-19, 1 Cor. 15:22, Eph. 2:3, 1 Jn. 1:8)

Open theists are aware of these objections and they do try to answer some of them. For instance, Alan Rhoda suggests in a relatively recent article that divine providence is still possible under open theism by appealing to game theory.[35] Citing the theory of moves (TOM)[36] proposed by Steven Bram, Rhoda suggests that God could structure cycles of rewards and punishments to guide the decisions of people, effecting a degree of divine providence.

His suggestion leaves me perplexed. Without much justification, Rhoda brushes aside the fundamental difference between the idealized rational actor in game theory and the libertarian free will actor in open theism. Think about the classic TV show Star Trek. Mr. Spock probably comes close to being an embodiment of the rational actor in game theory. Mr. Spock always tries to maximize the expected utility given all available information. Because of his rationality, an objective omniscient observer who knows all the information available to Mr. Spock ought to be able to predict exactly what Mr. Spock will do in a given situation. However, that is precisely what open theism says cannot be the case for their libertarian free will actor. Captain Kirk is perhaps a better personalization for their libertarian free will actor. Open theists believe that even if God knows everything up to the moment Captain Kirk makes a decision, God still cannot foreknow what Kirk will do with absolute accuracy. That leaves a game theorist no choice but to say that Captain Kirk is less than rational. Unfortunately, without the rational actor premise, game theory simply cannot apply.[37]

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[33]For discussion on classical theistic alternatives from Boethius, Aquinas, Ockham, Molina, Frankfurt, see Zagzebski, Linda, “Foreknowledge and human freedom” in Quinn, Philip L. and Taliaferro, Charles ed., A Companion to Philosophy of Religion. (Oxford, Blackwell Publishing, 1999), p.291-299.

[34]Roy, Stephen. How Much Does God Foreknow: A Comprehensive Biblical Study. (Downers Grove, IL: IVP, 2006) presents a detailed exegesis of all the key biblical passages. Erickson, Millard. What Does God Know and When Does He Know It? (Grand Rapids, MI: Zondervan, 2003) contrasts the open theist and traditional view of divine providence from the biblical, historical, and philosophical perspectives. See also Frame (2001), Ware (2000).

[35]Rhoda, Alan “Getting Beyond the Chessmaster Analogy: Game Theory and Divine Providence” in Oord, Thomas Jay, ed. Creation Made Free: Open Theology Engaging Science. (Eugene, OR: Pickwick Publications, 2009) p. 151-175.

[36]Brams, Steven. Theory of Moves. (Cambridge: Cambridge University Press, 1993). See also: Brams, Steven. Game Theory and the Humanities: Briding Two Worlds. (Cambridge, MA, MIT Press, 2011) Idem. Biblical Games: Game Theory and the Hebrew Bible. (Cambridge, MA: MIT Press, 2003)

[37]According to E. Roy Weintraub, all of neoclassical economics rest on three axioms: “1. People have rational preferences among outcomes. 2. Individuals maximize utility and firms maximize profits. 3. People act independently on the basis of full and relevant information.” from “Neoclassical Economics” in “the Concise Encyclopedia of Economics” [http://www.econlib.org/library/Enc1/NeoclassicalEconomics.html] In other words, rational actors choose the action

Therefore, given his open theism, I fail to see how Rhoda can take game theory to explain divine providence. The rational actor in game theory is objectively deterministic while the libertarian free will actor in open theism is objectively indeterministic. Appealing to game theory as an open theist is a case of having your cake and want to eat it too!

One may suggest that there are ways to fix his argument by re-defining free will and rationality in more compatible terms: a person can be both free and rational. That is exactly my point though. Alternative philosophical models of free human agency do exist. Open theists maintain that divine foreknowledge and human freedom are contradictory (point #3). But all they simply presuppose that as a philosophical definition without any biblical or scientific evidence. One could argue that with TOM, Rhoda demonstrates precisely the possibility of **compatibilism**[xii]. A rational actor might always choose to accomplish God's good intention even though in the actor's mind, he is acting purely from his selfish interest. That paradox is what Adam Smith meant by the invisible hand.[38]

It seems that open theists are less concerned about accepting or denying divine foreknowledge as such, more concerned about its consequence for divine culpability (point #4). They worry that any harmonization of divine foreknowledge and human will implies that either God allows evil when he could stop it, or God actively intends evil. But like it or not, the God in the bible does let people sin (Acts 7:42, Rom. 1:24, 26, 28), and does actively inflict pain and suffering for different reasons (Gen. 6:5-7, Gen. 50:20, Job 1-2, Isa. 53, John 9:1-3, 2 Tim. 1:8-12).

As the sons of this age from K Street would likely opine: keeping God in the dark only creates so much *plausible denialability* before making God looks like an aloof and incompetent fool – a God who not only gambles, but blames his loss on the people!

The real sting of evil resides in our present inability to see God's good intention (Rom. 8:28) in the midst of pain and suffering. As Joseph said to his brothers, “You intended to harm me, but God intended it for good to accomplish what is now being done, the saving of many lives.” (Gen. 50:20, NIV) However, Joseph only realized that in the end. The existential incongruity is even more poignantly displayed through the pain of his father (Gen. 37:34-35, 42:36-38, 45:25-28). Still, our present ignorance of God's good intention is not a proof that God has no good intention.

I often wonder if Christians are doing unbelievers a favor by seating them in a jury box and making the “case for God” as if God has hired us as his defense attorney. That is like trying to defend the judge before the criminal. There is only so much case you can make for the existence of light to the born blind. As Plato perceptively lamented, people might even murder you in a cave for insinuating that there is light out there! A case does need to be made, as I will argue in a following section. But it is not the case for God, but rather the case for humanity.

Humanity 2.0 in Googol-part Fugue

Let us examine point #3 in my logical outline of open theism: is it true that divine foreknowledge and human freedom are really necessarily contradictory? To refute that point, we need a counter example, a conceivable scenario where the two compatibly exist.

In 1 Cor. 13:9-10, the Apostle Paul writes, “For we know in part and we prophesy in part, but when completeness comes, what is in part disappears” (NIV). Then, in verse 12, “For now we see in a mirror

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yielding the highest utility function value given the information available to them. Knowing all the information available to a rational actor, an objective observer ought to be able to predict exactly what the actor would decide. See also the end note on the rational actor theory.

[38] Smith, Adam (1759), "Theory of Moral Sentiment", pp. 184–5 in: The Glasgow edition of the Works and Correspondence of Adam Smith, vol. I" (Oxford: Oxford University Press, 1982)

dimly, but then face to face. Now I know in part; then I shall know fully, even as I have been fully known.” (ESV)[39]

Traditionally, the text has been interpreted to reveal that in the new heavens and new earth, humanity will become perfect in knowledge and wisdom.[40] Every person will then freely choose what is good and optimal. As a result, everyone knows what to expect. And if even human beings have that kind of foresight, it becomes logically impossible for God to not have perfect foreknowledge himself.

Suppose we think of the new heavens and new earth in abstract as a continuous function that takes the free choice of every person as an independent input variable at each logical moment to compute the state of the next logical moment. And further suppose that the analytical form of the function is made known by God to all. And thirdly, suppose there is a goodness measurement for the state of the universe, call it the “cosmological utility function.” It is only rational for every person, equipped with perfect knowledge, to use one's perfect wisdom to coordinate their inputs so as to maximize the output of the cosmological utility function. No matter how the solution is found, the extreme value theorem guarantees the existence of the maximal and minimal value points if the function is bounded and continuous. To make things perfectly deterministic, all we need is a rule to pick from the equally best.

In terms of game theory, such an eternal reality may be seen as a **non-zero sum cooperative game**[xiii] where there is a predictable optimal for each move. It is also like a symphony with many players engaging in a *googol-part fugue*: Everyone performs superbly and nobody makes a mistake. The outcome is a perfect harmony.

It is therefore conceivable for divine foreknowledge and human freedom to be both true. All it takes is an imagination of what may be called an open *source* or open *access* God, a God who gives mature humanity all the necessary knowledge and wisdom so that human beings can make the perfect choice in eternity (cf. 1 Cor. 13:11).

When the open theist God looks into the future, all he sees is nothing. When the open source God looks from eternity into what we call the future, he sees his image and likeness working together to bring forth his glory. The open theist God can never quite “rest.” The open source God sits back, relaxes, and enjoys the show on the eternal day of sabbath. The open theists have to bring God down to our epistemological level. The open source God lifts us all up towards his.

A perfect picture of humanity “2.0” might seem incredible in this physical universe. True. Yet, there are biblical and scientific evidences to suggest that the present universe is not meant for eternity, but rather, destined for desolation. Within an error probability of less than 0.4%, latest cosmological evidences suggest that we live in an ever expanding **flat universe**[xiv] with an omega of 1. The most likely fate of this universe is a heat death or deep freeze.[41] Unless there is complete overhaul of all the

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[39]Note the future indicative middle (deponent) and the aorist indicative passive of the second and the third “know”. “The words bring out the inadequacy of man's present knowledge of God in contrast with God's knowledge of man now and the knowledge of God that man will have in the future.” Rogers, Cleon, jr. & IIII. The New Linguistic and Exegetical Key to the Greek New Testament. (Grand Rapids, MI: Zondervan, 1998), p. 380.

[40]While some cessationists have suggested that the perfection in 1 Cor. 13:8-12 refers to the closure of the scriptural canon at the end of the apostolic age, that has not been the traditional interpretation. McDougall, Donald G. “Cessationism in 1 Cor. 13:8-12” The Master's Seminary Journal 14/2 (Fall 2003) p. 177-213. Erickson, Millard J. Systematic Theology. (Grand Rapids, MI: Baker Books, 1985) p. 999-1002 sees the end state with perfect and complete knowledge. Grudem, Wayne. Systematic Theology: An Introduction to Biblical Doctrine. (Grand Rapids, MI: Zondervan, 2000), p. 1162 draws the distinction between perfection and completeness, suggesting that it is perfect, but not complete. Horton, Michael. The Christian Faith: A Systematic Theology for Pilgrims on the Way. (Grand Rapids, MI: Zondervan, 2011), p.697-698 observes the creator-creature distinction drawn by Francis Turretin noting that the understanding of God in the perfect state will be clear and intuitive.

[41]Recent measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) are consistent with a flat universe

fine-tuned physical constants, it is fairly inconceivable that this universe could be where humanity will reside eternally. Time, space, matter, and energy could all have different meanings in the universe of the new heavens and new earth.

The **best of all possible worlds**[xv] argument was originally put forth by Gottfried von Leibniz and is embraced by various Christian rationalists through some versions of divine middle knowledge.[42] Historically, what Leibniz put forth have been mostly ridiculed, never logically refuted. It is easy to ridicule because this world just doesn't feel like it could be the best possible. However, it may be helpful to reason backward from the eternal end. The best of all possible worlds is more intuitively conceivable if every human being is in the state of perfect freedom, knowledge, and wisdom, which I have argued to be possible in terms of biblical theology. Working backward, if there is a necessary development path for humanity to traverse in order to get to that state of perfection, then every step along that development path could also be said to be the best of all possible worlds for the moment.

Soul Making

Let us probe further into that development path for humanity. One may ask: why couldn't God just place us in that perfect world where we have perfect knowledge and wisdom to begin with? Why save the best for last? To ask it like my children: Why can't we skip the appetizers, the main entrée, and go right to the dessert? To give this question more biblical theological sophistication: If day is "good" (Gen. 1:3) and night is by implication not-as-good, what is the point of having six nights and six days before having an eternal seventh day?[43]

Before we attempt an answer, it is worth noting that the keyword is "why." *What is the point of all these?* The problem of evil does not fade away with Christians dancing around the one question that truly matters: the purpose of evil. Describing evil as an inconvenient possibility of libertarian free will, a self-originating cause, is a non-explanatory explanation. At most, it answers how evil happens, it does not address why it happens. It is like saying the Sandy Hook massacre took place because the

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theory. NASA, "Will the Universe expand forever?" (http://map.gsfc.nasa.gov/universe/uni_shape.html) See also Adams, Fred and Laughlin, Greg (1999).

[42]Erickson, Millard J. Christian Theology. (Grand Rapids, MI: Baker Books, 1985) p. 356-362 describes a "moderately Calvinistic model" that is "in many ways similar to the argument of Gottfried von Leibniz in his Theodicy." Erickson's proposal may be described as a compatibilist middle knowledge model, and is therefore different from Molinist middle knowledge, Occasionalism, and traditional Arminianism in their incompatibilist view of free will. A common problem with all the middle knowledge models is the lack of direct biblical support. A problem specific to the incompatibilist versions is the so-called "grounding objection" IEP, "Middle Knowledge" (see http://www.iep.utm.edu/middlekn/) Without committing to any middle knowledge model *for God*, it is possible to think of a compatibilist middle knowledge model *for humanity*, supposing the perfection of human wisdom and knowledge in eternity. True or not, the mere *conceivability* of sufficient human middle knowledge and perfectly rational choice requires the rejection of the claim that divine foreknowledge and human freedom are necessarily incompatible.

[43]The poetic literary structure of the Genesis 1 text has been emphatically observed by proponents of the framework interpretation. Blocher, Henri. In the Beginning: The Opening Chapters of Genesis. (Downer Grovers: Intervarsity, 1984). Wenham, Gordon J. Genesis 1-15. (Waco, Texas: Word Books, 1987), p. 39–40. Kline, Meredith. Kingdom Prologue: Genesis Foundations of a Covenant Worldview. (Eugene OR: Wipf and Stock, 2006). For an introductory comparison with the literal day and day-age views, see Hagopain. G. et. al. The Genesis Debate: Three Views on the Days Creation (Mission Viejo, CA: Crux, 2000). In all three prevailing views of the Genesis 1 account, the significance of the six "nights" and their obvious parallels with the primordial chaotic darkness (*Tohu wa bohu*) is not explored . The framework theory treats the nights as mostly wire frames. The other two views treat the nights as divine recesses, which seems counter-intuitive because of the lack of night on the day of eternal rest. Illuminations from the parallel John ch. 1 text also seem generally lacking in all three views. Without accepting or rejecting the text as literally historical, it seems plausible to see the seventh day as a poetic reference to the eternal day *yet* to come. The father is still at work and so is the son (John 5:17). The evil of the night is still present. But on the eternal seventh day, there will be just day light and no more evil. God can rest and let his creatures do some work for a change!

gunman had a gun. Open theism gives a greater good justification for free will, namely, so that people would love God freely. But that is still not an explanation for evil itself.

John Hick identified two traditions in the way problem of evil is handled: Augustine is representative of the Latin tradition which emphasizes the free will argument. Irenaeus is representative of the Greek tradition which stresses the soul making argument.[44] The distinction is too simplistic but is helpful nonetheless in contrasting two fairly different perspectives of sin: (a) sin as the result of the lack of will or self control, versus (b) sin as the result of the lack of true knowledge and wisdom. Assuming we can conceive of an eternity where there is perfection of human knowledge and wisdom, the soul making argument could be understood as saying that there are experiences such as humility, faith, hope, compassion, and forgiveness that are possibly obtainable only in the state of imperfect knowledge. And those experiences may be necessary in the maturity of the human soul. For example, without sinning, Jesus experienced the humility of being human (Phil. 2:7 *kenosis*, the emptying of himself) through hunger (Matt. 4:1-2), thirst (John 19:28), sadness (John 11:35), weariness (John 4:5-6), dependency (Luke 23:44-46), and death (Heb 2:14), giving meaning to his compassion (Heb 2:17-18, 4:15). So even though the state of imperfect knowledge and wisdom make people more prone to lapses in judgment because it may seem more "rational" (pleasurable) to sin, there is a greater good for the imperfect state of knowledge.

Kurt Gödel's incompleteness theorems may be applied to illustrate an important idea in the soul making argument. Even if a person has a consistent knowledge set of every true proposition, there will still be propositions in the set which the person cannot prove. Therefore, to have confidence in the veracity of those unprovable propositions, a person must rely on faith. To attain faith, one must come to a place of humility. The state of imperfect knowledge may be necessary in drawing out such humility. For example, the bible portrays sacrificial forgiveness as the greatest love of all (Num. 14:19, Luke 7:47, John 15:13). It is hard to conceive how anyone can experience forgiveness, whether as the forgiver or the one being forgiven, if the world begins in a state of perfection and nobody ever wrongs anyone. The *Felix Culpa* (literally, "happy mistake") argument says that failures often create the logically necessary conditions for some other experiences.[45] Wrongs make the experience of forgiveness possible.

In terms of the process of soul making, contemporary thinking in statistical learning, especially with ensemble methods[46], may help illustrate how the human mind combines competitive perspectives into an intuition that best explains the perceived reality.[47] Faith and hope could be understood as such fundamental perspectives, or "base learners," that are matured through interpretation of real life experiences. As a deliberative process that takes place in what could be called the "debate society of the mind,"[48] these perspectives compete, much like players in a game, in rounds of mental "debate

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[44]Hick, John. Evil and the God of Love (New York: Harper & Row, 1966)

[45]Helm, Paul. The Providence of God. (Downers Grove, IL: InterVarsity Press, 1994) p.213-215

[46]Elder, John and Semi Giovanni. Ensemble Methods in Data Mining: Improving Accuracy Through Combining Predictions. (San Rafael, CA: Morgan & Claypool Publishers, 2010) noted that ensemble methods have been "the most influential development in Data Mining and Machine Learning in the past decade." Elder identifies "importance sampling" as a common strategy in all classic ensemble methods. The synergy between ensemble methods and game theory has been explored by Robert Schapire and Yoav Freund, who themselves discovered boosting, a popular family of ensemble methods. Schapire and Freund see the ensemble classifier and the base classifier as competitors in a repeated online learning game. See Schapire, Robert E. and Freund, Yoav. Boosting: Foundations and Algorithms. (Cambridge, MA: MIT Press, 2012)

[47]Kahneman, Daniel. Thinking Fast and Slow. (New York, Farrar, Straus and Giroux, 2011) distinguishes two systems in the mind. System 1 is fast and intuitive while system 2 is slow and intentional. The judging audience and debater metaphor may find a certain mapping in the two system view.

[48]Minsky, Marvin. The Society of Mind. (New York: Simon & Schuster, 1986) famously explores a society view of the mind

games" until an optimal ensemble opinion is formed that makes the best-fit perspective given the gathered data. That optimum corresponds to the Nash equilibrium in game theory.[49]

There are practical spiritual encouragements that are found in the soul making argument. It reminds us that even in the toils and labors of the present life we are gathering experiences with eternal values (Matt. 6:20, Mark 10:21, Luke 12:33). It gives us the patience to make the most out of the days that are evil (Eph. 5:15-16). At the same time, the soul making argument also correctly focuses our hope in the glory of eternity when faced with adversity (Col. 3:2, Jam. 5:7-8, 1 John 2:15-17). Faith often becomes therapy rather than prophecy when Christians demand our best life *now*. The Christian church often becomes worldly when the otherwordly perspective is set aside.[50] The soul making perspective helps set our priority straight: we are aliens in this fallen world; a better place is being prepared for us.

Humanity on Trial

Greater good arguments such as the free will defense and the soul making defense are often called theodicy, the defense of God. But the word theodicy is quite misleading because God really doesn't need our defense. The biblical big picture is rather that *humanity is on trial*.[51] God brings glory to himself by demonstrating the worthiness of humanity as his servant to rule over all creation. And God sees fit to permit evils and inflict pain and sufferings in this world as part of the trial.

The trial of humanity could be a necessary step in the development of the collective human soul. An ancient prince often leads his army into battles to earn the respect from his subject before he ascends to the throne. Worthiness is attributed to the lamb that was slain (Rev. 5:2, 4, 9, 12). The prologue in the book of Job reveals Satan's jealousy for Job as the blessed "servant of God," hinting at the necessity for the demonstration among the heavenly host. The forbidden fruit in the garden (Gen. 2-3), the tests of Abraham (Gen 22), the trials of Joseph (Gen. 37, 39), the sufferings of Job (Job 1-2), the battles fought by Joshua and the judges, the temptations and the passion of Jesus Christ (Matt. 4:1-11, Mark 1:13, Luke 4:1-13, Matt. 26-27, Mark 14-15, Luke 22-23, John 18-19), and the persecutions of the early church (Acts 5:17-18, 8:1-3) are some biblical stories in which God demonstrates the worthiness of his

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which consists of many simple member parts that may not seem at all intelligent when each is by itself. The "debate society of mind" view may be seen as a higher level metaphor over Minsky's lower level picture.

[49] Thinking of the deliberative process in terms of a debate "game," a speech may be seen as a move with the goal of boosting the importance of a subset of the data samples, thereby strengthening the importance of the view held by a debater representing a certain ideological perspective. A debate cycle begins with a previously synthesized hypothesis in the mind of the judging audience. The hypothesis may be tested against any data samples. The debaters would then look for data samples that shine the best light on their views while demonstrating weakness of the current synthesized hypothesis. The debaters will then highlight, or "boost," certain data in the next debate speech, thus effecting a kind of importance sampling that may add to the weight of their represented view in the next synthesis.

[50] Wells, David. The Courage to be Protestant: Truth-lovers, Marketers and Emergents in the Postmodern World. (Grand Rapids, MI: Eerdmans, 2008) summarizes his previous four-volume critique of contemporary evangelicalism by arguing that "it takes no courage to sign up as a Protestant ... to live by the truths of historic Protestantism, however, is an entirely different matter. That takes courage in today's context." His choice of the word "courage" is very helpful here. If a constructive biblical definition of "free will" can be found, it cannot be in the possibility to oppose God's will. Rather, it ought to be the ability to defy the corrupted norms of the world. The apostle Paul speaks of true freedom among the redeemed. The unredeemed have a rational will, but it is not true freedom, as it is bound to sin. A biblical free will is the non-conformist courage to do what God expects (falls within the bound of God-given freedom) rather than what the worldly norm expects (which is no freedom at all).

[51] In May 2011, 20 Nobel laureates representing planet earth put humanity on a trial at the symposium of global sustainability. (http://newswatch.nationalgeographic.com/2011/05/18/at-stockholm-gathering-of-minds-planet-earth-vs-humanity/) The idea of putting humanity on trial is nothing new, as science fiction writers have often imagined the trial of humanity with advanced alien civilizations as the jury, like the Q Continuum in Star Trek: The Next Generation that decides to put humanity on trial. While Jesus Christ never seemed to present a theodicy, he interceded for mankind in the manner of *anthropodicy*, e.g. "Father, forgive them, for they do not know what they are doing." (Luke 23:34).

chosen servants to himself and to the heavenly host.

A trial can be seen as a zero-sum game where one side wins and the other side loses. The wager between Satan and God in the prologue of Job could be a microcosm for the trial of all humanity. Apparently, God has so much confidence in Job he practically dared Satan into the betting game (Job 1:8, 2:3).

A trial is ultimately an examination of authenticity and true understanding. Just as a multiple choice question has to contain both right and wrong answers, propositions that are true and false must be present in the state of imperfect knowledge to fashion a fallible trial. In the present age, choosing what is morally wrong may seem pleasurable while choosing what is morally correct may seem painful. That moral dilemma opens up a real possibility of failure because God creates people to rationally seek the pleasurable. And indeed, human failed the very first test. However, the soul making of humanity as a whole is occasioned with successes that do demonstrate the growth of humanity's total understanding of God, accumulating to the cross, which both a trial for Jesus, the representative of the new humanity, and a redemptive act for all creation.

Therefore, the apostle James exhort Christians to consider it pure joy when faced with trials of many kinds (Jam. 1:2). Trials and tribulations are often been blessings in disguise for Christians, resulting in praise, glory, and honor in the end (1 Pet. 1:6-7).

For any trials to be meaningful, there must be control samples. It may be unfair to compare humanity with anything but itself . Hence, God allows the enemy to plant an inauthentic humanity alongside the authentic humanity. The authentic humanity is the chosen eternal bearer of God's image and likeness. The inauthentic is meant to be destroyed at the end. The wheat grows together with the weed; only in maturity are they separated as the authentic humanity will be fully revealed (Matt. 13:24-30). The inauthentic will outnumber the authentic. “Many are called. Few are chosen.” (John 6:37, John 15:16, Acts 13:46, 48, Rom. 8:29, 9:1-12, 16, Eph. 1:4)

Scientific experiments involving human subjects are often double blind by design, so that the set up of the control samples is unknown both to the ones being tested and any agents administering the tests. Nobody besides God knows the eternal fate of any individual human being until the person confesses to faith, and even then only God knows with certainty if the confession is authentic (1 Cor. 2:11, Rom. 8:27, 1 Sam.16:7, Luke 16:15, 1 Pet. 3:4, 2 Cor. 10:7, Heb. 4:13). God must limit his intervention in order not to “tip his hand” to Satan and give up the double blind. Hence, the sun shines on both good and bad people (Matt. 5:45). What are Christians to do? “Do not be jealous of evil doers,” (Prov. 24:11, 19, Ps. 37:1, 7-8) but rather, “love your enemies.” (Matt. 5:44) Even though few are chosen, the members of the authentic humanity is scattered among all nations (Rev. 7:9). Christians must bring the gospel to the ends of the world (Matt. 28:19-20, Mark 16:15-16, Luke 24:46-49).

At the same time, the co-existence of the authentic and the inauthentic means that Christians must always be on guard against false teachings. “Test everything, hold fast what is good.” (1 Thes. 5:21, cf. 2 Pet. 3:17, 1 John 4:1-3). Much like the refinery of precious metal in a furnace where the pure will remain and the adulteration will burn away (Matt 8:11-12, Matt. 13:49, 1 Pet. 1-7, 2 Pet. 3:7), in the fullness of time, even the original heavens and earth will burn away (2 Pet. 3:10-13, Rev. 14:11, Rev. 20:10, 15). Only the authentically chosen ones with their experience of humility, faith, hope, and love are worthy of their presence in new heavens and earth (Rev. 21:1, 4). Those who are not worthy follow the fate of the natural course of this desolated universe, which is eternal darkness (Matt. 24).

Summary

The problem of evil presents an intellectual hurdle for many to believe in a good and omnipotent God.

The emergence of open theism could be seen as an apologetic response to lower the hurdle. While well intended, open theism contradicts biblical revelation, not only in its direct claims, but also when its logical implications are worked out. In this essay, I outlined several alternative perspectives to the problem of evil, employing illustrations that may appeal to the scientific minded and mathematically versed non-believers. I hope that my attempt may serve as an encouragement for Christian thinkers to find more creative ways to engage their intellectual gifts in witnessing the gospel.

It is my conviction that any biblical answer to the problem of evil must begin with a high view of humanity in eternity while recognizing its imperfection in the present. Faith is confidence in what we hope for. A low view of God and a bottom-up view of eternal life extrapolating from the present leave us little to hope for. Faith is not a restorative therapy of a paradise lost in this world, but a prophecy about a future perfection in a different world. Authentic humanity is on trial, in a qualifying examination for its glorious role in the eternal day. The deliberation is full of setbacks and pains but it will be over soon. God has a purpose in all these and everything will be revealed to us in the end.

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[i] **Game Theory** studies the mathematical models of cooperation and conflicts among rational agents. Used in economics, sociology, political science, psychology, and increasingly biological sciences, game theory is also a key cornerstone of the emerging discipline of decision science. John von Neumann co-founded the field of game theory with Oskar Morgenstern in their inaugural treatise "Theory of Games and Economic Behavior." Besides game theory, Von Neumann was also widely recognized for his work in quantum logic and the Monte Carlo method. He was often credited as the father of modern computer for inventing the Von Neumann ("Princeton") architecture. John Nash was another key figure in the field of game theory and received a 1994 Nobel prize in economics for his pioneering work. His remarkable life story was popularly retold in the biography "A Beautiful Mind" and the eponymous Oscar-winning movie adaptation. Von Neumann's minimax theorem has demonstrated constructively that zero-sum games with finite set of actions and rational players result in an equilibrium. Nash showed that an equilibrium also generally exists in any non-cooperative games with finite set of actions and rational players, provided that each player holds accurate beliefs of the strategies used by other players. Game theory rests on the axiom of the utility-maximizing rational actors. The idea of rational actors has been implied in economics since Adam Smith speaks of the "invisible hand" in his "Theory of Moral Sentiment."

Von Neumann, John and Morgenstern, Oskar. Theory of Games and Economic Behavior, 3rd ed. (Princeton, NJ: Princeton University Press, 1944)

Halmos, P.R.. (1973) "The Legend of von Neumann". The American Mathematical Monthly: volume 80. p. 382–394.

Nasar, Sylvia (1999). A Beautiful Mind: A Biography of John Forbes Nash, Jr., Winner of the Nobel Prize in Economics. (New York: Simon and Schuster, 1994)

Becker, Gary. The Economic Approach to Human Behavior. (Chicago: University of Chicago Press, 1976).

[ii] **Open Theism** is characterized by five basic claims according to David Basinger who writes the introductory chapter in The Openness of God: A Biblical Challenge to the Traditional Understanding of God. (Downer Groves, IL: InterVarsity Press, 1994):

1. God not only created this world ex nihilo, but can (and at times does) intervene unilaterally in earthly affairs.
2. God chose to create us with incompatibilistic (libertarian) freedom—freedom over which he cannot exercise total control.
3. God so values freedom—the moral integrity of free creatures and a world in which such integrity is possible—that he does not normally override such freedom, even if he sees that it is producing undesirable results.
4. God always desires our highest good, both individually and corporately, and thus is affected by what happens in our lives.
5. God does not possess exhaustive knowledge of exactly how we will utilize our freedom although he may at times be able to predict with great accuracy the choices we will freely make.

[iii] **Rational Actor** Theory, also known called the Rational Choice Theory provides a formal model for social and economic behavior of human beings. Rationality is understood as the behavior consistent with a choice that maximizes utility (or pleasure) while minimizing cost (pain) given perfect information. More complex models based on the probability of expected outcomes lead to the closely related Decision theory. Gary Becker was an early proponent of applying rational actor models more widely. Behavioral economics augment the pure rationalistic picture with theories that account for

the apparent irrational behaviors. Some notable ones include the bounded rationality model of Herbert A. Simon, the Allais paradox of Maurice Alice, and the prospect theory of Daniel Kahneman and Amos Tversky,. Simon in 1978, Allais in 1988, Becker in 1992, and Kahneman in 2002 received the Nobel prize in economics for their works.

[iv]**Libertarian Free Will** is the belief that: (1) The existence of alternative possibilities (or the agent's power to do otherwise) is a necessary condition for acting freely. (2) Determinism is not compatible with alternative possibilities (it precludes the power to do otherwise).

Kane, Robert. The Significance of Free Will. (New York: Oxford University Press, 1998)

The open theist sees the sovereign predestination of God as a kind of determinism, and is therefore incompatible with libertarian free will. Dualist theological positions, such as Occasionalism, combine a physical/primary/divine determinism and a metaphysical/secondary/human libertarian free will.

[v]**New heavens and new earth** (Isa. 65:17, 66:22, 2 Pet. 3:13, Rev. 21:1) is a phrase used throughout this essay to refer to what is commonly referred to as “heaven.” The bible says that the present heavens and earth will pass away (2 Pet. 3:10, Matt. 24:35, cf. Mark 13:31, Luke 21:33). Latest cosmology also seems to suggest that this physical universe is indeed uninhabitable in the very long run. The usage of the “heavens” and “earth” does not seem to correspond with the modern cosmological understandings of the outerspace and the planet earth. Ancient readers did not have a picture of a blue marble when they hear the word earth. Therefore, the phrase heavens and earth correspond closer to what some refer to as the spiritual and physical realms. The same distinction may carry over to the new heavens and new earth.

[vi]**Optimization theory** identifies the nature of functions and the conditions for which optimized input parameters can be efficiently found. It presumes the extreme value theorem which states that if a real-valued function f is continuous in the closed and bounded interval [a,b], then f must attain its maximum and minimum value, each at least once. An implication of the extreme value theorem is that given the analytical form of a function f and unlimited time, it is possible to identify all the points within the interval [a, b] where maximum or minimum values are found.

[vii]**The Incompleteness theorems** of Kurt Gödel are explained by Stephen Kleene (1967). Mathematical Logic (Dover, 2002), p.250 as:

1. Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory
2. For any formal effectively generated theory T including basic arithmetical truths and also certain truths about formal provability, if T includes a statement of its own consistency then T is inconsistent.

[viii]**Statistical learning** is a general discipline about data samples based machine learning. While machine learning is often assumed to be a problem that is computationally intractable, there are classes of learning problems where there is a good chance for finding efficient methods to train the learners. The probably approximately correct learning framework (PAC Learning) identifies the conditions for a machine learner to process the samples in polynomial time. Equivalent concepts are found in Vapnik–Chervonenkis dimension (VC dimension) which measures the capacity of a statistical classification algorithm and the Glivenko–Cantelli theorem in probability theory.

Poole, David and Mackworth, Alan. Artificial Intelligence: Foundations of Computational Agents. (New York, Cambridge University Press, 2010), p.350ff.

Russell, Stuart and Norvig, Peter. Artificial Intelligence: A Modern Approach. (Upper Saddle, NJ: Pearson, 2010), p.714.

[ix]**Control sample** is used in experiments to compare against the experimental subject sample. The two samples are identical except for the independent variable being tested.

Kerlinger, Fred and Lee, Howard. Foundations of Behavioral Research. (Orlando, FL: Harcourt, 2000), p.486ff.

[x]**Double-Blind** is an experimental design strategy for tests involving human subjects where the designer seeks to minimize unrecognizable effects due to psychological biases by keeping the control sample setup opaque to both the research administrators and the test subjects. In single-blind experiments, only the subjects are unaware of the their control status. In triple-blind experiments, the subjects, the research administrators, and the research evaluators are all unaware of the way the control is set up.

[xi]**Problem of evil** is stated in deductive (or logical) and inductive (or evidential) forms. William L. Rowe expresses one evidential version this way:

1. There exist instances of intense suffering which an omnipotent, omniscient being could have prevented without thereby losing some greater good or permitting some evil equally bad or worse.
2. An omniscient, wholly good being would prevent the occurrence of any intense suffering it could, unless it could not do so without thereby losing some greater good or permitting some evil equally bad or worse.
3. (Therefore) There does not exist an omnipotent, omniscient, wholly good being.

Another version is expressed by Paul Draper:

1. Gratuitous evils exist.
2. The hypothesis of indifference, i.e., that if there are supernatural beings they are indifferent to gratuitous evils, is a better explanation for (1) than theism.

3.Therefore, evidence prefers that no god, as commonly understood by theists, exists.

[xii]**Compatibilism** is the belief that free will and determinism are compatible ideas. Compatibilists understand free will as the freedom to act according to a person's desire. Therefore, one could be acting freely even when all actions are deterministic. It is necessary to distinguish between physical determinism and divine determinism. Physical compatibilism understands the determinism to be the result of materially causality. Divine compatibilism understands the determinism to be in accordance to the predetermined plan of God. A dualist view where an indeterministic physical universe is intervened by a divine determinism is also conceivable. A dualist view may allow some Christians to be divine compatibilist without being a deist.

[xiii]**Cooperative game** is a game where the competition is between coalitions of players rather than individual players. Coordination game is a kind of cooperative game where players arrive at decisions by a deliberative process of consensus building. A non-zero sum game is where the gain does not necessitate the loss of another player. Poole and David (2010), p. 424.

[xiv]**Flat universe** is the cosmological model that seems to best fit the observed data obtained from WMAP measurements. Without the presence of dark energy, a flat universe continues to expand but at a decreasing rate. However, the presence of dark energy makes it more likely that the expansion slows down initially but speeds up again eventually. It means that the flat universe has practically the same fate as an “open” universe, which will end up in heat death or deep freeze. Davies, Paul. Cosmic Jackpot. (New York: Houghton Mifflin Company, 2007), p.41

[xv]**The best of all possible worlds** argument comes from Gottfried Leibniz's work in 1970 Essais de Théodicée sur la bonté de Dieu, la liberté de l'homme et l'origine du mal (Essays on the Goodness of God, the Freedom of Man and the Origin of Evil). Leibniz suggests that God considered every possible world before choosing to actualize the present one. God chose to actualize this one because it is the best among them all. In his suggestion, Leibniz did not seem to worry about human free will and his view on free will might therefore be called compatibilist. Alvin Plantinga suggests that from a libertarian view of free will, it is conceivable that God might not be able to actualize the best of all possible worlds. Plantinga refers to that possibility as Leibniz's lapse and uses that to account for the existence of evil. This essay uses what can be seen as a mirror opposite suggestion to argue against open theism, namely that, from a compatibilist view of free will, it is conceivable that the perfect and authentic humanity might always freely choose the best of all possible worlds in eternity.

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Leading a Successful Missions Trip in Your Discipline

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Abstract

The global missions community goes wanting for skilled workers in almost every discipline. However, even students at a Christian institution that emphasizes global engagement remain largely unaware of the impact they can make in missions by leveraging their own academic specialty. In this paper, we draw on our experience leading discipline-specific missions trips as a means to encourage students to reframe their thinking about personal involvement in missions.

We discuss the need for students to experience missions firsthand, and the student outcomes we have observed in intercultural awareness and spiritual formation. A key student outcome is an increased willingness to consider vocational missions service in both internships and full-time service after graduation.

We also offer practical guidance for faculty or staff interested in leading discipline-specific missions trips with their students. Although our experience is with Computer Science missions trips, the majority of the material here is applicable across academic disciplines.

Key ideas: global engagement, discipline-specific missions, integration of faith and discipline, cross-cultural interaction, student outcomes, practical guidance.

1 Introduction

All Christians are familiar with the Great Commission:

> Go and make disciples of all nations, baptizing them in the name of the Father and of the Son and of the Holy Spirit, and teaching them to obey everything I have commanded you.[1]

Unfortunately, many Christians have a very narrow view of the Great Commission as a vocational calling, thinking it applies mostly to full-time evangelists, church planters, and Bible translators. Students share this narrow view. They are quick to assume that missions is only for "super Christians" having skill sets that they will never possess.

Of course, this narrow view could not be further from the reality on the ground. Modern missions organizations are not fundamentally different from any other organization in their need for people gifted in leadership, management, logistics, finance, analytics, marketing, publishing, communication, personnel, training, technology, and a host of other areas. In short, anyone with expertise of use in a secular organization is someone with expertise of use in a missions organization.

An effective antidote to this narrow view of missions is participation in a missions trip. Particularly for college students, who are forming their vocational goals and aspirations, a missions trip

[1] Matthew 28.19-20, NIV

can provide the impetus they need to expand their vocational planning beyond the almost purely economic goals emphasized by contemporary culture.

2 Types of Missions Trips

All missions trips should:

- Share the redemptive love of Jesus with a needy world.
- Minister to the spiritual and physical needs of the people being served.
- Welcome new Christians into the family of faith.
- Grow the vision and faith of trip participants through service and fellowship.

Beyond these general goals, however, there is considerable variation among missions trips, particularly as found in Christian higher education. In this section, we discuss three types of missions trips: *general-service*, *discipline-specific*, and *vocational*.

2.1 General Service

A *general-service* missions trip is open to students from any discipline. Work in the field—although valuable—is only accidentally related to the academic specialty of team members, although team leaders often have specific expertise. The key advantage of a general-service trip is that nearly anyone can be usefully engaged on the field, regardless of past experience, class standing, or declared major.

Typical focus areas for general-service missions trips include evangelism, vacation Bible school, music, drama, maintenance and construction projects, and public health education (e.g., infant care, clean water, sanitation, HIV).

2.2 Discipline Specific

A *discipline-specific* missions trip is normally restricted to students with a particular ability or those taking particular classes. Trip leaders are usually experts in the discipline. In contrast to a general-service trip, a discipline-specific trip can leverage the common interests and abilities of the team to provide focused ministry and service on the missions field.

Examples of discipline-specific missions trips include: a musical group collaborating with a local evangelistic outreach, a sports team that teaches kids their sport and their Christian worldview, a Christian Education class that helps a local church develop Sunday school curriculum, or an environmental science class that drills water wells and builds cisterns to provide clean water in a remote location.

Note that although participants in such trips have expertise in the same discipline, they need not all be in the same academic major or on the same vocational path. For example, not all members of a choral group are necessarily music majors. Nor must students in a general-education Geology class be planning careers as well drillers.

2.3 Vocational

A *vocational* missions trip welcomes students with a specific vocational calling, allowing them to engage their academic specialty *directly* in a missions context. The specificity of a vocational missions trip is key. In a discipline-specific trip, students from many majors may share an ability in music or be taking the same elective class. However, a vocational missions trip enlists students in the same (or closely related) majors to use *that specific discipline* on the field.[2]

[2]Another name for this type of trip might be a *major-specific* missions trip.

Examples of vocational missions trips include: business students working on a micro-finance program in a developing country, pre-medical students providing medical services at a mission, or Computer Science students developing computer software for a mission.

A vocational missions trip provides students with the most robust vision for how they could apply their own skills and gifting to serve *directly* in missions as a vocational calling. Take just one example. In our on-line, cloud-based, mobile-enabled, global technology ecosystem, the need in missions for skilled workers in Computer Science (CS) and Information Systems (IS) has never been greater. At Taylor, our CS and IS students have ample opportunity to apply their skills to missions computing while on campus through both class and volunteer projects. But it's when students experience *on-site work with full-time missions technologists* that they develop an understanding and a vision for how they can contribute vocationally to missions by leveraging their own skill and passion.

In this paper, our focus will be on discipline-specific and vocational missions trips. We draw on our experience leading two vocational missions trips.

1. Operation Mobilization (OM), Carlisle, England, 2011. Computer Science students developed features for Open Petra, an open-source Enterprise Resource Planning system that OM uses to operate its mission.

2. Logos Hope (OM's missions ship),[3] Hong Kong, 2013. Computer Science and Media Communications students developed a web-based, database-backed, on-board information system for the ship's company.

3 Success

We're interested in leading *successful* missions trips. Here we consider missions trip success from two vantage points: the *mission* and the *student*.

3.1 Mission Success

At trip's end, the mission should be better off than when the team arrived. To help ensure this outcome, identify a stakeholder within the mission who will help you select an appropriate project in advance of the trip and who will champion your trip within the mission. There are many additional subtleties in leaving the mission in a better state after your trip. We recommend the text by Corbett and Fikkert [**?**] for a thorough discussion of "helping without hurting."

Of course, measures of success will vary based on your discipline. As examples, our key measures of success for Computer Science missions trips are these:

1. Improved software functionality within the mission

2. Additional skills or expertise gained by missions technology staff

3. Additional tools and resources provided to the missions staff to help their ongoing work

4. Accurate and complete documentation of student deliverables

5. Clear understanding of post-trip interaction with students or our institution

6. Better connections between missions technologists and the "outside world"

7. Missions staff encouraged by exposure to student enthusiasm and energy

[3] `http://www.gbaships.org/`

Measures 6 and 7 are surprisingly important. Most missions technologists operate under severe resource limitations compared to their peers in the commercial world. Compounding this challenge, the technology role in which they serve tends to be undervalued by missions outsiders, sometimes even including supporters. For such missionaries, interaction with a vibrant student team that understands intuitively why the missionary's work is important can be a huge morale boost.

3.2 Student Success

For the student, a successful missions trip results in academic growth, cultural growth, spiritual growth, and changed attitudes.

Academic Growth Criteria for academic growth depend largely on the specific discipline in which students are engaged. For Computer Science trips, we look for the following indicators of academic growth:

- Interacted with stakeholders in the design, implementation, testing, and deployment of a software system. Coped well with stakeholders who: had a different cultural heritage, spoke a different language natively, employed a different style of communication, had different interpersonal relationship expectations, etc.
- Rose to novel changes not normally found in the campus lab or classroom (e.g., limited Internet availability, non-functional mobile device).
- Demonstrated flexibility by working on novel applications (e.g., scripture translation, non-profit resource planning, shipboard operations).

Other disciplines will have other academic success factors. All disciplines will find that success in the missions context will look very different from success in the classroom, in the lab, or in a domestic internship or practicum.

Cultural Growth Whether domestic or international, the majority of missions trips cross a student's cultural boundaries. The successful student is willing to engage the new culture. Prior to the trip, he or she will devote effort to learn the new culture's history and geography and obtain rudimentary language skills for the area. Real opportunities for growth, however, only appear when the trip is under way. Students grow by embracing the sights and sounds of unfamiliar places, the smells and tastes of unfamiliar foods, and the social practices of unfamiliar peoples. After returning home, a student should be able to contextualize the cross-cultural experience relative to their home culture. Many students have had limited international experience and uncritically see their home culture as "right" or "best." A key success factor is their willingness to escape this parochialism and see themselves in a global context.

Spiritual Growth For anyone involved in Christian higher education, our ultimate goal is the spiritual formation of our students. Missions trips are often "mountaintop" experiences for students. That's good, but we also want students to experience long-term spiritual growth.

To seed spiritual growth, encourage students to *prepare their hearts* in advance of the trip. The main focus of a missions trip should be the *mission*—it's for the benefit of the unreached people students will encounter. Academic outcomes should take a back seat to the mission itself.

Students have the opportunity to grow in their *flexibility* toward God's providential will. Not everything will go according to their plan—or yours. On our trip to write software aboard the Logos Hope missions ship, we arrived in Taipei, Taiwan, where the ship was scheduled to arrive the next day. While we waited in baggage claim, we learned by e-mail that the ship had experienced mechanical problems on her voyage from Hong Kong to Taiwan, and had returned to Hong Kong for repairs![4]

A missions trip makes manifest each team member's comfort zone. As one student wrote about his first missions computing trip:

> Students do not understand that their "comfort zone" is intentional, not something to be avoided; while they may be able do anything, some of their skills are better than the rest. Breaking students out of their comfort zone is moot; breaking them into their "ability zone" is paramount. Upon arrival, they will be able to develop and deploy God-given talent.

Thus, one measure of growth is *identifying and occupying* one's "ability zone." Conversely, once their comfort zone becomes clear, students also show growth in their willingness to *move out of their comfort zone*. They have an opportunity to experience the blessed uncertainty of living life with room for the Lord to show up.

Students also exhibit growth by willing *participation in ministry* on the field. On the Logos Hope, for example, every member of the ship's company devotes one day in seven to direct ministry, either aboard or on shore. We found that many of our computing students—natural introverts—were anxious about this aspect of the trip. In a post-trip reflection paper, one student wrote:

> During this trip, I truly loved my opportunity to work along side the crew in the ship bookstore. I was able to experience what it was like to work alongside others from different countries and serve the visitors from Hong Kong. When I first started the day off, I felt a little out of my comfort zone, but by the end of the day I was not only making announcements over the PA system, but I was talking to some of the visitors who came aboard the ship.

By praying together in advance of their missions days, and debriefing together afterwards, we found that everyone overcame their nerves and enjoyed the ministry experience.

Changed Attitudes As mentioned in Section 1, missions trips can help reverse the misconception that missions is the exclusive purview of a small group of specialists. Following a missions trip, students may exhibit one or more of the following new attitudes regarding the relevance and applicability of their discipline to global missions.

- Student recognizes the need for their academic discipline in missions.
- Student works with a missions organization for a college internship or practicum.
- Student includes missions as a live possibility in their post-graduate career planning.
- Student is more willing to support (financially and in prayer) missionaries working in his or her discipline after graduation.
- Student takes a full-time position with a mission upon graduation.

4 Guidance

This section presents guidance that we have found particularly helpful as we have led two international vocational missions trips.

[4] Yes, we caught up with the ship a couple of days later in Hong Kong.

4.1 Planning and Preparation

1. *Determine the size of your team.* Know in advance the size of your team. Some institutions set a minimum size, below which the trip is deemed "not worth it." The upper limit on team size is more likely to come from your missions partner, depending on the numbers they can house, transport, and feed. Be sure to include yourself and other team leaders in your count.

2. *Choose team leadership.* Choose the best leadership for the trip. It may be just you, or it may be a team. Our school encourages married couples to co-lead trips so that one partner doesn't accumulate significant life experiences not shared by the other. These trips have been life highlights for both of us. We have also found it helpful to have a leader of each gender for a mixed gender team.

3. *Determine the cost of the trip.* Know the approximate cost of the trip before you advertise for participants. Our experience has been that the mission itself will be a great help in this regard. Expect to pay a *per diem* charge for room and board for each team member, as well as a processing fee for administrative overhead at the mission. You will probably have to deal with transportation arrangements and payment yourself. We have had good experiences working with a travel agent who specializes in group travel.

4. *Conform to institutional requirements for participants.* Check on institutional requirements for international travel insurance for you and your team. It will probably be required by your school or missions partner and proves very helpful in the event a team member falls ill.

   If your institution has a policy about academic standing for trip participants (e.g., academic encumbrances, minimum credit hours, minimum GPA, current financial account), be sure to include this information in your invitation to participate. When students apply to participate, check with the registrar or other administrative office to be sure the student meets all institutional criteria.

5. *Identify a method to prioritize interested students.* Know in advance how you will prioritize should more students be interested than the maximum size of the team will allow. Possible criteria include student major, class standing, credit hours, and GPA. Consider whether veterans of previous trips should be prioritized above or below first-timers. Determine whether students need specific skills to be effective on the trip.

6. *Verify passports and visa requirements.* All team members need valid passports for international travel. Be sure passports are valid for six months after your intended date of return to your home country. Check on visa requirements *well* in advance of your trip. If team members hold passports from different countries, be sure to check visa requirements for each.

7. *Meet regularly before departure.* Meet with your team on a regular basis prior to the trip. We have traveled during our January term, so we scheduled weekly meetings most weeks of fall semester. Early meetings should focus on communicating the details of the trip and selecting participants. Later meetings should emphasize logistical details.

8. *Prepare the team spiritually.* Devote intentional time during each team meeting to prepare the team spiritually. Discuss candidly student expectations and concerns. Set an example by sharing *your own* expectations and concerns. Pray for each other during team meetings and throughout the week. Make sure students know that God will use the trip to *stretch* them, but will not *break* them.

4.2 Fund Raising

1. *Identify your institutional policy for fund raising.* Check with your administration regarding institutional policies on tax-deductible fund raising for a missions trip. At our school, a student trip qualifies as a missions trip if at least 50% of time on site is devoted to missions service; such trips are tax deductible for donors. Both our trips to date have far exceeded this criterion (our students wrote a *lot* of software).

2. *Determine which institutional office will process donations.* Work with the appropriate administrative office at your school to arrange for processing of donations, including receiving donations, routing funds to the appropriate account, and sending receipts to donors. Get advice on how to make this process easy (e.g., return envelope, instructions to donors to include in your mailing, etc.). Ask whether you will be assessed a fee for these services; if so, add it to the cost of the trip.

3. *Require participation in fund raising.* All students should participate in raising funds for the trip. Although some students may be in a financial position to simply "write a check" for the cost of the trip, there is great value in placing the team's finances before the Lord and trusting him for the outcome.

4. *Prepare a standard fund-raising letter.* The best way to provide consistent information to potential donors is for you to prepare a one-page fund-raising letter to be used by all students. Leave room on the page for the student to include a brief (but required) hand-written note to the recipient. Students should address the envelope by hand as well. Encourage the student to pray for the recipient while they write the note and address the envelope.

5. *Set expectations for fund-raising letters.* Not only should every student participate in fund raising, but they should all do so to the same extent. Based on advice from our campus office for general-service missions trips, we require that students each send 75 fund raising letters. Communicate a clear deadline for when all letters must be in the mail.

6. *Keep track of progress on fund raising.* Have students submit their letters to you (or your assistant) rather than posting them directly. This practice allows you to track student progress on getting letters sent. It also makes it easier to collaborate with your campus post office for bulk mailing and to charge back postage to the proper account for your trip.

7. *Consider use of e-mail for fund raising.* Students may prefer to contact donors by e-mail. Although this approach is convenient, it lacks the personal impact of a hand-written appeal. If you do allow e-mail funding letters, make sure students include your standard team letter (e.g., as a PDF attachment) and a personal note. Tracking progress is harder with e-mail than paper letters. Ask students to submit to you regularly a record by name and e-mail address of donors who they contacted by e-mail.

8. *Consider on-line giving.* Many potential donors will appreciate an option to contribute to your trip on the Web. Younger donors (e.g., recent alumni) may not ever have a checkbook! Be aware that on-line payment processors levy a fee of 2–3% of the donation amount.

   Your institution probably already has the ability to accept on-line donations and can set up a page for giving to your trip. One advantage of e-mail fund-raising is that an e-mail message can include a direct link to the donation web page for your trip.

4.3 Team Dynamics

1. *Make decisions as a team.* Within the broad confines of your leadership, let the team make decisions as a group. This practice allows them to take ownership of the trip and to hold one another accountable for decisions they've made together.

2. *Encourage open and honest communication.* Make sure everyone has a voice in important issues and in team decisions. Intentionally draw out contributions from quiet students so that the *entire* team is involved.

3. *Hold the teams' feet to the fire.* Give the team permission to enforce team decisions, including the authority to establish consequences. For example, the team may decide that all fund-raising letters must be sent by a given date—under threat of being cut from the team. Because everyone participates in such a decision, everyone is subject to being cut.

4. *Use leadership authority as a last resort.* Your team will be strong if granted broad responsibility for its own governance. However, as the leader, it's your responsibility to step in should the team head in a harmful direction, or as a last resort to resolve an intractable impasse.

4.4 On the Trip

1. *Keep important documents with you at all times.* Keep with you a photocopy of all important travel documents for the entire team, including passports, insurance cards, local contacts, and medical information (e.g., allergies, medications). We travel with a three-ring binder known affectionately as the "Book of Doom."

2. *Prepare to leave from the moment you arrive.* In order for your service to be sustainable, the mission must be able to take over where your team left off when you leave for home. Leave behind documentation, blueprints, operating instructions, and notes on progress. Don't wait until the end of your trip to create these artifacts—you will be too busy preparing to leave. Start creating this information from your first day on site and maintain it throughout the trip.

3. *Meet regularly as a team.* While adjusting to the many changes your team encounters during the trip, it is crucial that you meet together regularly. A team meeting is familiar and helps restore a sense of normality in the group. Talk over challenges, opportunities, successes, and failures with the same candor you established prior to departure. Share prayer requests and pray for each other.

4. *Gauge the state of each team member.* Gauge the degree to which each team member is adapting to the trip environment. One strategy we've employed during team meetings is to ask each student to rate their physical, emotional, and spiritual "health" on a scale of 1–10.

4.5 Back at Home

1. *Complete remaining administrative tasks.* Submit any forms or reports required by your institution after leading a student trip. If fund-raising fell short, deal with remaining costs (at our school, student accounts are billed for any shortfall). Close out fund-raising accounts.

2. *Share your trip with others.* Others on your campus and in your community will be excited to hear from you and your students about your missions experience. Schedule presentations at your school or in area churches to share what your team did and to encourage others to participate in missions. This is your chance to show your *own* "boring missionary slides!"

3. *Continue to interact with the team.* You and your team have participated in what will have been a life-altering experience for many. Provide students with opportunities to process their experience upon returning to campus. For example:

    - Meet as a team a few more times to debrief with each other on the experience. Encourage everyone to share what they learned, what was good and bad, how they see themselves in a new light, what God taught them during the trip, and how they are dealing with reverse culture shock. As you have done from the outset, continue to pray with and for each other.

- Ask students for a written reflection on their experience. An informal paper or written answers to questions you provide can help students integrate their experience to their lives back on campus.

To encourage post-trip participation, set expectations for students during trip planning. Don't treat these activities as optional—they are *integral* to trip.

4. *Plan for next time.* While the experience of the trip is still fresh in your memory, take time to evaluate all aspects of the trip so that you can improve the experience next time you go. One framework we use is to ask three questions: "What should we *keep* next time?" "What should we *add* next time?" "What should we *eliminate* next time?"

5 Conclusion

Leading vocational missions trips has been a wonderful experience both for us and for our students. We conclude with the story of one student.

As she later recalled, Ashley signed up for our first Computer Science missions trip to "check off the box labeled *missions trip* on my college to-do list." During our trip to England, Ashley saw first-hand the impact she could have on global missions by using her God-given computing and design skills. The following summer, she interned at a different international office of the same mission. When it came time to consider post-graduate employment, Ashley's short list included several missions. Earlier this year, she accepted a full-time position at one of the missions organizations just weeks before graduation.

We trust that as you engage your students in missions, God will grant you similarly successful outcomes as you seek to fulfill the Great Commission.

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[5] `http://www.lightsys.org/`
[6] `http://cmc.taylor.edu/`

Faith Integration Projects for First-year Students

Abstract: Faith Integration Projects for first-year Students (Doug Phillippy, Messiah College)

This talk will consider the use of projects to motivate students to think deeply about how their faith connects with mathematics. This talk will begin by describing what a faith integration project is, including the goals and objectives of such a project. The talk will briefly describe a number of projects written by the speaker, with a more detailed look at one of those projects. The talk will conclude by discussing how these projects are being used to assess how students are doing at articulating a maturing understanding of the connection between faith and mathematics.

This paper will describe projects that are intended to motivate first-year students to think deeply about how their faith connects with mathematics.   It is important to understand that these projects form one section of a 350-page text entitled, *The study of Mathematics: Developing a mature understanding of mathematical thought, with consideration of Christian faith and vocation*. This text, which has not yet been published, has an intended audience of first-year mathematics majors. As such the projects described in this paper are just one component of a larger work that is intended to be a tool to help first-year mathematics majors begin their study of mathematics from a Christian perspective.

This text, which is now in its third draft, has been a nine-year project which is nearing its completion. In fact, I am currently seeking a publisher for this text. There are three main themes woven throughout the text. Obviously, one of those themes is mathematics. I have included topics in the text that I believe will be of interest to first-year mathematics majors. Some of these topics are foundational to the study of mathematics, and others, though they are not typically found in the curriculum of a first-year student, were chosen because they are thought-provoking. The second major theme interwoven throughout the text is that of the Christian perspective. We will consider the study of mathematics in light of the Christian faith, asking how faith influences the study of mathematics and in turn how the study of mathematics can impact faith. The third theme is that of my life story. Throughout the text, I share my journey as a Christian mathematician, noting some of my struggles and highlighting what I have learned along the way.

The main body of the text is divided into four sections. Chapters 1 and 2 form the introduction to the text. The second section (chapters 3-9) of the text focuses on applied mathematics covering topics such as check digits, graph theory, number theory, and problem-solving. The third section (chapters 10-13) of the text focuses on theoretical mathematics covering topics such as logic, proof and small axiom systems. The fourth and final section of the text (chapters 14-21) consists of what I will call faith integration projects. These projects cover a variety of mathematical topics and are designed to promote discussion in the area of faith integration.   They are the focus of this paper.

Most of this paper consists of one example of a faith integration project that is included in my text. It will also include a brief description of several of the other projects. But before we consider these projects, I need to make a few introductory comments regarding the nature of these projects. I will do so in light of several key characteristics of faith integration mentioned throughout the text of which these projects are a part. My intent is not to develop these characteristics fully here. Still, I mention them here because they provide the framework for understanding the nature of these projects.   These characteristics include the two-way nature of faith integration, the importance of community in faith integration, and faith integration as a scholarly project. These characteristics form the foundation on which the projects are built.

One characteristic of faith integration that is emphasized in chapter 12 of the text is the two-way nature of faith integration. By this I mean that integrating faith with the study and practice of mathematics should include both the insight that a Christian worldview brings to the discipline of mathematics and the contributions of the discipline to a Christian view of reality. In the text, I suggest that of these two possibilities, it is often easier, especially at the undergraduate level, to consider the impact that the study of mathematics has on a Christian view of reality rather than vice versa. I do not want to develop the reasons for this here, but I mention it because the majority of the projects that I have written focus on the impact that mathematics has on faith. In this paper, I will consider one such project, a project that suggests our study of mathematics can help to develop our intuition with regard to the infinite, including our eternal God.

A second characteristic of faith integration is that it is most effective when it takes place in community. In my text I suggest that faith integration is a process, noting that initial attempts at faith integration may result in less than noteworthy results. Still, any attempt at faith integration, even a flawed attempt, can prove beneficial, if for no other reason than to promote discussion among those who actively seek to connect faith and learning. Not only does this allow the architect of the attempt to practice faith integration, but it draws from the knowledge of an entire community to sharpen that attempt.

The projects presented in my text attempt to model this characteristic of faith integration. Each project is meant to encourage dialogue. To accomplish this, I begin the conversation with some of my own thoughts on a particular topic. These thoughts are intended only to initiate the dialogue, not to provide the reader with an expert's final analysis of the topic. In particular, each project consists of a short essay that is an attempt on my part to relate faith and mathematics in some way. These essays discuss a variety of mathematical topics appropriate for undergraduate students. For the most part each essay is self-contained and no successive progression through the material is required.

Nevertheless, because the essays are designed to promote discussion and not provide an answer, my hope is that they will provide a basis for further work in the area of faith integration. So, the essay is only part of the project. Each project has the potential for reader participation. Each project will begin with a question and include some of my thoughts as to how that question might be answered. As such, my essays provide an opinion and not "the answer" to the question.

The key to these projects really is the reader's response. My role is only to begin the conversation. The reader's response may be a critique of the answer that I have provided in the essay or it may be the reader's own answer to the question, or it may be both. It may even be the reader's initial thoughts to some other question that the essay prompted her to consider. In any case, the goal of each essay is to engage the reader in connecting faith and mathematics.

While the primary goal of the essay portion of each project is to begin a conversation with the reader regarding faith and mathematics, my writing serves an additional purpose. In particular, the essays in the text are given to serve as a pattern of the type of work that is expected to enter into the dialogue. In the text, I emphasize that faith integration is **any** attempt by an educator or student to relate one or more of the academic disciplines (not necessarily the individual's major or specialty) to a biblical worldview. I argue that faith integration should not be limited to a narrow range of approaches. I note that this is especially true for those who are in the initial stages of thinking about faith integration. Still, I suggest some guidelines for work to be done by a student in response to the essay portion of the project.

My goal is to move the student toward William Hasker's definition of faith-learning integration. This definition describes faith integration as "a **scholarly** project whose goal is to ascertain and to develop integral relationships which exist between the Christian faith and human knowledge, particularly as expressed in the various academic disciplines". So while I place very few restrictions on my students in other settings with regard to faith integration, these projects raise the bar a bit. This is not to say that the reader's response need be as extensive as the initial essay itself, nor is the goal to produce some paper ready for publication. Instead, the goal is to think seriously about faith and mathematics, connecting the two in some fundamental way.

I offer some guidelines as to what it means to produce a project that is scholarly in nature. At a minimum, the dialogue should be a response to some of the work already done in the text. It might seek to answer one of the questions asked at the conclusion of an essay, or it may be a response to the essay itself. At a more serious level, the dialogue might be original work, not a follow-up to discussion in the text. As such the projects in the text do not provide the material for the work being done but serve as a guideline for the type of work that might be done. For example, the reader may choose to use the project presented at the end of this paper entitled "the Infinite and Intuition" to construct a multi-step exercise that promotes active learning, with the reader discovering a faith-learning principle as he or she works out the exercise.

Ultimately, the purpose of this section is to help the reader think deeply about mathematics and faith, whether by responding to the author's thoughts or by producing original work. In either case, the discussion should include appropriate worked-out mathematical examples as well as an overview of the topic being considered, including pertinent definitions and theorems. Discussion should include references to Scripture and appropriate faith-related definitions. It might seek to identify which of the faith integration approaches described in the text best fits the approach expressed in the dialogue. It also might include what others have written and said about the topic. A student project need not include all of the above elements, but it should consist of those that are necessary to make the dialogue appropriate for an academic discussion.

I will now turn to the projects. Currently my text includes 8 such projects. Before considering one of those projects in detail, I will give a short description of several of those projects.

Infinity and Time - This project asks the question "What role (if any) should a Christian perspective have in discussing the solution to a problem that requires time to approach infinity?" Several Christian perspectives of time are discussed including the classic Augustinian view that time is finite. The author argues that this viewpoint of time impacts the way Christians should think about infinite limits. Moreover, he argues that such a discussion has a place in the mathematics classroom. Though the focus of this project is on the nature of time and the concept of an infinite limit, the underlying theme is how Christian thought can be introduced into the mathematics classroom.

The next three projects discuss the role that the infinite plays in mathematical reasoning and suggest that by studying the infinite in mathematics we can gain insight into the Christian faith. These three projects are related through this common theme and are best understood if considered as a whole.

Overcoming Paradox - In this project, our emphasis will be on paradox and the role the infinite plays in both overcoming and creating paradox. We begin by exploring what can happen if the infinite is excluded from the reasoning process. We suggest that excluding the infinite from the reasoning process or even having an improper view of the infinite can lead to paradox. In particular, we discuss some of the paradoxes that came about from the Greek understanding of time and space. The Greek understanding of time and space was limited by the way the Greeks viewed the very small. We use the geometric series to expose some of their misconceptions. In so doing, we illustrate how a proper inclusion of the infinite in the reasoning process can overcome paradox, thus giving a better understanding of reality.

The Infinite and Intuition - In this project we ask how well our intuition does at answering questions related to the infinite. We explore the possibility that the study of the infinite in mathematics can develop our intuition with regard to God and things eternal. We note that the result of an infinite process can be counter-intuitive and sometimes even paradoxical. In this project, we develop the integral test and consider several problems that test our intuition and challenge our understanding of reality. This project is included in its entirety at the end of this paper.

Taming the Infinite - Because of the sometimes counter-intuitive nature of the infinite, we suggest in this project that careful consideration must be given to underlying assumptions when working with the infinite. In particular, we consider the alternating series and the conditions necessary to guarantee the uniqueness of a sum. Using this as a backdrop, we suggest that care must be taken when studying the infinite not to trivialize it. We suggest that a blind application of laws that hold in a finite realm to God can lead to a trivialization of God.

Higher Dimensions and Paradox – In this project we ask if the study of dimension in mathematics can help to resolve paradox within the Christian faith. In particular, we ask how the study of higher dimensions in mathematics might shape our view on the ideas of predestination and free-will that are important to the Christian faith. A sphere's attempt to convince a square that there is a third dimension (taken from Edwin A. Abbott's book *Flatland*) serves as the impetus for seeking an answer to these questions. In particular, the inability of the square to understand three-dimensional space is used to motivate how time and space-bound humanity might find it difficult to understand a God that is not bounded by space and time.

Numbers and the Bible – In this project we ask what role the idea of number plays in our faith. In particular, we ask if the study of number in Scripture is appropriate. We note a couple of claims about number in Scripture including the claim that Scripture is written in a numerically significant way, the claim that certain numbers have more than just quantitative meaning, and the claim that the use of number in Scripture is reason to believe that Scripture contains error.

I conclude this paper with one of my projects as it appears in my text.

The Infinite and Intuition

He has set eternity in the hearts of men; yet they cannot fathom what God has done from beginning to end. Ecclesiastes 3:11

Question: Can the study of mathematics help a Christian develop intuition with regard to understanding God and eternity?

In the project entitled "Overcoming Paradox", we noted that exclusion of the infinite in the reasoning process can create paradox. We saw that the exclusion of the infinite in Greek mathematics led to paradox and a flawed world view. Moreover, we also argued that in the absence of the infinite, many statements in the Bible lose much of their meaning or they become paradoxical. In this chapter we suggest that care must be taken when incorporating the infinite into the reasoning process. Infinite processes can yield results that are quite surprising and sometimes even paradoxical. In this chapter, we examine several infinite processes that will test our intuition and challenge our understanding of reality.

However, before we consider these examples, I need to define what I mean by intuition. Davis and Hersh note that the word intuition is used by mathematicians in many different ways. In their book, *The Mathematical Experience*, they list at least 6 different ways that this word can be used, including as a substitute for rigorous proof, a brilliant flash of insight, visual, relying on a physical model, and incomplete. *The American Heritage Dictionary* also lists several meanings of "intuition" including, "the act or faculty of knowing without the use of rational process" and "sharp insight". By intuition, I simply mean "insight into". I am not too concerned with how this insight may come about. It may be some mysterious insight that few others seem to possess or it may be an insight that has been developed through use.

When it comes to intuition about God, Scripture suggests that we as natural human beings at best have a flawed intuition. The prophet Isaiah recorded the following statement of God regarding the thoughts of God and humans, "'For my thoughts are not your thoughts, neither are your ways my ways,' declares the LORD. 'As the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts.'" God can be described as omniscient, omnipresent, omnipotent, and eternal. These attributes are closely related to his infinite nature and make him difficult to comprehend. Answers to questions regarding his triune nature are not easily formulated. Still, people have sought them out. Theologians search the Bible and interpret the Scriptures to gain an understanding of the eternal. Then they attempt to describe the God who existed before the world began (John 17:5), and is not bound by space (I Kings 8:27). Is it any wonder that their descriptions are incomplete? As Dorothy and Gabriel Fackre have noted, human beings are often limited or "tripped up" by the language of their experience. That is, the language, experience, and knowledge of finite beings are often inadequate in describing an infinite and eternal God. Nevertheless, in an attempt to understand God and His creation, many authors have written books on Christian theology trying to answer these and other questions.

Likewise, mathematicians have long sought to gain an understanding of the infinite. In fact, mathematics has been called the science of the infinite. Mathematicians construct axiomatic systems and use symbols and operations within the framework of those systems to gain an understanding of the infinite. They use the infinite in their reasoning and routinely perform processes of infinite length. As a result, these same mathematicians sometimes stumble over paradoxes that arise within their carefully constructed axiomatic systems. The uncertainty that these paradoxes raise has ramifications that reach to the very foundations of mathematics. Nevertheless, in an attempt to describe the world around them, mathematicians continue to produce work that has its basis in those foundations.

What happens when a process is repeated an infinite number of times? This question is itself of a paradoxical type, since we cannot answer it by doing what it asks, regardless of what the actual process is that is to be repeated. However, as we have already seen in our discussion of the geometric series, the result of an infinite process can be described quite precisely. As another example, consider the derivative. A study of calculus reveals that if the value of Δx moves infinitely close to 0, then the value of $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ approaches $f'(x)$ (the derivative) assuming that $f(x)$ is a differentiable function of *x*. In calculus, we study infinite processes such as summing an infinite number of terms or moving infinitely close to a point by using the concept of the limit. In other areas of mathematics, infinite processes are studied with different tools, such as the idea of a random variable distribution in probability.

The subject of probability is a good place to begin our study of intuition. Imagine, for instance, that two young children, Ben and Amy, are flipping a coin. Ben wins when heads comes up and Amy wins with tails. If the first flip comes up heads, both kids will consider that a natural event in the game. If the first 5 flips are heads, Ben will be excited and Amy dismayed. If the next five flips are also heads, Amy will start to get suspicious. And if the *next* five flips are still heads, even Ben will know that the coin is probably not fair. What underlies their suspicion is an intuitive understanding that a fair coin should land heads up in close to half of all flips, over the long run. Mathematically, we say that the proportion of heads should approach ½ as the number of trials approaches infinity. This is an example of the Law of Large Numbers at work in probability theory. It is a tool for considering an infinite process that involves randomness, unlike most problems in calculus.

Now, suppose I perform an experiment in which I flip a coin 5 times and ask the reader to predict the results. There are 32 possible outcomes to this experiment, each equally likely to occur. They are listed below:

HHHHH, TTTTT
HTTTTT, THTTT, TTHTT, TTTHT, TTTTH
THHHH, HTHHH, HHTHH, HHHTH, HHHHT
TTHHH, THTHH, THHTH, THHHT,HTTHH, HTHTH, HTHHT, HHTTH, HHTHT, HHHTT
HHTTT, HTHTT, HTTHT, HTTTH, THHTT, THTHT, THTTH,TTHHT, TTHTH, TTTHH

Although each of the above events is equally likely to occur, intuition may make some of the outcomes more likely to be chosen by the reader as a prediction. Let me explain. Our intuition tells us that flipping a fair coin is not very likely to result in five heads. This will happen only 1in 32 times. Our intuition also tells us, that though it is more likely to obtain four heads than five, it is also more likely to obtain three heads than four. Because of this, and our intuition that the flipping of a coin should produce random results, the events displaying “more randomness” may be more likely to be chosen as predictions by the reader than the events consisting of less random patterns. In other words, even though the events described by HHHHH and HTTHT are equally likely to occur, the appearance of randomness in the latter event makes it more likely to be chosen as a predicted result of flipping a fair coin five times.

The above examples illustrate that our intuition can be beneficial in understanding a problem, but it also can be misleading. The following exercise is meant to test your intuition.

Exercise: How Many Threes?

What percentage of whole numbers have at least one 3 in their base 10 representation (for example 127 does not have a 3 in its base ten representation and 333 does)?

Your Guess:____________

a. In an attempt to help answer this question, complete the following table (I have completed the first two rows; no entry is needed in the shaded cells). Complete each row before moving to the next row. Work from left to right across each row. Each cell should contain the number of whole numbers with a three in the appropriate digit. Do not count a number twice. For example when considering all the two-digit numbers (row 2 below), the whole number 33 should be counted in the 1st (leading) digit column and not again in the second column.

b. Try to identify a pattern in each column. In particular answer the following questions:

What happens to each entry as we move down the column to the next row? In other words what is the relationship between the entries in a given column?

What is the pattern for the rightmost entry in each row? In other words, how do the numbers increase along the diagonal adjacent to the shaded cells?

		Number of 3's in the: (from left To right)						
	Number of whole numbers in this range	1st digit	2nd digit	3rd digit	4th digit	5th digit	Total number of whole numbers with at least one 3 (sum of columns 3-7)	Percentage of whole numbers with at least one 3
All 1 digit numbers (0-9)	10	1					1	.1
All 2 digit numbers (0-99)	100	10	9				19	.19
All 3 digit numbers (0-999)	1000							
All 4 digit numbers (0-9999)	10000							
All 5 digit numbers (0-99999)	100000							

0	10	20	30	40	50	60	70	80	90
1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99

c. Consider the last row in the table. Write the sum of 5-digit whole numbers with at least one of their digits being a "3" by making use of the patterns established in part b).

d. Generalize your result in part c) to account for a number that is *n* digits long.

e. Let $p(n)$ represent the percentage of whole numbers with at least one digit being a 3. Use technology and blot $p(n)$ for n ranging from 1 to 50.

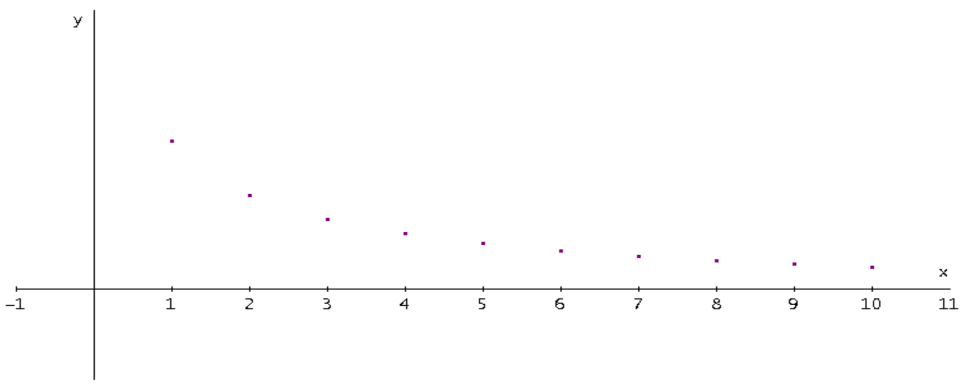
f. Let n approach infinity and make use of the sum of a geometric series to calculate the percentage of whole numbers with at least one 3 in their base 10 representation.

How did your intuition match up with reality in this exercise? If you are like most people, probably not too well. Perhaps this is because intuition is influenced by the experiences of everyday life. Most people don't experience numbers that are very large in magnitude on a regular basis. Therefore it is reasonable to expect that your guess was likely formulated with numbers that are relatively small in magnitude in mind.

We continue to test your intuition in this section by considering several more exercises that have surprising results. But before we turn to those exercises, we need to develop the integral test for infinite series. This is done in the next exercise.

Exercise: Discovering the Integral Test for Infinite Series

Consider the following graph of the first 10 terms (all positive) of an infinite series: $\sum_{n=1}^{\infty} a_n$.



a. Label each point with its x and y-coordinates (k, a_k).

b. Connect the points with a continuous decreasing function $f(x)$ that approaches the x-axis as x approaches infinity.

c. From each point construct a rectangle by drawing a horizontal line of length one to the right of each point. This line is the top of the rectangle, and the corresponding portion of the x-axis is the bottom of the rectangle. What is the area of the rectangle?

d. Write an expression that represents the sum of the areas of the rectangles.

e. Assume $\int_1^{\infty} f(x)dx$ is infinite (diverges), what can you conclude about $\sum_{n=1}^{\infty} a_n$?

f. Assume $\sum_{n=1}^{\infty} a_n$ is finite (converges), what can you conclude about $\int_1^{\infty} f(x)dx$?

g. Explain your reasoning for parts e and f.

By repeating the above exercise with line segments of length one drawn to the left of each of the points in the graph, it can be shown that the converse of each of the statements in parts e) and f) is also true. Thus we arrive at the integral test:

If $f(x)$ is a positive, continuous, and decreasing function for $x \geq 1$ and $a_n = f(n)$ for all *n*, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

In the following two exercises, we evaluate improper integrals and use the integral test for infinite series to obtain results that defy our intuition and maybe even cause us to question our view of reality.

Exercise: Shopping at the Infinity Toy Store: The p-series

Suppose you purchase a set of blocks at the Infinity Toy Store. This set of blocks is an infinite set of square blocks with dimensions $1x1x1$ inch, $\frac{1}{2}x\frac{1}{2}x\frac{1}{2}$ inch, $\frac{1}{3}x\frac{1}{3}x\frac{1}{3}$ inch, etc. for all positive integer denominators. The question is whether or not this set of blocks will fit in your dorm room. To complete this exercise use your knowledge of infinite series.

a. If the blocks are arranged by stacking them one upon another, beginning with the largest, write a summation that represents the height of the stack.

b. Determine the convergence or divergence of the infinite series found in (a) of this exercise by using the integral test (evaluate the integral). How high is the stack?

c. By completing (b) in this exercise, you should have found that the stack will eventually reach beyond the moon. What does your intuition tell you about the possibility of fitting this set of blocks in your dorm room?

d. Suppose instead of stacking the blocks one on another, you attempt to lay them out on the bottom of your top desk drawer. In this arrangement, no block is to be stacked on top of another. Write a summation that represents the total area of the bases of all the blocks.

e. Determine the convergence or divergence of the infinite series in part (d) by using the integral test. How much area is required in your top desk drawer to store the blocks? To answer this question, sketch a potential arrangement of the blocks.

f. Evaluate $\int_1^\infty \frac{1}{x^p}\,dx$. For what values of p does the integral converge? What can you conclude about $\sum_{n=1}^{\infty} \frac{1}{n^p}$?

Exercise: Another Toy at the Infinity Toy Store: Gabriel's Horn

Suppose you purchase a horn at the Infinity Toy Store that was made by revolving the function $f(x) = \frac{1}{x}$ about the x-axis on the interval $[1,\infty)$. (This object is known as Gabriel's Horn).

a. Show this horn can hold a finite amount of liquid by evaluating an appropriate integral.

b. Set up an integral that represents the amount of paint required to paint the exterior surface of this horn.

c. Explain why $\sqrt{1+\frac{1}{x^4}} > 1$, and use this fact to show that there is not enough paint in the universe to cover the outside of this horn.

The exercises in this project were offered to illustrate two principles regarding human intuition as it relates to the infinite. First, because human intuition is grounded in an experience in a finite world, and because that experience is often in the context of quantities that are relatively small, human intuition with respect to the infinite is unlikely to be something that has had opportunity to develop. Second, when it comes to the infinite, some outcomes don't seem to make sense, much less be intuitive. After all, how can cubes that stack higher than the moon fit inside my desk drawer? Or how can an object hold a finite amount of paint and yet be unable to be covered by any amount of paint?

With these exercises in mind, I am ready to answer the question posed at the beginning of this chapter: The study of mathematics can help a Christian develop intuition with regard to understanding God and eternity. The two principles mentioned in the previous paragraph can be applied to my theology. In these exercises, I see anew that my quest to understand an infinite God is hindered by my experience in a finite world. Moreover, my study of the infinite in mathematics enables me to experience the infinite in ways that no other discipline can offer. In other words, my intuition about things that are eternal has a chance to develop. My experience also teaches me to expect the unexpected when it comes to studying God. Outcomes that don't make sense in a finite reality are possible in the realm of the infinite. We will consider some of these outcomes in the next project.

Questions for Further Thought

1. How does the author answer the question, can the study of mathematics help a Christian develop intuition with regard to understanding God and eternity? Do you agree or disagree with his thoughts?

2. Identify one belief that you hold about God which you do not fully understand. In what ways is this belief related to God's infinite nature? Has the discussion in this chapter given you any insight regarding this belief?

3. Read 1 Corinthians 2. Analyze the claims the author makes in this chapter in light of what this passage says about understanding things related to God.

4. Identify one surprising mathematical result that you have encountered which is based in the infinite (not mentioned in this chapter). Does this result give you any insight into spiritual things?

5. Has your intuition ever failed you when it comes to thinking about God? In what ways is God's infinite nature related to this failure?

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Reading Assignments and Assessments: Are Your Students Reading Math Texts Before Class, After Class, Both, or Neither?

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Abstract

In his recent book *What the Best College Students Do* [Bain, 2012], Ken Bain defines a number of different types of students including "surface learners," "strategic learners," "routine experts," and finally, "deep learners." In our mathematics courses at Trinity, we have found examples of all of these student types. A major determinant of their preferred approach to learning appears to be the ways and degrees to which mathematical texts and other written materials are read prior to class sessions. Each full-time member of the department both assigns and assesses the reading of mathematical materials prior to class sessions. Assessment methods, as well as the corresponding pedagogical choices, vary significantly. We also discuss the results of a related survey of over 100 Trinity undergraduates enrolled in mathematics courses during fall 2012.

1 Introduction

Do you assign sections of the mathematics text to be read by your students prior to class? If so, do you assess your students' completion and/or understanding of these assigned readings? What percent of your students actually read the material before class? Do any of your students read the material after class, perhaps in the process of doing homework or reviewing for an upcoming exam? At Trinity Christian College, the mathematics faculty members are interested in all of these questions. One of the authors is an expert in literacy and regularly teaches and does research in the area of reading in different disciplines. Since all of the mathematics colleagues regularly assign texts, articles, book chapters, and other readings, and we also hold our students accountable for these reading assignments with related assessments, we decided to delve more deeply into the general topic of reading assignments and assessments in mathematics courses.

2 Insights from Research

Recent studies have confirmed the need for instructors to apprentice students into ways of reading that are particular to different academic disciplines (See e.g., [Hynd, 1999], [Shanahan, 2004], [Stahl et al., 1996], [VanSledright, 1995], and [Wineburg, 1991]). Some specific studies have looked at reading in mathematics. Schwartz and Kenney [Schwartz and Kenney, 1995], Fuson, Kalchman, and Bransford [Fuson et al., 2005], and Martinez and Martinez [Martinez and Martinez, 2001, page 47] delineated specific ways of thinking that are unique to mathematics.

Shanahan and Shanahan [Shanahan and Shanahan, 2008] and with Misischia [Shanahan et al., 2011] compared expert and novice studies to delineate how mathematicians read. Weber and Mejia-Ramos [Weber and Mejia-Ramos, 2013] responded to that study with an investigation of the use of sources in reading for mathematics. Johnson et al. [Johnson et al., 2011] contrasted different literacy practices in language arts and mathematics and determined that the different disciplines required different sorts of literacy practices and strategies. Hersch [Hersch, 1997] argued that students need to be taught how words are used differently in mathematics class. Mathematics may look and sound exactly like everyday English, but words like <u>similar</u> and <u>number</u> have different meanings in everyday English than they do in mathematics.

In our study, though, we were interested in the perceptions of our students. Do college students find a disciplinary literacy approach to teaching mathematics to be helpful? Do they find required disciplinary reading in mathematics to be useful to their understanding? Do regular reading assignments in mathematics courses, accompanied by appropriate assessments, result in students reading the material prior to class? Are these pedagogical approaches viewed as effective by these students?

3 Pedagogical Approaches for Assigning and Assessing Mathematical Readings

Although each of the mathematical colleagues assigns and assesses readings in her or his mathematics courses, there are a variety of methods currently being used. In several introductory level courses (including differential calculus, finite mathematics, and statistics), students are required to read section(s) of the textbook prior to a class session, sometimes with the assistance of key questions to focus the reading. Assessment during the following class takes the form of a short quiz. In one class, students work in (randomly assigned) pairs and may refer to notes that they took from the reading. In another class, the format of the quiz is randomly determined and could be individual, small group, entire class, or possibly omitted. In later semesters of calculus, these reading assessments are continued but sometimes with less guidance prior to the assigned reading.

In "Math Concepts for Teachers I," active reading strategies are modeled and students regularly complete short online quizzes prior to class. In "Mathematics Within a Liberal Arts Tradition," students are assigned to reading groups for a specific chapter from *Mathematics in a Postmodern Age: A Christian Perspective* [Bradley and Howell, 2001]. Students answer assigned questions and share the responsibility of leading the class discussion of this chapter. In the same course, students typically work in pairs to research a famous mathematician and present the resulting "math cameo" during class.

At the advanced level, students are encouraged to use active reading strategies to solve simple practice problems in discrete mathematics. In linear algebra, true/false questions related to the assigned readings enable students to self-assess their understanding and address common misconceptions. Generally, pairs of students are responsible for sharing their answers to these questions with the entire class. If their classmates disagree with any of their answers, or if any of their answers are incorrect, then those questions are explored more fully in class. In geometry, students read chapters and complete related response sheets from *Journey Through Genius* [Dunham, 1990] and *Mathematics in a Postmodern Age: A Christian Perspective* [Bradley and Howell, 2001] and lead discussions during subsequent classes, either individually or as part of a larger group. In history of mathematics, students read assigned material, work on related exercises, and submit a 1-2 page synthesis accompanied by several questions for class discussion. These questions are submitted prior to class sessions and result in a flipped classroom model for instruction. Finally, students in the senior capstone course read *A Certain Ambiguity: A Mathematical Novel* [Suri and Bal, 2007] and *Mathematics Through the Eyes of Faith* [Bradley and Howell, 2011] in preparation for class discussions and a later midterm paper that requires students to articulate a Christian worldview and its implications for their chosen major and vocation. (See [Klanderman and Robbert, 2012] for additional details on how the latter is accomplished.)

In summary, the types of readings assigned, as well as the nature of the corresponding reading assessments, vary from course to course. However, the common goal is to encourage students to read mathematics texts, books, and articles for understanding and to complete these readings prior to subsequent class sessions. In turn, our pedagogical approaches seek to build on the students' already developing understanding of the concepts rather than to simply repeat the major concepts from the assigned sections in a routine lecture.

4 Results from a fall 2012 Survey of Trinity Students

A total of 114 undergraduate students enrolled in a mathematics course at Trinity Christian College completed a short survey at the end of the semester in fall 2012. Student participation was voluntary and no data from the survey were analyzed before course grades were submitted. Although this convenience sample is not random, it is nonetheless representative in several important ways. Approximately 65% of the respondents were female, essentially matching the college enrollment as a whole. (See Figure 1.) The proportions of respondents from various academic majors are reason-

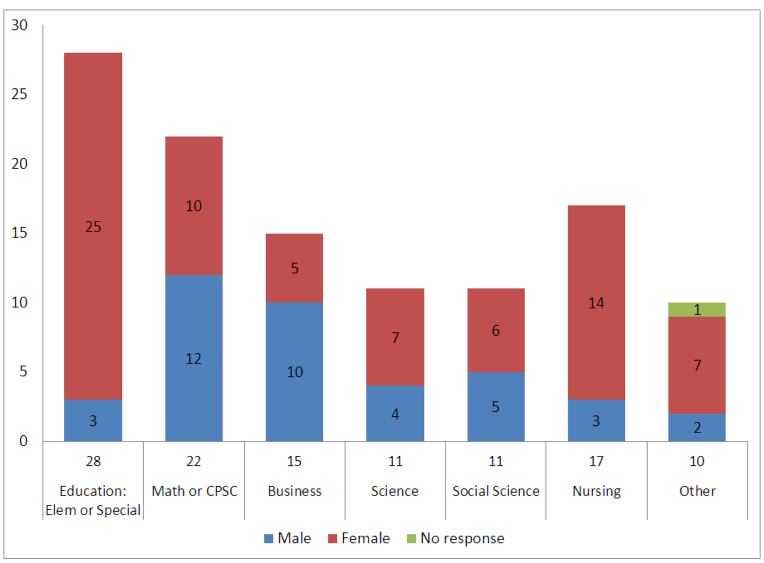

Figure 1: Gender Distribution by Major Cluster

able approximations of the entire student body with two notable exceptions. (See Figure 2.) Due to the mathematics courses included in the sample (introductory statistics, differential calculus, Math Concepts for Teachers I, and advanced level courses in geometry, history of mathematics, and a senior capstone seminar for math majors), very few students with majors in the humanities or fine arts completed the survey. Also, not surprisingly, the sample has a much higher proportion of mathematics majors (16% of the sample vs. 2.1% for the entire student body). In fact, the mathematics majors in this sample represent 72% of all mathematics majors at Trinity.

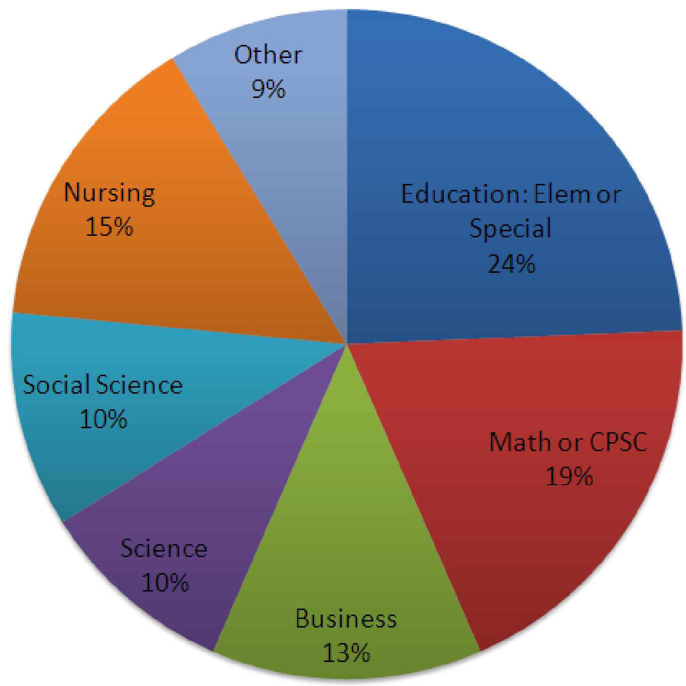

Figure 2: Distribution of Respondents among Major Clusters

In addition to demographic questions discussed above, these anonymous surveys asked students to give a Likert-scale rating (strongly agree, agree, disagree, or strongly disagree) to the following two statements:

Statement 1: *The reading assessments encouraged me to read the assigned material prior to class.*

Statement 2: *The reading assessments were an effective learning tool in the course.*

Students were also asked to identify positive and negative aspects of the reading assessments and to discuss any changes in their approaches to reading that occurred during the semester. Finally, students with a major in mathematics were also asked if their self-perception as a mathematics major had changed as a result of the course.

As we analyzed the data from the surveys, we decided to group the Likert-scale responses into two categories, combining "strongly agree" with "agree" and combining "disagree" with "strongly disagree." For Statement 1, a total of 75% of the 114 students agreed that the reading assessments encouraged them to read the assigned material prior to class. The results were not uniform across major clusters, with 96% of mathematics/mathematics education/computer science majors and 100% of biology/chemistry/exercise science majors responding in agreement. By contrast, only 46% of elementary education majors agreed with Statement 1. Given the gender imbalance among elementary education majors, it follows that the level of agreement for Statement 1 was lower among females (72%) than males (80%). (See Table 1.)

Major Cluster	All	Male	Female	No Response
Education: Elem or Special	13 (46.4%)	2 (66.7%)	11 (44.0%)	0
Math or CPSC	21 (95.5%)	11 (91.7%)	10 (100%)	0
Business	12 (80.0%)	8 (80.0%)	4 (80.0%)	0
Science	11 (100%)	4 (100%)	7 (100%)	0
Social Science	7 (63.6%)	2 (40.0%)	5 (83.3%)	0
Nursing	15 (88.2%)	3 (100%)	12 (85.7%)	0
Other	7 (70.0%)	2 (100%)	4 (57.1%)	1 (100%)
All	86 (75.4%)	32 (80.0%)	53 (71.6%)	1 (100%)

Table 1: Strongly Agree or Agree for Statement 1

For Statement 2, a total of 68% agreed that the reading assessments were an effective learning tool in the course. Among the major clusters, the agreement level was highest among math/math ed./computer science majors (91%). Interestingly, for students who had completed multiple math courses at Trinity and were now enrolled in advanced level courses, there was 100% agreement with Statement 2. Other majors had lower levels of agreement, the lowest being 53% for the business/accounting majors. Once again, the relative gender imbalance among these latter majors resulted in lower agreement levels for males (55%) than females (74%). (See Table 2.)

Shifting to the qualitative data, there were many positive aspects related to reading assessments that were noted by the students. A total of 40 (35%) of students mentioned that reading in preparation for class enabled them to better understand the professor's explanation or their exploration of the content in class. Of the respondents, 23 (20%) noted that the reading assessments forced them to actually read the material, which perhaps indicates that students need to be held accountable for these readings. A smaller proportion (18 students or 16%) cited the earning of points as a positive aspect, even though none of us count the reading assessments for more than 4% of their total semester grade. A few students mentioned that the readings allowed them to learn concepts in greater depth and to consider different ideas or methods for solving problems. Finally, one student remarked that it also proved valuable to return to the assigned readings after class and as a review prior to the unit exam.

As for the negative aspects of reading assignments, some of the most frequently occurring responses were not viewed as negative by their professors, including "time consuming" (28 responses or 25%) and "no response or none" (21 responses or 18%). A total of 27 students (24%) noted that the readings were often confusing to understand and required further discussion during class. This

Major Cluster	All	Male	Female	No Response
Education: Elem or Special	15 (53.6%)	1 (33.3%)	14 (56.0%)	0
Math or CPSC	20 (90.9%)	10 (83.3%)	10 (100%)	0
Business	8 (53.3%)	4 (40.0%)	4 (80.0%)	0
Science	6 (54.5%)	1 (25.0%)	5 (71.4%)	0
Social Science	9 (81.8%)	3 (60.0%)	6 (100%)	0
Nursing	12 (70.6%)	1 (33.3%)	11 (78.6%)	0
Other	8 (80.0%)	2 (100%)	5 (71.4%)	1 (100%)
All	78 (68.4%)	22 (55.0%)	55 (74.3%)	1 (100%)

Table 2: Strongly Agree or Agree for Statement 2

response provides us with impetus to examine more fully the complexities of reading mathematics texts even as it confirms the value of input from the professor in the learning process. Other negative reactions that were cited less often but multiple times included "readings were boring," "hard to focus on key ideas," and "not motivated to read the <u>entire</u> section."

With regard to whether the inclusion of reading assessments affected the students' approach to reading, the plurality of students (43 responses or 38%) stated that there was no change. However, a total of 42 students (37%) indicated a positive impact on their approach to reading, including the realization that reading is imperative to understanding, that reading is worth doing and helps in learning math, that previewing is helpful, and that "math books can actually make sense." Only 7 students (6%) indicated that they still hate reading or do not want to read textbooks as a result of the course.

As a way of sorting through all of the aforementioned data, we decided to group the students into one of four categories based upon their perceived attitude to reading assessments and their effectiveness as a learning tool. The categories were based primarily on their qualitative responses. We titled the first of these four categories "Learners." Students in this category emphasized positive aspects of reading assessments and their role in learning the material in the course. A total of 69 students (61%) fit in this category. A total of 83% of this group agreed that the reading assessments encouraged them to read the material and 80% agreed that the reading assessments were a valuable learning tool. Few negative responses were observed in this group, although a few noted that readings could be confusing and a few others commented that in cases where they fully understood the material prior to class, the class sessions were not as beneficial. Overall, we were very encouraged by the large size of this category of students who clearly benefited from the assigned readings and related assessments.

We classified a second group as "Obeyers," a group that included 22 students (19%) of the sample. Students in this group obeyed the professor and read the assigned materials, resulting in 100% agreement that the reading assessments encouraged them to read the texts. However, only 68% of these students agreed that the reading assessments were an effective learning tool. Not surprisingly, the most frequently cited positive aspect was that the reading assessments encouraged them to read the material. Few negative comments were offered and those listed either cited the length of time required to read or the need for the assessments to count for more credit in the course grade. Overall, we were satisfied that students in this category also benefited from the assigned readings, perhaps due to the related assessments.

A third group we named "Point Maximizers." As the name implies, these 17 students (15%) were motivated by the (relatively insignificant) credit awarded for the reading assessments. Positive aspects typically pointed to the ability to increase their course grade through these assessments, while negative comments typically noted that the readings were often confusing. Interestingly, only 41% of these students agreed that the reading assessments encouraged them to read (as opposed to the credit earned by these same assessments) and only 35% of these students agreed that the reading assessments were an effective learning tool. Rather, the reading assessments appeared to be a means to the end of a higher course grade. Overall, we are satisfied with the result of students reading the material, even if the primary motivation was the earning of a small amount of points toward the course grade.

Finally, there is a group that we chose to call the "Unhappy Campers." Fortunately, only 6 students (5%) fell into this response category. None of these students agreed that the reading assessments encouraged them to read the text, and only 33% of them agreed that the reading assessments were an effective learning tool. For these students, there were no positive aspects, and the common negative aspect was that the readings were confusing. While we were disappointed with these respondents, we were relieved that only 5% of our students were placed into this category.

5 Conclusion

Overall, the survey data confirm what we had hoped. Namely, reading assignments in mathematics courses, when accompanied by appropriate assessments, can promote learning and motivate students to be prepared to learn when they arrive at our mathematics classes. It was interesting to us that different students seem to be motivated by different factors but that 75% of the students self-reported that they completed the assigned readings, even if some may have resorted to skimming the material to answer assigned questions or prepare for a reading quiz rather than reading the entire section. The number of students that remarked that the readings were at least sometimes confusing and difficult to understand highlights the need for further study into the distinctive features of reading mathematics texts and a related need for professors to offer guidance and scaffolding, particularly in the introductory mathematics courses.

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The Structures of the Actual World.

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Abstract

Scripture teaches that God has a plan *for* the universe. Given creation *ex nihilo*, the universe is *nothing*—nothing but God's acting in a multifaceted, coordinated way according to his composite plan, which we refer to as *the actual world*. In order for it to *function* as a plan, the actual world must have a *temporal* structure, a *representational* structure, and a *proto-causal structure*. This paper presents a formal model of these three structures of the actual world.

Scripture teaches that God has a plan *for* the universe. Given creation *ex nihilo*, the universe is *nothing*—nothing but God's acting in a multifaceted, coordinated way according to his composite plan, which we refer to as *the actual world*.[1] In order for it to *function* as a plan, the actual world must have a *temporal* structure, a *representational* structure, and a *proto-causal structure*. This paper presents a formal model of these three structures of the actual world.

Treating the actual world as God's plan is neither new nor widely-accepted by Christian philosophers. Leibniz, of course, is credited with first proposing the idea back in the 17th century, but his view differs from our view developed in this paper.[2] Among contemporary theistic philosophers and theologians there seems to be an implicit preference for Alvin Plantinga's modal realism, which is a metaphysics of modality according to which the actual world is a maximal, temporally-invariant *state of affairs*.[3] Our view differs from this, too. As we show in this paper and others, recovering the actual world

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[1] We are grateful for the helpful comments we have received on this paper from Bill Eppright, Jonathan Zderad, and Brad Sickler.

[2] While the actual world *functions* as God's plan according to Leibniz, what it *is* is a set of mutually-compossible, complete individual concepts. See Mates (1968). An interesting question is the logical relationship between Leibniz's view and the one presented here. According to what is presented in this paper, a set of divine commitments relating to the transition and co-existence of world states within a discrete 3+1 dimensional grid produce, when enacted, what appear to us to be objects, properties and relations. Tracing the history of such an object with all of its relations yields (we think) an object in a Leibniz actual world.

[3] However, Alexander Pruss has recently argued that Plantinga's platonic view lacks several desiderata and that his Aristotelian/Leibnizian view of modality is superior to both of these even though admits that the details of his view have yet to be worked out. See Alexander R. Pruss, *Actuality, Possibility, and Worlds* (New

as God's plan for the universe provides a theologically-faithful, logically-consistent, theoretically-useful, and comprehensive metaphysics of modality, mathematics, and science. (It is worth mentioning here that our view of the actual world precludes the incoherence that plagues set-theoretic constructions of platonic entities such as Roberts Adams' maximal propositions and Alvin Plantinga's "book" on the actual world.[4] In the end, they must be emended so as to avoid the iterative conception of a set and to preclude there being a power set of the "world-story" *S* or "book on a world" *S*.[5] As far as we know, no one (until now) has revised these set-theoretic accounts to avoid this problem. Our view treats the actual world as a complex relational structure, some of whose parts are themselves mereological sums defined predicatively, so that global self-reference is eliminated.[6])

Three preliminary issues.

Before we present the formal model, clarification of three interconnecting issues may prove helpful. The first is about *representations* and *models*, the second is our view of *events*, and the third is about *time*. All descriptions and models are representations, but not all representations are models, and no models are descriptions. Recognizing these differences is crucial to understanding what we are trying to do. As Anjan Chakravarty explains, while there is a *symmetric similarity* relation between the described and the description, there is an *asymmetric intentional* relation between the thing represented (an object, event, etc.) and the representation (*n*). That is, the communicating agent may intend a thing to represent something even though the representation may bear very little resemblance to the thing represented.[7] My point is that a model need not be isomorphic to the represented in every detail. Whatever isomorphism there may be, it is usually at

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York, New York: The Continuum International Publishing Group, 2011): 205.

[4] Adams acknowledges the "threat of paradox" and hopes that a formulation of a "true-story" account of actuality may be found. See Adams (1974): 229. Bringford (1985) shows the incoherence.

[5] Menzel (1986): 72. Bringsjord (1985).

[6] A *predicative* definition of a mathematical object (i.e., the *definiens*) does not include terms whose reference is the object to be defined (i.e., the *definiendum*), nor does it quantify over a domain of which the object is a member, or over sets of things of which the object being defined is a member. A *predicative* definition of a collection refers only to objects that exist independently of the collection.

[7] Chakravarty (2007): 70-73.

the structural level. This is the key notion in *Ontic Structural Realism* (OSR) in the philosophy of science.[8] It is the view that scientific theories (mathematical models) do not inform us about the *nature* of what is modeled, but rather its *structure*. Accordingly, the model of the actual world being constructed in this paper is not intended as a description of the actual world, but rather only a representation of its structures. Consider figure 1.0 below.

The structures of the actual world	→	**The actual world**
		↓
Mathematical models / scientific theories	→	**The universe**

Figure 1.0

The model proposed here lies in the upper left quadrant. It is not the actual world itself (upper right quadrant), which is God's plan for the universe. Nor does it mathematically model the universe (lower quadrants).

event

If the actual world is indeed a divine representation *for* the universe, then some view the nature of events must be assumed. As they are *perceived*—an *event* is a change in an object or a physical system over a duration. As we *conceive* them for the purposes of this paper, an event is the mutual manifestation of disposition partners over a duration. However, we hold that an event—*objectively considered*—is God's

[8] Ladyman and Ross (1998): 130.

compositionally-conferring existence over a sequence of Planck moments, creating Planck regions of the universe according to plan. (A *Planck moment*, according to quantum physics, is the smallest, physically-possible duration, 10^{-43} seconds. A *process* is a sequence of events standing in some causal relation; a *situation* or a *state* of a physical system is the composite of all the invariant features of a complex process. As God *enacts* his plan we perceive some of these processes as various objects, having various dispositional properties, standing in various relations and infer from the manifestation of these properties various laws of nature.[9]) This, then, is a metaphysical account of a basic concept in science. What an event is is not a scientific question.

Even so, the model of the structures of the actual world being proposed in this paper cannot ignore science. If the actual world is indeed a divine representation *for* the universe, then some view of the *structure* of the physical world must also be assumed. We assume that *The base units of space and time are given by Planck-scale physics* (PL), and that *The neo-Lorentzian physical interpretation of the mathematical formalism of the Special Theory of Relativity is correct* (NL). With these in mind consider—as Carlo Rovelli claims—that, while the present knowledge of the elementary dynamical laws of physics are given by Quantum Field Theory (QFT), the Standard Model of Particle Physics (SM) and General Relativity (GR), yet, taken together, they comprise an inconsistent set. GR entails that space-time is not a fixed metric background as assumed in SR, but rather a *dynamical field*. However, Quantum Mechanics (QM) entails that all dynamical fields are quantized. Rovelli concludes that

> What Newton called "space" and Minkowski called "spacetime," is [. . .] nothing but a dynamical object – the gravitational field . . . [which has a] quantized, discrete structure at the Planck scale. [. . .] [10]

A theory of *Quantum Gravity* seeks a synthesis of GR and QM by providing an account of this "quantized, discrete" structure of space-time at the Planck scale. The main competing theories of QG are black hole thermodynamics, string theory, loop quantum gravity. Other, possibly complementary, theories that are noted in the literature are non-commutative geometry and causal sets.[11] Of these, the *Causal Set Hypothesis*

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[9] Events and so-called concrete states of affairs (as objects, property and/or relations) are *processes.* See Smolin (2001).

[10] Rovelli (2009): 8,9.

[11] The *Holographic Principle* may be the key to unifying these. For an introduction accessible to non-specialists, see Smolin (2001).

(CSH) is assumed here. It holds that space-time at the Planck scale is a locally-finite, partially-ordered set of elementary events.[12] These elementary events are the members of causal sets. The *Causal Set Hypothesis* (CSH) is a mathematical model of the dynamics of sequential growth.[13] "Becoming" is a real process due to the continuing "birth" of new elements of the causal set.[14] Also, William Lane Craig has argued from our second assumption (NL) for the reality of absolute simultaneity.[15] Putting these together, (CSH) together with (NL) and (PL) permits us to hypothesize that, for each Planck moment, there is a single, total state of the universe. It follows, first, that the universe is a 3+1 discrete manifold.[16] It also follows that if a representation *for* the universe has an order—an order, that is, of discrete total states—then absolute simultaneity exists first at this ontologically basic level. As such God's plan is truly *the privileged* frame of reference. The remaining issue, is to introduce a *model* of the structures of God's plan, which we call the actual world. So now, taking the actual world to be God's plan *for* the universe, the question before us is this:

> What must the structures of the *actual world* be if (1) the universe is a discrete 3+1 *discrete* manifold at the Planck scale and if (2) the *absolute simultaneity* consistent with the neo-Lorentzian physical interpretation of *Special Relativity* is the case?

In short, the problem is to model a representation *for* the universe over time. Here is an analogy. In addressing the question of how the apparent temporal sequence of the physical world should be represented mathematically, Tim Maudlin thinks that "some intrinsically time-directed representational medium . . ." such as music might be useful.[17] We agree, music is a heuristically fruitful analogy. Since the world states comprising the actual world are God's plans, we may think of the totality of these plans on the analogy of a

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[12] Kinematics. See Dowker 2011,2 .

[13] Dynamics. See Dowker 2011,2.

[14] See Craig (2001) on the importance of the notion of "becoming" for Christian theology.

[15] Noting that both Poincaré and Lorentz acknowledge the logical possibility of absolute simultaneity, Craig notes that mathematical models *of* the universe and its causal structure, such as Minkowski space-time, lack the concepts or resources for a notion of absolute simultaneity by virtue of their epistemological commitment to logical empiricism. See Craig (2008).

[16] See Ostoma and Trushyk (1999).

[17] Maudlin (2007): 135-142.

complex musical score—perhaps a symphony, for example. Three features of music are pertinent. The first is that it is measured in terms of a sequence of "beats". Second, notes being sounded from different instruments stand in differing relations to those being sounded from other instruments. This second point itself suggests a third. There is a sequence of sounds issuing from each instrument. To reiterate, the actual world as God's plan for the universe is a *like* a musical score, having these three structures. If so, the mass-energy distribution constituting physical space and time is nothing other than God's acting according to plan and perceived by us as manifesting dispositional properties and described by laws of nature.

To put all of this another way, scripture entitles us to hold that, given creation *ex nihilo* each *realized total state* of the universe is a matter of God's acting (i.e., willing, speaking)—God's acting at that moment, in a multifaceted, coordinated way, according to his plan. Thus, it is not that the "heavens declare the glory of God" in a way analogous, say, to the way Joshua Bell's performance of Chopin's *Nocturne* in C-sharp minor reflects Chopin and Bell. Rather, Joshua Bell's performance's reflecting Chopin and Bell is the analogue! It can only be a weak analogue to the heavens' declaring the glory of God. Furthermore, for creation, God is both composer and performer. The actual world is the composition. The actual world is a complete *representation for* the universe over time.[18] It has a temporal structure, a representational structure and a proto-causal structure. The model we are constructing is, therefore, a *model* of a *God's representation for the universe* and that representation shares the structural features of a piece of music.

time

This construal, provides a way to synthesize McTaggart's two views of time referred to as the "A-theory" and the "B-theory".[19] The A-theory (sometimes called the *presentist*, *dynamic* or *tensed* view of time) holds that the apparent distinction between the past, the present, and the future is objectively real, though only the present moment is real.[20] The B-theory (sometimes called the *eternalist*, *static*, or *tenseless* view of time) holds that all times and their contents are equally real and stand in a *earlier-than* relation to each other. The model proposed here treats "times" as real entities in so far as they are *frames for* physical events. Both the plan and the plan-realized are real. They are not merely mental constructs. Like sheet music to a

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[18] There is the further question of whether the plan is infinite or only potentially infinite. Moreover, there are at least two construals, given a deterministic assumption, and two construals, given libertarian freedom and indeterminism. These are not addressed in this paper.

[19] McTaggart (1980).

[20] Some presentists hold that both the past and present are real.

musician: on the page, the music is B-theoretical; as performed the music is A-theoretical. Like a play to its director: the script is B-theoretical; as performed: A-theoretical. Like a recording to a music lover: on the recording B-theoretical; as played, A-theoretical. Physical time is God's sequential acting and is A-theoretical. However, though God's being constitutes metaphysical time[21], God's plan is B-theoretical. Therefore, times and durations are real as segments of God's plan. Therefore, modeling the structures of the actual world cannot be a merely constructing a *set* or a *conjunction* of states of affairs. It is like modeling music, say a symphony. Let us now consider the structures of the actual world.

The Temporal Structure

Since the actual world is a representation *for* the universe over time, it will have a temporal structure. Typically, the construction of a formal model involves three choices, choices as to which kinds of temporal *individuals* to use (e.g., points, periods or events), which kinds of *relations* that hold between individuals (e.g., precedence, inclusion, overlap, etc.), and which kinds of *conditions* should be incorporated so as to adequately model what is to be modeled.[22] One might think of the actual world as a book with blank pages. Since creation is assumed to have a temporal beginning, the book has a first page but no last page. The content of "pages" are *world states*. Let us pause here to clarify this notion.

world states

A *world state* is the representational content *for* Planck cells and regions. There are several conceptually possible types. Let us suppose that an ***atomic*** world state represents the content of a *Planck cell* (a 3-dimensional irregular hodon) at a *Planck moment* (a chronon). A representation of the content of the entire world at a Planck moment is a ***total*** world state. It *represents* a discrete, irregular "cube" of simultaneity which is the entire universe at a Planck moment. A ***composite*** world state is any combination of atomic world states without a regional or temporal gap. An example is the composite world state picked out as *Lincoln's giving the Gettysburg Address*. Such *composite* world states are *like* the standard view of propositions or states of affairs, but are *not* conceptually identical to them. Unlike the standard view of propositions, they are not the *content* of a person's thought. No person could think a *complete* representation.

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[21] DeWeese (2004:252). God's temporal mode of being is best characterized as *omni-temporality*, namely God is *metaphysically temporal* and exists necessarily so that (1) (in this sense) God exists at all times and is everlasting and (2) God's existence in metaphysical time grounds causality, and therefore physical time.

[22] See van Benthem (1984): 1.

The informational content of a person's belief cannot be as detailed as a world-state, which is a *complete* representation. Propositions are *abbreviated* representations; the content of occurrent propositional attitudes. Propositions represent either particular world states *per se* (not yet "actualized"), world states achieved ("actualized"), or a quantifications over world states (i.e., they may represent all, some, most, world states.)

With the concept of a *world state* in mind, let us return to the notion of temporal structure. We suggested that the temporal structure of the actual world could be conceived as a "book with blank pages." The content of such "pages" are *world states*. Alternatively, the temporal structure may be conceptualized as a blank film tape, which is infinitely long in one "direction." An "exposed frame" on the film tape is a world state. On these analogies we have *pages* and the *contents* of pages and *frames* of film and *contents* of frames. The difference is crucial. On our model, neither concrete events nor the so-called "states of affairs" are the ontological components of the temporal structure we are attempting to model, nor are the more common abstract points or instants (and intervals of them). Rather, the primary ontological components are *world states*, which are *plans.* Plans are temporally framed or demarcated by *discrete frames for time*. They are thereby temporally-located relative to other plans. Events (according to the model being developed) are plans achieved. In sum, the model constructed here takes *discrete frames for time* as its individuals and *precedence* as the basic relation. The temporal structure of the actual world is an infinite, strict linear order of discrete total *frames* for time. Let us formalize this.

Definition. Let an ***a-frame*** be a real, minimal *frame for* time.

What is meant by, 'minimal', is that an *a-frame* demarcates a "snapshot" of the universe at a Planck moment (i.e., 10^{-43} seconds). It is not the content of the "snapshot", nor is it the universe itself. It is not a segment of physical time itself, because it has no *physical* content. Moreover, though it *has* representational content, an *a-frame* is nothing but the frame. An *a-frame* demarcates a temporal "slice" of the mass-energy distribution or a neo-Lorentzian "cube" of simultaneity. Each a-frame's representational content is a total world-state. Since a *total* world state is a finite composite of world states[23] and a *world state* is a state of affairs to be achieved, *a-frames* demarcate *plans*, in so far as they are segments of an overall *plan*. The overall plan could be understood as a complete course of action.[24]

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[23] A "tensed universal description"describes this composite state. See Yandell (1999: 308).

[24] Thus, this is related to an event ontology, simply because it is a sequence of plans and realized plans are events. Concrete states of affairs are derivative. This is consistent with Smolin (2001).

There are two approaches to representing and reasoning about time and temporal relations: the *first-order approach* (initiated by Russell (1903) and continued by Quine (1960)) and the *modal approach* (initiated by Prior (1955) (1967)). The choice of approaches may be merely a matter of a theorist's preference, but some have argued cogently for the superiority of the former.[25] As Meyer (2009) acknowledges, "There are temporal claims that resist regimentation in terms of tense operators, but which are easily accounted for by rival theories that make use of quantification over times [. . .]."[26] However, just as quantifying over worlds implicitly treats them as real entities, quantifying over *frames for time* takes them as real entities. The quantification is intended to objectual, because the objects are assumed to be real *frames* for physical time. This means that something should be said regarding what "times" are supposed to be. In this model, "times' are, first of all, *a-frames* and convex sequences of them. They frame segments of the temporal structure of the actual world. Physical "times" (i.e., time *per se*) just are demarcated periods of God's realizing his plans. Therefore, even though the approach taken here is similar to treating times as platonic abstract objects or as events, *a-frames* are clearly different.

Definition. An ***a-frame structure*** is an ordered couple $\langle \boldsymbol{A}, \prec \rangle$ where A is an infinite set of a-frames (denoted by lower-case, italicized letters x, y and z) and where $\prec$ is the binary relation *precedence* ('$x \prec y$' means "x precedes y") . The binary relation $\prec$ satisfies the conditions IRREF, TRANS, DET, INF, and DISC defined below.[27] These conditions are treated axiomatically, permitting further deductions regarding the structure of the actual world. The first condition that $\prec$ should model is that no *a-frame* precedes itself, which means that $\prec$ is *irreflexive*.

IRREF $\forall x \sim (x \prec x)$ (irreflexive)

Secondly, if one *a-frame* precedes another, and that one a third, then the first precedes the third. In other words, $\prec$ is *transitive*.

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[25] See van Benthem (1983) and Mellor (1981).

[26] Meyer (2009): 201.

[27] Compare to van Benthem (1983): 12-18.

TRANS $\forall x \forall y \forall z\ ((x \prec y \wedge y \prec z) \rightarrow (x \prec z))$ (transitive)

IRREF and TRANS are intended to construct $\langle A,\prec\rangle$ so as to reflect time as having (or being) a constant rate of "flow" in one "direction" only. These conditions entail asymmetry, i.e., if one *a-frame* precedes another, the second does not precede the first: $\forall x \forall y\ (x \prec y \rightarrow \sim y \prec x)$. Thus, $\prec$ is *asymmetric*.

Thm1. $\vdash_{FL} \forall x \forall y\ (x \prec y \rightarrow \sim y \prec x)$ **ASYM**
Proof. Suppose $x \prec y$. Now, for *RAA* suppose $y \prec x$. It follows that $x \prec x$, contradicting IRREF.

To ensure that $\langle A,\prec\rangle$ does not "branch", we stipulate that any two a-frames are either identical or one precedes the other. There are no other alternatives. Thus, $\prec$ is *determinate* and *linear*.

DET $\forall x \forall y\ (x = y \vee x \prec y \vee y \prec x)$ (determinate)

By imposing conditions IRREF, TRANS, and DET on relation $\prec$, $\langle A,\prec\rangle$ is a strict linear order (i.e., a chain). It is "strict" because of IRREF. It is "linear" (or "total") because DET rules out branching, since all elements are involved and there are no alternatives and because IRREF rules out circularity.

God's *plan* has a first moment, but no last. We capture both of these features by stipulating that one a-frame has no predecessor and every a-frame has a successor.

INF $\exists x \sim \exists y\ (y \prec x) \wedge \forall x \exists y\ (x \prec y)$ (infinite)

Thus, INF ensures that $\langle A,\prec\rangle$ has a first moment, but no last moment.

We want to model a representation for time as a sequence of discrete entities.

DISC[28] $\forall x \forall y\ (x \prec y \rightarrow \exists z\ ((x \prec z) \wedge \sim \exists u\ (x \prec u \wedge u \prec z)) \wedge$ (discrete)
$\forall x \forall y\ (x \prec y \rightarrow \exists z\ ((z \prec y) \wedge \sim \exists u\ (z \prec u \wedge u \prec y))$.

[28] Same as van Benthem's DISC, (1983): 18.

To recapitulate, the *temporal* (or sequential) structure of the actual world is an *a-frame structure*, which is an ordered couple $\langle A, \prec \rangle$, where A is an infinite set of a-frames and where $\prec$ is the binary *precedence* relation defined on D so that '$x \prec y$' means "x precedes y". Thus, $\langle A, \prec \rangle$ is an infinite, strict, linear order of discrete entities, which we call, '*a-frames*'. Since *a-frames* demarcate Planck moments, they must be subliminal. Hence, the model is in keeping with our sense of the smooth continuity of time even though it is discrete.[29] Also, since, by definition, every *a-frame* demarcates a Planck moment, $\langle A, \prec \rangle$ is *homogenous*: the pattern of time is invariant. In sum, the *temporal* structure of the actual world is an *a-frame structure* $\langle A, \prec \rangle$, satisfying the five axiom conditions, IRREF, TRANS, DET, INF , and DISC, where each *a-frame* demarcates a discrete moment of the entire universe.

As was said earlier, if a representation *for* the universe is ordered sequence of "discrete total *frames for* time", then absolute simultaneity exists first at this ontologically basic level. And now if this representation is God's plan, then God's plan is truly *the privileged* frame of reference. Moreover, if what we have so far is accurate, then the quantum states of the gravitational field are nothing but God's acting according to plan.

Summary

In sum, the temporal structure of the actual world is an ***a-frame structure***, which is an ordered couple $\langle \boldsymbol{A}, \prec \rangle$ where A is an infinite set of a-frames (denoted by lower-case, italicized letters x, y and z) and where $\prec$ is the binary relation *precedence* ('$x \prec y$' means "x precedes y") . The binary relation $\prec$ satisfies the conditions IRREF, TRANS, DET, INF, and DISC .

The representational structure

Taking stock, the actual world is the infinite history of the universe. It is a totality of discrete frames *for* time with representational content, called "total world states". The *temporal structure* of the actual world

[29] See Chew (1985) who proposed that "multiple emission and absorption of soft photons in a discrete quantum world (implicate order) generates the continuous Cartesian-Newtonian-Einsteinian space-time world of localizable objects and conscious observers with measuring rods and clocks (explicate order)." "[. . .] explicate order together with space-time is an approximation emerging from complex but coherent collections of "gentle" quantum events—the emission and absorption of soft photons." (59) "The complexity responsible for objective reality within an apparently-continuous space-time we propose to associate with multitudes of coherent low-energy electromagnetic quanta." (59)

is characterized by a precedence relation satisfying certain conditions. Fully-stated, the *temporal structure* of the actual world—God's plan for the universe—is an infinite, strict linear-order of *a-frames*, which when filled with content are called *world states*, which are representations *for* events.

Distinct events vary in duration. This fact underlies the seven types of *temporal* relations between any *two* events. Representations *for* those events will mirror these seven types of relations. Some are *disjoint*. One may *overlap* another and one may *meet* another. One may *begin simultaneously* with another and one may *end simultaneously* with another. One may occur *entirely within the duration* of another or both may *occur simultaneously*. These various relations are the *representational structure* of the actual world.

Three issues must be emphasized and clarified before we proceed. First, the model being proposed is not a model of *time* in any of the familiar senses. Standard tense logic involves an ontology of times which stand in some temporal relation. However, the temporal structure offered in the previous section is a model of time in which *a-frames* are the fundamental ontological entities, as opposed to the more standard points, periods or events. So, although the model presented in *this* section *resembles* a model of time in terms of period, interval or an event structure, it is a model of the *structure of representations*.

Second, since *world states* are representations *for* physical reality, they cannot stand in temporal relations *per se*. Only *realized* world states, which are events and states, stand in temporal relations. Hence, strictly speaking, world states stand in *proto-temporal* relations. To put this another way, since a *world state* (i.e., a theistic plan) is the representational content *for* an event or process, then while events and processes have *temporal duration* (are *in* time), world states have *representational duration* (are *for* time). Intervals of a-frames *demarcate* the *temporal duration* of events and processes (which again are realized plans or world states). But the *representational duration* of every world state *is* an interval of *a-frames* or interval of the *temporal* structure $\langle A,\prec\rangle$.

Third, there are no "vague" concepts or inaccurate perceptions in the actual world.[30] The reason is that the actual world is understood to be God's plan for the universe. God is not apprehending an already existing object.[31] However, since each world state is a complete in every detail, no human could cognitively grasp a world state. Propositions, which are the information content of declarative sentences, can only represent them. Propositions are abbreviated representations of world states. Thus, what is needed is a *model* of the *structure of representations* constituting the actual world.

[30] See Kamp (1979).

[31] This underscores the difference between *propositions* and *world states* mentioned earlier.

Definition. Let a ***duration*** $\boldsymbol{\delta_n}$ ***of a world state*** be a non-empty convex subset of $\langle A,\prec\rangle$, which has (1) a temporal location and (2) an absolute value, which is its "length" in *a-frames*.

Thus, first of all, each *a-frame* correlates to exactly one natural number. (They may be set in a one-to-one correspondence.) The set of *a-frames* A is an ω-sequence satisfying the axioms of PA^2 (2nd-order Peano Arithmetic). Moreover, each *a-frame* has a "name" or "place", which is some natural number for every member of the set $\mathbb{N}$. Second, since *a-frames* are real, durations as intervals of *a-frames* are real; independent of human conception. Third, every world state correlates to a temporally-invariant proposition. The representational structure is a mereological sum. The representational structure of the actual world is thus "mirrored" by a structure of propositions.[32]

Definition. Let a structure **A** be ***generated from*** **B** if

1. **A** and **B** have the same signature, σ(**A**)= σ(**B**), that is, the meaning of the relation symbols of **A** are the same as those of **B**,
2. The *domain* of **A** is constructed from the domain of **B**, and
3. The *interpretation* of the relation symbols of **A** are the same as those for **B**.

Definition. The ***duration structure*** $\langle \boldsymbol{D},\prec\rangle$ is generated from $\langle \boldsymbol{A},\prec\rangle$, where

(i) $\boldsymbol{D}$ is the set of all subsets $\boldsymbol{\delta_n}$ of $\langle A,\prec\rangle$,

(ii) $\prec$ is a ***precedence*** relation on D defined as follows:

$$\boldsymbol{\delta_n} \prec \boldsymbol{\delta_m} \leftrightarrow \forall x\in\delta_n, \forall y\in\delta_m\ x \prec y.$$

Definition. ***inclusion***

1. $\boldsymbol{\delta_n \sqsupset x}$ A duration δ_n ***includes* an *a-frame* x** iff x is a member of δ_n.

$$\delta_n \sqsupset x \leftrightarrow x\in\delta_n$$

2. $\boldsymbol{\delta_n \sqsupset \delta_m}$ A duration δ_n ***includes*** another duration $\boldsymbol{\delta_m}$ iff every *a-frame* that is a member δ_m of

[32] See Sider (2006).

is also a member of δ_n.

$$\delta_n \sqsupset \delta_m \leftrightarrow \forall y\ (y \in \delta_m \rightarrow y \in \delta_n)$$

Since *durations* demarcate and locate events and concrete states of affairs, they must be "uninterrupted" sequence of *a-frames*. This feature is expressed formally by the notion of *convexity*.

Definition. ***convexity*** A *duration* δ_n is a convex set iff for any two *a-frames* x,y, there is a third, z, such that, if x precedes z and z precedes y, then δ_n includes z.

CONV δ_n iff $\forall x \forall y\ ((\delta_n \sqsupset x \wedge \delta_n \sqsupset y \wedge \exists z \in \langle A, \prec \rangle\ (x \prec z \wedge z \prec y)) \rightarrow \delta_n \sqsupset z)$.

Most, but not all, durations are have first and last members. When they do, we say they are *bounded.*

Definition. ***boundedness*** A duration δ_n is ***bounded by a-frames x and y***, $B(x\ \delta_n\ y)$ iff every other *a-frame*, z, that is included in δ_n is preceded by x and precedes y.

B(x δ_n y) iff

(i) $x = y \wedge \delta_n \sqsupset x \wedge \delta_n \sqsupset y \wedge {\sim}\exists z\ \delta_n \sqsupset z$, or

(ii) $(x \neq y \wedge \delta_n \sqsupset x \wedge \delta_n \sqsupset y \wedge \forall z\ (\delta_n \sqsupset z \rightarrow (x \prec z \wedge z \prec y))$

Definition. ***identity*** Any two durations are identical iff each includes all of the *a-frames* included by the other.

$\delta_n = \delta_m$ iff $\forall x\ (\delta_n \sqsupset x \leftrightarrow \delta_m \sqsupset x))$

The aim now is to model the representational structure of the actual world. This involves accounting for the relevant *features* of the durations of world states and for the seven types of *relations* that may obtain be world states and thus between the durations that demarcate and locate them.

AXIOMS.[33]

Since the actual world is a representation *for* events and concrete states of affairs, the first thing to ensure is that the model does not permit the inference that there are temporal gaps within some durations.

A1 **CONV**(δ) Every duration is a convex subset of $\langle A,\prec\rangle$.

$\forall\delta_n\forall x\forall y\ ((\delta_n \sqsupset x\ \wedge \delta_n \sqsupset y \wedge \exists z\in\langle A,\prec\rangle\ (x\prec z \wedge z\prec y)) \rightarrow \delta_n \sqsupset z)$.

Some states of affairs are perpetually sustained so that they are unbounded into the future.

We need also to ensure that non-perpetual durations (except single *a-frames*) have distinct first and last elements.

A2 **BOUND**(δ) Every finite duration δ_n of D is bounded.

$\forall\delta_n\exists x\exists y\ B(x\ \delta_n\ y)$

Thm 1. ***D*** is partitioned into two subsets: the set of all *finite* subsets and the set of all *unbounded* subsets.

A3 **DET**(δ) Every two distinct *a-frames* determine a duration.

$\forall x\forall y\ (x \neq y \rightarrow \exists\delta_n\ B(x\ \delta_n\ y))$

A4 **CONT**(δ) For every two durations having some *a-frame* in common, there is a duration containing all and only these *a-frames.*

$\forall\delta_n\ \forall\delta_m\ \forall x((\delta_n \sqsupset x \wedge \delta_m \sqsupset x) \rightarrow \exists\delta_k\ \forall y\ (\ \delta_k \sqsupset x \leftrightarrow (\delta_n \sqsupset y \vee \delta_m \sqsupset y))$.

Thm 2. Every *a-frame x* is included some *duration* $\boldsymbol{\delta_n}$,

i.e., $\forall x\in\langle A,\prec\rangle\ \exists\delta_n\ (\delta_n \sqsupset x)$.

---

[33] This axiomatic formulation has been influenced by van Benthem, (1983), (1984), Bochman (1990), and Allen (1984). For any two distinct *a-frames*, there is a duration containing a third distinct *a-frame* containing exactly one of the pair.

$\forall x\forall y(x \neq y\ \rightarrow\ \exists\delta_n\ \exists z(\delta_n \sqsupset z \wedge (\delta_n \sqsupset x \leftrightarrow \delta_n \not\sqsupset y)))$.

Thm 3. Every duration δ_n is a strict linear order of *a-frames*,

i.e., $\forall x \forall y \forall z \in \delta_n(\sim(x \prec x) \wedge ((x \prec y \wedge y \prec z) \rightarrow (x \prec z)) \wedge (x{=}y \vee x \prec y \vee y \prec x))$.

There are seven types of relations between any *two* world states. Some are *disjoint*. One *overlaps* another and one *meets* another. One begins at the same time as another (*co-starts*) and one ends at the same time as another (*co-finishes*). One occurs entirely within the duration of another (*during*) or at the same time as the other (*simultaneous*). This is graphically represented in *Figure 2.0*, where the letters, *i, j, k, l, m,* and *n* represent durations.

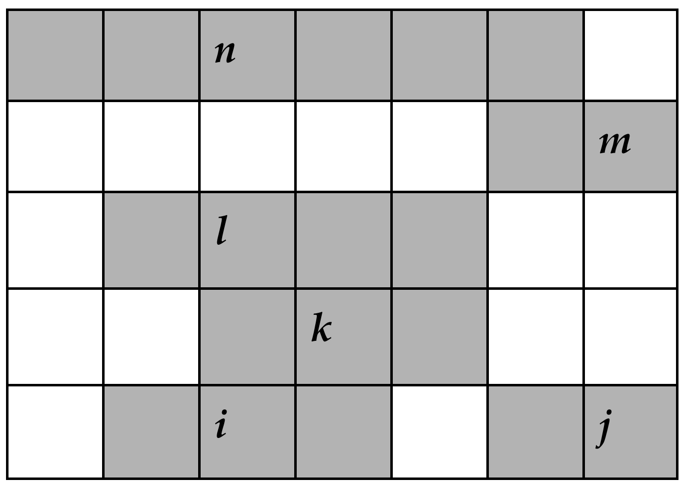

Figure 2.0

A5 ***overlaps*($\boldsymbol{\delta_i, \delta_k}$)** $\boldsymbol{\delta_i \circ \delta_k} \leftrightarrow ((\exists x\, (\delta_i \sqsupset x \wedge \delta_k \sqsupset x)) \wedge (\exists z\, \delta_i \sqsupset z \wedge \sim\delta_k \sqsupset z) \wedge (\exists u\, \delta_k \sqsupset u \wedge \sim\delta_i \sqsupset u)$.

A6 ***meets* ($\boldsymbol{\delta_k, \delta_j}$)** $\boldsymbol{\delta_k \mid \delta_j} \leftrightarrow \delta_k \prec \delta_j \wedge \sim\exists z\, (\delta_k \prec z \wedge z \prec \delta_j))$.

A7 ***disjoint* ($\boldsymbol{\delta_i, \delta_j}$)** $\boldsymbol{\delta_i}$**, DIS** $\boldsymbol{\delta_j} \leftrightarrow \delta_i \prec \delta_j \wedge \exists z\, (\delta_i \prec z \wedge z \prec \delta_j))$.

A8 ***co-starts* ($\boldsymbol{\delta_i, \delta_l}$)** $\boldsymbol{\delta_i}$ **CS** $\boldsymbol{\delta_l} \leftrightarrow \exists x\, (B(x\ \delta_i\ y) \wedge B(x\ \delta_l\ z))$.

A9 ***co-finishes*** **(δ_k, δ_l)** $\boldsymbol{\delta_k}$ **CF** $\boldsymbol{\delta_l}$ $\leftrightarrow\ \exists z\ (B(x\ \delta_k\ z) \wedge B(y\ \delta_l\ z))$.

A10 ***during*** **(δ_i, δ_n)** $\boldsymbol{\delta_i}$ **DUR** $\boldsymbol{\delta_n}$ $\leftrightarrow\ \forall x\ (\delta_i \sqsupset x \rightarrow \delta_n \sqsupset x) \wedge \exists y\ \exists z(y \prec \delta_i\ \wedge \delta_i \prec z \wedge \delta_i \sqsupset y \wedge \delta_i \sqsupset z)$.

A11 ***simultaneous*** **(δ_j, δ_m)** $\boldsymbol{\delta_j}$ **SIM** $\boldsymbol{\delta_m}$ $\leftrightarrow\ \delta_j = \delta_m$.

Thm 4 Every finite duration meets and succeeds another.

Summary

In sum, the representational structure of the actual world is a *representational duration structure* $\langle D,\prec\rangle$ generated from the temporal structure $\langle A,\prec\rangle$, where D is the set of all subsets δ_n of $\langle A,\prec\rangle$, and $\prec$ is a *precedence* relation defined on D such that $\delta_n \prec \delta_m \leftrightarrow \forall x{\in}\delta_n, \forall y{\in}\delta_m\ x \prec y$. The set D satisfies the four axiom conditions, CONV, BOUND, DET, and CONT. Every temporal relation between events is represented.[34]

The proto-causal structure

Some *world states* stand in a *proto-causal* relationship so that *when* they are physically realized, these become *events* which stand in a *causal* relationship. To put this another way, if *causality* a relation between events (so understood), then a *causal process* is a sequence of such events. Whenever two events occur simultaneously, neither one causes, or is caused by, the other. Therefore, a causal process does *not* include any event that stands in a simultaneity relation with any of its constituents. Furthermore, one event *can* precede another distinct event and yet not stand in a causal relation to the latter. But not every sequence of atomic world states represents a causal process. Therefore, these two types of sequences must be distinguished, which means that the proto-causal structure of the universe must be correlated to a relational substructure of $\langle D,\prec\rangle$. We say "correlated to" because what stand in proto-causal relations are those world states that *have* representational duration, but not temporal duration.

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[34] While $\langle A,\prec\rangle$ is a topological structure, $\langle D,\prec\rangle$ is not.

A. A. Robb (1914:371) showed that "the elements of time form a system in conical order."[35] He was able to duplicate the geometric structure of Minkowski Spacetime on the basis of a *before* relation (and conditions) among points. Malament (1977) and Winnie (1977) *take* Robb's *conical* order to be a causal order. The *Causal Set Hypothesis* is an advancement on these views. As we said earlier, it holds that space-time at the Planck scale is a locally-finite, partially-ordered set of discrete elementary events.[36] These elementary events are the members of causal sets. A *causal set* is a sequence of elementary events that stand in a causal relation. Therefore, a *proto-causal set* is a sequence of world states that stand in a proto-causal relation. Causal sets constitute a mathematical model of the dynamics of sequential growth and change at the local level.

Given (CSH), the proto-causal structure of the actual world $\langle C, \prec \rangle$ is a *proto-causal set structure,* where C is the set of all proto-causal sets $_{cN}$ and $\prec$ is a *precedence* relation defined on C. The binary relation $\prec$ satisfies the three axiom conditions TRANS, NCIRC and FIN below:[37]

TRANS	$\forall x \forall y \forall z \in C\ ((x \prec y \wedge y \prec z) \rightarrow (x \prec z))$	(transitivity)
NCIRC	$\forall x \forall y \forall z \in C\ ((x \prec y \wedge y \prec z) \rightarrow (x = z))$	(non-circularity)
FIN	$\forall x \forall z \in C$, the set $\{y: x \prec y \prec z\}$ is finite	(finiteness)

How does this *proto-causal* structure relate to the *temporal* and *representational* structures of the actual world? The proto-causal structure of the universe must be correlated to a relational substructure of $\langle D, \prec \rangle$. Therefore, causal structure is not identical to, but depends on (related to) temporal structure. God's plan has *temporal structure*. God's acting according to plan is the observed *causal structure* of space-time. *Precedence* is the basic relation in the temporal structure of the actual world, but *causal precedence* (i.e., the Minkowski relation of "possible causal precedence") is the basic relation for the causal structure of the actual world.

[35] Robb (1914): 371. It may be that William R. Hamilton (1805-1865) deserves the credit for the first mathematical model and metaphysical interpretation of time and causality.

[36] "The causal set hypothesis is that in the deep quantum regime of very small scales, spacetime is no longer described by a metric on a differentiable manifold, but by a causal set. Just as ordinary matter appears smooth and continuous to us, but is really made of atoms and molecules, so, it is proposed, spacetime appears continuous at large scales but is really a causal set and the continuum spacetime of our experience is just an approximation to the discrete order." Fay Dowker. *Contemporary Physics*, Vol. 47, No. 1, January–February (2006): 5.

[37] Dowker (2006): 4.

Minkowski spacetime and the Causal Set Hypothesis of QG are mathematical models of the universe. They should be—and aim to be—consistent with other physical theories. An account of the actual world has subtle, but crucially different aims. It should be consistent with physical theory—not *as* a physical theory—but as a metaphysical theory. The consistency of a metaphysical theory with a physical theory is a matter of explanatory power.[38] To what extent does the metaphysical theory account for physically-unaccountable features of the physical theory? However, even a theory of the causal nature of time must accept that some issues cannot be accounted for physically. As Robb put almost 100 years ago,

> Though space may be analyzable in terms of time relations, yet these remain in their ultimate nature as mysterious as ever ; and though events occur in time, yet any logical theory of time itself must always imply the Unchangeable.
>
> Thus may I conclude in the words of Carlyle :
>
> > 'Know of a truth that only the Time-shadows have perished, or are perishable ; that the real Being of whatever was, and whatever is, and whatever will be, *is* even now and forever. '

Concluding observations:

With this model of the structure of the actual world we have an ontological grounding of a logically-consistent theory of modality, the basis for an ontology of science and a modal-structuralist philosophy of mathematics,[39] in addition to the basis for a *world-inclusion semantics* for systems of formal logic—all of which are rigorously rooted in the biblical doctrine of God, whose acting is the dynamic underlying reality.

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[38] Maudlin, e.g., takes laws of nature as ontological primitives.

[39] See Walter Schultz, Towards a Realist Modal Structuralism: A Christian Philosophy of Mathematics *Philosophia Christi*, 12.1 (2010): 139-154.

Life Lessons from Leibniz

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The year 2016 is the tri-centennial of Leibniz's death (1646–1716). And 2013 is not too early to begin a special celebration of this man of mathematics. Besides being the co-discoverer of calculus and the implementer of binary numbers, formal logic, and formal languages, all of which foreshadowed the computer age, Leibniz is said to be one of the last to know almost everything that was known about almost anything. Professionally, his occupation was librarian in the princely court of Hanover in old Germany. Serving under a succession of three dukes, the last of whom became George I of England, Leibniz had to continually re-invent himself—somewhat like us older teachers and professors who have continually re-invented ourselves over the years as classroom technology changed from slide rules to hand-held calculators to computers to a profusion of computational schema and distance-learning on the web—under changing administrations and expectations. Throughout his long life, he traveled extensively, maintained a vibrant, voluminous correspondence with theologians, savants, politicians, and friends. In fact, Leibniz is said to have "fine-tuned" the notion and practice of "the balance of power" among nations and pioneered the idea and practice of ecumenicalism within the fragmented church universal. He has much to teach us about math, life, and faith. In the space given us—we give a few life lessons from this giant of a man.

Figure 1: Leibniz as a young man

Why Leibniz?

Besides the tri-centennial, there is a personal reason for talking about Gottfried Wilhelm Leibniz. One afternoon, a colleague at King—Bill Linderman—and I explored the AMS's web-site on mathematical genealogies [1]. As we traced his ancestry from student to advisor, we oohed and aahed at the succession of distinguished names. When encountering Isaac Newton, I was downright envious. Bill was the great-great-something of the most famous mathematician since Archimedes! Then we explored my roots. Some names had multiple advisors. When we came to C. Felix Klein we opted for Rudolf Lipshitz rather than Julius Plücker who would have led us to Gauss, and we passed through Dirichlet; Poisson; Lagrange; the venerable Euler; the Bernoulli brothers: Johann the younger and Jacob the elder; Nicholas Malebranche; and THEN Leibniz, who in turn was mentored by Christiaan Huygens. Wow!

Since this news had magically made these men less distant to me, I resolved to learn more about each of them, and started reading. In 2009, a new biography by Maria Antognazza of Leibniz was published, [2]. Its back cover cited praise from various reviewers. One of these said,

> The first modern biography of Leibniz. . . was published by Guhrauer in 1842, and, dull as it is, it has remained authoritative ever since. Antognazza's labours [in THIS new biography] mean that Guhrauer can at last be sent back to the stacks: she has. . . constructed a lively and thoroughly documented story...even on matters of detail.

From 2009 to early 2013, her book sat on my nightstand. More often than not, I read a few pages each night before falling asleep, ever grateful that this biography was not the more dull one mentioned above. As I read, I realized that some of Leibniz's goals, life-turns, and zest give helpful insights in contemplating our own lives as students and teachers of mathematics. At the risk of over-simplifying Leibniz, we break these into five life lessons.

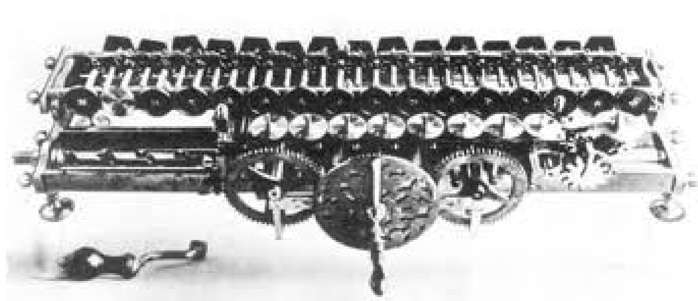


Figure 2: A multiplication machine

Read

Leibniz's father was a philosophy professor at the University of Leipzig; as such he maintained a large private library in which he often read selections to a very young son. Unfortunately, his father died when Gottfried was 6. But Gottfried was allowed continued access to this library, reading texts in Latin and Greek. By the time he himself was ready for student university life, he had already read the masters in philosophy and letters, and could engage his professors as a veritable equal—and earned a degree in philosophy at age 16, a degree in law at 19, and a doctoral degree in law at 20.

So extensive was Leibniz's approach to reading that he became an expert in numerous fields:

> Philosophy, mathematics, astronomy, physics, chemistry, geology, botany, psychology, medicine, natural history, jurisprudence, ethics, political science, history, antiquities, languages (German, European, Chinese), linguistics, etymology, philology, poetry, theology, church reunification, diplomacy, technology, structure of scientific societies, libraries, the book trade. [2, p. 2]

And his resultant writings, as re-published and published-for-the-first-time by the Berlin Academy of Sciences, "will eventually extend to one hundred twenty large quarto volumes."

As he described himself,

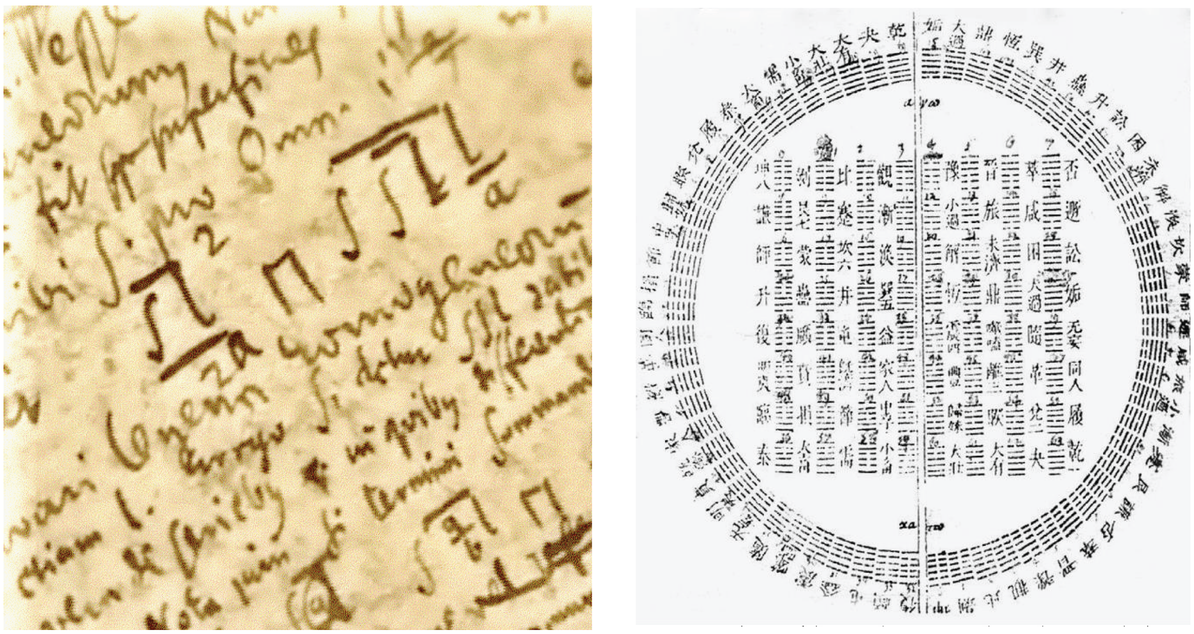
> So many thoughts occur to me in the morning during an hour in which I am still in bed, that it takes me all morning, and sometimes all day and more, to write them down. [2, p. vi]

Dream

What motivated Leibniz?

Gottfried had been born in Leipzig in 1646, two years before the end of the Thirty Years' War, a religious conflict within Christendom that devastated central Europe for an entire generation. Whereas Leibniz was a Lutheran, he had freely read the entire gamut of extant Christian ideas in his father's study. He came to believe that if people could truly communicate on conflicting issues, genuine agreement could be found. Thus, he championed the idea of a natural language in which ideas could be formulated without confusion and misunderstanding. Of course, a small-scale setting for developing such a language was mathematics, and he pioneered what we now know as formal logic—primarily in the hope that this language could grow so as to encompass theology and diplomacy, which in turn might lead to the abolition of war, and, in particular, religious strife. He spent much of his life trying to harmonize Protestant and Catholic perspectives globally, and, more locally, Lutheran and Calvinist perspectives, on what was truly substantive.

As an example of Leibniz's ideas on harmonizing Christendom, he advocated a strategy of what is now called the *balance of power*. For instance, Louis XIV, Leibniz thought, needed to be kept in check lest another Thirty Years' War break out afresh. Leibniz advanced the idea that if Louis really wanted war—and the commensurate material gain and prestige of such activity—instead of devastating Christendom, why not conquer Egypt which at that time had been controlled for the past several hundred years by the Mamelukes—a caste of Muslim warrior-slave-rulers, continuously refurbishing itself by child-trafficking in eastern Europe and western Asia.



a. The first integral sign [3, disc 2]   b. The first binary numbers

Figure 3: Manuscript fragments

Travel

In part to seek audience with Louis's advisors so as to promote this plan, Leibniz traveled to Paris. While there he met Christiaan Huygens—the leader of the French Academy of Sciences—who in turn mentored Leibniz in mathematics, giving him challenging exercises in infinite series. Meanwhile, Leibniz had become intrigued with Pascal's addition machine, and improved it so as to be a multiplication machine, as illustrated in FIGURE 2. He demonstrated this device at the French Academy of Sciences and, in London, at the Royal Academy. The Leibniz Society has re-created a working model of Leibniz's machine wherein the computations are performed with steel bearings in binary, a system he gleaned from reports sent to him from China by Jesuit scholar-missionary Joachim Bouvet concerning the ancient Chinese text, the *I Ching*, that is, the *Book of Changes*. In particular, these texts represented the *yin*—the zero—as a broken line (– –) and the *yang*—the one—as a solid line (—); when these symbols are stacked, successive layers correspond with higher multiples of two; thus, 2 is represented as $\begin{smallmatrix}\text{—}\\ \text{– –}\end{smallmatrix}$, 3 as $\begin{smallmatrix}\text{—}\\ \text{—}\end{smallmatrix}$, and so on, as indicated in FIGURE 3.b.

FIGURE 4 is a map showing Leibniz's wanderings over the years to professional societies, libraries, and courts.

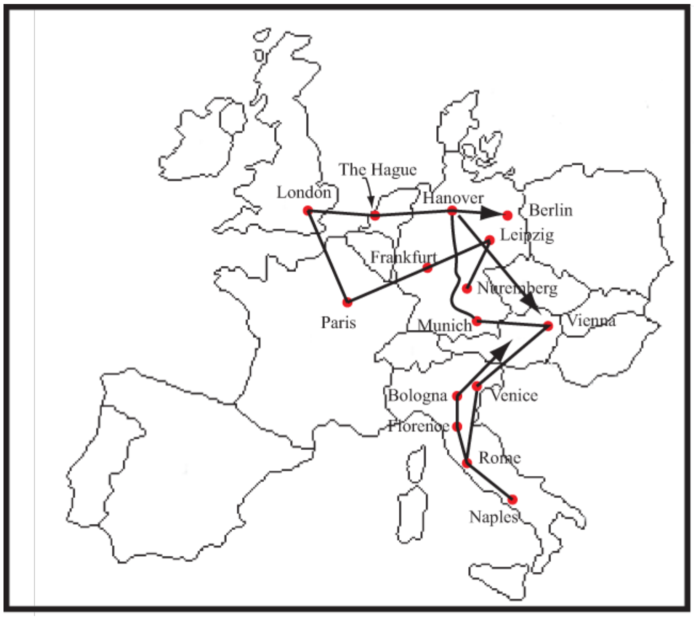


Figure 4: Leibniz travel itinerary

Hang-Out

Many of my students are experts at what is popularly called *hanging out*—enjoying each others company and talking about many things. Leibniz, too, practiced this art, both in person and in letters. He was a member of the Royal Society and the French Academy of Sciences. He established the Berlin Academy and served as its president. He advised Peter the Great in person, which in due course led to the Russian Academy. Indeed, Leibniz had *hung-out* with Huygens, and, in a two-month spurt of creativity at age 29 (1675), he more or less discovered what we now call calculus. When he formally wrote the details nine years later, he communicated some of these pertinent papers to the French Academy through Nicholas Malebranche, a scholar and priest whom he himself had mentored, and who in turn recruited the Bernoulli brothers (Jacob and Johann) to make even greater sense of these early discoveries. FIGURE 3.a shows perhaps the first instance of an integral sign from an old Leibniz manuscript.

Improvise

For the last 41 years of his life, Leibniz served as librarian to a succession of three dukes at Hanover. Of course, Hanover was not a hot-bed of new ideas; and each new duke had their own priorities. As Leibniz reminded himself,

> [It] bothers me that I am not in a great city like Paris or London, where there are plenty of learned men from whom one can benefit and receive assistance. Here [in Hanover] one scarcely finds anyone to talk to; it is not regarded as appropriate for a courtier [like me] to speak of learned matters. [2, p. 196]

Duke-1's primary focus for Leibniz was to establish a fine legal library, Duke-2's focus was on improving silver mining operations—the primary revenue source for the duchy, and Duke-3's focus was on documenting the long lineage of the House of Hanover and its origins in the old Roman Empire, so that in due course the prince at Hanover became an elector in the Holy Roman Empire, and ultimately was chosen as England's sovereign, the successor to Queen Ann.

For each of these transitions of power and vision in a new duke, Leibniz needed to re-invent himself, and make himself useful. Indeed, for those of us who teach at small institutions and have weathered a series of administration transitions, we too have had to re-invent ourselves, retooling ourselves as waves of new technology and administrative innovations challenge the value, utility, and effectiveness of what we do. In Leibniz's case, imagine returning from researching a genealogical lead in some remote corner of the Empire to find that all the library's books had been crated and stored in an attic. What would you do? Leibniz could have moved on to other opportunities. Various math greats have second-guessed him, and suggested that he should have moved on at different junctures. Yet I am impressed with his steadfastness—and his ability to be so creative and productive for so long a time in a place and position which many of his contemporaries and present fans would consider a place of almost hopeless opportunity.

I close with my favorite Leibniz quote:

> Providence puts right the mistakes of human beings, so that often things which are thrown badly fall well. [2, p. 65]

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Perspectives on Chaos: Reflections of a Mathematical Physicist

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Chaos (*n.*)

1. (obsolete) chasm, abyss;
2. (a) (often capitalized) a state of things in which chance is supreme;
   (especially) the confused, unorganized state of primordial matter before the creation of distinct forms;
3. (a) a state of utter confusion;
   (b) a confused mass of mixture.[1]

Etymology: from the Greek χάος meaning "abyss" or "chasm," referring to a primordial state of existence.

1 Introduction

In many ancient civilizations, creation stories tell how God (or god, or gods) creates the world as we know it by forming "order out of chaos". Here, chaos is understood to be the original state of the universe, and was, naturally, disordered and inhospitable to life.[2] This understanding of an original χάος leads naturally to today's popular usage of the word, in which the typical meaning is one of disorder or confusion. For this reason, it is not too surprising that mathematical discoveries, primarily in the 20th century, that produced completely unexpected and oftentimes confusing results became known as chaos, and the underlying bases of it called Chaos Theory.

This paper deals with the popular understanding of Chaos Theory and with theologians' use of Chaos Theory. I focus on how sober second thought and an orthodox Christian worldview may allow us to temper our reaction to popular overstatements of the power and influence of Chaos Theory in the material world, whether dealing with secular or theological aspects of life. In order to do

[1] The definitions for chaos were taken from the Merriam-Webster Online Dictionary: `http://www.merriam-webster.com`, accessed March 29, 2012.

[2] As examples: the biblical creation story speaks of a "formless and void" earth (Gen. 1:2) after God creates the heavens and the earth in Gen. 1:1; in Babylonian myth, Marduk becomes the supreme god after defeating Tiamat, who, Davis notes, was "the personification of the forces of chaos" (83).

this, I will begin with a (very) brief introduction to Chaos Theory by considering two well-known chaotic systems that give a flavour of this field that suddenly developed with the advent of the (super-) computer.[3] This leads to a discussion of how chaos is popularly understood and how its "power" can be easily overstated. In particular, the near ubiquity of chaotic systems can lead to a view of creation that, spurred on by a post-modern outlook, sees the world as being inherently unpredictable. I also consider a couple of theological issues—human free will and "natural" divine action—that spring from the emergence of Chaos Theory and its spread beyond mathematics. I then sketch both practical as well as philosophical and theological problems with popularized chaos theory in society and in theology, problems which I believe are insurmountable. Finally, I will outline how an orthodox Christian perspective of who God is and how he interacts with his creation, both animate and inanimate, can allow us also to view chaos in more appropriate perspectives.

2 Chaos Theory

I will present a very short introduction to chaos through brief descriptions of two models. These models provide simple examples of some of the characteristics of chaotic systems, and allow for a consideration of a few philosophical issues that arise when considering chaos and predictability of deterministic systems. Very rich and engaging material that can supplement this brief introduction can be found in a variety of sources, a few of which I use in this section without further reference: James Gleick's *Chaos: Making a New Science*, Wesley Wildman and Robert John Russell's "Chaos: A Mathematical Introduction with Philosophical Reflections,"[4] and James Crutchfield *et al*'s "Chaos."

Deterministic systems, sometimes called dynamical systems, are systems for which a mathematical function can describe the future motion of an object, or objects, in space (be it real or phase space). For example, the (approximate) motion and position of a cannonball after it is shot can be given by a couple of short mathematical equations. Much of first-year physics deals with simple physical systems like this. Higher-level physics classes and physics research in general look at more complicated systems, usually dealing with more complicated motions and equations. Some systems are called "chaotic." These chaotic systems are a subset of the general class of dynamical, deterministic systems.

The first example of a chaotic system does not come from the field of physics but instead from biology: the Logistic Map. It is an iterative equation designed to model the population of a species that lives generation-by-generation (i.e., only one generation of that species is alive at one time). The model is quite simple: If x_n represents a scaled value (from 0 to 1) of the species' nth-generation population, then the next generation's population, x_{n+1}, is modelled as

$$x_{n+1} = rx_n(1 - x_n),$$

in which r, called the "biotic potential," accounts for how well the species can live in the current

[3]It is important to note that the late arrival of adequate computing power is the popular reason for Chaos Theory's delayed development, but it is not the only, or possibly not even the main, reason. Henri Poincaré, now dubbed the "Father of the mathematics of Chaos Theory" (Kellert, 1993, p. 121), introduced many mathematical methods still used to study chaotic systems in the 1890s while studying the (in-)stability of planets' orbits. Kellert has a valuable chapter (Chapter 5, Beyond the Clockwork Hegemony) exploring the many reasons why Chaos Theory took so long to develop.

[4]A significant amount of the material I present at the end of this section regarding philosophical issues are presented in the cited article by Wildman and Russell.

environment. It is a single-number accounting for variables like the size of an individual female's litter, the gestation period, number of predators, food sources and the like. The values of r range from 0 to 4, where 0 means that the species is not at all fitted to this environment (and so the species will die out in a single generation), and 4 means that the species is exceptionally suited to the environment. The scaled population, x, which is x_0 for the beginning generation, x_1 for the next generation, etc., is a fraction of the maximum possible population for the species.

As long as x_0 is positive, we can predict the population of future generations by continually solving this simple equation (mathematically, this is called iterating). Take, for example, $r = 0.5$ and $x_0 = 0.1$, or one-tenth its maximum value:

$$\begin{aligned} x_0 &= && 0.1 \\ x_1 &= rx_0(1-x_0) &=& 0.045 \\ x_2 &= rx_1(1-x_1) &=& 0.0012375 \\ x_3 &= rx_2(1-x_2) &=& 0.0006179842969 \end{aligned}$$

Things do not look good for this population. In fact, by the tenth generation the population has dropped to 0.008% of its original value, and this number keeps dropping as the generations go on. Further analysis shows that this population is doomed to extinction—regardless of the initial population—if r is less than 1.

If we consider a species with larger biotic potentials, there is hope for the population. For r values between 1 and 3, the long-term population approaches $x_n = (r-1)/r$ as $n \to \infty$. In other words, after a number of generations, the population will stabilize: the preceding generation's population will persist as the population for the next generation, with a predictable value. One way to understand this better is through visualization. It is possible to plot the long-term behaviour of a species' population as it depends on the species' r value. Without any advanced analysis of the equation, this can be done using a computer graphing program and the following algorithm:

1. For the r value you are interested in, start with an arbitrary x_0 between 0 and 1.

2. Solve/Iterate the Logistic Map equation many times, to get values for x_1 up to, say, x_{400}.

3. Starting with x_{400}, iterate one hundred times more, so that you also have $x_{401}, x_{402}, \ldots, x_{500}$ and plot all these points on the x vs. r graph.

   Note that, if the long-term behaviour of the population is constant, then all the points you just plotted will be right on top of one another, making them appear as a single point.

4. Repeat steps 1 through 3 with a new value of r.

Using this algorithm for many r values between 1 and 4 gives us the plot shown below in Figure 2.[5] What we see is what would be expected for r values less than 3: the long-term value of the populations—which is a constant value for any specific r value—settles down at a value of $(r-1)/r$.

However, when r is larger than 3, there is no constant long-term population! Between 3 and about 3.45, the population "bounces" between two discrete values; the solution curve bifurcates. The population one year is, say 0.81; the next year it is 0.42; the next it is 0.81 again, etc., etc.

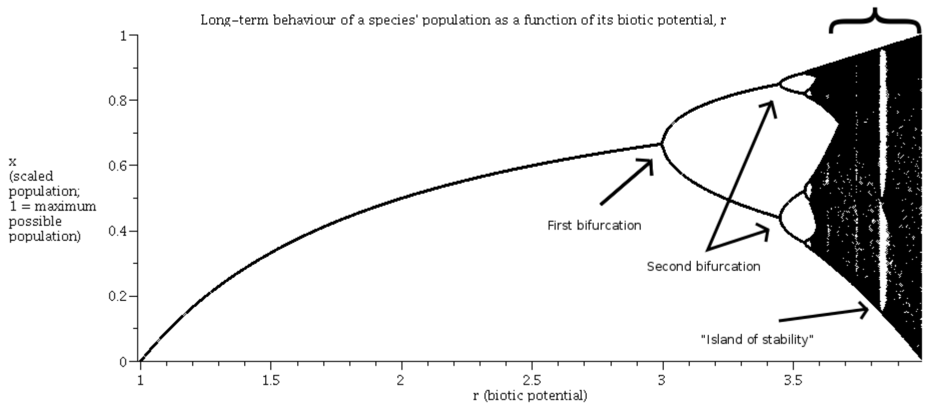

Figure 1: The Logistic Map, showing long-term solutions to the logistic equation for different values of r. The region below the brace is the chaotic region. The region in which all solutions lie is called the "solution manifold."

At slightly higher values of r, the curve bifurcates again, meaning the population moves between four values; then eight, 16, etc., until, for values of r higher than 3.55 or so, the population from one generation to the next change pretty much arbitrarily, so that we get the smear we see in the graph below the brace. The smears of solutions can't be just anything; they remain within a definite range. This area on the graph holding all the chaotic results is called the solution manifold.

What is important to keep in mind while looking at the chaotic results is not just that we have produced a smear of results, but that, if you are given the exact population of the n^{th} generation, and the exact r value of the species, you will be able to predict the exact population of the $(n+1)^{\text{th}}$ generation. The theoretical system is still deterministic.

At the same time though, if you are even a little bit unsure of the population or the r value, your prediction for population for future generations is not going to be exact and it will get less and less accurate as you go more generations into the future. This is a manifestation of the system's sensitive dependence on initial conditions.

What is occurring in this deterministic system is something that is perhaps easier to explain using an analogy to bread-making. Suppose you had a lump of dough and kneaded it using an electric bread-mixer. The bread-mixer's beaters have a known geometry and a known rotational speed. If the exact makeup of the bread was also known, it would be theoretically possible to predict exactly where every piece in the dough will end up after the kneading is finished. The system is deterministic. However, if you were to mark two tiny pieces of the dough that are initially extremely close together (with tiny amounts of food colouring, perhaps), you would find that as the kneading occurs, these two pieces would eventually get separated and would end up in essentially arbitrary places in the dough at the end of the mixing process. Theory notwithstanding, there is no *practical* way these ending points could have been predicted.

These two tiny pieces of dough correspond to two initially-close values of r in the Logistic Map, values that are so similar that perhaps even a computer can't tell them apart because of the

[5]Unattributed plots were made by the author, using the computer algebra software package Maple 14.

finite size of the computer's memory. In theory, we should be able to predict the evolution of the system; in practice, we cannot. The processes of stretching and folding of the dough represent the continual evolution of this system, and lead to subsequent values that are for all intents and purposes unpredictable. These sorts of systems and problems make up the field of mathematics now called Chaos Theory. There is much more that can be said about the Logistic Map, as it is well-studied and is representative of many other chaotic systems. For example, there is "order in the chaos," as is hinted at in the figure above: consider the "island of stability" for r values near 3.8, for which the smear is replaced by just a few possible values that the population will bounce about as generations go by. What is important for this paper, however, are the following properties found in the logistic map, and in other chaotic systems:

1. They are deterministic in theory, in that the model predicts new values based on existing values. Thus, these models are *not* random.
2. There is a sensitive dependence on initial conditions; slightly different starting values can result in completely different evolutions. The "trajectories of the system" are eventually very different. We see this to be true for most values of r greater than 3.45 or so in the Logistic Map.
3. The system is still constrained; although the trajectories may be different, they are not different in the sense that one or both "blow up to infinity." Instead, the trajectories remain within a certain limit, or manifold. For the Logistic Map, as an example, the x_n values never exceed 1.

Many mathematical systems can be chaotic; one of the few requirements for chaos is that the system be non-linear, as is the case for the Logistic Map. Another non-linear and chaotic system is called the Lorenz Attractor, named after Edward Lorenz, who proposed it as a simple model of convection in the upper atmosphere. He developed it after finding odd (at the time) results in his primitive weather model he was developing to help improve weather forecasting. The Lorenz Attractor is a three-variable system of partial differential equations (Lorenz, 1963, p. 135), much more complicated than the Logistic Map, but far less complicated than the actual equations that describe motions in the upper atmosphere (we will look at systems like that later in this paper). The value of the Lorenz Attractor, beyond its aesthetics, lies in its place in history at the start of serious study of chaos and predictability, and the beginning of Chaos Theory. Figure 2 shows the plot of two trajectories in the Lorenz Attractor. Without knowing what exactly is being plotted and what the x, y and z axes represent (they are the "phase space" of the system), it is clear that there is a manifold upon which the trajectories lie, and that there are two attractive basins about which the trajectories spend much of their time circling. In this case, the two trajectories both began the model simulation near the upper circle, but as time continued, their paths diverged. By the end of the model run, the red (dot-dashes) trajectory had been most recently circling the bottom attractor while the blue (dashes) had been circling the top.

One implication drawn by Lorenz was that predicting the weather wasn't going to be as simple as he and others had first thought: continued improvement in the amount and accuracy of the data going into a weather prediction model wasn't going to improve accuracy and extend its forecasting ability to arbitrarily long periods of time as he had hoped. No matter how accurate the data were, they were never going to be perfectly accurate. Sensitive dependence on initial conditions was going to lead to chaotic trajectories, with the forecaster unable to make any reasonable predictions after a certain amount of time. It is in this sense that we can understand chaotic systems as representing, in the

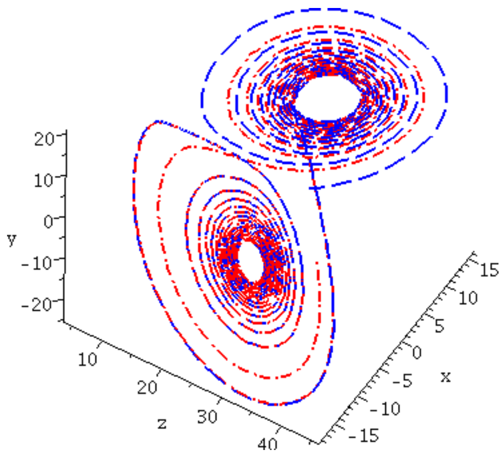

Figure 2: Two Lorenz Attractor trajectories plotted in red (dot-dashed) and blue (dashed), with two slightly-different initial starting points. Both are integrated for the same length of time.

words of Wildman and Russell, "a *tertium quid*" between the strict randomness of a mathematically-unpredictable system and the strict determinism of a simple, non-chaotic deterministic system (like an object undergoing simple harmonic motion, such as the pendulum of a grandfather clock). Chaotic systems are in-principle deterministic, but are in-practice (eventually) unpredictable.

After Lorenz discovered this chaotic unpredictability in his simple weather model, others found chaos lurking in the systems they were studying: the patterns of heartbeats, the oscillations of the stock market, the dripping of water fountains. Things typically thought of as being well-understood and modelled by simple linear equations were now seen to be more complicated and fraught with chaotic signals. Many times, what had originally been written off as noise in the data and subsequently ignored was now being interpreted as being a real signal in the data. The signal was chaotic, but there was value in studying it (and, thanks to the rapid growth in computing technology, systematic study was possible) and writing books about it. Fascinating new and fantastic truths about how the world actually works have been discovered thanks to Chaos Theory.

3 Popularized Chaos Theory

Books like Gleick's *Chaos: Making a New Science* are invaluable aides in getting the message out to the general public concerning what Chaos Theory is and the potential it has to help further scientific enquiry in a wide range of disciplines. Chaos has been popularized through books, websites and even t-shirt prints. People are now much more aware of the mathematically chaotic. However, the understanding is limited and not necessarily based on the mathematics, but on the popular (mis)-conceptions of what mathematical chaos is actually. Let's call this Popularized Chaos Theory. A mainstay of Popularized Chaos Theory is the common explanation (actually a major extrapolation) of what the Lorenz Attractor implies: the flutter of a butterfly's wing in Brazil can cause a tornado in Alabama two weeks later.Some claim that this is the source of the term, "the Butterfly effect."[6] It is worth noting that this was not always chaos. The same story line feature about small differences leading to big changes later on has shown up many time previously in Hollywood. Prime examples include the *Back to the Future* trilogy of the 1980s and '90s (at a time when Chaos Theory was barely known outside of mathematical academia) and the pre-Chaos-Theory classic, *It's a Wonderful Life*, from 1946.

There is an additional way Popularized Chaos Theory has affected mass society. The underlying message, apparent even in the subtitle of James Gleick's book, is that *chaos changes everything.* Chaos is everywhere once you know what to look for. "A new science" is emerging and if it weren't for scientists' general preference for easy models, we'd have realized long ago that things are much more complicated than they've let on. This new science is fascinating, chaotic, strange and hard to really understand. It portrays the world in ways other than how scientists have been telling us the world works.

This is all true: the deeper one dives into any field of scientific study, the more unanswered, and maybe unanswerable, questions one finds; the more we see how the approximating assumptions we've used to organize the phenomena into nice categories break down; the more grey there is.

It is worthwhile to note that this popularized idea of what Chaos Theory can spread even by means of the scientific community. A scientist whose area of research is not Chaos Theory may be just as likely as a regular member of the public to absorb Popularized Chaos Theory into their worldview and even apply that viewpoint when working in their own field. The intersection of public and scientific spheres has allowed Popularized Chaos Theory to spread naturally and slowly to take its place as part of the treasury of assumed scientific common knowledge.

Popularized Chaos Theory can actually lend itself to a distrust of science. There appears little hope for accurate predictions of phenomena if it requires a knowledge of the positions of all the world's butterflies! From there, it is a small step to conclude that our world and its processes are not predictable and those who try to tell otherwise are ideological optimists (or pessimists, depending on perspective) at best, or snake-oil salesmen at worst. This may be most true for some of my nearest research colleagues, weather forecasters and climate scientists.

4 Chaos Theory's Use in Theology

Chaos theory has also attracted the attention of theologians trying to explain or defend the existence of human free will from Pierre-Simon Laplace's fore-ordained, deterministic view, presented in the early 19$^{\text{th}}$ Century:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the beings which compose it, if moreover this intelligence were vast enough to submit these data to analysis, it would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom; to it nothing would be uncertain, and the future as the past would be present to its eyes.[7]

[6] Another explanation for the origin of the name is that a long-time view of the trajectory of an object in a Lorenz Attractor results in a figure which, when viewed from the proper angle, looks somewhat like butterfly's wings.

[7] Translated from "Introduction to Oeuvres," vol. VII, *Theorie Analytique de Probabilites* (1812-1820).

Efforts to refute this view got a major boost after the formulation of quantum mechanics in the early 1900s. Quantum mechanics is very different from classical mechanics and "sees" the world in a vastly different way. By considering all elementary particles as being represented by probability waves which have no discernible properties until they are measured, it introduces an element of unpredictability into physical systems at the smallest scales. A particle's position and momentum are represented by probability functions rather than by actual values, at least until that quantity is measured (and immediately after the measurement, the quantity becomes progressively more unknown once again).

Because of this, there is a non-zero probability that normally-impossible ("classically forbidden") phenomena can actually occur. For example, *tunnelling* occurs when a particle (usually an electron) escapes from a region that it doesn't have the energy to get out of normally. A macroscopic equivalent to this would be having people pass through a fence that they have no ability to climb over: one moment they are on one side of the fence and some time later they've magically transported to the other. This tunnelling phenomenon is not just some interesting sideshow, but has applications in real life. It is used in scanning tunnelling microscopes, which are able to map out the fine contours of materials at the molecular level by making use of the tunnelling properties of electrons. Events like these occur at the quantum level and may not be likely, but they can and do occur. We can only calculate the probabilities that certain events occur and we cannot say with any certainty that it will occur at any particular time.

Because predictions made at the quantum mechanical level are only probabilities, there is a causal openness at this level that Laplace was unaware of. This openness was then exploited by theologians who proposed that, since there is openness at the quantum level, strict causal determinism was not entirely valid and human free will could be explained easily in light of this (Wildman and Russel 84-86). Furthermore, this openness was also proposed to give God an opportunity to act silently within his creation using natural means,[8] that is, without violating any laws of nature, simply by encouraging certain allowable (but possibly unlikely) events at the quantum level to occur.[9] In doing so, miracles, which might be seen as occurrences of highly-unlikely events (spontaneous healing of terminal cancer, droves of quail settling down in a desert day after day after day, etc.[10]), could be explained by natural means. If entirely successful, all God's acts could be seen as acts through and within nature, with quantum-mechanical nudging being the "causal joint" through which God's actions and will became events in history.[11] This view, called Quantum Determination, can describe a universe that is still divinely deterministic—God still knows the future, because he is actively participating in the world through nature—but causally open as far has humanity is concerned.

Quantum Determination becomes, in the words of the Anglican theologian and physicist, John Polkinghorne (1995, p. 152–154), the "ontological gap" in a reductionistic, "bottom-up" description of the physical world. This leaves room for what he describes as a "top-down" description, similar to the concept of emergence used to try to understand the actions of complex systems that do not

[8]From this point in this article, "natural" and associated words will be used for occurrences that can be explained using generally-accepted descriptive laws of nature (e.g. Law of Gravity, Maxwell's Equations, Newton's Second Law, etc.).

[9]This silent activity is understood as acting or influencing events without adding energy to the universe, which would violate the First Law of Thermodynamics (which states that the total amount of energy in the universe remains constant). Therefore, God, to act silently in nature, would only perform works that make no measurable energy differences.

[10]Not all theologians, and certainly not all Christians, would be willing to limit their definition of miracles to only these sorts of events, I hasten to point out.

[11]See, for example, the description of Quantum Determination given in Koperski (2000).

seem to be acting simply as the sum of their constituent parts. For example, our human brains may be more than just a collection of neurons, made up of atom-containing molecules, subject to deterministic natural laws. Since quantum mechanics shows that deterministic natural laws do not result in exact predictions, there is room for an understanding of complex systems, like brains or people, to have effects not based simply on their atomistic constituents, which is the reductionistic argument. Human agency, or free will, could be an example of this, as could be, at a larger scale, God's influence on his creation.

Although this seems to be an appealing conclusion for those with deep philosophical issues with a causally-closed universe, sober scientific second thought has given many theologians pause. The main thrust of the argument against Quantum Determination is that it "does not provide sufficient freedom for God to significantly influence macroscopic events" (Koperski, 2000, p. 546). Although the microscopic, quantum-mechanical reality may be unpredictable, there is good reason to believe that a finite number of little nudges at the quantum-mechanical level cannot do what the Quantum Determination supporters were hoping it would. This amplification-effect problem makes purported divine agency at the quantum-mechanical level irrelevant at larger scales. Additionally, the quantum-mechanical effects are only felt when measurements occur. These measurements are discrete events, and so limiting God's actions to those at the quantum level leads to a God who only intervenes in creation discretely. "Such an episodic account of providential agency does not seem altogether satisfactory theologically" (Polkinghorne, 1995, p. 153), since the many scripture passages that deal with God's care and upholding of creation (such as John 5:17, Colossians 1:17 and Hebrews 1:3) imply a continual action rather than an episodic set of interventions.[12]

With the emergence of chaos theory, and its resulting macroscopic unpredictability, Quantum Determination was augmented to become Chaotic Quantum Determination. It posits that unpredictable quantum mechanical effects at the microscopic levels can result in small changes to macroscopic chaotic systems that then can lead to macroscopic unpredictability. In essence, the sensitive dependence on initial conditions in chaotic systems becomes a possible solution to the amplification effect problem. Since there is no way to measure the infinitesimal differences in a system, tiny nudges that make no change in energy to a system could lead to different trajectories within the system that are completely unpredictable over time. Tiny nudges could be due to divine action or could even represent (freely-made) human choices. Either way, these nudges would be explainable as entirely natural processes. In this sense, chaos and quantum mechanics together present a more scientifically-pleasing possibility for the ontological gap needed for a top-down description of the physical world.

Polkinghorne (1995, p. 154–156) suggests that we should heuristically consider reality using both bottom-up and top-down descriptions in a complementary fashion, in much the same way that quantum mechanics uses the term complementarity to describe how microscopic particles can be described as both/either a wave and/or a particle. Additionally, he notes that this complementarity should also extend to God himself: God knows things as they really are and this surely implies that God knows the temporal in its temporality. Because of this, Polkinghorne argues that God is also the God of a "world of becoming" and therefore must be a God who possess not only an "eternal

[12] In fact, the original Greek verb for "hold together" (συνέστηκεν) in the Colossians passage is in the perfect indicative, which is the tense to use when talking about a action in the past that continues to be true. In the Hebrews passage, "sustains" (φερών) is a present participle, which implies continuing action in the Greek.
(Augustine of Hippo, 1982, p. 171; Book V, Ch. 20, ¶40) comments: "Let us, therefore, believe and, if possible, also understand, that God is working even now, so that if His action should be withdrawn from His creatures, they would perish."

pole" but also a "temporal pole" as well. "Because the future of [this world of becoming] is not yet formed, even God cannot know it."

This bold statement regarding God's predictive ignorance is challenged by Denis Edwards (1995, p. 163–164), based largely on a consideration of how God's Trinitarian characteristics deepen and challenge our understanding of who God is and how he acts. Although united as One, a Trinitarian understanding allows us to clarify different ways in which God continuously creates or recreates his creation. For example, efficient causality, or "how" things occur, Edwards posits, is an example of God the Father's creative power. God the Son is reflected as the exemplar, the Wisdom, by which things are created; this is God's exemplary causality. God's final causality is found in the Holy Spirit, which is reflected as the Goodness which allows the created thing to do what it is created to do.

Against this Trinitarian backdrop, Edwards suggests that Polkinghorne is correct that the universe needs both bottom-up and top-down descriptions to be fully understandable. Both randomness, self-organization and chaos as well as deterministic natural laws are integral tools in God's strategy for creating and maintaining the world; the sets are "two components of the same Logos immanent in the structure of the universe" (Heller, 1995, p. 121), understood to be divine actions that are free acts of love, overflowing from the Trinity's mutual love, which allows freedom to its creations, since they are reflections of the Trinity's personhoods themselves (Edwards, 1995, p. 167–169).

However, even if the Trinitarian God allows for, respects, and is responsive to human free will and the contingency of natural processes, it does not logically follow that the Trinity itself cannot know the future. In fact, Trinitarian theology stresses the mystery and incomprehensibility of God; we should not presume to limit God's eternity by over-stressing his temporality. Scripture speaks of both together: God the Son did not know the time of the final judgment, but God the Father does (Matt. 24:26). Thus I would join with Edwards (1995, p. 170–171), in saying that "God's eternity embraces all of time and so we can assume that future contingent events are present to God" (Edwards 170-171).

5 Chaos and Meteorology

My interest in Chaos Theory and unpredictability arose from my time in graduate studies in the Department of Atmospheric and Oceanic Studies (formerly the Department of Meteorology) at McGill University. I have studied the growth of "unpredictable energy" in simple fluid systems, the same field in which Edward Lorenz was working when he introduced his chaotic attractor. Inasmuch as the system Lorenz presented was simple, a model for the actual atmosphere is more complicated, by orders of magnitude. Not surprisingly, it has been numerically found that, if the simple system was chaotic, the more complex system was even more so.

Atmospheric turbulence research, and weather prediction, is predicated on appropriateness of modelling the flow of fluid (in this case, mostly air) with the Navier-Stokes (N-S) equations. These are a set of non-linear[13] partial differential equations based on Newton's Second Law, that an object with mass m will change its velocity by an acceleration $\vec{\boldsymbol{a}}$, when a force $\vec{\boldsymbol{F}}$ acts on it, following the equation

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}.$$

[13] Its nonlinearity is due to the "advection term," $(\vec{\boldsymbol{u}} \cdot \nabla)\vec{\boldsymbol{u}}$, which contains the velocity field, $\vec{\boldsymbol{u}}$, more than once.

The N-S equations are developed for a continuous field of objects (volumes of air), rather than for an individual object.

It is worthwhile noting that this set of equations has proven to be more than a match for those interested in it purely for mathematical reasons. In order to encourage a better mathematical understanding of the N-S equations, the Clay Mathematical Institute included a problem related to the existence and smoothness of solutions to the N-S equations as one of seven unsolved problems for which there is a $1 million prize. In fact, the problem is seen as hard enough that the problem does not need to be solved in order for the money to be given out; the winner needs *only* to shed more light on the particular problem![14]

In atmospheric fluid dynamics, the N-S equations are also approximated and added to, in order to make problems more easily adaptable for computation (making simplifying assumptions), to include other effects and forces that can act on the air in the atmosphere (such as the Coriolis force that explains the effect of the earth's rotation), and also to prevent spurious solutions.[15] Because of this, the actual N-S equations are not normally used in computer models. However, the equations that are used are still non-linear and still exhibit sensitive dependence on initial conditions. Research has tended to indicate that, like the simpler Lorenz attractor system, an atmospheric model based on the N-S equations is also chaotic and there is a limit on how long we can accurately predict future weather.

After Lorenz' pioneering work in the 1960s, another major advance in the study of atmospheric predictability came in 1972 when C. E. Leith and R. H. Kraichnan attempted to measure how predictable the atmospheric wind might be. They assumed the entire atmosphere acted like a perfect three-dimensional, isotropic[16] turbulent flow, described by only the N-S equations and very simple initial and boun-dary conditions. Using this model, they measured the length of time for "error energy" from the smallest lengthscales of their model to infect the largest lengthscales. The error energy is a measure of the energy that is due to air flow at lengthscales that we know we did not measure accurately at the start because observations are only made at specific points. As this air moves around, it interacts with itself at these small scales, but, after a time, these interactions end up changing how the air moves at larger lengthscales as well. Eventually, even the largest and most energetic lengthscales are affected by this initially-small-scale error energy, which means that even the largest atmospheric formations, the jet streams, will no longer be modelled correctly. They found that this occurs in about two weeks (Leith and Kraichnan, 1972, p. 1056). Lorenz (1969, p. 305) himself came up with a value of 16.4–17.6 days in further investigations. These times have long been held to be accurate as an upper limit on the abilities of weather forecasting (the weeks part, at least; some estimates have since gone as low as one week), and no one in the industry currently produces a forecast longer than 15 days (and most people do not—or should not!—trust forecasts beyond day five).

Strangely enough, this upper limit is independent of the accuracy of our data readings and the numerical accuracy of our model. In fact, it is even independent of the extensiveness of our measurement network (that is, our weather stations that take temperature, wind-speed and rainfall

[14]See `http://www.claymath.org/millennium/Navier-Stokes_Equations/` for a description of the actual problem.

[15]For example, many sets of equations explicitly prevent the fluid (air) from travelling faster than the speed of sound, which would lead to shocks (such as those that cause sonic booms from fighter jets travelling faster than Mach 1) that are numerically difficult to model. If you are investigating situations in which this sort of motion is unlikely, it is numerically safer to completely remove the possibility of it happening.

[16]*Isotropic* describes a phenomenon that looks the same in whatever direction you look; there is no preferred direction of flow in isotropic turbulence.

rate data). As long as we have a discrete observing network (and this is an obvious physical necessity), there will be error in the lengthscales smaller than the network spacing and these errors will propagate to larger scales. The smaller the initial error lengthscale, the faster the initial rate of error propagation to larger scales! The amount of improvement in long-term predictability times due to better modelling and initial conditions (leading to better "day-one" forecasts) is offset by increased error-growth overall, leading to only marginally better long-term forecasts (Straus and Shukla, 2005, p. 10).

Measuring the precise amount of unpredictability in atmospheric flow has been the goal of many researchers since the 1970s, with different studies done that consider different types of atmospheric flows: differing length- and timescales, various weather regimes (calm air vs. winter storms vs. summer thunderstorms, etc.), and the like. My own contribution to the unpredictability-measurement knowledge base was a study that considered the effect of condensation and evaporation, and the resulting latent heat release or capture, on the predictability in convective plumes.[17]

In 2003, a Workshop was held in Savannah, GA, entitled "The Known, the Unknown and the Unknowable in the Predictability of Weather."[18] The resulting report (Straus and Shukla, 2005) focused on the issues of how good we can expect weather forecasting to become, based on the "knowability" of different factors involved in the process. The most important result discussed at this workshop was actually one first noted by Lorenz in 1969, that the intrinsic predictability time of an isotropic fluid-dynamical system is dependent on the way in which the kinetic energy is arranged in the system. Given what is now known about the dynamics of the system, there is actually a chance that weather could be more predictable than originally thought: better models combined with more numerous and accurate observations might result in weather forecasts that remain valid even beyond two weeks.[19] This has yet to be borne out in practice, but there is at least a hope that it could be the case.

6 Practical Critiques of Popularized Chaos Theory

The above consideration of best-case forecast times, and the realization that long-range weather forecasting isn't necessarily as doomed to failure as is normally thought, is an example of a general

[17]In a nutshell, we found that more condensation = less predictability, but by a predictable amount. See Spyksma and Bartello (2008).

[18]It is unknown to this author whether the name for the workshop was thought up before or after Donald Rumsfeld's memorable February 2002 speech.

[19]A large key to the problem lies in the slope of the system's kinetic energy (KE) spectrum. If the slope is less than k^{-3} (k is inversely proportional to lengthscale), there is a finite maximum predictability time, while KE spectra steeper than k^{-3} can result in an arbitrarily large range of predictability.

In general, three-dimensional isotropic turbulence has a KE spectrum with a slope of approximately $k^{-5/3}$, and so there ought to be a finite predictability time, if the assumption that the atmosphere can be modelled as three-dimensional isotropic turbulence is correct and the atmosphere is a chaotic system. Our own experience however, suggests the assumption is not correct: the wind is generally a more horizontal than vertical motion of air. In fact, at large scales, the atmosphere can be thought of as layers of two-dimensional isotropic turbulence, whose KE spectrum's slope is k^{-3}. This result puts the hope of longer-term predictability "on a knife's edge" (Straus 3), as it were, since that slope is the boundary between the two regimes. Note how this is quite unlike what earlier studies were implying; with some sober second thought, there may be far more hope for forecast improvements than has been assumed.

At smaller scales (to lengths of 200 km or less), although the flow is still not three-dimensionally isotropic, it generally has the same $k^{-5/3}$ KE spectral slope. The cause of this slope is one of the unknowns the Sloan Conference listed. On a positive note, until the reasons for this spectrum are known, there is hope that, even at these scales, there could be room for major improvements in forecasts, given better models, more accurate observations and a denser observation grid. At any rate, the outlook isn't necessarily as bleak as it is generally made out to be.

practical critique of worldviews that incorporate a Popular Chaos Theory outlook. It is true that chaos is all around us, and that scientists have, as a general rule, tried to ignore or explain away chaotic data in order to find nice regularities and structures in nature. It is also true that many complex physical systems worth studying in the world appear to be non-linear and extremely likely to exhibit chaos. However, these two considerations do not imply that the world is chaotic, in the sense in which the word chaotic is normally understood. Popularized Chaos Theory overstates the case for chaos and a distrust of old science.

First of all, in many cases, the chaotic system is only chaotic in certain cases. For example, the Logistic Map (Figure 2) is chaotic only when the biotic potential value, r, is between 3.57 and 4; there is no chaos for lower values of r.

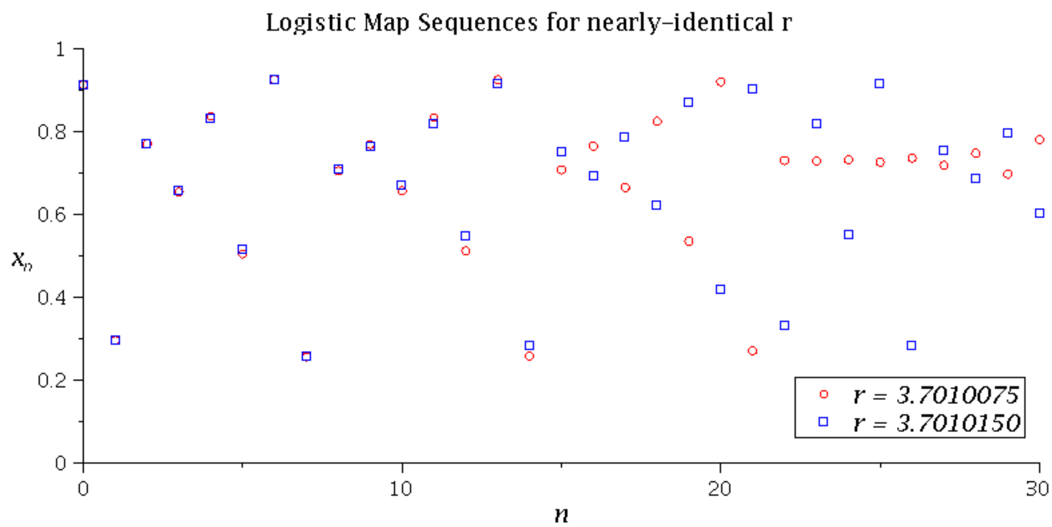

Figure 3: Two 30-generation populations series, one in red circles and the other in blue squares, using slightly-different r values in the Logistic Map. n represents the number of generations after the starting population values, x_0, for each series.

Secondly, there is the previously-mentioned "eventual unpredictability" issue: even when the system exhibits chaos, the unpredictability of the system is initially very small and takes time to grow. For example, consider the plot (Figure 3) of subsequent iterations of the logistic map for two nearly-identical r values, beginning with nearly the same x_0 value. For the first 10 or so iterations, the differences between the two sequences remains relatively small (note how the squares and circles very nearly overlap). This means that, in many chaotic systems, the chaotic signature (this eventual divergence of initially-similar values) may not develop until well after close study of the system is over, or after the system has been disrupted for some other reason.[20]

[20] Consider, for example, being able to find almost exactly the value of r for a certain species. However, it would be very unlikely that, as the generations go by and your logistic-map simulation and the actual species populations are being tracked, the environment remains so constant that your original estimated value for r remains valid. Does the divergence of your model from the actual populations for later generations reflect chaotic divergence or simply a no-longer-correct r value for the model?

Thirdly, many chaotic systems are chaotic only at a low background level. For example, the chaos in heartbeat periodicity is one which can only really be seen once the mean heartbeat frequency has been removed from the analysis. Only then is the chaos noticeable.

All three concerns expressed here are linked in a specific way. The system being studied may be non-linear, and may even be chaotic, but the importance of the chaos may be so low that there is a very good justification in approximating the non-linear system with a linear one that can be studied more simply and more in-depth. The only caveat is that we must not then forget that the system is not the linear system we've approximated it as, but it is, in fact, richer and more complicated. As a physics professor, this is a serious and deep point that I try to make to my students, even at the first-year level.

7 Philosophical Critiques of Popularized Chaos Theory

Using Popularized Chaos Theory as a framework through which to see the world is problematic also because it holds to a belief that nature and models of it are qualitatively similar. But there is an open ontological question regarding chaos: does chaos actually exist in nature as the models describe, or is chaos simply a mathematical by-product of the model itself? Mathematical models can be chaotic. But the real world is not a mathematical model; the model is an attempt to approximate and describe the real world. In so doing, many times, more complicated effects are removed from the governing equations, which both describe and explain the system's actions (Wildman and Russell, 1995, p. 78), and replaced with statistical approximations.[21] These approximations result in equations which only describe approximately and statistically what is occurring and no longer give the same sort of physical explanation (consider the use of r in the Logistic Map to summarize everything about a species and its relationship to its environment). Thus, there is no logical reason to believe that, simply because these models are chaotic, the real-world phenomena they are approximately describing are chaotic as well.

Keep in mind that these objections do not mean that the models themselves are untrustworthy or unhelpful. In fact, physics is the art of using incorrect models wisely, remembering what assumptions were made when considering if the results arrived at are or are not useful or trustworthy (I humbly submit that, in general, physicists do this fairly well).

This naturally leads to the objection that many of these incorrect models do a good job of describing the phenomena they are supposed to describe. This is true; if it weren't, weather forecasts would be even more unreliable than they are currently. But is doing a good job the same as being exactly correct? No. To fully validate the results, we would need to compare the model to actual data that ought to be in a chaotic regime, and this is a non-trivial task. "Testing such models in detail is impossible unless the system exhibits equilibrated or simple periodic behaviour" (Wildman and Russell, 1995, p. 79). In other words, in-depth testing of the model against nature can only occur when chaos is absent, leaving the ontological question unanswered (and potentially unanswerable).

[21] In the case of the Navier-Stokes equations, while they are based on Newton's Second Law, which both describes and explains motion of a fluid in terms of forces, they are approximated using the *continuity hypothesis,* which assumes the individual particles act together as a fluid, relieving us of the need to apply Newton's Second Law on each and every molecule in the fluid. As well, the viscosity term is a statistical description of what role molecular friction plays in the motion of the fluid.

8 Practical Critiques of Chaos Theory's Use in Theology

In addition to a critique of Popularized Chaos Theory's worldview and assumptions regarding the trustworthiness of science and utility of using simplified systems to investigate physical phenomena, we can also reconsider the proposals made regarding the intersections of Chaos Theory with human free will and divine action. With regard to the free-will issue, it is important to note that both the original problem (a deterministic universe leaves no room for real individual free will) and the proposed solution using Chaos Theory (unmeasurable differences in initial conditions means that the determinism exists but the predictability is lost) both arise from *philosophical reductionism,* that is, reductionism as part of one's worldview. Philosophical reductionism states that every large effect is the culmination of innumerable small effects: *all* macroscopic activities can be explained by the movements of molecules, atoms, quarks, and the like, as well as by the fundamental forces driving these movements.

Methodological reductionism, that is, studying a system by breaking it up into its constituent parts, has long been used by scientists, as it works very well to explain much of what is being and has been studied in the material world, and encourages systematization. However, its success has led many scientists, and others, to subscribe to philosophical reductionism, rejecting possible explanations for phenomena that cannot be derived from "within," as it were.

However, there are problems in science that are proving difficult to explain from an exclusively-reductionistic viewpoint. For example, Polkinghorne (1995) points out that the physical sciences have had, and continue to have, difficulty in explaining mental processes. Consider as well the human feelings of love, loyalty, or anger. Reducing thoughts and emotions to atoms, their actions and their configurations, is fraught with difficulty.

More generally, methodological reductionism works "admirably when confined to first-order scientific investigations", but it is dangerous "when extended to second-order methodological considerations, or to normative questions of being, truth, and value" (Aiken, 2012, p. 234). In other words, reductionism works well (as a method) in the confines of an idealized scientific method inquiry (observe, hypothesize, measure, test, confirm, etc., in a well-ordered manner), but much less so when aspects of reality that science has no ability to measure or integrate are included. Studies of mental processes, emotions, and the like, may take on some of the physical aspects of reality that make methodological reductionism difficult to use successfully.

Some of this is due to the non-locality that is implicit in quantum mechanics, which means that individual atoms and molecules are never completely acting on their own (and chaos' sensitive dependence on initial conditions only worsens things). This is not to say that we will never be able to understand mental processes from a purely reductionistic frame, but that we should not be surprised or disappointed if we cannot (Polkinghorne, 1995, p. 153). Therefore, philosophical reductionism might not be appropriate as a part of one's worldview.

With the understanding that reductionism may not be able to explain all physical occurrences, we can be open to other options for solving the "free-will problem." Polkinghorne (1995, p. 153–155) argues for a "top-down" understanding of the universe to complement the "bottom-up" understanding gleaned from methodological reductionism. The existence of top-down explanations for phenomena, like mental processes, which cannot (currently, and potentially forever) be understood in an exclusively-reductionistic manner, opens the door for human free will to be something that is not necessarily deterministic and predictable.[22]

[22] Polkinghorne is quick to warn against arguing for a top-down understanding in most instances; scientifically, it is

An extra-reductionistic viewpoint is not an exclusively Christian option. Many of the first people investigating mathematical chaos and the chaotic patterns found throughout physical phenomena saw themselves seeing patterns that seemed to have little in common with the underlying physics driving the system (e.g. Houghton, 1989, p. 81). Gleick (1988, p. 5) describes these chaos pioneers:

> They had an eye for pattern, especially pattern that appeared on different scales at the same time. They had a taste for randomness and complexity, for jagged edges and sudden leaps. Believers in chaos—and they sometimes called themselves believers, or converts, or evangelists—speculate about determinism and free will, about evolution, about the nature of conscious intelligence. They feel they are turning their back on a trend in science toward reductionism, the analysis of a system in terms of their constituent parts: quarks, chromosomes, or neurons. They believe that they are looking for the whole.

The whole may not always be the sum of its parts. The Christian meteorologist John Houghton (1989, p. 50) remarks that "there is no sense therefore in which it can be said that physics is complete where there is a detailed understanding of fundamental forces and particles. The interaction of components on one scale can lead to complex behaviour on a larger scale that in general cannot be deduced from knowledge of the individual components." Because of this, the proposal that Chaos Theory is needed to make human free will possible is not needed once philosophical reductionism is rejected.

We can also focus on the Chaotic Quantum Determination view of divine action, namely that divine action may be explainable naturally if we see God nudging chaotic events in such a way that there is no energy change from the nudges. A first objection to this was already mentioned earlier. Polkinghorne notes that these nudges would be at the quantum-mechanical level and would have to be discrete events, rather than a continuous creation/re-creation interaction that many Christians understand God's action in his cosmos to be. In additional to this, there are other practical issues that make Chaotic Quantum Determination an unappealing option.

First of all, "quantum mechanics and chaos theory cannot bear the heavy load that advocates of divine action place on them" (Peterson, 2000, p. 882). Consider the following logical progression that makes the argument for ubiquitous chaos (Koperski, 2000, p. 549):

1. Linear differential equations never lead to chaos, but non-linear ones can.
2. There are infinitely more non-linear differential equations than linear ones.
3. If nature is governed by differential equations, one would expect most systems to be non-linear and therefore chaotic.
4. Therefore, nearly all real physical systems can be chaotic.

The objections to this line of reasoning are plentiful. First of all, differential equations don't *govern* nature; instead, we *describe* (and sometimes explain) natural phenomena using differential equations.

Second, if nearly all systems in nature are supposed to be non-linear, why is it that a large number of important physical systems are described by linear differential equations (Koperski, 2000, p. 549–550)? Examples of these linear differential equations—that ought to be exceedingly rare—include

preferable to find explanations that fit into a bottom-up understanding of nature, if there is one. But there may be phenomena that may continue to defy a satisfactory bottom-up explanation. This may be due to "gaps" or "a degree of under-determinism in the account of the bottom-up description alone," or because the range of the causation is too large to be explainable due to localized events and reductionistic arguments.

some fairly widely-used equations: Newton's Second Law, Schrödinger's equation, and Maxwell's equations for electrical and magnetic waves.

There is a third objection, an inherent phenomenological weakness, in the argument as well. Chaos may be ubiquitous, but it actually isn't all that important in many systems (Ruelle, 1994, p. 26–28). Only if the system is mostly chaotic does the chaos really matter. In the vast majority of cases, the long-held practice of looking at the general trend in the data and ignoring the chaotic static is quite justifiable.

A fourth, practical, argument Christians may have against Chaotic Quantum Determination is one I have alluded to earlier. In order for all miracles to be explained in a natural fashion, we must reject the validity of most miracles found in the Bible. These are the sorts of miracles that cannot fit into a Chaotic Quantum Determination explanation, the type we must be willing to say never took place at all, are misunderstandings of what actually happened, or are "pious legends" (Bonting, 2002, p. 55). Recall the solution manifold on which chaotic trajectories exist: a system may be chaotic, but it does not therefore imply that "anything goes" for this system. The solution manifold for the time evolution of a dead body does not include points in the future where the body is once again alive. No amount of nudging at the quantum-mechanical level, divine or not, can make dead people rise from the grave nor virgins get pregnant. Chaotic Quantum Determination is "too constraining for the God who fed the five thousand" (Peterson, 2000, p. 882). Therefore, unless we are willing to accommodate to the hypo-miraculism required by Chaotic Quantum Determination, Christians must reject it.

Koperski (2000, p. 552) summarizes his critique of Chaotic Quantum Determination this way:

> [T]he idea that nature is overwhelmingly chaotic is easily detached from the mathematically grounded science that first introduced it. Like 'artificial intelligence' and 'virtual reality', 'chaos' is a highly suggestive rubric. It is common to think of it as complete turmoil, disorder, and unpredictability. The truth is somewhat disappointing. Chaos comes in degrees and often is found in the midst of stable structures. But this kind of circumscribed chaos works against CQD [Chaotic Quantum Determination]. If CQD is correct, then in whatever manner sensitive dependence on initial conditions is restricted in nature, God's influence would be restricted to the same degree.[23] Most theists will take this consequent to be unacceptable. Simple modus tollens tells us that something must be wrong with CQD.

9 Theological Critiques

John Houghton (1989, p. 50) suggests that Christians ought to be "looking for a double consistency," both scientific and providential, when it comes to understanding God's infinite and intimate interaction with his creation. When looking at the practical problems that crop up with Popularized Chaos Theory and Chaos Theory as an opening for human free will and divine action, we have already seen that these views fail the scientific-consistency test. It is important to recognize that they also fail the providential test, as they do not fit within an orthodox Christian worldview.

First of all, there is what John Jefferson Davis (1997, p. 70) calls a "serious category mistake" at play in these worldviews, limiting physical realities to only the physical substances and the physical

[23] [My footnote; not in the original quotation.] Note that God may willingly *choose* to have his influence restricted in this manner as a general rule by covenantal choice. QCD however, implies that God's influence *must* be restricted.

laws of nature. Dooyeweerdians would describe it as an unwarranted reduction of the aspects of creation to only those modes dealing with scientific truth (the quantitative, spatial, kinematic and physical modalities), while denying the existence or relevance of the other modalities, such as the pistic, ethical, social, and even the biotic.[24] Humans, made in the image of God, are more than simply physical beings. "The biblical doctrine of the *imago Dei* places a fundamental barrier (from a Christian viewpoint) against all attempts to explain the human person completely or exclusively in terms of scientific law" (Davis, 1997, p. 79).

An additional category mistake is made when freedom or free will is equated with randomness or unpredictability. Human free will is not necessarily so random or unpredictable, but instead is useful in order for people to work out that which they propose to do. In this context, actions occurring due to free will are neither random nor unpredictable (Davis, 1997, p. 79).

Finally, a major error is made when we assume God must act solely through nature. God is not bound to natural laws; God created and perpetually sustains the creation and the laws that guide it (Houghton (1989, p. 49); Gen. 1; Job 38-40; Ps. 104; etc.). It is part of God's providence that the universe acts the way it does; God is not breaking laws if he chooses to uphold creation in a different way than normal. The ultimate law is God's, not nature's. Abraham Kuyper (1953, p. 70) puts it this way:

> [A]ll created life necessarily bears in itself a law for its existence, instituted by God Himself. There is no life outside us in Nature, without such divine ordinances, ordinances which are called the laws of Nature—a term which we are willing to accept, provided we understand thereby, not laws originating *from* Nature, but laws imposed *upon* Nature. (emphasis in the original)

10 Chaos Theory in Orthodox Christian Perspective

In light of these considerations, I would like to propose a few ways in which our current understanding of Chaos Theory can aid in our understanding of God, his creation, and our place in it. These suggestions are not uniquely my own and they are proposed in the context of the warning given by Peterson (2000, p. 884) and many others, including Augustine of Hippo, that it is never wise to allow the current scientific understanding of an issue to be the ultimate authority. With that warning noted, there are some things that chaos theory can help us understand and appreciate in a better way, as scientists, as Christians, and as joint members of God's creation.

First of all, we need to improve our understanding of the place of chaos in creation. Chaos does appear to be all around us, manifesting itself in innumerable physical phenomena, easy to see if we have the eyes to see it. This does not mean that creation is chaotic in the popular sense, however. Coining the term "chaos" was a successful marketing ploy, allowing researchers to hype their findings and impress the public (Kellert, 1993, p. ix). Much confusion may have been avoided had chaos theory been named something else. A chaotic nature does not imply that the world is an unpredictable jumble about which we are hopelessly being tossed; instead, we must keep in mind the mathematical concept of the solution manifold. As well, we are reminded that "chaos comes in degrees and often is found in the midst of stable structures" (Edwards, 1995, p. 552).

One positive change chaos theory has brought to scientific studies has been the growing realization that there may be more to nature than can be understood using a reductionistic framework that

[24] An helpful introduction to Dooyeweerdian modal aspects can be found in Basden (2008), among others.

has been used with great success in most scientific studies. There is a place for more than physics and chemistry when studying complex (and even not-so-complex) systems, including life and mental processes. This is not to say that methodological reductionism must be rejected—it is a valuable tool in many scientific disciplines—but we must consider the whole as well as the contributing parts to gain a better understanding of our natural world.

It is important to remember that chaos theory is, first and foremost, mathematical. Adopting a view of mathematics as a language made and used by people, we need to keep in mind that these mathematical results are just that: mathematical. "The equations are maps, not reality. It could be that the apparent determinism is an artifact of the particular way in which we have chosen to map reality—in terms of mathematical physics" (Southgate et al., 1999, p. 135). This point is easy to forget, especially when working in the physical sciences, but it is vital to retain perspective.

Our view of ourselves in the world and in relation to God can also be addressed in the light of chaos theory. Granting the existence of chaos in nature (or at least humanity's perception of chaos in nature), we can make a few points. Wildman and Russell (1995, p. 83–85), who, after concluding that chaos is not necessarily the key to granting humans their free will, note that chaos can also limit the strength of the argument for a strict deterministic view of the world. The world might not be ontologically open,[25] but chaos makes a strong case for epistemic openness[26] at the human level, since chaos leads to eventual unpredictability, even of strictly deterministic systems.

This epistemic openness is at the human level, and I, with Edwards, would not claim it to be true at the divine level as well. Proposing that God, in his temporality, cannot know the future is contrary both to Scripture (Matt. 24:36, Jer. 29:11, etc.) and historical Church understanding, especially her acceptance of the mystery of the interrelationships of the Triune God. Instead, I see this lack of clear knowledge of the future as being one major distinction between God and all his creation. The limits of human predictive ability that Chaos Theory presents allows us once again to exchange epistemic hubris—a product of a Laplacian-based deterministic view—with a more appropriate epistemic humility (Davis 1997, p. 81; Job 40:3-5, 42:1-6).

In fact, recent advances in mathematics and physics have done much to emphasize this understanding of humanity's limitations within creation. Gödel's Incompleteness Theorem implies that the logical systems upon which our mathematics and scientific studies are based are incapable of showing consistency. Special Relativity places an upper bound on the speeds at which physical objects can travel and therefore places a limit on how quickly we can communicate. Heisenberg's Uncertainty Principle from quantum mechanics identifies fundamental limits on the accuracy of measuring multiple microscopic properties of a system at the same time. The Second Law of Thermodynamics exposes the limits on the efficiencies of mechanisms and useful energy conversions. Chaos Theory and eventual unpredictability fits in with the rest of these limitations, marking the great difference between "an *infinite Creator* and a *finite and limited creation, including man*" (Davis, 1997, p. 80, emphasis in the original).

The rejection of a Laplacian agenda for scientific studies and the realization that our scientific truth is not certain can be a boon in another way for Christians and those of other faiths (including atheism). Epistemic humility on all parts can open up opportunities for varying interpretations of data as well as for renewed interactions between scientific and faith communities.

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[25] *Open,* in this sense, refers to unpredictability or unknowability of future events. A *closed* universe is one in which all future events could be, in theory, predicted; there is no openness for different future outcomes.

[26] *Ontological openness* implies the system is open (openness exists, in fact); *epistemic openness* means that there there is openness *as far as what we can know/* is concerned.

11 Conclusion

Chaos theory has been used to present openings, or "ontological gaps" into which God could step and interact directly with nature without violating any natural laws. Although this possibility may be dismissed as a prescription for divine action, several theologians have suggested it may prove to be a useful metaphorical way to understand better how God acts to create and sustain his creation. For example, Polkinghorne (1995, p. 155) proposed that we view the mental (something not explainable through reductionistic means) and the material as "opposite poles of a single reality." This "largely conjectural and heuristic" means of understanding creation opens our understanding of divine action to include both natural-law sustenance as well as some sort of special meta-natural-law intervention of God's own choosing, beyond our understanding or comprehension.

Going further, I believe that we need to see chaos as being one aspect of created reality. The statistician David J. Bartholemw (quoted in Davis, 1997, p. 78) notes that, "since chance [and therefore also chaos] is such an integral part of creation, it must be part of God's plan ... [and] ... grist for the providential mill rather than as an obstacle to providential action." God can use both regular natural laws and chaotic events to cause his creation to unfold and to serve his redemptive plan.

Once again our epistemic humility ought to make us recognize that in some cases, such as in our understanding of the Trinity, a metaphorical understanding is all we are capable of as humans. In terms of divine action, the idea that God works through and alongside chaos is an apt metaphor for how God can work in the world, but it is not a prescription as to how God *does* or *ought to* work (Edwards 1995, p. 172–175; Polkinghorne 1995, p. 154–155; Davis 1997, p. 81–83; Peacocke 1995, p. 141–143). Perhaps, in that context, we may be bold to humbly suggest, in the spirit of Bartholemew (as quoted in Peacocke, 1995, p. 142), that God chose to make a world containing chaos "because it would have the properties necessary for producing beings fit for fellowship with himself."

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Forming the Analytical Society at Cambridge University

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The Analytical Society, an organization begun by students at Cambridge, was founded in 1812. Even though it was entirely student-led, the society was responsible for significant changes in the Cambridge mathematics curriculum and in the way mathematics was perceived in Britain throughout the nineteenth century. Its success was likely due to the outstanding students who formed the group, some of whom went on to become leaders in British science and mathematics for the next fifty years. In this paper we will briefly look at several of those who played important roles in forming and leading the society and we will consider the circumstances leading to its formation.

In the fall of 1809, John Frederick William Herschel (1792 – 1871) matriculated at St. John's College, one of the two largest colleges at Cambridge University. A serious student of mathematics, Herschel came from a privileged, upper-class family. When it didn't work out for Herschel to be away at school, his family was able to hire a private tutor and allow Herschel to finish his education at home. Because his father, William Herschel, was a world-renowned astronomer--he discovered the planet Uranus--John Herschel likely grew up accustomed to scientific talk and held to high expectations. At Cambridge Herschel would distinguish himself as a superior student, finishing as the senior wrangler in 1813. After graduating from Cambridge Herschel was elected to the Royal Society and became a fellow of St. John's. Although he continued to do mathematics for several years, primarily as an avocation, Herschel eventually followed in his father's footsteps and became a distinguished astronomer.

Like Herschel, George Peacock (1791 – 1858) began his studies at Cambridge in the fall of 1809, entering Trinity College, the rival of St. John's in terms of size and influence at the university. Peacock's background, however, was much different from that of Herschel. Born in the Yorkshire village of Denton, Peacock's father was the parish rector and ran a small school, where Peacock received his early education. Before entering Cambridge, he also attended the Richmond School, where he seemed to blossom under the tutelage of The Reverend James Tate, a Cambridge graduate and a fellow of Sidney Sussex College. Apparently thinking a great deal of Tate, Peacock dedicated his most famous work, *A Treatise on Algebra*, to him and refers to Tate in glowing and affectionate terms.

Because of his relatively simple background, Peacock's prospects were somewhat limited. Distinguishing himself at Cambridge would be his best hope for an academic career and Peacock certainly succeeded. He was a talented student of mathematics and eventually rose to a position of leadership among an elite circle of talented students. Peacock finished second on the tripos exam of 1813, bested only by Herschel. After that impressive performance he became a fellow of Trinity and went on to have an important career at Cambridge. By all accounts he was a popular and influential tutor at Trinity, where two of England's most important mathematicians, Augustus DeMorgan and Arthur Cayley were among his students. While at Cambridge, Peacock made a number of contributions, including his work in reforming the Tripos exam, and by extension, the entire mathematics curriculum. Most people are aware of Peacock as the author of *A Treatise on Algebra*, a book that is credited with transforming the way algebra was perceived, freeing it from simply being thought of as a generalized arithmetic. In 1839 Peacock followed the path of several other Cambridge faculty members and left his faculty post at Cambridge to become Dean of the Cathedral at the nearby town of Ely, where he served until his death in 1858. Throughout his life, even after moving to Ely, Peacock was actively involved in the affairs of the university, serving on various university commissions and committees, many charged with instituting significant reforms.

Coming to Cambridge a year after Herschel and Peacock, Charles Babbage enrolled at Trinity in 1810, later moving to Peterhouse College. Since his father was a wealthy banker, Babbage wasn't under the pressure of having to earn honors to make a place for himself in academia. Given that sense of freedom, Babbage chose to venture outside the established system and did not take the Tripos exam, passing up any opportunity for honors. However, by deciding to forego the exam, he also freed himself from the need to laboriously prepare for the exam and instead was able to focus on things he wanted to study. Primary among these was mathematics. His decision did not adversely affect his later life as he went on to become famous as the developer of the analytical engine and wrote extensively in a variety of areas, including philosophy and apologetics. Although he was not actively involved with the university, Babbage was given the honor of being appointed Lucasian professor in 1828. As Ball states, he "held the chair until 1839, but by an abuse which was then possible he neither resided nor taught"[1], but that's a story for another day.

Although Herschel, Peacock and Babbage were talented mathematicians, the fact that they studied mathematics is not surprising. In the early nineteenth century, everyone at Cambridge studied a lot of mathematics and there are several reasons why. A young man went to Cambridge to be liberally educated not to prepare for a profession. Perfecting an ability to think logically and reason well was seen as an important part of that process, and studying mathematics was considered to be the best way to acquire those skills. Because it was the central part of a truly liberal education, everyone studied a lot of mathematics, even though it was expected that most graduates would never use the particulars of the subject after graduation. In fact, most students went on to have careers in the church, but still they studied mathematics and very little theology. Since no effort was made to prepare students to become professional mathematicians, there was also no need to revise the curriculum to include the latest theories or to prepare students to do research.

Another reason why mathematics played such a central role at Cambridge was the Tripos, or Senate House Examination. used to rank students for honors. It was necessary to have an objective exam to obtain precise rankings and mathematics certainly qualifies as an objective subject, so the exam was a difficult and comprehensive mathematics test. And, since mathematics formed the basis of this important exam, mathematics was also primary in the curriculum.

Even though mathematics played such a central role in the curriculum, by the time Herschel, Babbage, and Peacock matriculated at the university, the Cambridge mathematical community had become fairly stagnant. On the continent mathematicians, using the analytical methods they preferred, continued to develop new results. At Cambridge, these methods were seen as not helping accomplish the goal of training young men to think logically and be educated as gentlemen, so they continued to emphasize the synthetic methods that are typical of areas such as geometry. Over the years, mathematics in Britain got further and further behind what was being done in places like France and Germany. Despite the efforts of one professor, Robert Woodhouse, who, prior to 1810 was the lone voice on the Cambridge faculty advocating for the analytical methods, the gap between what was being done on the continent and what was being taught at Cambridge continued to widen. Consequently, students, like Herschel, Peacock, and Babbage were not exposed to these new methods.

A second factor separating Cambridge mathematics from that of the continent was the use of different notation. Cambridge professors continued to use the notation introduced by Newton, while the continent was largely using the calculus notation introduced by Leibnitz, making it difficult for students to access foreign works. For example, the calculus text by Lacroix, which was popular in the era around 1810, was not being read by Cambridge students.

Finally, there was the continuing and looming presence of the Tripos exam, which ruled the curriculum. For example, as long as Newton's notation was used on the exam, it would be the primary notation used in instruction. Faculty also discouraged students, even the best students, from considering any topics or problems which would not be covered on the exam. Some students found this quite discouraging. For instance, in a letter to his father, Herschel reports that he is frustrated by not being able to read books that would be beneficial to his advancement, because the topics they covered were not pertinent to the exam material.[2]

In a similar vein, Babbage gives the following disappointing introduction to his life at Cambridge:

> "Thus it happened that when I went to Cambridge I could work out such questions as the very moderate amount of mathematics which I then possessed admitted, with equal facility, in the dots of Newton, the d's of Leibnitz, or the dashes of Lagrange. I had, however, met with many difficulties, and looked forward with intense delight to the certainty of having them all removed on my arrival at Cambridge. I had in my imagination formed a plan for the institution amongst my friends of a chess club and also of another club for the discussion of mathematical subjects.
>
> In 1811, during the war, it was very difficult to procure foreign books. I had heard of the great work of Lacroix, on the "Differential and Integral Calculus," which I longed to possess, and being misinformed that its price was two guineas, I resolved to purchase it in London on passage to Cambridge. As soon as I arrived I went to the French bookseller, Dulau, and to my great surprise found that the price of the book was seven guineas. After much thought I made the costly purchase, went on immediately to Cambridge, saw my tutor Hudson, got lodgings, and then spent the greater part of the night in turning over the pages of my newly-acquired purchase. After a few days, I went to my public tutor Hudson, to ask the explanation of one of my mathematical difficulties. He listened to my question, said it would not be asked in the Senate House, and was of no sort of

> consequence, and advised me to get up the earlier subjects of the university studies.
>
> After some little while I went to ask the explanation of another difficulty from one of the lecturers. He treated the question in just the same way. I made a third effort to be enlightened about what was really a doubtful question, and felt satisfied that the person I addressed knew nothing of the matter, although he took some pains to disguise his ignorance.
>
> I thus acquired a distaste for the routine of the studies of the place, and devoured the papers of Euler and other mathematicians, scattered through innumerable volumes of the academies of Pertersburgh, Berlin, and Paris, which the libraries I had recourse to contained.
>
> Under these circumstances it was not surprising that I should perceive and be penetrated with the superior power of the notation of Leibnitz.[3]

There is yet another person who deserves mention in this story, someone who is neither a mathematician or even directly associated with the university. From 1783 until his death in 1836, the Rev. Charles Simeon was the rector of Holy Trinity Church in Cambridge. Simeon, a graduate of Kings College, was known for his inspirational preaching and had a vibrant and important ministry among Cambridge undergraduates. As D. A. Winstanley notes in his book *Early Victorian Cambridge*, "Many undergraduates were brought by Simeon's preaching to think seriously about religion for the first time in their lives; and sometimes these youthful disciples were a source of anxiety to the older members of their party."[4] The anxiety mentioned here is likely the result of an on-going discussion between the conservative members of the established church and the zealous evangelicals. These discussions were common with vocal supporters on both sides of the issue at Cambridge. For instance, Isaac Milner, who was president of Queens College from 1788 to 1820, was a strong evangelical.

As an evangelical, Simeon thought that Christians should find ways to use their inward piety to affect outward changes in society. Consequently, he was a strong advocate for various missionary ventures, inspiring the students who sat under his preaching to do the same. In 1804, Simeon was one of the founders of The British and Foreign Bible Society, organized with the goal of publishing and distributing Bibles without additional comment, allowing anyone to read and interpret the scriptures for themselves. While this sounds like a worthy goal today, at the time it was widely viewed as a threat to the Anglican Church, partly because it was supported by people from a variety of denominations. Endorsing the proper form of a popular version of the Bible became a symbol of the internal struggles between the established church and the evangelicals, including people like William Wilberforce, who continued to press for an outward expression of inward piety. Simeon's strong influence on undergraduates even extended to this new missionary venture, which the students approached with unusual enthusiasm. As Winstanley reports,

> "With the generosity of youth they longed to give others the happiness and peace of mind which they themselves had obtained; and, not having yet learned the value of caution, they were perhaps too ready to think that their seniors were not

> sufficiently active in spreading the good news of the Gospel. With such a vast vineyard crying for cultivation, it was almost impossible for the more enthusiastic of them to linger in the market-place, waiting to be hired; and under the influence of a few pious undergraduates in the autumn of 1811 conceived the idea of establishing an auxiliary branch of the British and Foreign Bible Society at Cambridge."[5]

This proposal, or any proposal initiated by students, was quite unusual. However, this would be the first of several such initiatives that were proposed in the next decade. This unusual proposal significantly combined the existing theological discussions within the church with the radical idea that students could even have a voice in forming a new group. Once the proposal was made, Winstanley describes a period of intense debate, with student advocates on one side and most of the senior leadership of the university on the other. Eventually the students gave way to Simeon's leadership who worked out an agreement with the leaders in the university to establish a Cambridge branch of the Bible Society in December of 1811. The fact that the impetus for this agreement came from students was itself a significant fact and an important prelude to the Analytical Society. In commenting about this Winstanley mentions that this episode, as well as a second incident which took place several years later, marked a turning point in the relationship between students and those in authority. He states ". . . but when peace returned to Europe and authority began to be seriously challenged, it was inevitable that undergraduates should attempt to cast off the shackles of the previous age."[6]

Once the students had succeeded in forming their branch of the Bible Society, a second debate immediately followed, about whether the Bible should be published alone or with the prayer book. This new discussion was just as intense as the first, involving many students and others in the Cambridge community. With this backdrop, it is interesting to read Babbage's account of the founding of the Analytical Society, taken from his manuscript *Passages from the Life of a Philosopher*.

> "At an early period, probably at the commencement of the second year of my residence at Cambridge, a friend of mine, Michael Slegg, of Trinity, was taking wine with me, discussing mathematical subjects, to which he also was enthusiastically attached. Hearing the chapel bell ring, he took leave of me, promising to return for a cup of coffee.
>
> At this point Cambridge was agitated by a fierce controversy. Societies had been formed for printing and circulating the Bible. One party proposed to circulate it with notes, in order to make it intelligible; whilst the other scornfully rejected all explanations of the word of God as profane attempts to mend that which was perfect."[7]

Babbage goes on to report that after seeing one of their advertisements his first thought was that it might be a good idea to parody this group and organize a society for translating the small work of Lacroix on the calculus. He proposed "periodic meetings for the propagation of d's; and consigned to perdition all who supported the heresy of dots. It maintained that the work of Lacroix was so perfect that any comment was unnecessary."[8]

Babbage shares this idea of a parody of the Bible Society controversy with Slegg who, in turn, mentioned it to one of his mathematical friends, Edward French Bromhead. Soon, what appeared to start out as joke was seen to be a good idea, and a meeting was proposed for the "purpose of forming a society for the promotion of analysis."[9] Herschel and Peacock are reported to have been present at that first meeting, along with several others, and it was decided to form a group called "The Analytical Society." Among the goals of the group was the translation of Lacroix's text, that regular meetings would be held and papers would be read, and to publish a volume of their transactions. Babbage reports that they were "much ridiculed by the Dons; and, not being put down, it was darkly hinted that we were young infidels, and that no good would come of us."[10]

Unlike the conscientious work and serious spirit that preceded the formation of the Cambridge branch of the Bible Society, the formation of the Analytical Society was almost accidental, conceived as a whim. There was apparently no group of students that were having serious discussions about the advantages of the analytical methods used by continental mathematicians or the deficiencies in the current Cambridge curriculum and how these could be rectified. It is interesting that Enros comments that those present at the first meeting were unacquainted with each other[11] although Babbage claims that they were all known to Bromhead.

What these young men could soon agree on was the need to incorporate more analytical methods and the continental notation in their mathematical study at Cambridge. This was not an entirely new idea at the university. As mentioned above, one member of the mathematics faculty, Robert Woodhouse had already been advocating for the use of analytical methods, but was not able to make any headway with his fellow faculty members. Because Woodhouse was a voice on the faculty for making these changes, some students, probably a few of the better students, were familiar with them, even though the new notation was not used on the all-important examinations.

In 1812, Herschel, Babbage, and Peacock were three of the best students in Cambridge. While it's difficult to believe that they did not at first know each other before the initial meeting in Bromhead's room, they soon became good friends. As Ball says, ". . . in 1812, three undergraduates—Peacock, Herschel, and Babbage—who were impressed by the force of Woodhouse's remarks and were in the habit of breakfasting together every Sunday morning, agreed to form an Analytical Society with the object of advocating the general use in the university of analytical methods and of the differential notation, and thus, as Herschel said, 'do their best to leave the world wiser than they found it.'"[12] Whether they had known each other prior to their first meeting or not, it is reasonable to conclude that the well-known debates about the distribution of the scriptures taking place in Cambridge throughout 1811 and into 1812, paved the way for other student-led initiatives and were a significant factor in planting the idea to form the Analytical Society.

With their initial enthusiasm, the leaders of the Analytical Society set out a number of goals, some of which were accomplished and some were not. For instance, their anticipated publication, which came to be known as the *Memoirs of the Analytical Society*, had contributions from only two authors, Babbage and Herschel, and soon ceased to exist. The translation of Lacroix, however, was accomplished by Herschel and Peacock, with Peacock also publishing an extensive text of examples to illuminate the material in Lacroix. Perhaps the most enduring legacy was the reformation of the methods and notation used on the Tripos exam, an effort led by George Peacock. By the early 1820's the Tripos exam had been reformed to use the analytical notation and once that was accomplished, the nature and essence of the mathematics curriculum was changed for good. Even though Peacock, who was responsible for many subsequent changes, had a strong personality, the existence of this wider group of student associates had to have been a major influence on his work and on changing the nature of the mathematical culture in Britain for the rest of the century.

Footnotes
1. Ball, page 126.
2. Enros, page 110.
3. Babbage, pages 26, 27
4. Winstanley, page 18.
5. Winstanley, pages 18, 19.
6. Winstanley, page 25.
7. Babbage, pages 27, 28.
8. Babbage, page 28.
9. Babbage, page 29.
10. Babbage, page 29.
11. Enros, page 107.
12. Ball, page 120.

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Pedagogical Enhancements to the DeSymbol Logic Translator

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Abstract

DeSymbol is a program that translates first-order predicate logic expressions into English. It is intended to be a practice tool for students who are learning logic for the first time or who are trying to refresh their memories if they need to use symbolic logic for an upper-level course. Students start with an English sentence and translate it by hand into symbolic logic notation; then they can check their work by using DeSymbol to translate their notation back into English. If the English sentence produced by DeSymbol differs significantly from the original English sentence, this helps the student to see what error was made in the logic expression.

The latest version of DeSymbol adds support for prepositions, so that the student can now test expressions such as on(a, b) and $\forall$ x $\forall$ y (on(x, y) $\rightarrow$ under(y, x)). It also now supports a wider variety of idiomatic translations, including improved translations of common student mistakes. For example, the student who begins with the English sentence *All cats are mammals* and writes the expression $\forall$ x(cat(x) $\wedge$ mammal(x)) will see DeSymbol re-translate the expression as *Everything is a cat and a mammal*, which helps the student to see why the expression is incorrect.

1 Introduction

Symbolic logic is important in many areas of computer science, but students sometimes struggle when learning to use the language of symbolic logic to represent their ideas. The DeSymbol logic translator is intended as an aid to students who are learning to translate English sentences into symbolic logic notation.

In a typical homework assignment, a student may be asked to provide the symbolic logic notation for sentences like these:

> All bats are little.
> Some birds swim.
> No poodles are amphibians.

Correct answers look like this:

$\forall$ x (bat(x) $\rightarrow$ little(x))
$\exists$ x (bird(x) $\wedge$ swims(x))
$\neg$ $\exists$ x (poodle(x) $\wedge$ amphibian(x))

Some students are able to learn how to do these translations easily. Others, however, struggle. They may confuse $\rightarrow$ (*implication*) and $\wedge$ (*and*):

$\forall$ x (bat(x) $\wedge$ little(x))
$\exists$ x (bird(x) $\rightarrow$ swims(x))

or be unsure about where a negation symbol goes:

$\exists$ x (poodle(x) $\wedge$ $\neg$ amphibian(x))

Often, translating the logic sentences back into English is enough to let students know that they have made a mistake:

Everything is a little bat.
There exists an x such that if x is a bird, then x swims.
Some poodles are not amphibians.

A teacher can give this feedback during exercises in class, and students can give feedback to each other if they are working in a study group. DeSymbol is meant to give this same kind of feedback, but since DeSymbol is a computer program, it is available at all hours of the day or night, and it has infinite patience, if a student needs to practice for several hours or days.

2 DeSymbol overview

DeSymbol is a web-based program which, given a symbolic logic expression, translates that expression into English. The user enters an expression like this

$\forall$ x (bat(x) $\rightarrow$ little(x))

into a text box and asks for a translation; DeSymbol provides an English equivalent like this

All bats are little.

The user can then check whether that was what he or she really intended to say. When entering the logic expression, regular ASCII characters can be typed into the text box from the keyboard, while special symbols such as $\forall$ and $\rightarrow$ are entered using buttons on the form.

The translation is done using definite clause grammars in Prolog, with the vocabulary stored as a collection of Prolog facts. (See [1], [2], [3], and [6] for more information on Prolog grammars.) The web interface is written in JavaScript and communicates with the server using Ajax. A default vocabulary is provided, but users can add their own vocabulary words as desired. [4], [5]

These are some sample DeSymbol translations:

Logic expressions	**Translation**
$\forall$ x (poodle(x) $\rightarrow$ dog(x))	All poodles are dogs.
$\forall$ x (dog(x) $\rightarrow \neg$ green(x))	No dogs are green.
$\neg \exists$ x (little(x) $\wedge$ frog(x) $\wedge$ barks(x))	No little frogs bark.
$\exists$ x (insect(x) $\wedge$ purple(x))	Some insects are purple
$\forall$ x (lizard(x) $\rightarrow$ small(x) $\wedge$ reptile(x))	All lizards are small reptiles
$\forall$ x (bat(x) $\rightarrow$ brown(x) $\vee$ black(x))	All bats are brown or black
loves(arthur, guinevere)	Arthur loves Guinevere.
likes(bob, snow) $\wedge \neg$ likes(bob, ice)	Bob likes snow and Bob does not like ice.

3 Improvements in the new version

The newest version of DeSymbol includes improvements both to the user interface and to the actual translations.

3.1 User interface

The new DeSymbol user interface includes tabs that provide the user with general instructions, example translations, some sample exercises, and a history of previous expressions entered by the user in the current session. Editing expressions and adding new vocabulary words has also been made easier.

3.2 Translation

The translation ability of DeSymbol has been improved. Prepositions have been added to DeSymbol's vocabulary, so that the user can enter expressions like these:

on(a, b)
near(sam, frodo)
$\forall$ x $\forall$ y $\forall$ z(above(x, y) $\wedge$ above(y, z) $\rightarrow$ above(x, z))

Uncountable nouns like *water* have also been added:

likes(kermit, water)

More kinds of sentences now have idiomatic translations. For example, the expression

$\forall$ x (cat(x) $\wedge$ furry(x))

instead of being translated symbol-for-symbol as

For all x, x is a cat and x is furry

now gets the simpler translation

Everything is a furry cat.

The new system will be used in the Fall 2013 *Foundations of Computer Science* class at Rowan, in which we teach the students introductory symbolic logic, and feedback from the students will be used to evaluate the new features. DeSymbol is free software covered by the GNU Public License, and it is available on the web at http://elvis.rowan.edu/desymbol/ for anyone to use.

4 Acknowledgements

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Insights on the Neyman - Pearson Lemma : Alternative critical regions, and their power.

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Abstract

The Neyman - Pearson Lemma is a powerful fundamental lemma in the area of hypothesis testing in Statistics.It gives the best test when testing simple vs.simple hypotheses.In this talk we would like to investigate testing a population mean
$H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1 > \mu_0$.
As a result of the N - P Lemma, the best test is of the form, “Reject H_0 if $\overline{X} > c$ " , where c is chosen so that the Type I error probability is α. Let n be small. What are some alternative decision rules of size α, what is their power in comparison to the best test? The talk should be of interest to a person who has had a first course in Probability and Statistics.

Introduction

In Probability and Statistics, we often teach hypothesis testing, including the test of a population mean:

H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$

We make the following assumptions:
1. Random Sample
2. Population ~ N[μ, σ^2] with σ^2 known.

We use the Test Statistic

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

If H_0 is true, then $Z \sim N[0,1]$.
Decision Rule: Reject H_0 if $Z > z_\alpha$

which is equivalent to Reject H_0 if $\overline{X} > c$ $\left(= \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}.\right)$

This decision rule is shown to be the “best test” by the N-P Lemma.

Definition - Best Test (Best Critical Region)

$C \subseteq R^n$ is the best test if it has more power to correctly reject H_0.
Let $A \subseteq R^n$ be any other test with
$\alpha = \Pr[(X_1, X_2, ..., X_n) \in C; H_0 \text{ true}] = \Pr[(X_1, X_2, ..., X_n) \in A; H_0 \text{ true}]$
Then
$\Pr[(X_1, X_2, ..., X_n) \in C; H_1 \text{ true}] > \Pr[(X_1, X_2, ..., X_n) \in A; H_1 \text{ true}]$

Neyman - Pearson Lemma.

Let $X_1, X_2, ..., X_n$, where n is a fixed positive integer, denote a random sample from a distribution that has p.d.f. $f(x; \theta)$. Then the joint p.d.f.of $X_1, X_2, ..., X_n$ is

$L[\theta; x_1, x_2, ..., x_n] = f(x_1; \theta) \cdot f(x_2; \theta) \cdots f(x_n; \theta)$.

Let θ_0 and θ_1 be distinct fixed values of θ so that $\Omega = \{\theta;\ \theta = \theta_0, \theta_1\}$, and let k be a positive number.

Let C be a subset of the sample space such that :

- $\frac{L[\theta_0; x_1, x_2, ..., x_n]}{L[\theta_1; x_1, x_2, ..., x_n]} \leqslant k$ for each point $(x_1, x_2, ..., x_n) \in C$
- $\frac{L[\theta_0; x_1, x_2, ..., x_n]}{L[\theta_1; x_1, x_2, ..., x_n]} > k$ for each point $(x_1, x_2, ..., x_n) \in C^*$
- $\alpha = \Pr[(X_1, X_2, ..., X_n) \in C; H_0]$

Then C is a best critical region of size alpha for testing the simple hypothesis
H_0: $\theta = \theta_0$
Against the alternative simple hypothesis
H_1: $\theta = \theta_1$.

N - P Lemma Simplified and Applied

Let $\theta = \mu$, $\sigma^2 = 1$, n=2 and $\alpha = .20$.

So we are testing
H_0: $\mu = \mu_0 = 0$
H_1: $\mu = \mu_1 = 1$

The p.d.f. is
$f(x; \mu) = \frac{1}{\sqrt{2\pi}} \exp\{ \frac{-1}{2}(x - \mu)^2\}$
so the Likelihood function is

$L[\mu;\, x_1, x_2] = f(x_1;\, \mu) \cdot f(x_2;\, \mu)$

$$= \frac{1}{\sqrt{2\pi}} \exp\{ \tfrac{-1}{2}(x_1 - \mu)^2\} \cdot \frac{1}{\sqrt{2\pi}} \exp\{ \tfrac{-1}{2}(x_2 - \mu)^2\}$$

$$= \frac{1}{2\pi} \exp\{ \tfrac{-1}{2}\left[(x_1 - \mu)^2 + (x_2 - \mu)^2\right] \}$$

Therefore the Likelood Ratio is

$$LR = (L[\mu_0;\ x_1,\ x_2] / L[\mu_1;\ x_1,\ x_2]) = L[0;\ x_1,\ x_2] / L[1;\ x_1,\ x_2] = \frac{\frac{1}{2\pi} \exp\left\{\frac{-1}{2}\left[x_1{}^2 + x_2{}^2\right]\right\}}{\frac{1}{2\pi} \exp\left\{\frac{-1}{2}\left[(x_1-1)^2 + (x_2-1)^2\right]\right\}}$$

$$= \exp\{\tfrac{-1}{2}\left[x_1{}^2 + x_2{}^2 - (x_1 - 1)^2 + (x_2 - 1)^2\right] \}$$

$$= \exp\{\tfrac{-1}{2}\left[x_1{}^2 + x_2{}^2 - (x_1^2 - 2x_1 + 1) - (x_2^2 - 2x_2 + 1) \right] \}$$

$$= \exp\{\tfrac{-1}{2}[\ 2x_1 + 2x_2 - 2]\}$$

$$= \exp\{1 - (x_1 + x_2) \}$$

So we reject H_0 if

$LR \le k$

or $\ell n(LR) \le \ell n(k)$

or $1-(x_1+x_2) \le \ell n(k)$

or $(x_1+x_2) \ge 1 - \ell n(k) = c$

or $\overline{X} \ge \frac{c}{2} = c'$

c is chosen so that $P[\ (x_1+x_2) \ge c \mid H_0 \text{ true }] = \alpha = .20$

Bottom Line - N-P Lemma says....

... that the best test is to reject H_0 if $x_1+x_2 \ge c$.

c is chosen so that $P[\ (x_1+x_2) \ge c \mid H_0 \text{ true }] = \alpha = .20$

(We find c to be 1.188) (see next section for details)

What is the Power for the test?

Power = $P[\text{ Reject } H_0 \mid H_1 \text{ True }]$

$= P[\ (x_1+x_2) \ge c \mid \mu = 1\ ] = .7170$ (see next section for details)

$\beta = P[\text{Type II Error }] = P[\text{ Accept } H_0 \mid H_1 \text{ True }] = 1 - \text{Power} = .2830$

According to the N-P Lemma, this is the best that we can do.
We would like to consider some other test regions of size α=.20, and see what their power is.

Details

$X_i \sim N(\mu, 1)$, $i = 1, 2$

$X_1 + X_2 \sim N(2\mu, 2)$

If H_0 True, $\mu = 0$, so $X_1 + X_2 \sim N(0, 2)$

$$\begin{aligned}
\alpha = .20 &= P[\text{Type I Error}] \\
&= P[(x_1 + x_2) \geq c \mid \mu = 0] \\
&= P\left[\frac{(x_1 + x_2) - 0}{\sqrt{2}} \geq \frac{c - 0}{\sqrt{2}}\right] \\
&= P\left[Z \geq \frac{c}{\sqrt{2}}\right] = .20
\end{aligned}$$

so $\dfrac{c}{\sqrt{2}} = z_{.20} = 0.84 \longrightarrow c = \sqrt{2}\ 0.84 = 1.188$

If H_1 True, $\mu = 1$, so $X_1 + X_2 \sim N(2, 2)$

$$\begin{aligned}
\text{Power} &= P[\text{Reject } H_0 \mid H_1 \text{ True}] \\
&= P[(x_1 + x_2) \geq 1.188 \mid \mu = 1] \\
&= P\left[\frac{(x_1 + x_2) - 2}{\sqrt{2}} \geq \frac{1.188 - 2}{\sqrt{2}}\right] \\
&= P[Z \geq -0.574] = .7170
\end{aligned}$$

$\beta = 1 - \text{Power} = 1 - .7170 = .2830$

Alternative Critical Regions

A1 - Random

A2 - Single Observation

A3 - Maximum of (X_1, X_2)

A4 - Minimum of (X_1, X_2)

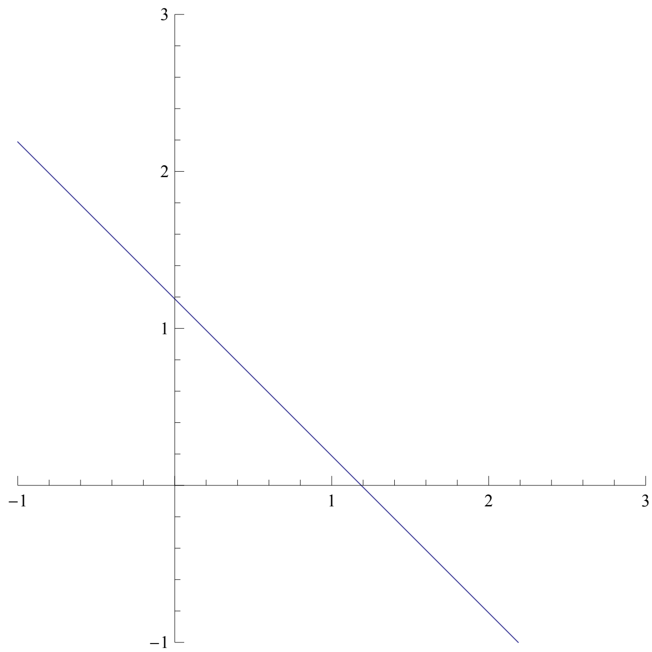
A5 - Chi Square

A6 -Chi Square & Positive

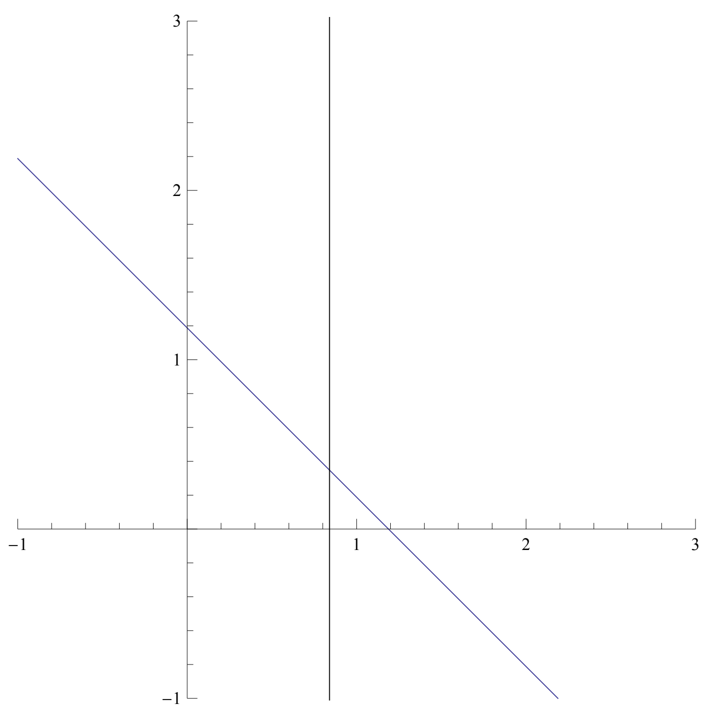
A7 - $X_1 + X_2$ & Positive

Graphs of Critical regins in R 2

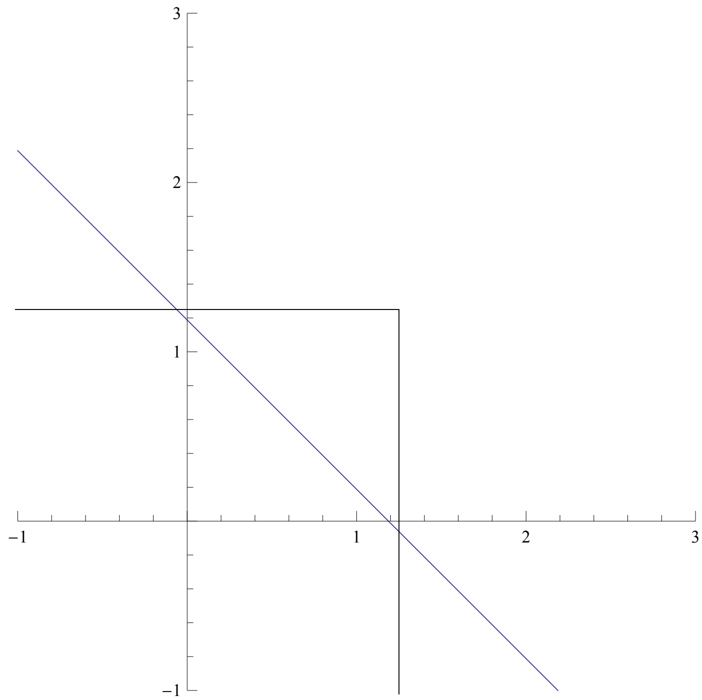
C



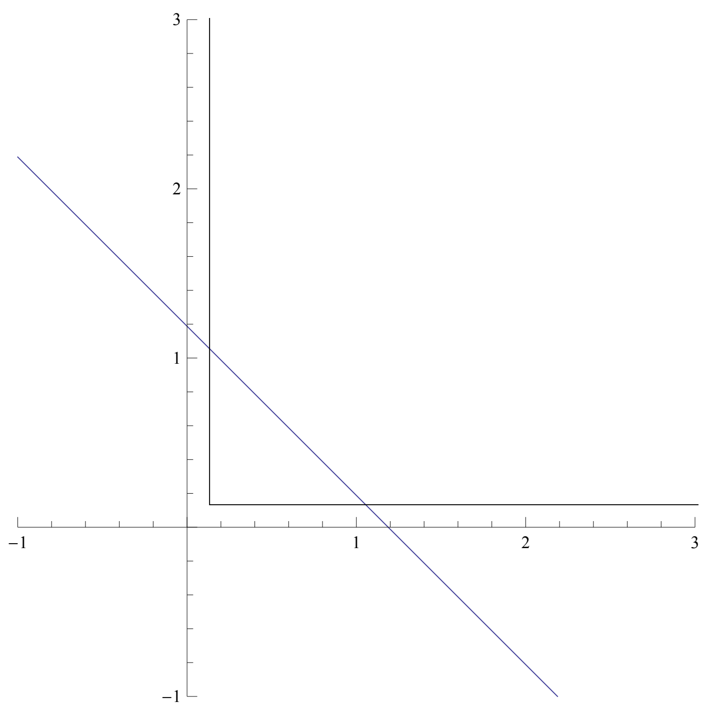
A2



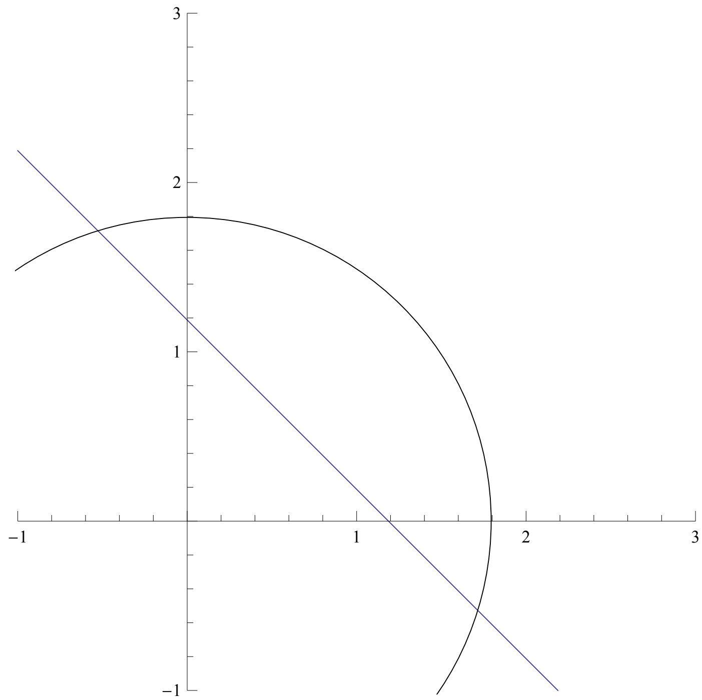
A3



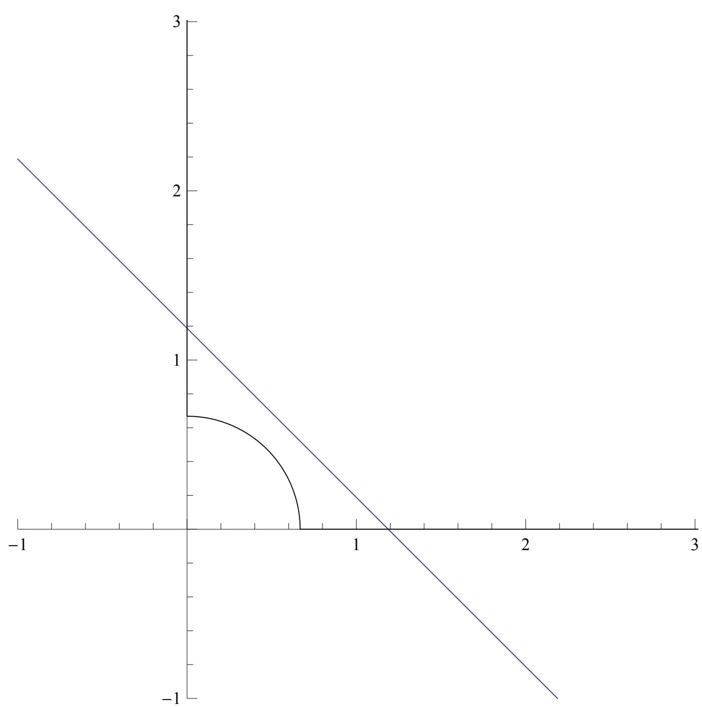
A4



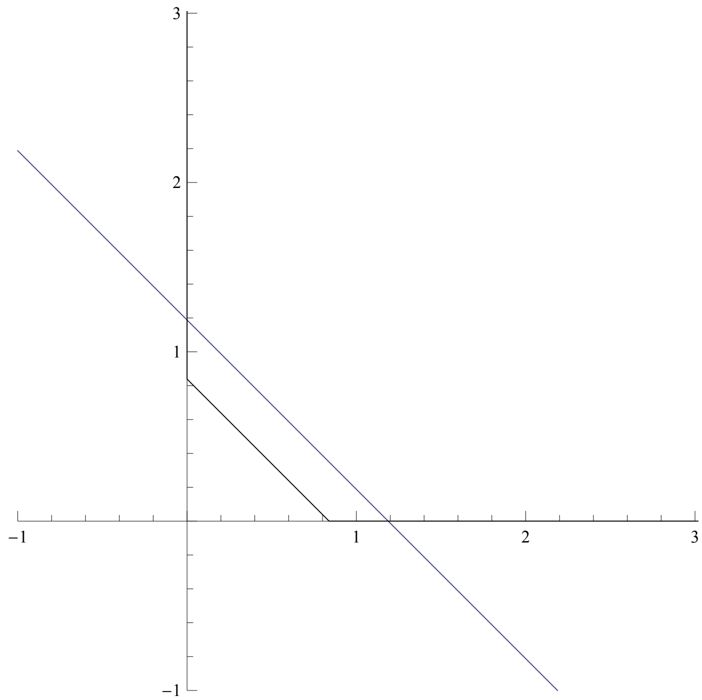
A5



A6



A7



Results

Test	Power	β	Difference in Power
C	.7170	.2830	-
A1	.2000	.8000	.5170
A2	.5636	.4364	.1534
A3	.6416	.3584	.0757
A4	.6512	.3488	.0658
A5	.5158	.4842	.2012
A6	.6751	.3249	.0419
A7	.6754	.3246	.0416

Conclusions

While the Neyman-Pearson critical region is best, there are several candidates that are nearly as good, especially those involving order statistics. This can also be viewed as a thresholding process, which is fairly easy to implement. It can also lead to some alternative sequential methods, where the sample size is not determined in advance, but instead is a random variable.

It would be interesting to look at some cases where n= 3 or 4 in the future.

Philosophy Motivates Undergraduates in Mathematics
Dusty Wilson, Highline Community College
2013 ACMS Conference, St. Paul, MN

Teaching seminars on the philosophy of mathematics is good for students and good for educators. I will explain a model I have used for a discussion seminar around the philosophy of mathematics including its impact on the students directly involved as well as in my broader duties as a mathematics teacher. These projects have been very helpful in pursuing my research interests as well as having an avenue to explore connections between faith and mathematics. I will conclude with a few of the risks and rewards that you should be aware of should you consider taking on similar activities.

Over the last five years, I have organized discussion seminars and research projects for students. I did this to reach a specific group of students and to provide impetus for my own growth. The students I work with are those seeking opportunities beyond the limited class offerings our college has in mathematics. Typically students solve this by pressing through the sophomore level courses and only then branch out after transfer to university. But I thought I could do more by including students in my personal interest in the philosophy of mathematics. Through reading good books, asking insightful questions, and pushing higher order thinking through writing, students' lives are being changed and I am growing intellectually.

The next three paragraphs outline the way I have organized my seminars. For those more interested in the outcomes than the how-to, come back to these paragraphs if and when you decide to explore a project like this for yourself.

I did my undergraduate studies in a discussion seminar format and have continued using this familiar structure.[1] I meet with 4-10 students for a book discussion every Friday after other classes have ended. Everyone is expected to contribute with participation being measured by students coming prepared with typed questions based upon the readings. Since questions are written in advance, even quiet students are able to contribute to the discussion. In terms of work load, I expect students to read 1-2 books per quarter credit. I have found this ample material for discussion while laying the groundwork for students to write a related essay. We always set aside a week for students to share their writing and provide peer feedback.

In putting together these seminars, I have taken advantage of two flexible programs at my college: Special Studies courses and our Honors Program. Like many schools, we have special studies courses in our catalog. However, these are rarely used. I have found a topics class a great avenue to reach students and simple to arrange logistically. In addition to using the special topics heading, I have used the same class structure in working with students on projects for an honors option. Our honors program allows students to do a research project connected to one of their classes. The level is intended to be similar to what a student might face in upper division courses (recall, I teach at a two year college).

[1] Following two years of community college, I completed my undergraduate studies at The Evergreen State College in Olympia, WA.

More importantly, students are given opportunity to discover knowledge for themselves. As one student asked, "Do you think creativity is a requirement for doing math?"[2] These projects give students a chance to answer that for themselves by exploring mathematics outside the boundaries of their standard textbooks.

From my vantage point, there is little difference between the two programs (honors and special studies). Most seminars include honors and credit students. However, it can make a difference to the students. Limiting it to a credit class would (necessarily) add to the tuition expenses. On the other hand, working strictly under the honors heading would exclude all but the best students from eligibility. The most important aspect is to find a framework that works for you and provides tangible motivations to students.

About 45% of mathematics majors attend community college. However, most do not make the conscious decision to pursue mathematics until after they transfer. This impacts the type of students attracted to these seminars. The students I attract are those interested in a challenge, those with an interest in philosophy, and those few decided mathematics majors who are looking for a space of their own apart from the droves of engineering majors in their calculus classes.

Because the class is completely optional, all participants tend to be fully invested. The benefit to mathematics majors is that the seminars give them a leg up after transfer because they have a deeper understanding of the historical and philosophical context in which the mathematics came to be. That is, they better understand the value and application that is behind their studies. More interesting are those that come in without a stated desire to major in mathematics. These students typically are talented in mathematics, but have never considered it their primary focus. It's not uncommon for the seminar to change their focus as a result of these projects.

Harry Kim is an example of such a student. Harry was a new calculus student to our College. As an international student from Korea, he was the only new face to join my philosophy seminar. All the others were former students of mine. While he quickly established himself in the calculus, the seminar was a greater challenge. Readings I expected to take students two hours, took Harry more than six. With a shy laugh, Harry said, "I learned more English than philosophy in that class."

But Harry did learn about mathematics and mathematics history. Specifically, he expanded his picture of mathematics as being more challenging than merely textbook calculus. This led him to spend months fixated on the error function; he believed that he could find a clever trick that would could make the unintegratable integrateable. This in turn led him to play with Taylor Series months before his peers … but these were not the elegant answer he sought. As if one unsolvable problem was not enough, Harry joined the throngs who have sought to unravel the mystery of the distribution of prime numbers. Most

[2] Alison Mehlhaff, education major. Unless otherwise indicated, student quotations are from note cards collected on the first day of each term. On the card, I ask students for their name, birthplace, native language, an interesting fact about themselves, and one question for me (any topic so long as it is at most PG rated).

recently, he took it upon himself to generalize a solution and in so doing stumbled upon the Fourier series. But to his chagrin, he discovered that someone else had done this first.

When Harry came to me, he was intending to major in chemistry. However, the complexity and mystery of mathematics enticed him. Our reading (The Loss of Certainty by Morris Kline) gave him a glimpse at the beauty of mathematics that textbooks hide behind an inscrutable veil. This project (and others we have worked on) provided Harry a direction and purpose for his studies.

Because of these, my college is experiencing a rise in students (like Harry) who are declaring a mathematics major while still at community college. While not my original goal, I do see it as a success for the philosophy of mathematics in that students are drawn to the field of mathematics when they understand its power and mystery.

In addition to helping those students directly involved, the advent of seminars on the philosophy of mathematics correlates well with my broader growth as a teacher. This might seem counterintuitive because the seminars cause a tug-a-war over my time and attention. On the one hand, the seminars take focus and energy that could be allocated to an algebra or calculus class. They also cause me to focus more attention on the strong and interested students, perhaps leaving less for those most in need of care.

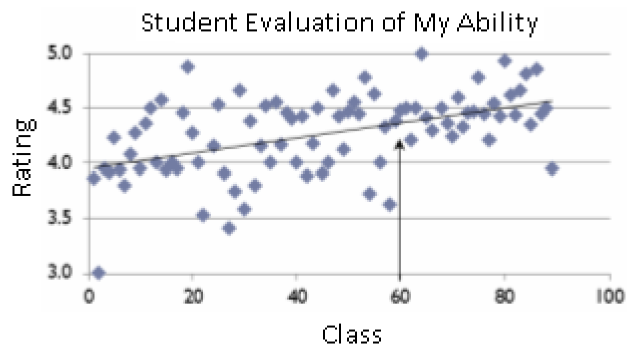
On the other hand, the seminars provide a rich foundation that can be woven into my other classes. A student wrote, "He [Dusty] showed the great background story for math. The story makes me understand and interested in the topic."[3] Rather than each class being a collection of tricks and techniques, the philosophy (and corresponding history) allows me to teach much more holistically. As another wrote, "I like how once in a while he [Dusty] would educate us on the philosophical aspects of math."[4] The availability of these projects also shows that I am willing to go above and beyond expectations (even if students don't avail themselves). Students have begun to look forward to and ask about future offerings. All together, I believe that I am a better overall teacher for having spent this time focused on a few select students.

A surprise to me is that algebra students are just as likely as those in the calculus to ask questions. One student asked, "What is math?"[5] and another, "Did you find out what the purpose of math is in philosophy?"[6] I believe this may be because the lower level students are more resistant to typical math presentations and thus more intrigued that there might be alternative motivations.

To support the hypothesis that this has helped me broadly, I did a more comprehensive analysis of my student evaluations over my tenure at Highline up through 2012. I wanted to know if there was evidence that these seminars were impacting the quality of my teaching. Students were asked: "The

[3] Anonymous comment from a student evaluation, multivariable calculus. This comment was edited for its grammar.
[4] Anonymous comment from a student evaluation, multivariable calculus.
[5] Parker Wilson, a business major.
[6] Claudia Gaudia, an undecided student in a business algebra course.

ability of the instructor to communicate the subject matter has been …" on a 5 point scale with 1 being very poor and 5 being excellent. The results are given in the graph below with the arrow indicating where I taught my first seminar. The upward trend is obvious as well as a much tighter clustering in the latter third of my career that coincides with when I taught my first seminar.[7]



To be clear, I am not claiming that correlation implies causation or that these seminars represent the only variable impacting my ability as a teacher. For example, the birth of our third child may well have been a major factor in the early variation. Perhaps as likely, I was very raw and it took me a number of years to mature into an effective teacher. But whatever the reason, I believe this shows conclusively that the addition of these seminars not only has not had an adverse effect on my other teaching duties, but may have actually had a positive impact.

As I have shown, these seminars have had a positive impact upon the students directly involved as well as helped me grow into a better overall teacher. But their value to me extends beyond the classroom. There are few (if any) faculty at my college who share my interest in the philosophy of mathematics. Furthermore my community college provides little incentive for professional growth in this field. But these seminars allow me to explore my own interests alongside others. Working with students allows me to develop my understanding in a cooperative setting.[8]

Because the seminars center on reading and dialogue, I have opportunity to practice articulating ideas. Perhaps as valuable, I get to hear the unfiltered questions of students which help me know the issues I must address. As an example, I have had numerous students ask about Godel's incompleteness proofs. The reoccurrence of the question pushes me to learn. I want to articulate clear and insightful answers.

[7] This graph requires a brief explanation. I taught about 90 classes (through 2012). Each data point represents the average rating given by a single class. As I teach three classes most terms, the data can be read (roughly) three points at a time. Prior to the first project, the mean ranking was 4.2 while afterward it rose to 4.5. I used a chi squared test and verified at the alpha = 0.05 level that this change was not due to random variation. I should also note that there is a small amount of data missing from this sample from a handful of classes for which I did not collect evaluations. These were almost exclusively before my first seminar and their inclusion (I dare say) would have only strengthened my case that I am a better teacher today than a decade ago.

[8] Never having worked at a research university, I speculate that this may be a key factor in why researchers like to work with graduate students.

Seminars cannot provide a complete solution to this as the concepts are too new to students for them to ask nuanced questions. But it is a stepping stone.

In addition to purely philosophical questions, students also push me toward integration. One student asked, “How has your understanding of mathematics changed your view or understanding of your religious beliefs?”[9] This kind of question is powerful as it pushes me to explore the impact of my faith on my discipline (and vice versa). Where is there time for such analysis and synthetic thinking? At any university, time is at a premium and this question has little credence at the public community college where I am tenured. But students want to know, “How can you connect God and math?”[10] I have found that these seminars give the reason, motivation, and space for me to think in an integrated manner.

Most of the books and articles we discuss do not have an overtly religious tone. If anything, the authors are skeptics who deify mathematics. But the philosophical themes, coupled with my known beliefs and the authors’ clear presuppositions, put some of life’s enduring questions on the table for conversation. A student asked, “Did higher education change your view on religion in any way?”[11] There are some students who want to know, and these discussions provide a platform for them to air their questions.

While nearly all the student feedback I have had regarding philosophy as well as the integration of math and faith has been positive, the work load itself is taxing. Since these projects are not part of my official teaching load, I like to think of it as tithing a credit. Thus I have been selective of the topics and timing. As I get good resources that I want to discuss, I begin to seek out students with whom to partner. By the end of the term, I tell myself “never again.” But like the woman giving birth ‘forgets’ the pain, I find a new book and am inspired to begin planning another one. Practically speaking, my college operates on quarters and I most often teach these seminars in Fall and Spring – Fall because I have more energy and Spring because I want to reach out to graduating students. In a semester system, I speculate that these would be well suited for the January term.

I have also found that spending more time with students feeds itself. Seminars allow me to work with students like Harry (the chemistry turned mathematics major I mentioned earlier). Such students are a pleasure to work with on a personal and mathematical level. But building meaningful relationships through projects and in the classroom includes an inherent risk. It requires opening your heart, which includes the risk of pain. This most recent term has been especially challenging for me. I can vividly recall a young woman sitting in my office pouring out hopes and dreams only to die a few months later in an auto accident. This tragedy was on top of a former student murdering four before being killed by the police. These incidents were very difficult and sent me to repeatedly read the Book of Job. But it has also given me a stark reminder of the importance and urgency of reaching students by any means possible, including through studies in the philosophy of mathematics.

[9] Evan Pfister, engineering major
[10] Audrey Chavarria, engineering major
[11] Daven Camacho, engineering major

Even with the demands and risks of conducting seminars on the philosophy of mathematics, there is no doubt in my mind that they are part of what has made me the teacher I am today. I began teaching at the ripe age of 22. I was excited about mathematics, students, and education. One of my fears was that my passion for teaching would fade when the honeymoon ended. Now after twelve years of teaching, over 100 classes, and around 3,000 students, I would say my fear was unfounded. I credit these projects for part of what has helped me grow in my passion and commitment.

While I hope that others will consider offering seminars in the philosophy of mathematics, my broader desire is that you will reflect upon those aspects of mathematics you find most intriguing and then develop ways for you to pursue truth *with* students. Too often we are beaten down by a ceaseless parade of committee meetings, too much material, too little time, and piles of exams to grade. But we are more effective as teachers when we ourselves are challenged and learning. We can motivate others when we are inspired. And working closely with students allows us to build meaningful relationships and thereby transform the lives of the next generation.

Expanding Jonathan Edwards' Typology Program: The Bell Curve as a Type of Christ

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Abstract

In 1993 an unfinished notebook of Jonathan Edwards called Types *was published for the first time since his death in 1758. It contains a more explicit argument than any of his previous works for extending biblical typology to nature in a biblically grounded manner. This paper is an attempt to extend that research program into mathematics/statistics. The thesis is that the normal distribution (the graph of which is the bell curve) is a type of Christ. All relevant mathematical concepts are defined and a description of thirteen different typological significations of Christ are given. The primary signification is that the celebrated Central Limit Theorem, in which the normal distribution is found to be the center of modern Statistics, typifies that Christ is the center of the plan of God.*

Introduction

> *He made known to us the mystery of His will, according to His kind intention which He purposed in Him with a view to an administration suitable to the fullness of the times, that is, the summing up of all things in Christ, things in the heavens and things on the earth.*
> *~ Ephesians 1:9-10*[1]

In the eighteenth century, Jonathan Edwards, the "most brilliant of all American theologians,"[2] set forth a research program for a novel view of biblical typology. In particular, he was brought up in the contemporary Puritan typology where in the Old Testament an object or event was a type of something to be revealed in the New Testament, most notably Christ. Taking it a step farther, though, Edwards wrote that not only in the Bible, but throughout the entire creation, in nature and history, God expressed Himself in divine communication, a typological language. Furthermore, this language could be learned by man in order to learn more about the God who communicated it. Unfortunately, Edwards' ideas on the subject were not picked up by his successors because they were not published after his untimely passing in 1758. It would not be until 1948 that the first of his notebooks on typology was published,[3] and not until 1993 for the one which succinctly articulated his views.[4] Follow-up work on Edwards' thought has sought to analyze it, rather than advance it, until now.

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[1] All quotations are taken from the New American Standard Bible (NASB), Copyright © 1960, 1962, 1963, 1968, 1971, 1972, 1973, 1975, 1977, 1995 by The Lockman Foundation.

[2] Marsden, George; 2004; *Jonathan Edwards: A Life*; Yale University Press, p. 1.

[3] Miller, Perry; 1948; *Images or Shadows of Divine Things*; New Haven: Yale University Press.

[4] *Typological Writings*; Works of Jonathan Edwards, Vol. 11; New Haven: Yale University Press; p. 146-153. In the same volume, Anderson Wallace in the "Note on the Manuscript of 'Types,'" p. 145 confirms that this was the first time 'Types' was published.

In this paper, I attempt to advance the research program[5] set forth by Edwards from the standpoint of (i) a Christian who largely shares Edwards' theological convictions, and (ii) a mathematician/statistician who is interested in exploring how the abstract objects of mathematics might typify Christ. My thesis is that the normal probability distribution, which is the center of statistical theory, is a type of Christ. Thirteen individual typological features are described. In writing to an educated but non-mathematical audience, this paper is organized in the following sections: Section 1 is a review of the relevant literature, where some of Edwards' most prescriptive statements for the research program are quoted. To my knowledge, no modern adherent of Edwards' thought has carried out his research program on nature types. Section 2 describes all of the theory regarding the normal distribution necessary to understanding the typological claims made in subsequent sections. Section 3 presents four typological features regarding Christ and the univariate normal distribution. Section 4 presents four typological features regarding Christ and the multivariate normal distribution. Section 5 presents three typological features regarding Christ and the limiting nature of the normal distribution. Section 6 is the conclusion, which includes Table 5, a summary of the thirteen typological features presented in Sections 3-5.

1. Literature Review

There is a lot of material written on the typology of Jonathan Edwards. Since the purpose of this of this paper is to present the normal distribution as a type of Christ in the sense Edwards advocated, my literature review will be selective. In particular, it will focus on framing this paper within the broader context of the literature, and then quoting the relevant passages from Edwards as a foundation for the type presented in subsequent sections.

The starting point for Edwardsian typology necessarily begins with the first publication of Edwards' manuscript *Images or Shadows of Divine Things*[6] in 1948 by Perry Miller, with his controversial introduction.[7] In it, he argued that the relevant background for Edwards' system was (i) the Puritan typology of his day was getting out of hand and that, (ii), Edwards took from the Newtonian "New Science" an affirmation of a set of coherent ontological principles, and (iii) Edwards adopted Locke's epistemological idea of distinguishing perception from thing perceived. With this background in Edwards' mind, Miller argues that Edwards recast "the universe in such a manner that nature and history might be viewed as infinite repetitions of a few eternal rules"[8] in order to "attempt… a second reformation."[9] The result, according to Miller, was a system of uniting nature, history, and Scripture, placing nature on equal footing with Scripture.

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[5] At several points in the paper, I refer to Edwards' "research program." By this I mean Edwards' desire to achieve a unified account of his theology, which would include both biblical types, as well as his nature and history types. I do not mean to imply that he sought adherents to extend his research and views in this area. Indeed, he sought to produce such a work himself. In fact, Edwards appears to have intentionally avoided publication on this issue until his ideas had matured and he could offer a robust apologetic with it. See Wallace, "Note", p. 11ff. for a detailed discussion. Nevertheless, Edwards wrote that the whole world was filled with this typological language of God. So, by "research program" I affirm the merit of Edwards' work and believe it points in a fruitful direction for others to extend. I choose to identify myself with his work in *Types* and seek to make an extension within the parameters he originally set forth.

[6] Hereafter referred to as *Images*.

[7] Miller, "Introduction," p. 1-41. For the record, Edwards' original title was *Shadows of Divine Things*. This was later changed to *Images of Divine Things*. Two other titles Edwards considered were *The Book of Nature and Common Providence* and *The Language and Lessons of Nature*. Miller's title reflects a combination of the first two. In 1993, the 11th volume of the Works of Jonathan Edwards was published, in which editor Anderson Wallace used the title *Images of Divine Things*; p. 50-142.

[8] Ibid., p. 30.

[9] Ibid., p. 24.

This was a novel break from tradition and opened the door for the eventual emergence of Emerson's transcendentalism, a legacy which undid the theology Edwards so carefully sought to construct.[10]

Regarding the Newtonian and Lockean influences, some have uncritically accepted them,[11] while others have undermined or even argued against them.[12] Perry Miller framed Edwards' historical and nature uses of typology as intentionally breaking with his Puritan tradition. Later scholars generally agree that Miller overrstated the case.[13] In particular, Edwards did not place nature and history on the level with Scripture, but gave explicit rules subjecting extra-biblical types to Scripture.[14] Miller's thesis has led to antitheses and, although they have improved our knowledge of Edwards, many questions remain. In the best review of the literature to date, Janice Knight wrote, "This difficulty arises at least in part from the focus of the studies; debate over Miller's claims too often sidetracks examination of Edwards' own writings. The alternative here proposed evaluates Edwards' typology within the context of his own first principles."[15]

Knight's modern approach is the one I seek to follow in this paper. In particular, not enough attention has been given to Edwards' brief notebook *Types*, which, as opposed to his more lengthy *Images* and *Types of the Messiah*[16], which expounded numerous scriptural examples, it succinctly contains much of his philosophy of typology with only the most general Scriptural quotations. I would like to bring more attention to it with some generous quotations.

To preface the quotations, a word about Edwards' view is in order. Explicit biblical types consist of two parts, a type and an anti-type. The type is a real thing in the world which in some way points to a greater reality beyond itself. The anti-type is a spiritual reality that is being pointed to by the type. The anti-type is often, but not always, Christ. Edwardsian examples include, "the sun signifies Christ," and "marriage… is a designed type of the union between Christ and the church."[17] Types, or emblems, are to be distinguished from analogies, or tropes, in that analogies are merely a comparison between two things that do not have an essential relationship, often for the purpose of illustration. For example, 'the sun is like Christ in that it is the chief of all heavenly bodies and rules the world' and 'the level of intimacy between a husband and a wife is a good illustration of the level of intimacy Christ desires with the church.' Types are designed by God. Therefore typological language includes words like "signifies," and "represents" to indicate Authorial intention. Analogies are designed by people in order to convey concepts. Analogies often use "like" or "as."

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[10] Ibid., p. 37. See Knight, Janice; 2005; "Typology;" *The Princeton Companion to Jonathan Edwards*; Lee, Sang Hyun, ed.; Princeton, NJ: Princeton University Press; 190-209, on p. 194. See also Feidelson, Charles, Jr; 1953; *Symbolism and American Literature*; Chicago: University of Chicago Press, p. 99-101, who picks up Miller's idea.

[11] For example, see Hunt, Richard; 2003; "Refiguring an Angry God: The Nature of Jonathan Edwards;" *Interdisciplinary Literary Studies*; 4:21-35.

[12] See Davis, Thomas M; 1968; "The Traditions of Puritan Typology;" Ph.D Dissertation, University of Missouri, p. 385, 387. On p. 527 of Wainwright, William; 1980; "Jonathan Edwards and the Language of God;" Journal of the American Academy of Religion; 48:4, p. 519-530; he explicitly argues against Miller's thesis. In Anderson Wallace's authoritative "Editor's Introduction to 'Images of Divine Things and 'Types;"' p. 3-33; without addressing Miller's theses directly, he provides additional data which undermine the extent of Miller's theses, for example, Newton on p. 16, and Locke on p. 23-24.

[13] See Davis, p. 374, 383, 390-391 and Wainwright, p. 527 and Lowance, Mason I., Jr; "Typology and Millennial Eschatology in Early New England;" in *Literary Uses of Typology: from the Late Middle Ages to the Present*; Ed. Earl Miner; Princeton, NJ: Princeton University Press; p. 273.

[14] See "Proposition #2" below.

[15] Knight, p. 194. This is also the approach of Wallace, "Introduction."

[16] *Typological Writings*; Works of Jonathan Edwards, Vol. 11; New Haven: Yale University Press; p. 191-328.

[17] Edwards, Images no. 5 (p. 52) and no. 9 (p. 53).

Types are right or wrong, to the degree that they reflect God's intention whereas analogies are merely good or bad, to the degree to which they communicate the human's idea. The innovation of Edwards was that he provided a theological and philosophical basis for expanding the set of types from explicitly biblical types to include extra-biblical types, particularly from nature and history, of other spiritual realities.[18] Wainwright gives a particularly lucid discussion of this issue.[19]

Below are three key quotations from *Types*, along with a proposition derived from each, which form the grounds for the type presented in later sections.

> *"Types are a certain sort of language, as it were, in which God is wont to speak to us. And there is, as it were a certain idiom in that language which is to be learnt the same that the idiom of any language is…. Great care should be used, and we should endeavor to be well and thoroughly acquainted, or we shall never understand [or] have a right notion of the idiom of the language. If we go to interpret divine types without this, we shall be just like one that pretends to speak any language that han't thoroughly learnt it…. God han't expressly explained all the types of Scriptures, but has done so much as is sufficient to teach us the language."*[20]

Proposition 1: Types are a certain sort of language which God speaks and can be learned by man.

> *"First, to lay down that persons ought to be exceeding careful in interpreting of types, that they don't give way to a wild fancy; not to fix an interpretation unless warranted by some hint in the New Testament of its being the true interpretation, or a lively figure and representation contained or warranted by an analogy to other types that we interpret on sure grounds."*[21]

Proposition 2: Extra-biblical types are permitted if an analogy can be made to a sure biblical type.

> *"I expect by very ridicule and contempt to be called a man of a very fruitful brain and copious fancy, but they are welcome to it. I am not ashamed to own that I believe the whole universe, heaven and earth, air and seas, and the divine constitution and history of the holy Scriptures, be full of images of divine things, as full as a language is of words; and that the multitude of those things that I have mentioned are but a very small part of what is really intended to be signified and typified by these things: but that there is room for persons to be learning more and more of this language and seeing more of that which is declared in it to the end of the world without discovering all."*[22]

Proposition 3: There are many natural and historical types God intended to signify that await discovery.

While I may not fully understand all of the nuances of Edwards' theology as a foundation for the best appreciation of his typology, my reading of Edwards' Types and Images has been an eye-opening

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[18] Nevertheless, Edwards' catalog was primarily biblical types (on his view), though he did include some extra-biblical, including nos. 76, 95, 142, 146, 156, 196, and 147.

[19] Wainwright, p. 524.

[20] Edwards, *Types*, 151.

[21] Edwards, *Types*, 148.

[22] Edwards, *Types*, 152. Throughout the work, Edwards discusses Ps 78:2; 125:1-2; John 9:7; 6:31-32; Rom 5:14; Gal 4:21-23; 1 Cor 9:9-10; 10:1-4, 6, 11; 13:2; Heb 4:3, 5:6, 11; 7; 8:2, 4-5; 9:1-4, 5, 8-11, 22-24; 10:1; 11:19; 13:11-13; 2 Cor 3:13-14; and Col 2:16-17. This list includes all Bible references in *Types* except those in his argument from the permutation of names, p. 150.

experience.[23] Upon reading, I have converted from seeing the world through analogy, to seeing it through typology—including my field of mathematics/statistics. For the purposes of this paper, I assume the above three propositions on typology. What follows is an attempt to explain one such remarkable phenomenon within this framework.[24]

2. Normal distribution Theory

The normal distribution is by far the most special probability distribution. In order to appreciate its remarkable status, we must first understand the class of objects among which the normal distribution is special: probability distributions. *A probability distribution is a graph, table, or formula that pairs each value of the thing measured, denoted by the variable* X, *with its associated probability*. For example, *Table 1* shows the number of *1's*, *2's*, and so on which were obtained when my kids rolled a fair *6*-sided die *60* times. The probabilities are obtained by dividing the frequency by *60*, e.g. The number *1* was rolled *8* times, so the probability of rolling a *1* was *8/60 = 0.13*. Notice that the probabilities sum to one, or *100%*. Figure 1a shows the same distribution, except in the form of a *graph*. It is not possible to write a *formula* for this exact distribution, since it was obtained empirically and would vary if the experiment were repeated. However, it is possible to write the formula of the theoretical distribution, which is, *probability=1/6* for all *x=1,2,…,6*. The graph of this theoretical distribution is shown in Figure 1b.

X	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
frequency	8	11	7	9	13	12	60
probability	8/60 = 0.13	11/60 = 0.18	7/60 = 0.12	9/60 = 0.15	13/60 = 0.22	12/60 = 0.20	60/60 =1.00

Table 1: Empirical probability distribution table of *60* tosses of a fair 6-sided die.

As another example, Figure 2 shows the empirical distribution of *67* students from one of my statistics classes.[25] This is a *graph*, like Figure 1a, that contains the probability a student in my class was at different heights. This probability distribution could also be expressed in a *table*, like Table 1. While a formula cannot be given for this empirical distribution (like Figure 1a), the *formula* for a theoretical model approximating it can be written.[26] A graph of this formula is the superimposed curve on Figure 2. Note again that the sum of the probabilities, which is the area of all of the bars, sums to one, or *100%*. Thus, we have two examples of common probability distributions, both of which have been considered in the three equivalent forms of *graph*, *table,* and *formula*.

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[23] See Lee, *Contemporary Theology;* Anderson, *Introduction;* and Knight, *Typology,* for three recent, sophisticated, and different attempts at synthesizing Edwards' theology as pertains to typology.

[24] As a Christian, I would like to acknowledge the following quotation from Wainwright, in support of my work, "If classical Christianity is correct, similar clues are available to those who wish to interpret the emblematic discourse constituted by the world of nature. Since God is the speaker, we can assume that what is expressed by types and emblems is truthful, significant, internally consistent, and coherent with reason and Scripture." Wainwright, 526.

[25] Probability and Statistics for Engineering and the Sciences, UC Riverside, Summer 2004.

[26] See Figure 8a for the formula.

Now that we understand what a probability distribution is, we turn to the number of them. On my bookshelf, the Handbook of Statistical Distributions[27] saw fit to include *40* distributions because of their importance for practical statistical work. On my wall is a map of *75* probability distributions and their theoretical relationships to one another.[28] The normal distribution is the most special distribution among all of the more than *75* known probability distributions.

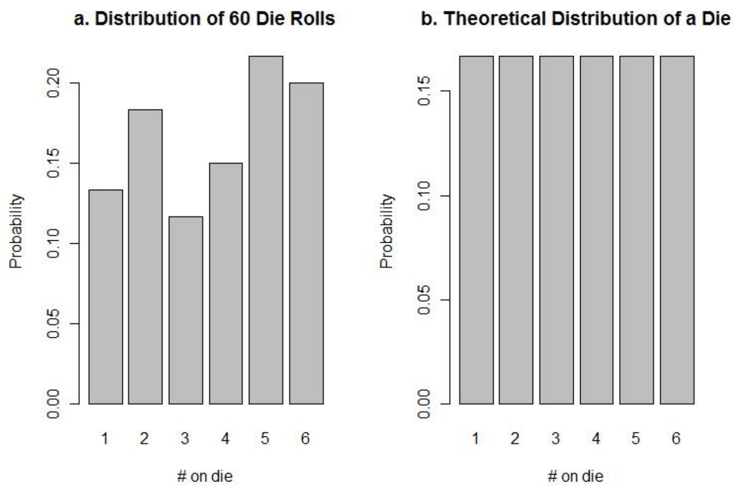

Figure 1: Graphs of the empirical and theoretical distribution of a *6*-sided die. a. Empirical probability distribution graph of *60* tosses of a fair *6*-sided die. b. Theoretical distribution graph of a fair *6*-sided die. Note that the number of tosses does not matter, as the expected probability is always *1/6* for each face.

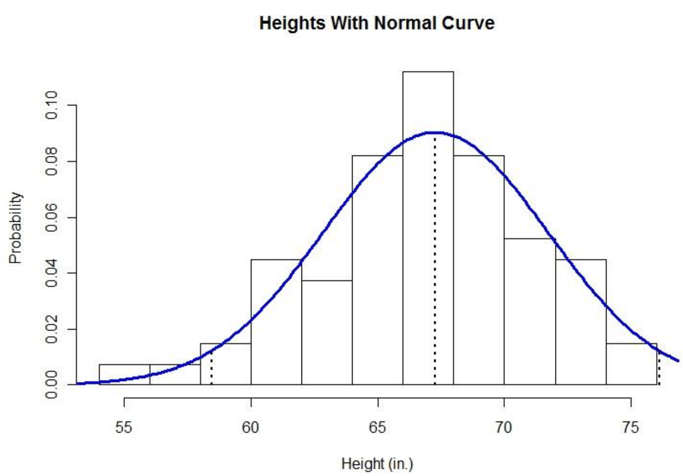

Figure 2: Empirical probability distribution of the heights of *67* students, in inches, with the theoretical normal distribution curve superimposed. The dotted lines mark the *mean* and the points two *standard deviations* above and below it.

[27] Evans, Merran; Nicholas Hastings; and Brian Peacock; 2000; *Statistical Distributions*, 3rd Ed; New York, NY: John Wiley & Sons.

[28] Special insert in *The American Statistician*; Vol. 62; Feb 2008; of Figure 1 from Leemis, Lawrence M. and Jacquelyn t. McQueston; Univariate Distribution Relationships, p. 45-53.

In order to understand what makes the normal distribution so special, we next consider some of the most common probability distributions, for comparison. Figure 3 shows the graph of six of them, along with their parameters. The parameter(s) of a distribution are the features which are used to uniquely define the distribution. Parameters themselves are variables. The distribution for the *6*-sided die is called the uniform distribution, with one parameter, *k*, the number of categories. For the *6*-sided die in Table 1 and Figure 1, *k=6*. If one were to apply this distribution to the California State lottery with *48* balls, we would use *k=48*. Each of the distributions in Figure 3 has its parameter(s) listed. The meaning of each parameter is not important for our purposes. What is important is that each distribution has one or more parameters and the parameters' nature (as expressed by its name(s)) is different for each distribution.

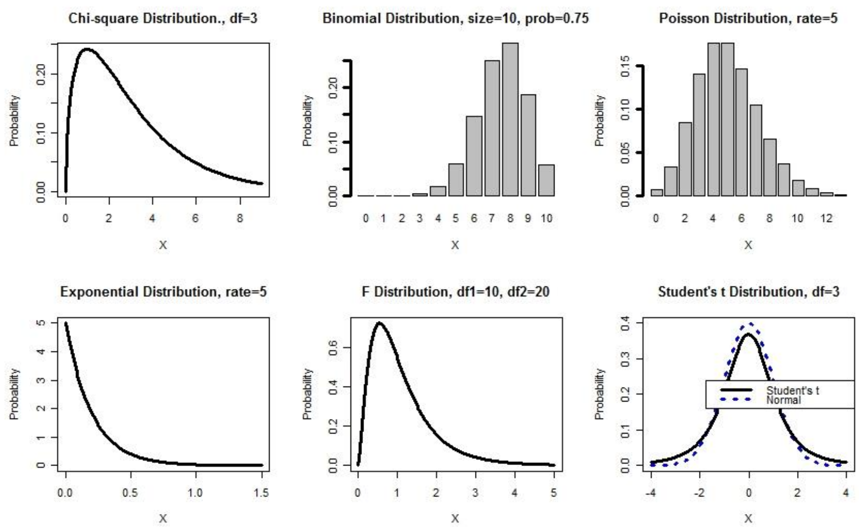

Figure 3: Six common probability distributions. The name is given at the top of each distribution, followed by the parameter(s) used to create the distribution. The *x*-axis label is simply "*x*" denoting the possible values the variable can take on. The Student's *t* distribution appears like the normal distribution, but has slightly fatter tails and a shorter mound, so the normal distribution has been added to show this.

The normal distribution has two parameters, *mean* and *standard deviation*, which we will need later. The *mean* is the average of the distribution. The *standard deviation* is an average distance away from the mean. For example, the mean of the height data is *67.3* inches, which is the exact center of the graph (Figure 2). The standard deviation of the height data is *4.4* inches, meaning that if we made a list of the number of inches away from *67.3* each student's height was, then the 'average' of these deviations would be *4.4* inches per student. If you go two standard deviations above and below the mean of any distribution, then at least *75%* of the data will lie in this range.[29] In Figure 2, *2*4.4=8.8* inches above and below the *mean* of *67.3* inches is *58.5* to *76.1* inches. These lines are marked on the graph, where it is clear that much more than *75%* of the data is contained within this interval.[30]

In order to frame the typology of the normal curve, I will begin with a summary of the features of the normal distribution which will be referenced in subsequent sections. For ease of reference, the relevant features are in a numbered list. I recommend you attempt to understand these concepts on the first read, but

[29] This result is result follows from Chebyshev's Theorem.

[30] In fact, *65* out of *67* points, or *97%* of the data is within the interval.

do not spend too much time on them. Note the main ideas and reference them as needed during the typological portions.

1. **Parameters**. Almost all of the probability distributions in use today are defined in terms of their *natural parameters*, which are the parameters which occur in the simplest possible version of the probability distribution formula. All of the parameters in Figure 3 are natural parameters. The *mean* and *standard deviation* of the normal curve are its natural parameters.[31]

2. **Univariate probability distribution**. A probability distribution involving only one variable. The tables and figures considered thus far are all univariate distributions (Table 1, Figure 1, Figure 2, and Figure 3).

3. **Joint probability distribution**. When k variables are measured simultaneously, then the joint probability of the different combinations of the variables' occurrence can be obtained in the same way as a single variable. This is a *k-variate distribution*. There is no limit to how large k can be. When $k=2$, the graph of the distribution becomes three-dimensional, with the two variables on the x and y axes, and the joint probability on a third axis. For example, consider a triple coin toss wherein three coins are tossed. Let X be the sum of the number of heads, which could be $0,1,2$, or 3. Let Y be the [absolute] difference between the number of heads, which would be 3 if there were 3 or 0 heads ($3-0=3$) and 1 if there were 1 or 2 heads ($2-1=1$). Table 2 shows the joint probability distribution of the two variables, *X=sum of heads* and *Y=difference of heads and tails*.[32] Figure 4 shows the distribution of Table 2 in the form of a graph. In order to understand joint probability distributions, the reader should study this example until it is understood.

4. **Marginal distribution**. A k-variate joint probability distribution has k marginal distributions, one for each $i=1,2,,...,k$. The marginal distribution is the univariate probability distribution of one variable, independent of the other $k-1$ joint variables. For example, in Table 2 the marginal distribution of the sum of the number of heads is given in the bottom row. It can be obtained by summing the rows labeled 1 and 3. The marginal distribution of the difference of the number of heads is given in the last column. It can be obtained by summing across the columns labeled $0,1,2$, and 3.[33]

5. **Conditional distribution**. A univariate probability distribution which depends on one or more other variables. When $k=2$, we write, *the conditional distribution of X, given Y*. For example, Table 3

[31] Technically, all of the above distributions could be defined in different, but equivalent, ways. If they were, then the parameters would change according to the modified definition, while retaining the same probability distribution. The use of natural parameters allows an essentially unique formulation.

[32] To fully understand Table 2, let H = *heads* and T = *tails*. The eight possible outcomes are: *HHH, HHT, HTH, THH, HTT, THT, TTH*, and *TTT*. Each of these events is equally likely and so there is a $1/8$ probability of obtaining *HHH*, which is $X=3$ and $Y=3-0=3$. There is a $1/8$ probability of obtaining *HHT*, which is $X=2$ and $Y=2-1=1$. However, *HTH* and *THH* are similar, giving a $1/8+1/8+1/8=3/8$ probability for $Y=1$. The other cases are similar.

[33] For example, to obtain the marginal distribution value $Y=1$, there is a $3/8$ probability of obtaining $Y=1$ when $X=1$ and a $3/8$ probability of obtaining $Y=1$ when $X=2$. When $X=0$ and $X=3$, there is zero probability. Therefore, the cumulative probability for $Y=1$, independent of X, is $0+3/8+3/8+0=6/8$.

gives the conditional probability distribution of the sum of heads, given the difference of heads. Thus, there are two conditional distributions shown in the table: '*X*, given *Y=1*' and '*X*, given *Y=3*.' The conditional distribution is obtained by taking the row from Table 2 and dividing it by the marginal probability of that row.[34]

6. **Correlation**. The natural parameter for the bivariate normal distribution which describes the strength of linear association between the two variables is the *correlation coefficient*. *Correlation* is a number between *0* and *1*, where zero is no linear correlation and *1* is perfect linear correlation. As an additional example of a joint distribution, see Figure 5. Here are three bivariate (*k=2*) normal distribution plots with different correlations. *Correlation=0* appears as a classic, three-dimensional bell. Keep in mind that the *x* and *y* axes on the bottom are the normally distributed variables (see the contour plots), while the *z*-axis coming up from them is the probability a particular realization is within a specified range of *x* and *y*. *Correlation=0.50* and *0.90* show the effect on the shape of the distribution. In particular, the closer correlation is to *1*, the more likely *x* and *y* are to both be large or both be small. It is also possible for *correlation* to be negative, at which case it is between *-1* (perfect negative linear correlation) and *0* (no correlation). If *correlation* were negative, then the tendency reverses to *x* being large while *y* is small, and vice versa.

diff \ sum	**0**	**1**	**2**	**3**	**marginal dist$_{diff}$**
1	0	3/8=0.375	3/8=0.375	0	6/8=0.75
3	1/8=0.125	0	0	1/8=0.125	2/8=0.25
marginal dist$_{sum}$	1/8=0.125	3/8=0.375	3/8=0.375	1/8=0.125	8/8=1.00

Table 2: Joint Probability Distribution of Triple Coin Toss Sums and Differences. The entries in rows labeled *1* and *3* paired with the columns labeled *0,1,2,* and *3* are the joint probabilities of the difference and sum, respectively, of the triple coin toss. Notice that they sum to *1.00*, or *100%*, which they must in order to be a distribution. The bottom row is the marginal distribution of the sum. The last column is the marginal distribution of the difference. Notice also that both marginal distributions sum to *1.00*.

[34] For example, the way to obtain the conditional$_1$ row of Table 3, is to take each of the entries of row 1 from Table 2, {*0, 3/8, 3/8, 0*}, and divide each of them by *6/8*, which is the marginal sum. This gives {*0, 1/2, 1/2, 0*}, which is the probabilities of the conditional distribution given in Table 3.

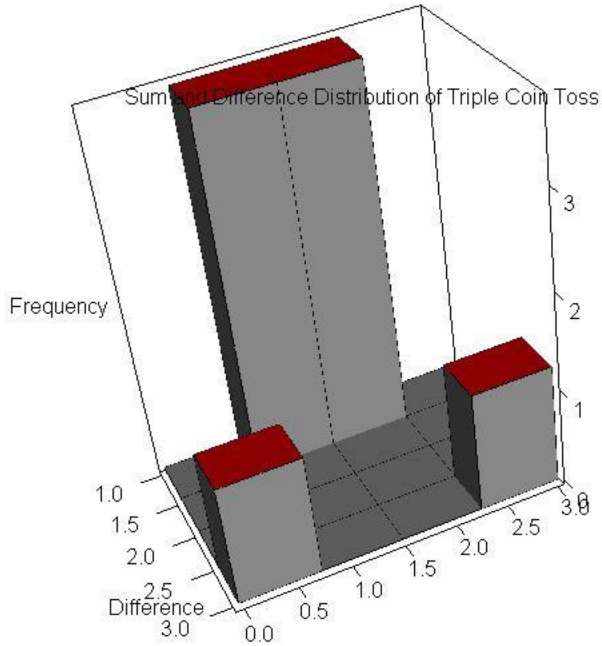

Figure 4: Sums and Differences of Triple Coin Toss. Three dimensional graph of the joint distribution of the triple coin toss from Table 2.

sum	**0**	**1**	**2**	**3**	**Prob$_{diff}$**
conditional$_1$	(0) / (6/8) = 0	(3/8) / (6/8) = 1/2	(3/8) / (6/8) = ½	(0) / (6/8) = 0	(6/8) / (6/8) = 1
conditional$_3$	(1/8) / (2/8) = 1/2	(0) / (2/8) = 0	(0) / (2/8) = 0	(1/8) / (2/8) = 1/2	(2/8) / (2/8) = 1

Table 3: Conditional Probability Distribution of Differences 1 and 3 of the Triple Coin Toss. The conditional$_1$ row is the conditional distribution of the sum of the dice, given that the difference is *1*. It is obtained by taking the *1* row of Table 2 and dividing it by its row sum (marginal probability), which is *6/8*. The conditional$_3$ row is the conditional distribution of the sum of the dice, given that the difference is *3*. It is obtained by taking the *3* row of Table 2 and dividing it by its row sum (marginal probability), which is *2/8*.

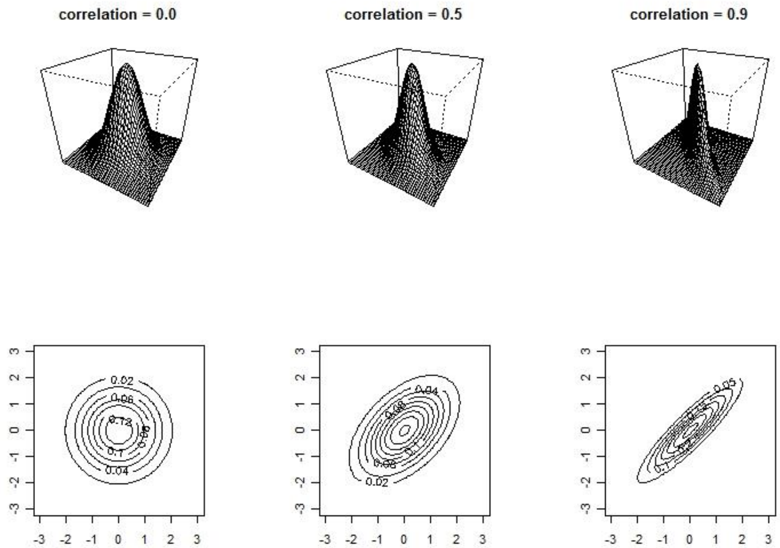

Figure 5: Bivariate Normal Distributions with Contour Plots. The first row contains three bivariate normal distribution plots, with correlations *0, 0.5*, and *0.9*. The second row contains a contour plot (horizontal cross sections of distribution plots) for each of the three distributions. Notice that the contours of the *correlation=0* plot are circles, whereas the contours of the non-zero correlation distributions are ellipses. The ellipses narrow as the correlation increases. The contours shown do not line up vertically with the distributions because the graphs of the three-dimensional distributions have been rotated in order to provide the best view of the effect of the correlation.

The normal curve is widely regarded as the most important probability distribution in all of mathematics. Here are three reasons. First is community recognition. Browsing the Table of Contents of over thirty introductory Probability and Statistics textbooks on my bookshelf revealed the majority of texts have an entire chapter devoted to the normal distribution, with no other chapters devoted to any other probability distribution. A Google search reveals 4.99 million hits for a search of "normal distribution," which is more than five times that of the second most common distribution, chi-square (see Table 4). Second is parameters. The natural parameters of the normal distribution are the mean and standard deviation. The mean is such a commonly used parameter that the fact that it is a natural parameter of the normal distribution, but not a natural parameter of almost all other distributions, is virtually unknown and unappreciated by non-statisticians. It is so fundamental to our thinking that we calculate the mean of other distributions from their natural parameters. The standard deviation, while not as commonly known as mean, is also routinely computed from the parameters of other distributions when working with them. Again, for the bivariate normal distribution, the natural parameter between two jointly distributed variables is the correlation coefficient. This is precisely the relationship of interest between numeric jointly distributed variables, yet it is the parameter of no others. Nevertheless, the correlation coefficient is still routinely calculated between non-normal jointly distributed variables precisely because it is so useful. Third is frequency of occurrence. When it comes to nature, the normal distribution is ubiquitous. Many physical measurements (e.g. heights), non-physical measurements (e.g. IQ scores), random phenomena (e.g. gambling), and errors (e.g. mismeasurements), are normally distributed. Thus, both statisticians and nature give the greatest attention to the normal distribution.

This attention may be summarized in the name, **normal.** All other distribution names are specifically related to either their nature or their discoverer (see the third column of Table 4). The normal distribution has both, but even more. Popularized by the famous mathematician Friedrich Gauss, the normal distribution is sometimes referred to as the "Gaussian" distribution (1.61 million Google hits). On the other hand, its nature is such that it is a symmetric mound with specific proportions. However, rather than being named as such, its frequency and universality has led to the name "normal" – which goes beyond its nature, or its frequency, but extends to what is expected from any unknown distribution – the default, if you will.[35]

Thus we have attempted to describe the salient features of the normal distribution in preparation for showing its special place among the distributions, which is reflected by its name. It is fitting, therefore, to make the first typological point with its name. Philippians 2:9 says, "God highly exalted [Jesus] and bestowed on Him the name which is above every name." **That the name "normal" is unique and above all other distribution names is an emblem of the uniqueness of the name of Jesus and His name which is above every other name.** The case for of the special place and name will follow from the evidence of the next sections.

Google Search Entry	Number of hits	Note on Name
"normal distribution"	4,990,000	Frequency
"chi square distribution"	1,040,000	Nature: square of the normal (X~Normal, χ^2~chi-square)
"binomial distribution"	1,010,000	Nature: two categories—success and fail (binary)
"Poisson distribution"	987,000	Discoverer: Poisson
"exponential distribution"	728,000	Nature: shape follows exponential curve
"F distribution"	529,000	Discoverer: R.A. Fisher
"Student's t distribution"	406,000	Discoverer: Pseudonymous author named "Student"

Table 4: Number of hits for the most commonly used probability distributions. Numbers were obtained by typing the entry in the first column into Google on 6/20/2012. The second column records the reported number of hits. The third column notes the origin of the name. In particular, there are two origins for the names of probability distributions, namely the discoverer or the nature of the distribution.

3. Graphs of the Univariate Normal Distribution

In this section, I propose four ways in which the graphs of the univariate normal distribution typify Christ, one per subsection. The format of each subsection is the same: a Scripture quotation followed by a brief explanation of the type, which will be given somewhere in **bold**. The Scripture(s) are intended to indicate the manner in which Proposition #2 in Section 1 is, or could be, satisfied, which would justify the type. It is assumed the reader is familiar with the biblical context and meaning of the passages quoted, as only brief explanations for less obvious cases are given. Explanations are not intended to be exhaustive, but

[35] When considering a distribution, this allows us to say things like, "Is it normal, or not?" or "It's close to normal."

only to incline the reader to the proposed type in order that they might "hear" it with their own spiritual ears and discern for themselves whether or not it resonates with the divine tone.

Beauty

One thing I have asked from the LORD, that I shall seek:
That I may dwell in the house of the LORD all the days of my life,
To behold the beauty of the LORD
And to meditate in His temple.
~Psalm 27:4

And He is the radiance of His glory and the exact representation of His nature…
~Hebrews 1:3

The beauty of the graph of the normal curve signifies the beauty of Christ (Figure 6). In fact, the shape of the graph is so noteworthy that it has been given a special name, "the bell curve." Imagine steadily pouring out a bucket of sand on a flat surface. A mound emerges. Next, imagine a knife slicing through the mound perpendicular to the bottom, from any position on the mound. The cross section of the cut would be a bell curve. If you were to compress this mound on two sides by two large parallel pieces of glass very close to one another, like an ant farm, then the resultant shape would also be a bell curve. [36] This amazing three dimensional mound turns out to be the shape upon which bells used to be cast, as they give the greatest resonance.

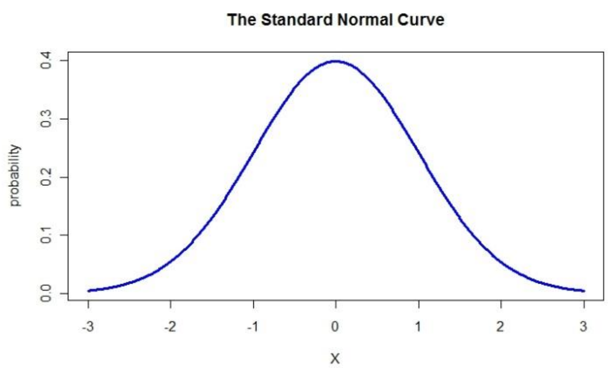

Figure 6: Graph of the standard normal curve, also known as the bell curve.

Finite and infinite

In the beginning was the Word, and the Word was with God, and the Word was God…. And the Word became flesh, and dwelt among us, and we saw His glory, glory as of the only begotten from the Father, full of grace and truth.
~John 1:1,14

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[36] The shape formed is a bivariate normal distribution and the cross sections are conditional distributions while the compressed version is the marginal distribution. See the top row of Figure 5. Many of the univariate distributions have a multivariate extension, with corresponding conditional and marginal distributions, so this phenomenon is not part of the typological argument. The type is the beauty. Typological features of the multivariate normal will be proposed in subsequent sections.

Therefore, following the holy fathers, we all with one accord teach men to acknowledge one and the same Son, our Lord Jesus Christ, at once complete in Godhead and complete in manhood, truly God and truly man, consisting also of a reasonable soul and body; of one substance with the Father as regards his Godhead, and at the same time of one substance with us as regards his manhood….
~Definition of the Council of Chalcedon (A.D. 451)[37]

According to orthodox Christian doctrine, Christ is one person, yet with two natures, fully God and fully man simultaneously. The graph of the normal curve has no beginning, and no end, as it never touches the *x*-axis on either side, continuing infinitely on both sides of the mean (Figure 6). This represents the divine nature of Christ. Nevertheless, the area under the normal curve is one, which is finite.[38] This represents the human nature of Christ.

Multi-faceted

Then the kingdom of heaven will be comparable to ten virgins, who took their lamps and went out to meet the bridegroom.
~Matthew 25:1

Then the King will say to those on His right, 'Come, you who are blessed of My Father, inherit the kingdom prepared for you from the foundation of the world.'
~Matthew 25:34

For we must all appear before the judgment seat of Christ, so that each one may be recompensed for his deeds in the body, according to what he has done, whether good or bad.
~ 2 Corinthians 5:10

Christ, the second person of the Trinity, is given different names in the Bible to emphasize different aspects of His nature, His functioning in that instance. In passages regarding the end times, a favorite topic of Edwards, Christ is referred to as Bridegroom, King, and Judge. For a particular use of the normal curve, it is defined by a particular configuration of parameters, of which three are shown in Figure 7. A model with large variance would look more like the 'Judge' curve whereas a model with more certainty would look like the 'King' curve.[39] **The way in which there are many functions of the normal distribution, yet they are all referred to as the one normal distribution, represents how Christ has many names and yet is one. The way that different parameter configurations show the various facets of the normal distribution signifies how different names show the various facets of Christ.**

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[37] http://www.reformed.org/documents/index.html

[38] Q: How can this be? A: The tails of the curve grow increasingly closer and closer to the line, without ever touching it. As such, they have no end, yet the area underneath the tails decreases at such a rapid rate that there is so little underneath that it sums to a finite number. This phenomenon is not unique to the normal curve. Many probability distributions, as well as other mathematical objects have this property. In Edwardsian language, if I am correct in this typology, then they are also images of divine things (in this case, the two natures of Christ), but they are images further removed from the type. Similar remarks could be made for some of the other typological connections in this section. Note, however, that while other mathematical phenomena may exhibit the features collected here for the normal curve, they are disparate phenomena, whereas here they are united in the one remarkable normal distribution. In particular, the features described in Section 5 are not exhibited by other mathematical phenomena.

[39] In fact, when working with the normal distribution without a computer, it is common practice to convert all normal distributions to 'the standard normal distribution,' and use the one table for the standard normal distribution for calculations. All normal distributions can be put into one-to-one correspondence with the standard normal.

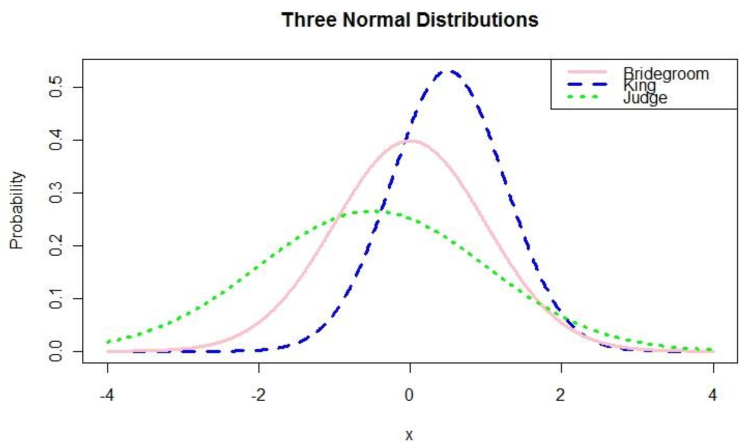


Figure 7: Three different normal distributions, Bridegroom~Normal(*mean* = *0.5*, *SD* = *0.75*), King~Normal(*mean* = *0*, *SD* = *1*), and Judge~Normal(*mean* = *-0.5*, *SD* = *1.5*).

Diversely expressed

> *Now it came about when Joshua was by Jericho, that he lifted up his eyes and looked, and behold, a man was standing opposite him with his sword drawn in his hand, and Joshua went to him and said to him, "Are you for us or for our adversaries?" He said, "No; rather I indeed come now as captain of the host of the LORD." And Joshua fell on his face to the earth, and bowed down, and said to him, "What has my lord to say to his servant?" The captain of the LORD'S host said to Joshua, "Remove your sandals from your feet, for the place where you are standing is holy." And Joshua did so.*
> *~Joshua 5:13-15*

> *Does not wisdom call,*
> *And understanding lift up her voice? …*
> *"To you, O men, I call,*
> *And my voice is to the sons of men….*
> *"The LORD possessed me at the beginning of His way,*
> *Before His works of old….*
> *"When He established the heavens, I was there,*
> *When He inscribed a circle on the face of the deep,…*
> *"Then I was beside Him, as a master workman;*
> *And I was daily His delight,*
> *Rejoicing always before Him…."*
> *~Proverbs 8:1,4,22,27,30*

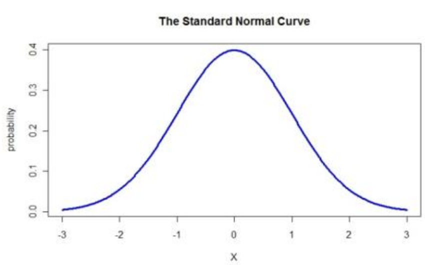
> *And the Word became flesh, and dwelt among us, and we saw His glory, glory as of the only begotten from the Father, full of grace and truth.*
> *~John 1:14*

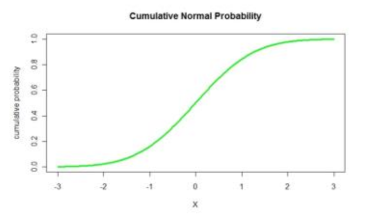
> *"I kept looking in the night visions,*
> *And behold, with the clouds of heaven*
> *One like a Son of Man was coming,*

And He came up to the Ancient of Days
And was presented before Him.

"And to Him was given dominion,
Glory and a kingdom,
That all the peoples, nations and men of every language
Might serve Him.
His dominion is an everlasting dominion
Which will not pass away;
And His kingdom is one
Which will not be destroyed.
~Daniel 7:13-14

Christ's manifesting Himself in different ways is signified by the manner in which the normal curve may be expressed in different ways. It is the same thing, but in a different form. Four examples of manifestations of the second person of the Trinity, Christ, are quoted above. They are the Captain of the LORD's host who received worship, the Wisdom of God which created the world, the God-man Jesus of Nazareth , and the Son of Man who will be given dominion over God's everlasting kingdom. Four examples of substantively different expressions of the one normal distribution are shown in Figure 8.

a. Normal density function	b. Graph of normal density function
$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma}}$	
c. Graph of cumulative normal dist. function	d. Key regions of normal distribution function



Interval	Probability
mean-3SD to mean-2SD	0.02
mean-2SD to mean-SD	0.14
mean-SD to mean	0.34
mean to mean+SD	0.34
mean+SD to mean+2SD	0.14
mean+2SD to mean+3SD	0.02

Figure 8: Different expressions of the standard normal distribution. a. The functional form of the normal distribution, in terms of mean μ and standard deviation σ. Note that *e* is the base of the natural logarithm, approximately *2.718*. b. The graph of the normal density function given in (a) with $\mu=0$ and $\sigma=1$. c. The graph of the cumulative normal density, where the *y*-axis which is obtained by taking the area of the normal probability distribution in (c), of all values less than *x*. d. Tablular form of some of the common areas under the normal curve.

4. Multivariate Normal Distribution

In this section, I propose four ways in which the multivariate normal distribution typifies Christ. The format is similar to Section 3. The features of the normal distribution described in Section 2 will be drawn upon heavily.

Multi-dimensional

> *Great is our Lord and abundant in strength;*
> *His understanding is infinite.*
> *~Psalm 147:5*

> *[Christ] is the image of the invisible God, the firstborn of all creation. For by Him all things were created, both in the heavens and on earth, visible and invisible, whether thrones or dominions or rulers or authorities--all things have been created through Him and for Him. He is before all things, and in Him all things hold together.*
> *~Colossians 1:15-17*

The infinite dimensions of the multivariate normal distribution reflect the infinite dimensions of Christ. The classic attributes of deity are omnipotence, omniscience, and omnipresence; all these and more are in view. Furthermore, the way in which the multivariate normal density expresses exactly the distribution which cannot be graphed in dimensions higher than four[40] signifies how Christ expressed in finite form the exact representation of the Father who cannot be seen.[41] If X is a vector of k random variables, μ is a vector of their k means, and Σ is a variance-covariance matrix,[42] then $f(x)$ is the exact functional form of the multivariate normal distribution in Figure 9. The graph of some bivariate normal

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[40] Color can be added to a three dimensional graph in order to represent a fourth dimension, although this is difficult to interpret.

[41] John 1:18, "No one has seen God at any time; the only begotten God who is in the bosom of the Father, He has explained *Him*."

[42] A *variance-covariance matrix* is a natural parameter of the multivariate normal distribution. It is the multivariate analog of the SD of the univariate normal distribution. It is a matrix of the *correlation coefficient* between each pair of the *k* variables of *X*, scaled by the *standard deviations* of each of the *k* variables.

distributions ($k=2$) are given in Figure 5. No matter how large k becomes, the functional form *f(x)* in Figure 9 remains the same.

$$f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp[-\frac{1}{2}(x-\mu)^T \Sigma^{-1/2} (x-\mu)]$$

Figure 9: Multivariate normal distribution function. If *x* is a vector of *k* variables, μ is a vector of their *means*, and Σ is a *variance-covariance matrix*, then *f(x)* is the functional form of the multivariate normal distribution.

Fullness (conditional)

> *That Christ may dwell in your hearts through faith; and that you, being rooted and grounded in love, may be able to comprehend with all the saints what is the breadth and length and height and depth, and to know the love of Christ which surpasses knowledge, that you may be filled up to all the fullness of God.*
> *~Ephesians 3:17-19*

Any conditional distribution of the multivariate normal distribution is itself normal. Furthermore, its parameters are completely defined in terms of the original parameters. This elegant and beautiful property of the normal distribution is not true of multivariate distributions in general.[43] **That the conditional is a cross-section for an individual variable of *x*, itself fully normal but dependent upon *x*, typifies how Christ reveals the fullness of God**.

For example, let ***x*** follow a multivariate normal distribution with $k=3$ variables, perhaps x_1 = "breadth" and x_2 = "length" and x_3 = "height." Now, each of the three variables of ***x***,has a conditional distribution, e.g. height (x_3), given fixed values of breadth (x_1) and length (x_2). This variable, 'height given fixed breadth and length,' is normally distributed. Graphically, this could be seen in Figure 5 where a cross section andy of the 3D curves would itself be a bell curve. Thus, not only is the whole normal, but each dimension (variable) within it is normal as well. A 'fullness' of the normal distribution is revealed

Complete (marginal)

> *God, after He spoke long ago to the fathers in the prophets in many portions and in many ways, in these last days has spoken to us in His Son, whom He appointed heir of all things, through whom also He made the world. And He is the radiance of His glory and the exact representation of His nature, and upholds all things by the word of His power.*
> *~Hebrews 1:1-2*

The *k* marginal distributions of a *k*-variate normal distribution are all normal. Furthermore, their parameters are completely defined in terms of the original parameters. Like the conditionals, this elegant and beautiful property is shared by few multivariate distributions. **The fact that the marginal is fully normal (the full curve) is emblematic of the person of Jesus being fully God**.

5. Limits

In this section, I propose three final ways in which the normal distribution typifies Christ. It differs importantly from the previous sections. Whereas Sections 3 and 4 proposed specific features of the normal distribution as typifying Christ, the phenomena in this section are more fundamental and comprehensive. They refer to mathematical theorems regarding the nature of the normal distribution and its relation to many

[43] It is true of a few. The multinomial and the multivariate Student's t come to mind.

other distributions. While some of the previous items proposed are interesting and provocative, in the absence of the phenomena of this section, the case is incomplete. Given the enormity of the Central Limit Theorem in Statistics and the world, I find the case becomes compelling.

The Central Limit Theorem

> *He is before all things, and in Him all things hold together. He is also head of the body, the church; and He is the beginning, the firstborn from the dead, so that He Himself will come to have first place in everything. For it was the Father's good pleasure for all the fullness to dwell in Him, and through Him to reconcile all things to Himself, having made peace through the blood of His cross; through Him, I say, whether things on earth or things in heaven.*
> *~Colossians 1:17-20*

> *The center of anything always exerts a very powerful drawing force. The fact is even more true in the spiritual realm. On the one hand, there is a drawing force in the center of your being; it is powerful and irresistible. And on the other hand, there is also a very strong tendency in every man to be reunited to his center. The center is not only drawing the object away from the surface, but the object itself tends toward its center! As you become more perfected in Christ, this tendency to be drawn within to the Lord becomes stronger and more active.*
> *~ Jean Guyon (c. 1685)*[44]

The most important theorem in all of statistics is called the Central Limit Theorem (CLT). It says that, for any single variable, if you take a bunch of random samples, and then take their means, that the distribution of those means (not the individual data points in all of the samples) tends to the normal distribution.[45] This is for a single variable. If you collect k variables simultaneously, and take the k means of each of a bunch of such samples, then the distribution of those sets of k means tends to the k-variate normal distribution. This theorem is surprising and remarkable. It is very easy to envision a world where the distribution of means for one distribution is one thing, and for another distribution is another. But this is not our world. In our world, every empirical distribution's means converge, inexorably, to the normal distribution.

Consider a world where there was no CLT -- no regular pattern followed by sample means or other measures of center. The business world would lose the forecasting power of time series techniques, industrial quality control would be hamstrung without control charts, scientific research would grope in the dark without regression and t-tests, economists would be back to the drawing board without their normal models, agriculture research would not have gotten off the ground without ANOVA, and I could go on – each of these foundational techniques resting on the assumption of normality. This is not to imply that there are no alternative approaches, because there are. However, the foundation of the statistical theory behind these and similar applications is the normal distribution. The statistical world has been built upon a mountain whose shape is the bell.

The idea of the CLT can be shown graphically. Figure 10 shows it for two distributions we have already seen: the rectangular distribution of the fair 6-sided die (Figure 1) and the exponential distribution

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[44] Madame Guyon was a seventeenth century mystic, writing before Edwards was born. Guyon, Jean; 1981; *Experiencing the Depths of Jesus Christ*, 3rd Ed.; Christian Books Publishing House; p. 55-56.

[45] This assumes the standard deviation of the distribution is finite. It does not apply to distributions whose theoretical standard deviation does not exist, such as the Cauchy distribution. Such distributions are theoretical only. Any distribution in the real world (empirical distribution) has a finite standard deviation, and so the CLT applies to it.

(Figure 3). Neither distribution begins anywhere near to the bell shape (Figure 10a,d), but at samples of size 30, the means of these samples are already close to normal (Figure 10b,e) and at size 100, it is clear that the bell shape has been achieved (Figure 10c,f).[46] A graphical interpretation of the CLT is that you could take *any* empirical probability distribution, no matter what its original shape, and put it on the left (e.g. Figure 10a,d). Then, take numerous samples from it. The larger the sample size, the closer the graph of the *means* will be to the bell curve (Figure 10b,e is closer and Figure 10c,f is closer still). This is not some esoteric trick that is only works when the sample sizes are virtually infinite. As shown in Figure 10, the shape is approximated by the bell curve for very modest sample sizes.

The CLT is remarkable because it does not matter what distribution the original variable has! It always works. This theorem is in many ways the foundation modern statistics.[47] **That the normal distribution is the center of the discipline of Statistics signifies that Christ is the center of the plan of God**.

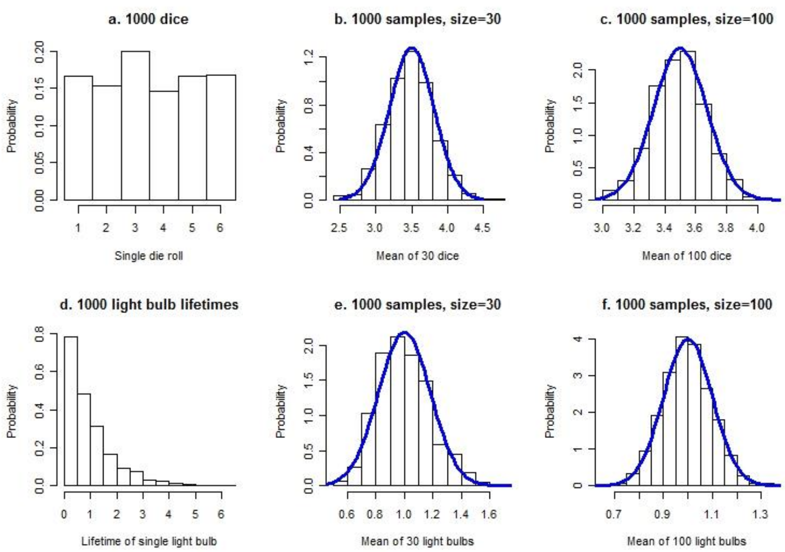

Figure 10: Graphs demonstrating the central limit theorem. a. Distribution of *1000* observations of a die roll, which fall into the expected rectangular shape. b. Distribution of the mean of *1000* samples of *30* dice each, which become approximately normally distributed. c. Distribution of the mean of *1000* samples of *100* dice each, which become approximately normally distributed. d. Distribution of *1000* observations of the lifetime of a certain brand of light bulb (in *1000's* of hours). e. Distribution of the mean of *1000* samples of the lifetime of *30* light bulbs per sample, which begins to look pretty normal, although notice the slight gaps below the normal curve on the left side and the corresponding bars above the blue curve on the right, which are the remnants of the original skew right light bulb distribution. f. Distribution of the mean of *1000* samples of the lifetime of *100* light bulbs per sample. Compared to (e), the distribution is even closer to the expected normal curve.

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[46] Means of the rectangular distribution become normal very rapidly, as do all symmetric distributions. The 'worst case scenario' is extremely skewed distributions, like the exponential, which still shows some skewness at samples of size 30, but has come into close agreement at sample size 100. Technical hypothesis tests can be run on the level of fit of the means to the bell curve to confirm this claim.

[47] With the advent of modern computers, resampling techniques can be used to avoid the use of the Central Limit Theorem. Nevertheless, even with modern computing, statistical methods developed for normally distributed data are the most powerful (technical term meaning able to successfully make statistical discoveries) and are therefore still the primary focus in most Introduction to Probability and Statistics courses in the United States today.

Limiting distribution

> *For Christ is the end of the law for righteousness to everyone who believes.*
> *~Romans 10:4*
>
> *There is neither Jew nor Greek, there is neither slave nor free man, there is neither male nor female; for you are all one in Christ Jesus.*
> *~Galatians 3:28*
>
> *The glory which You have given Me I have given to them, that they may be one, just as We are one.*
> *~John 17:22*
>
> *For from Him and through Him and to Him are all things. To Him be the glory forever. Amen.*
> *~Romans 11:36*

Many probability distributions have the normal distribution as their limiting distribution.[48] A limiting distribution of distribution *X* is the distribution *Y* that *X* approaches when the sample is large enough. It turns out that *Y* is normal for many different *X*. Consider the binomial distribution, for example (Figure 3). It can be used to model the number of correct responses on a multiple choice exam with *n* questions where the probability of getting each question correct is fixed at *100p%*. For example, let us start with $n=20$ questions with $p=0.10$ probability of getting each right, as shown in Figure 11a. This distribution is clearly skewed to the right, [49] and this is the typical sort of binomial graph when *p* is not *0.50*, at which case the graph is symmetric. When the number of questions increases, however, while the probability of success stays the same, then the graph becomes bell-shaped ($n=100$, *250* in Figure 11b,c). This example has shown that the normal distribution is the limiting distribution of the binomial.

As another example, consider the gamma distribution which can be used for modeling the number of minutes it takes until *n* cars pass a certain marker, if the traffic rate is *5* cars/minute. The distribution of minutes until $n=5,20$, and *50* cars is shown in Figure 11d,e,f. The number of minutes until $n=5$ cars pass is clearly not normal, whereas the time for $n=50$ cars has become rather normal. This illustrates a second limiting distribution. The normal distribution is, by far, the most common limiting distribution as so many other variables have it as their limiting distribution.

The regularity and the ubiquity with which probability distributions have as their limit, or end, to become one with the normal typifies the way in which things find their end or become one with Christ. Just as there are a great number of distributions which have the normal as their limiting distribution, so there are a great number of senses in which things find their end or become one in Christ. "The law" (Rom 10:4) and "all things" (Rom 11:36)[50] are quoted as examples which end in Christ. Ethnicity, socio-economic status, gender, (Gal 3:28) and mystical union (Jn 17:22) are cited as examples of oneness in Christ.

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[48] An exact percentage statement would be difficult, but of the six non-normal distributions in Table 4, five of them have the normal distribution as their limiting distribution. The exponential does not, because it maintains its same skew shape for all parameters.

[49] There are several ways to see this: (i) The bars on the left side of the curve are well above the superimposed bell curve. (ii) The left side of the curve begins at *0* questions correct, which has over a *0.10* probability of occurring, whereas the right side skews off with decreasing probability all the way to *20* questions correct, which is virtually impossible.

[50] "All things" should not be read to mean literally *100%* of all conceivable things. The obvious exemption is God. In fact, Paul is explicit about this in a similar passage, 1 Cor 15:27, "For He has put all things in subjection under his feet. But when He says, 'All things are put in subjection,' it is evident that He is excepted who put all things in subjection to Him." Possible other exempted things could include non-existent things (pink unicorns), logically contradictory things (rocks so big God cannot lift them), and other

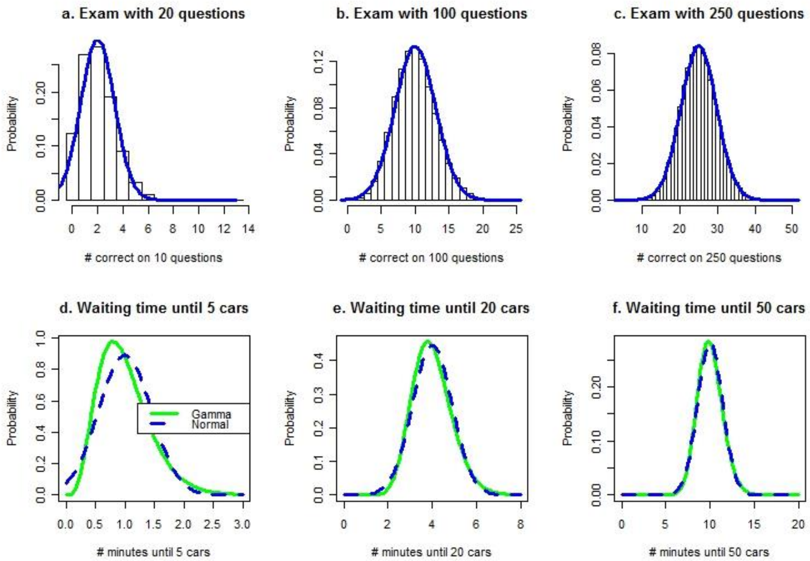

Figure 11: Limiting distribution plots. a-c. Distribution of the number of exam questions correct out of 10, 100, and 250 questions, respectively, with a normal curve superimposed. These distributions are based on the assumption of a 10% chance of getting each question correct. As the number of questions increases, the distribution becomes increasingly normal. d-f. Distribution of the waiting time until n (= 5, 20, and 50, respectively) cars cross a certain measuring strip. These distributions are based on the assumption that the average rate of traffic on this road is 5 cars per minute. As the number of cars waited for increases, the waiting time correspondingly increases, and it becomes increasingly normal.

6. Conclusion

And He is the radiance of His glory and the exact representation of His nature….
~Hebrews 1:3

In Section 2, we asserted that the normal distribution is the most special of all distributions. Sections 3-4 displayed eight features of the normal distribution. Different distributions share different features of the first eight, but none of them all.[51] Most importantly, none of the other distributions have anything even close to the remarkable features presented in Section 5. These have earned it the preeminent place in the theory of Statistics. **The normal curve is the glory of statistics. This fact may be taken as signifying that Christ is the glory of the kingdom of God**. That has been the theme of this paper, namely Jesus Christ, and his centrality in Christian theology and practice—both in Edwards and his Christian descendants today. This is clearly expressed in the theme verse quoted at the beginning, "He made known to us the mystery of His will, according to His kind intention which He purposed in Him with a view to an administration suitable to the

theologically inappropriate things (those cast into the lake of fire, Rev 20:14-15). The point is that the fact that all distributions do not have the normal as their limiting distribution does not nullify the argument for this type.

[51] In fact, Edwards argues that God not only made superior things in nature as images of spiritual realities, but He also made inferior things as images of the superior realities. "So it is God's way in the natural world to make inferior things in conformity and analogy to the superior, so as to be images of them. Thus the beasts are made like men: in all kinds of them there is an evident respect had to the body of man, in the formation and contrivance of their bodies, though the superior are more in conformity and the inferior less. Thus they have the same senses, the same sensitive organs, the same members—head, teeth, tongues, nostrils, heart, lungs, bowels, feet, etc. And from the lowest animal to the highest you will find an analogy, though the nearer you come to the highest, the more you may observe of analogy…." Images no. 19, p. 55-56. See also no. 86, p. 85. In this manner, the normal distribution should be construed as the superior image and the other distributions as inferior.

fullness of the times, that is, the summing up of all things in Christ, things in the heavens and things on the earth (Eph 1:9-10)."

In the eighteenth century, Jonathan Edwards articulated a vision of typology which embraced nature itself as a language in which God revealed divine things. This was not a liberal, break-from-tradition kind of view, but rather it was a grand expression of a brilliant mind attempting to capture the effulgence of the infinite Creator-God.[52] While his ideas were taken in various directions, to my knowledge there have been no followers who conscientiously attempted to carry out the program of deciphering 'the language of God' which Edwards began in *Images* and sketched out the ground rules for in *Types* (Proposition #1).[53] In so doing, Edwards stressed that extra-biblical types were permitted only if an analogy could be made to a sure biblical type (Proposition #2). This paper has been an attempt to do extend Edwards' program into the discipline of mathematics, particularly statistics (Proposition #3). As a summary of the work, and an aid in comprehension, Table 5 contains every normal curve type and anti-type of Christ presented in the paper, with support.

In the preceding, I have attempted to provide generous biblical quotations in order to indicate to the reader the direction of theological justification of every signification claimed. In the process I never intended to provide elaborate or exhaustive justification, but only an indication. This is in keeping with Edwards' own *Images of Divine Things*. This approach reflects the belief that the type is real, and its evidence is primarily 'seen' by the reader who looks at it, rather than intellectually convinced by reasoning on the analogies, though that is not out of place. Although the original ideas came during a time of prayer (not study), I do not claim divine inspiration for the specifics. While I have become increasingly persuaded by the totality of the type, namely the normal curve signifying Christ, I am open to revision of the specifics set forth here. The intent has been to explore an idea, meditate on the person of Christ, and open a dialogue—as opposed to develop new theology. It is my hope and prayer that this will spark for you some of the inspiration which Edwards expressed when he wrote, "The enjoyment of [God] is the only happiness with which our souls can be satisfied…. Fathers and mothers, husbands, wives, or children, or the company of earthly friends, are but shadows; but God is the substance. These are but scattered beams, but God is the sun. These are but streams. But God is the ocean." Only time will tell the success of this enterprise.

[52] Knight, p. 194 captures the richness of Edwards when she emphasizes that Edwards's core theology is about God as an infinitely complete and *effulgent* being who delights in communicating Himself to His creation. She adds that this flows out through Scripture, history, nature, and any other category one might come up with, although Edwards didn't care to categorize it all out, as for him it was all connected.

[53] This is not to say, or diminish, the work of Christians since Edwards who have recognized that nature teaches us about God. There are academic contemporary examples, like Gonzalez, Guillermo and Jay W. Richards; *The Privileged Planet*; Washington, D.C.: Regnery Publishing, Inc. In it, Gonzalez and Richards describe features of our planet and galaxy which they argue exhibit the precision of a Grand Engineer. There are also contemporary examples like Gothard, Bill; 1976; *Character Sketches*, Vol. 1; Institute in Basic Life Principles Publishing. In it, Gothard sees character traits of God expressed through animals, like the orderliness of the beaver. This is similar to Edwards' Images no. 102, p. 90 describing the industriousness of ants and bees. What is fundamentally different about works such as these is that they are only exploring a particular aspect of nature to learn a particular thing about God. Edwards program, by contrast, is to learn the *entire* language, which must necessarily encompass both the specifics of things like the aforementioned work (should it pass the biblical tests) as well as the higher order principles inherent to the language itself. It is this latter task which I find no evidence of in the literature, not the former.

	Type *Feature of Normal Curve*	Anti-type *Feature of Christ*	Support
Over-view	Name unique and above all other distribution names	"Name which is above every name"	Phil 2:9
	Glory of statistics	Glory of the kingdom of God	Heb 1:3
Graph of the Univariate Normal Distribution	Bell curve graph	Beauty	Ps 27:4; Heb 1:3
	Simultaneous finitude and infinity of the bell curve	Simultaneous human and divine natures	Definition of the Council of Chalcedon
	Different parameter configurations	Different facets	Mt 25:1; 25:34; 2 Cor 5:10
	Different manifestations	Different manifestations	Jos 5:13-15; Prov 8:1,4,22,27,30; Jn 1:14; Dan 7:13-14
Multivariate Normal Distribution	Infinite dimensional	Omniscience, omnipresence, omnipotence, etc.	Ps 147:5
	Finite and elegant formula	Human form	Col 1:15-17
	Conditional distribution	Fullness of God	Eph 3:17-19
	Marginal distribution	Fully God	Heb 1:1-2
Limits	Center of statistics	Center of the plan of God	Col 1:17-20
	Limiting distributions are+ normal	End of all things	Rom 10:4; 11:36
	Limiting distributions become one with normal	Mystical union with God	Gal 3:28; Jn 17:22

Table 5: Summary of all types and anti-types of the normal curve and Christ in the paper.

Postscript

It has been about one year since I wrote this paper, and about one month after presenting it at the 2013 ACMS conference. After reflection, and invaluable feedback with those whom I have discussed it with, particularly Rick Langer, Jim Bradley and Russ Howell, I personally retain belief in the thesis. However, it is not as clear or convincing to other people and I have been seeking ways to improve the argument. In particular, each of the phenomena claimed to have typological significance need a great deal of both clarification and further support. Beyond that, it seems desirable to develop the hermeneutic which Edwards sketched, in order to lay biblical hermeneutical criteria by which these and other alleged cases of nature types may be judged. This is the direction I am currently inclined. If successful, for future research, I have a collection of other remarkable mathematical phenomena which might also be mined for further typological connections.

An Investigation of *Hi Ho! Cherry-O* Using Markov Chains

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Abstract

In the children's board game *Hi Ho! Cherry-O*, players attempt to move 10 cherries from a tree to a bucket in the center of the game board. A spinner determines whether a turn includes moving cherries from tree to bucket or bucket to tree. The winner of the game is the first player to move all of her cherries from her tree to the bucket. We model the game play using a Markov chain and calculate the expected number of turns needed to complete one game. Then we investigate what happens when the rules are changed. We discover that rules changes designed to either increase or decrease the length of the game have the desired effect. However, when rules changes are combined, we find that rules changes designed to decrease the length of a game can hide the effect of rules changes designed to increase the length of a game.

1 Introduction

Hi Ho! Cherry-O is a children's board game for 2 - 4 players manufactured by Hasbro. Each player begins the game with 10 plastic cherries on a tree. The object of the game is to put all of the cherries into the player's plastic bucket. On each turn, the player spins a circular spinner to determine whether cherries are taken from his tree and placed into the bucket or removed from the bucket and put back on his tree. The first player to remove all cherries from his tree is the winner.

The spinner is divided into 7 equal area sectors. Table 1 describes the picture on each sector and the event that takes place when the spinner lands on that sector. We will refer to the four sectors that display 1, 2, 3, or 4 cherries as *cherry sectors*. We call the sectors that display a bird, dog, or spilled bucket *penalty sectors*. There is no bonus for spinning a cherry sector that shows more cherries than the number remaining on a player's tree. For example, if a player has 2 cherries on the tree and spins 4 cherries, the player simply wins. Similarly, there is no penalty for spinning a penalty sector if the total penalty is more than the number of cherries in the player's bucket. For example, if a player has 1 cherry in the bucket and spins the bird, then the player puts 1 cherry back on her tree and ends up with 0 cherries in her bucket.

Sector picture	Action
1 cherry	Take 1 cherry from tree and add it to bucket
2 cherries	Take 2 cherries from tree and add them to bucket
3 cherries	Take 3 cherries from tree and add them to bucket
4 cherries	Take 4 cherries from tree and add them to bucket
Bird	Take 2 cherries from bucket and add them to tree
Dog	Take 2 cherries from bucket and add them to tree
Spilled bucket	Take all cherries in bucket and add them to tree

Table 1: Description of spinner sectors

One might ask how long a game of *Hi Ho! Cherry-O* lasts "on average." We can answer this question by modeling the game as a Markov chain.

2 A Markov Chain Model

Consider a game of *Hi Ho! Cherry-O* through the eyes of one player. There are only 11 possible states that the player may encounter, depending on the number of cherries on his tree before the next turn begins. The probability of moving from one state to the next is completely determined by the outcome of spinning the spinner. Moreover, these probabilities are independent of the previous states occupied by the player. Thus, one player's game of *Hi Ho! Cherry-O* may be modeled as a Markov chain.

This insight is not original to the author. Indeed, in [1], [2] and [4] we find other children's board games modeled as Markov chains. Humpherys offers the most in-depth study of *Hi*

Ho! Cherry-O [3]. He shows that the expected length of a game of *Hi Ho! Cherry-O* is 15.8 turns. Cheteyan, Hengeveld, and Jones [2] perform a similar calculation for *Chutes and Ladders* and proceed to ask the following question: How does the expected length change after one modifies the rules? Following their lead, we ask the same question with respect to *Hi Ho! Cherry-O*.

2.1 Building the model

First, we assemble a Markov chain model of *Hi Ho! Cherry-O*. We record the probability of moving from one state to another using the following 11×11 stochastic matrix P:

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} &
\begin{bmatrix}
\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{7} & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 \\
\frac{1}{7} & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 \\
\frac{1}{7} & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 \\
\frac{1}{7} & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\
\frac{1}{7} & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{3}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{4}{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}$$

Each row and column is indexed by the number of cherries in the player's bucket. The entry in row i and column j is the probability that a player begins a turn with i cherries in his bucket and completes the turn with j cherries in his bucket. For example, at the beginning of the game, a player has no cherries in his bucket, which corresponds to row 0. On his first turn, he may add between one and four cherries by spinning one of the cherry sections, or he may end his turn with no cherries in his bucket because he spins one of the three penalty sections.

For another example, consider row 9. The $\dfrac{1}{7}$ entry in column 0 represents the probability that the spinner land on the spilled bucket, the $\dfrac{2}{7}$ entry in column 7 represents the probability that the spinner lands on the bird or dog, and the $\dfrac{4}{7}$ entry in column 10 represents the probability that the spinner lands on any of the cherry sections. Finally, notice that because the game ends when a player has 10 cherries in his bucket, there is a single 1 at the right end of row 10. In the language of Markov chains, state 10 is known as an *absorbing state*.

2.2 Calculating expected length of a game

An advantage of using a Markov chain to model *Hi Ho! Cherry-O* is that it is known how to use the information contained in P to determine the expected number of turns needed to reach state 10 (i.e. to finish a game). We outline the process here. The interested reader may consult [2] or an undergraduate probability textbook for more of the details. Maple 14 was used to perform all of the matrix calculations for this research project.

First, form the matrix Q from P by removing row 10 and column 10. Then let $N = (I - Q)^{-1}$. The entry in row i and column j of N is the expected number of times that a player moves from state i to state j during the course of a game. Since we are interested in the total number of turns needed to complete a game and since each player begins the game in state 0, the expected number of turns is the sum of the entries in row 0 (i.e. the top row) of N.

3 Effects of Rules Changes on Expected Game Length

3.1 An intriguing example

Now we return to our motivating question: How does the expected length of a game change after one modifies the rules? We begin by considering two rules changes. First, suppose that we double the values of the cherry sections. That is, if the spinner lands on a sector with n cherries, then the player puts $2n$ cherries in his bucket. This rules change should decrease the length of the game. Second, suppose that we double the bird penalty so that a player must take 4 cherries from her bucket and put them back on her tree. This rules change should increase the length of the game. Table 2 below summarizes the expected game lengths for each of these rules changes.

Rule Change	Mean Game Length
Double Cherry Values	5.5
Bird is 4 Cherry Penalty	17.6
Double Cherry Values and Bird is 4 Cherry Penalty	5.7
Normal Game	15.8

Table 2: Analysis of two rules changes

The expected game lengths show that the rules changes have the desired effects by themselves. However, when they are combined, the effect of doubling the cherry values appears to hide the effect of doubling the bird penalty. In particular, we note that the effect of combining the rules changes is not additive. The effect of doubling the cherry values produces an expected game length of about 10 fewer turns than normal, while the effect of doubling the bird penalty produces an expected game length of about 2 more turns than normal. If the combined effects were additive, then the expected game length for both of the rules changes together should be about 7.8 turns (i.e. 8 fewer turns). In an effort to understand why this occurs, we now consider the cherry sections and the penalty sections separately.

3.2 Effect of cherry sections

In order to study the effect of the cherry sections on the expected game length, we modify the cherry sections so that only one cherry can be added at a time. We also consider what happens if some of the cherry sections are deactivated; that is, if the spinner lands on a deactivated cherry sector, then no reward or penalty is given. Table 3 below shows the results of the analysis.

Rule Change	Ratio to Next Rule Change	Mean Game Length
1 Cherry	140.5	1957865
1 and 1 Cherry	10.7	13933.2
1, 1, and 1 Cherry	4	1305.2
1, 1, 1, and 1 Cherry		326.4

Table 3: Effect of cherry sections

We note that as expected, if a player can only add 1 cherry at a time and if only cherry sector is activated, then the expected game length is almost 2 million turns. However, as more cherry sectors are activated, the expected game length decreases roughly exponentially, as shown by the middle column. Thus, it appears that the cherry sectors increase the expected game length via exponential growth.

3.3 Effect of penalty sections

Next, we turn our attention to the penalty sections. We begin by isolating the bird and dog sectors. We imagine first that the bird and dog sectors are neutral. Then we add a 1-cherry penalty to each sector until we arrive at the normal game, in which the dog and bird penalties are each 2 cherries. It appears from Table 4 that as each penalty is added, the expected game length also increases by one. Thus, the bird and dog sectors affect the expected game length in a roughly linear fashion.

Rule Change	Mean Game Length	Increase in Length
No Penalty for Bird or Dog	11.3	
No Penalty for Bird, 1 Cherry Penalty for Dog	12.3	1.0
1 Cherry Penalty for Bird and Dog	13.4	1.1
1 Cherry Penalty for Bird, 2 Cherry Penalty for Dog	14.6	1.2
Normal Game	15.8	1.2

Table 4: Effect of bird and dog penalties

To examine the effect of the spilled bucket, we change the penalty that occurs when the spinner lands on that sector. Instead of forcing a player to remove all cherries from her bucket, we consider what happens if the spilled bucket require a player to remove 0, 1, or 2 cherries from her bucket. This modification of the spilled bucket forces it to behave like the bird and dog penalties. Thus, we can identify a modified spinner by the number of penalties assigned to each of the 3 penalty sectors. Each combination of penalties for these sectors is identified in the "Rule Change" column in Table 5 below.

We include the "Number of Penalties" column to group together spinners that have the same total number of penalties available. Notice that the expected game lengths for spinners with the same total number of penalties have identical or nearly identical expected game lengths. Moreover, it appears that expected game length increases by about 1 turn when the number of penalties increases by 1 turn.

Rule Change	Number of Penalties	Mean Game Length
0,0,0	0	7.7
0,0,1	1	8.3
0,0,2	2	9.0
0,1,1	2	9.0
0,1,2	3	9.8
1,1,1	3	9.9
0,2,2	4	10.7
1,1,2	4	10.8
1,2,2	5	11.8
2,2,2	6	13.0

Table 5: Effect of limiting the spilled bucket penalty

4 Conclusion

Now we have enough information to understand what we found in our first example. Recall that when we combined the effects of doubling the cherry values and doubling the bird penalty, we discovered that the effect was virtually the same as if we had not doubled the bird penalty. Our analysis shows that the cherry sections affect the expected game length in an exponential fashion, while the penalty sections affect the expected game length in a linear fashion. Since exponential growth always wins out over linear growth, the combined effect of the two rules changes is not additive.

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