Association of Christians in the Mathematical Sciences

JOURNAL and PROCEEDINGS

ACMS

May 2025, Volume 24

Dordt University Conference Attendees, May 28—June 1, 2024

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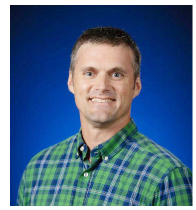
Introduction

The twenty-fourth biennial conference of the Association of Christians in the Mathematical Sciences (ACMS) was held at Dordt University from May 28 until June 1, 2024. Thanks go to Tom Clark and his colleagues for all their efforts that went into hosting it.

Four keynote speakers collectively gave a balanced series of presentations:

- Joel Amidon (The University of Mississippi)
  "Teaching Mathematics as Agape"

  Joel Amidon (BS, MS, PhD, University of Wisconsin-Madison) prepares teachers of mathematics, mentors doctoral students, and does research to consider what it looks like to teach mathematics/teach mathematics methods as agape, an unconditional act of love. His specialization is in mathematics education and the improvement of pedagogy to address issues of equity and justice. Joel is also the host/co-host of several podcasts focused on the improvement of teaching.

- Judith Canner (California State University Monterrey Bay)
  "Putting on Compassion: Ethics in a Data-Driven World"

  Judith Canner (BS, Shippensburg University; PhD, North Carolina State University) is a Board Member of the California Alliance for Data Science Education. She enjoys teaching courses on modeling, data visualization, and design/analysis and her research areas include quantitative reasoning, education, ecology, psychology, and the health sciences. Her professional interests include expanding access to statistics and data science education, data literacy, and data for good.

- Jason Douma (University of Sioux Falls)
  "Does 3 belong to 17? (and Other Absurd Questions)"

  Jason Douma (BA, Gustavus Adolphus College; PhD, Northwestern University) has taught at three liberal arts colleges: Lake Forest College, Carthage College and (since 1998) the University of Sioux Falls. He also serves as chair of the MAA's Council for Teaching and Learning, and chair of the Philosophy of Mathematics Special Interest Group of the MAA. Jason sings in the choir of First Presbyterian Church of Sioux Falls, where he serves as an ordained elder.

- Lydia Manakonda (Rensselaer Polytechnic Institute)
  "The Pursuit of AI and the Future of Health"

  Lydia Manikonda (BTech, MS, IIIT-Hyderabad, India; PhD, Arizona State University) builds intelligent decision-making models from multimodal data that are capable of learning and reasoning to address problems related to human factors in AI. She has over 10 years of experience in developing machine learning and data-analytics related models. Outside of work, Lydia is a busy mother of two young children and enjoys running and exploring new places with them.

Three pre-conference workshops took place on Tuesday, May 28:

- *Teaching Calculus with Infinitesimals*, led by Bryan Dawson
- *Early Career Professional Development*, led by Kristin Camenga, Amanda Harsy, and Derek Schuurman
- *Teaching Introductory Statistics with GAISE: Statistical Investigation Process, Multi-Variable Thinking, Simulation-Based Inference*, led by Nathan Tintle

The conference schedule is presented in Appendix 1, Appendix 2 gives the Parallel Session schedule, Appendix 3 presents abstracts for the various sessions, and Appendix 4 lists information for the individual participants.

The contributed paper sessions had a total of 69 presentations from the 117 conference attendees. Josh Wilkerson (ACMS President) has placed some presentation slides and photos taken by ACMS members on Google Drive. Click here to access them.

Not every paper presented at the conference was submitted to this *Journal and Proceedings*, and the journal has a protocol of accepting papers for consideration that were not presented at the corresponding conference. The following pages contain all submissions that made their way through the single-blind review process, each having been scrutinized by a minimum of two reviewers. Thanks go to the authors for their good work. Too numerous to mention are all the referees that were involved, but heartfelt thanks go to them for their diligence. In production matters a huge shout-out goes to Cal Jongsma, who carefully proofread all the pages and spotted many errors. Of course, any remaining ones are the fault of this editor!

The ACMS Board has decided that, beginning with this journal, an award will be given for the best paper appearing in it. Details regarding the criteria and selection process can be found on the ACMS website: https://acmsonline.org.

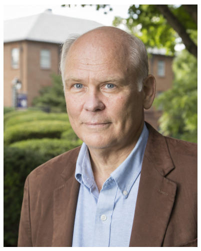
As reminder, ACMS is a 501(c)(3) organization, so any monetary gifts to it are tax deductible. Its bi-laws stipulate that conferences will be held on evenly-numbered years. The twenty-fifth biennial conference is scheduled for May 27–30, 2026, with Calvin University as the host institution. Further conference details can be found on the ACMS website.

Russell W. Howell (Westmont College)
ACMS *Journal and Proceedings* Editor

—— Articles ——

Trammel-Crafting the Quadratrix

Andrew J. Simoson (King University)



Over the years Andy Simoson has attended more than a dozen ACMS conferences, and this academic year marks his 45^{th} year at King University—a small Presbyterian-based school in the lovely Appalachian hills of east Tennessee—albeit that tenure was amazingly punctuated by two year-long Fulbright professorships to Gaborone, Botswana, and Dar es Salaam, Tanzania. On that last jaunt, he, along with his wife and two sons, hiked to the crest of Kilimanjaro.

Abstract

Hippias of Ellis (fifth century BC) is credited with inventing/discovering the quadratrix curve, a curve he used to solve one of the three famous impossibility constructions from Euclidean geometry, and which also solves a second one. How did he do this, and how can we fashion a physical trammel using easily available material so as to approximate, in a continuous manner, an ideal quadratrix curve?

1 Introduction

What's the oldest mathematical curve, after the line and circle; and how was it created? To answer, let's go back nearly 2500 years ago to the days of Hippias of Elis[1] (460–400 BC).

In his dialogue, *Hippias Minor*, Plato (c. 428–347 BC) records, courtesy of Socrates (c. 470–399 BC), Hippias of Elis boasting "I have never found any man who was my superior in anything." [10]

The noted nineteenth century mathematical historian Moritz Cantor's assessment was that[2] "Hippias's vanity was extreme even for a sophist. Nevertheless his learning was at the highest level of education of his time in scientific, mathematical, and astronomical knowledge" [2, pp. 182–183].

With respect to geometry, many propositions that Euclid organized into *The Elements* in about 300 BC dated to at least the days of Thales (626–545 BC); and a lively geometry question in the days of Hippias was, "Whereas we can bisect both line segments and angles, and trisect any segment, might we similarly trisect any angle?"

Rising to the challenge, Hippias may have reasoned in this fashion.[3] Within a unit square $ABCD$, as in the *trammel*[4] sketch of Figure 1, imagine a unit-length line segment $\mathcal{L}$, ever parallel to AB

[1] Elis is a region due west of Athens on Greece's western shores where the first Olympic games were held.

[2] My own German translation. With respect to whether Hippias himself created the quadratrix, Cantor, after examining pertinent texts from Pappus (290–350 AD) and Proclus (412–485 AD), concludes, "It would be difficult to derive any other meaning than the following. Hippias, namely Hippias of Elis, invented a curve around 420 BC [now called the quadratrix]." He goes on to say, "The first fully verified name of a curve different from the circle will be encountered at the beginning of the second third of the fourth century, approximately thirty years before Dinostratus (c. 390–320 BC) when Eudoxus (c. 390-340 BC) invented his Hippopede." [2, p. 183]

[3] This mind experiment, as we have it, was equivalently described by Pappus [1, pp. 105–108], long after the writings of many Greek geometers were lost or absorbed into other geometers' works through the centuries.

[4] A geometric trammel is a tool, the motion of whose parts are constrained in some way so as to generate a specific curve.

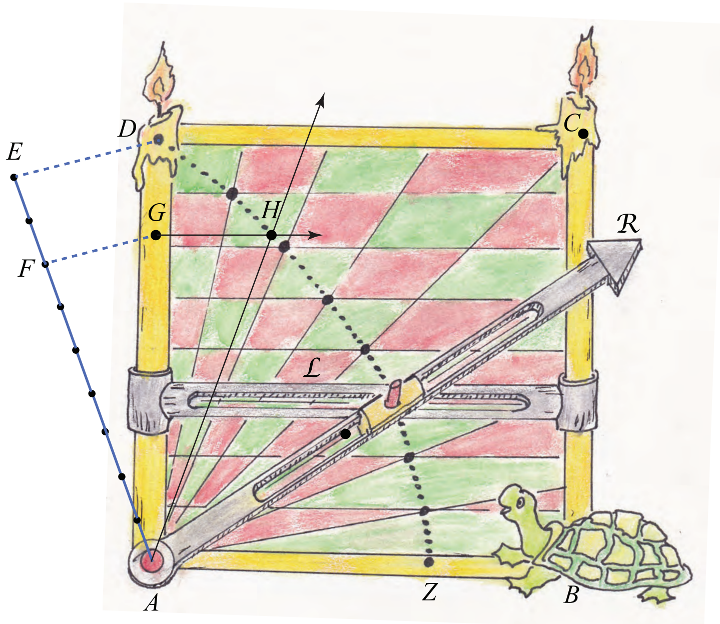


Figure 1: Whimsical author sketch of a quadratrix trammel.

with its endpoints ever along sides AD and BC, sliding uniformly from DC to AB in unit time while a ray $\mathcal{R}$ with tail ever at A rotates uniformly from initial alignment with AD at time $t = 0$, across the square, to final alignment with AB at $t = 1$.

At any time t, $0 \leq t \leq 1$, if $\mathcal{L}$ has traveled distance t from C to A, then $\mathcal{R}$ has rotated t of a right angle. The quadratrix curve is the locus of all points of intersection between $\mathcal{L}$ and $\mathcal{R}$ as t ranges from 0 to 1.

To illustrate the utility of an ideal quadratrix curve, let's trisect 60°. In terms of the notation of Figure 1, construct a ray from A into the second quadrant; proceed nine equal steps along this ray, ending at point E; let F be the point two steps backwards from E; construct segment ED; construct a ray through F parallel to ED intersecting side AD at point G; construct a ray through G parallel to AB intersecting the quadratrix curve at point H. Since DG has length $\frac{2}{9}$, then $\angle DAH = 20°$, an angle that cannot be constructed via compass and straight edge alone.[5]

1.1 An Interlude—Which Came First, the Quadratrix or a Conic?

In [6, p. 84] Julian Havil claimed that "in terms of the genesis of mathematical curves, the list begins straight line, circle, quadratrix." Yet some might object, saying, people must have noticed the distinctive elliptical intersection shape of a plane and cylinder when, for example, transforming trees into lumber.

Moreover, in early architecture, the base for many a tepee, igloo, yurt, or the Botswanan rondavel of Figure 2 is a circle, as Zaslavsky [16, p. 158] described the tradition from northern Tanzania:

> To mark off the circumference [of the circular hut], the builder tied a hoe to a rope of length

[5] Of course, since the motion of $\mathcal{R}$ is uniform across its range, it's also uniform across any subinterval. Thus, we may use the ratio 1:3 of an arbitrary angle rather than 2:9 of a right angle, as in the given example.

Figure 2: A traditional rondavel, Botswana; photo by author, 1991.

> equal to the desired radius. The rope was attached to a peg, and as he walked around this peg, he drew a circle with his hoe.

Similarly, it's an easy matter to fasten the ends of a rope to two pegs, with the rope length L greater than the distance between the pegs. Keeping the rope taut with a hoe—a veritable trammel—the builder drags the hoe along, so tracing a closed curve. For each point P along the resulting trail, the sum of the distances from P to the two pegs is L, a relation defining an ellipse.

Demonstrating that this elliptical building trick was a tradition, Weule [14, p. 65], in a state-sponsored ethnological study of the then German colony Tanganyika in 1910, described a southern Tanzanian dwelling:

> My abode is a hut in the purest Yao style, built by the [Yas people]. This hut is some forty feet by twenty. It is an oval structure whose ... interior forms an undivided whole, only interrupted by the two posts standing as it were in the foci of the ellipse, and supporting the heavy thatched roof.

A similar tradition is documented in Easter Island, possibly dating back to about 400 AD [3].

Thus, ellipse usage is very old, perhaps predating the quadratrix, although the earliest written account appears to date to Abud ben Muhamad in the ninth century AD [4, pp. 150–151]. Nevertheless, nearly a hundred years after Hippias, Menaechmus (380–320 BC) is credited with discovering the conic sections, using them for a solution to the classic third impossibility problem, that of doubling the cube. Yet with respect to actual mathematical analysis of curves themselves, Havil is probably correct.

1.2 Whence the Name *Quadratrix*?

Another common geometric question in the days of Hippias—and a notion foreshadowing the definite integral—was, given a shape such as a polygon or a section of the circle (a *lune*), could one construct a square of side length q whose area—its *quadrature* q in Latin and, equivalently, its $\tau\epsilon\tau\rho\alpha\gamma\omega\nu\iota\sigma\mu$ός in Greek from *four-sided*[6]—matched that of the figure?

Delighted with an exotic new figure to *quadrate*, literally, to square a shape, Hippocrates of Chios (470–410 BC), a contemporary of Hippias, coined[7] the term $\tau\epsilon\tau\rho\alpha\gamma\omega\nu$ίζ$o\upsilon\sigma\alpha$ (transliterated *tetrag-*

[6] The Greek prefix for *four* is *tetra*: $\tau\epsilon\tau\rho\alpha$. For example, the Greek word for quadrilateral is $\tau\epsilon\tau\rho\alpha\gamma\omega\nu$, *tetragon* when transliterated into English.

[7] Credit is generally given to Hippocrates, perhaps because he succeeded in squaring several lunes [6, p. 78].

onizousa) which was later rendered *quadratrix* in Latin[8] [1, p. 105]. Perhaps, the name seemed appropriate to Hippocrates since the motion of a descending bar and a rotating dial in Hippias's mind experiment occurs within a square, a quadrilateral, a tetragon.

Serendipitously, long after its coinage, the quadratrix curve fully earned and deserves its name, because a hundred years after Hippias, Dinostratus used it to square the circle[9].

2 Quadratrix Formula Derivation

Following Hippias's mind experiment outlined on page 1, Descartes's Cartesian coordinates, and radian measure, let's find a parametric equation representation for the quadratrix where the square's vertices are $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, and $D = (0, 1)$, in terms of a time parameter t. With our horizontal segment or descending bar $\mathcal{L}$ completing its motion in unit time, ray $\mathcal{R}$'s angular velocity must be $\frac{\pi}{2}$ radians per unit time for it to coincide with $\mathcal{L}$ at $t = 1$. Thus at any intermediate time t, $\mathcal{L}$ and $\mathcal{R}$ coincide at a point P written dually as

$$P = \lambda\left(\sin\frac{\pi t}{2}, \cos\frac{\pi t}{2}\right) = (x, 1 - t), \tag{1}$$

for some number λ since P lies along the unit vector $(\sin\frac{\pi t}{2}, \cos\frac{\pi t}{2})$, and for some positive number x since $\mathcal{L}$ lies $1 - t$ distance above AB. See Figure 3.

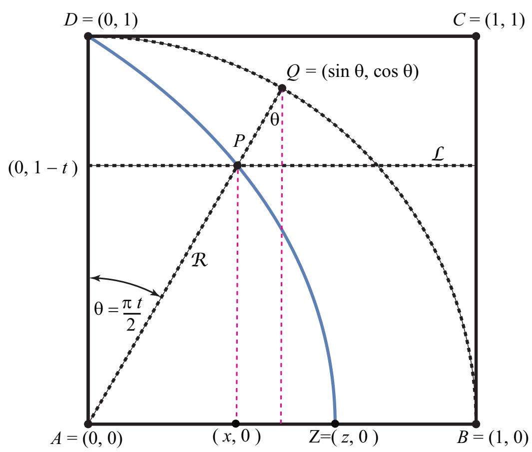


Figure 3: The quadratrix of Hippias: $P = (1 - t)(\tan\frac{\pi t}{2}, 1)$, $0 \le t < 1$.

Equating the second coordinates of Equation (1), and solving for λ gives

$$\lambda = \frac{1 - t}{\cos\frac{\pi t}{2}}. \tag{2}$$

Inserting this λ value of (2) in (1) and simplifying characterizes the quadratrix

$$P = (1 - t)(\tan\frac{\pi t}{2}, 1), \tag{3}$$

[8] The Latin suffix *trix* is feminine, whereas *tor* is masculine. Thus, for example, we have the word *matrix* rather than, say, *mator* from the colloquial *ma* in both English and Latin.

[9] With respect to note 7, some contend that Nicomedes (c. 280–210 BC) coined the term and was the first to square the circle via the quadratrix [8, pp. 226–233], [12, p. 165]

albeit P is undefined at $t = 1$.

Nevertheless, using L'Hopital's rule, we see that the quadratrix curve strikes AB at a point Z whose first coordinate z is

$$z = \lim_{t \to 1^-} \frac{1-t}{\cos \frac{\pi t}{2}} = \frac{2}{\pi}, \tag{4}$$

a result Dinostratus discovered, as we will see, without calculus. With compass and straight edge, for a given positive length z (namely, $z = \frac{2}{\pi}$), we can construct its reciprocal length (namely, $\pi/2$); its double (namely, π); and its square root (namely; $\sqrt{\pi}$), which means that, yes, given an ideal quadratrix curve, the unit circle can be squared and has quadrature $q = \sqrt{\pi}$.

3 Trammels

With respect to trammels and the art of applying mathematics to societal needs in general, Plutarch (46–119 AD) wrote [11, vol. v, pp. 471–473],

> The art of mechanics... originated by Eudoxus and Archytas[10] ... gave to problems incapable of proof by word and diagram, a support ... that [was] patent to the senses. Plato ... inveighed against them as corrupters and destroyers of the pure excellence of geometry, which thus turned her back upon ... abstract thought ... descend[ing to the] use of objects which required much mean and manual labour. For this reason mechanics [and the art of making trammels] was made entirely distinct from geometry.

If Plutarch's account[11] is correct, then Hippias had no quadratrix trammel. Instead, he generated the curve "mechanically by means of a series of points, which must then be joined by a steady stroke of the free hand" [1], much as illustrated by the series of dots along an imagined quadratrix curve in Figure 1, where each dot represents the intersection of the dyadic horizontal line $y = 1 - \frac{j}{2^n}$ and the dyadic polar ray $\theta = \frac{j\pi}{2n}$, for some integer j, $0 \le j \le n$, where $n = 3$ and θ increases clockwise in Figure 3.

Yet since the very nature of the quadratrix curve involves motion, a reasonable conjecture is that Hippias may have asked artisans to craft a trammel for it, just as later geometers were to do for other curves. So let's try it!

3.1 A second interlude: the conchoid of Nicomedes.

Before doing so, let's consider a classic physical trammel used by Nicomedes, and then contrast it with one for a quadratrix.

To generate the conchoid's equation, consider a variable length segment OP, where $O = (0, 0)$ in the x-y coordinate plane, and OP crosses the line $y = 1$ at the point $(\tan\theta, 1)$ so that the segment between $(\tan\theta, 1)$ and P has fixed length a for all radian values θ, $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Then the set

[10] Despite Plutarch's scathing words, Plato's friend Archytas (circa 420–350 BC) is credited with doubling the cube. And his student Eudoxus (circa 390–340 BC) served as headmaster of the *Academy* during Plato's sabbatical in Syracuse, and tutored Aristotle.

[11] In 1940, G. H. Hardy voiced similar sentiments [5, pp. 64,117,139], saying, "There is the real mathematics of the real mathematicians, and there is what I call the trivial mathematics... which includes its practical application, the bridges and steam engines and dynamos; [and] engineering is not a useful study for ordinary men."

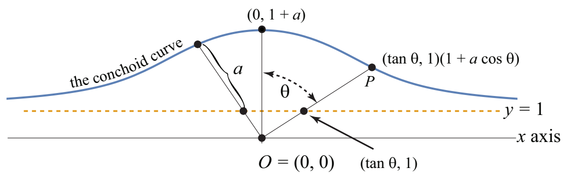


Figure 4: The conchoid of Nicomedes.

of all locus points P is the conchoid curve of Figure 4. Recall that the *unit* vector in a non-zero $(p,\, q)$ direction is the vector divided by the *magnitude* of the vector, denoted $||(p,\, q)|| = \sqrt{p^2 + q^2}$. Since $||(\tan\theta,\, 1)|| = \sqrt{\tan^2\theta + 1} = \sec\theta$, the unit vector in that direction is $\cos\theta(\tan\theta,\, 1)$. Thus P's position in the plane is given by $P = (1 + a\cos\theta)(\tan\theta,\, 1)$.

To construct a trammel for the conchoid, as in Figure 5, create a slot along the midline of a narrow rectangular wooden slat, positioning it so that the slot lies along the line $y = 1$. Embed a fixed pivot pin in the plane at the origin O. Call this slat and the pivot pin the *base*. Using a second slat, form an arrow-tip at one end; insert a fixed pin along the slot a units from the arrow-tip; then create a slot along this arrow slat from the arrow's tail to just shy of the fixed pin. Set the arrow slat down upon the base, so that its fixed pin can slide along the base's slot, and so that the fixed pin at O in the base lies in the arrow's slot. The locus of all points of the arrow-tip as the arrow slides along the two slots is a conchoid.

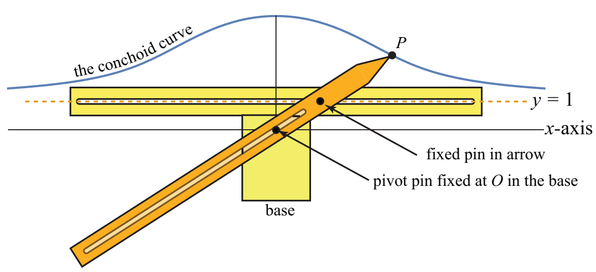


Figure 5: A conchoid trammel.

Figure 6[12] shows three different conchoid curves for three different pin positions in the arrow slat.

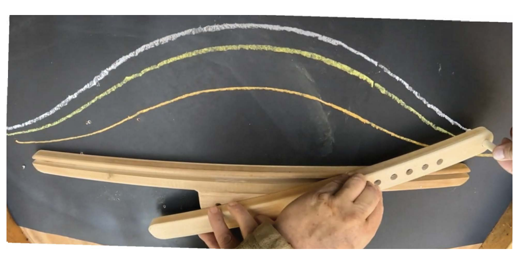


Figure 6: Conchoid trammel of Nicomedes.

Finally, let's use the conchoid curve to trisect a given angle θ. Arrange this angle so that $\theta = \angle AOB$ where O is the origin, A is up along the y axis, and B lies in the first quadrant.

Let Q be the intersection point between $\overrightarrow{OB}$ and the line $y = 1$. Let $b = |OQ|$. Then construct that ideal conchoid curve where $a = 2b$, and let A and B lie on the conchoid with $A = (0,\, 1 + a) =$

[12]This image is a frame from a *YouTube* video of the author demonstrating a half dozen trammels. The video can be found on *YouTube* by searching for the phrase "geometry trammel".

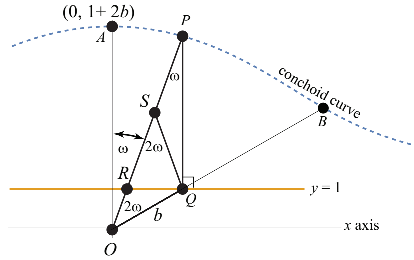


Figure 7: Trisecting $\theta = \angle AOB$.

$(0,\ 1+2b)$, as shown in Figure 7. Construct a perpendicular to the line $y = 1$ at Q which intersects the conchoid curve at point P. Let $\omega = \mu(\angle OPQ)$, the *measure* of the angle. Since PQ and the y axis are parallel lines cut by the transversal OP, then $\omega = \mu(\angle OPQ) = \mu(\angle AOP)$. Let R be the intersection of OP and $y = 1$. Let S be the midpoint of segment PR. Consider the circle with center S and diameter PR. This circle circumscribes the right triangle $\triangle PQR$. Construct segment SQ. We know that $b = |SP| = |SQ| = |SR|$, since SP, SQ, SR are radii of the the circle. The central angle $\angle RSQ$'s measure is twice that of the inscribed angle $\angle RPQ$. Therefore $2\omega = \mu(\angle OSQ)$. Since $\triangle OQS$ is isosceles, then $\angle POB = 2\mu(\omega)$, which means that$\mu(\theta) = \omega + 2\omega = 3\omega$. That is, $\angle AOR$ trisects $\angle AOQ$.

4 A quadratrix trammel

In experiments with my first few attempts at quadratrix trammels, I was sorely disappointed! I had been hoping that they would behave much like the conchoid trammel of Figure 6. That trammel faithfully generated the curves for which it had been created. That is, at any particular stylus position P, the stylus's tip had but one degree of motion or freedom: either forward or backward along the current tangent line to the curve at P.

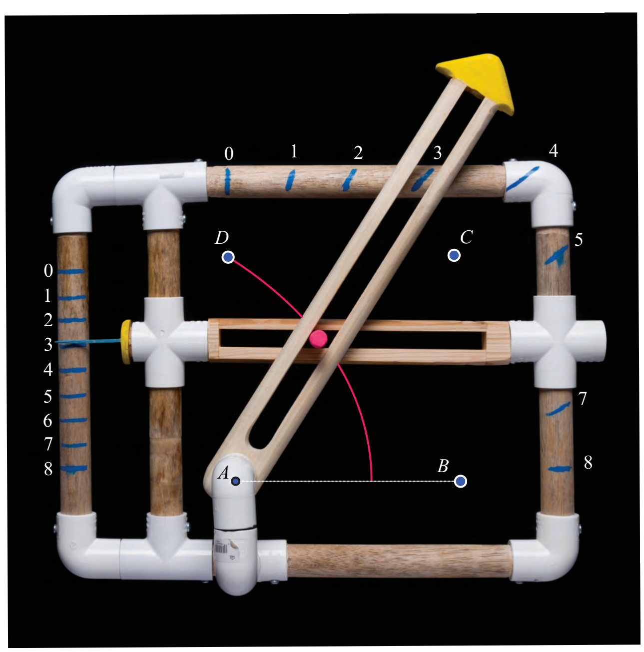


Figure 8: A calibrated quadratrix trammel.

But my new trammels appeared to have minds of their own. In retrospect, it's easy to identify the confusion. See Figure 1. With dial $\mathcal{R}$ fixed, bar $\mathcal{L}$ is free to slide up and down, so tracing a line angled along $\mathcal{R}$'s current orientation; and with $\mathcal{L}$ fixed, we see that $\mathcal{R}$ is free to rotate, and the stylus traces a horizontal line, namely, along the current orientation of $\mathcal{L}$ itself. That is, the quadratrix trammel has two degrees of freedom.

One approach to constrain this freedom of motion is by implementing a *duet* metaphor rather rather than a *solo* metaphor. We need four hands, not two, on this veritable piano!

With this idea in mind, and with respect to the notation of Figure 1 or 8, consider a square whose sides are rendered as wooden dowels joined by plastic PVC elbow connectors, secured by screws. The horizontal bar $\mathcal{L}$ is a slotted dowel to accommodate chalk; the dowel's ends are capped with PVC cross tees so that the tees slide along the left and right hand dowels. Dial $\mathcal{R}$ is a slotted slat to accommodate chalk. A PVC tee and two elbows form a gooseneck bridge from the lower dowel to a point A inside the square of dowels, so linking the doweled framing square square to $\mathcal{R}$'s tail. Vertex D of the unit square $ABCD$ is the length of a PVC tee below the midline of the upper dowel. Note that to the left of AD, a calibrated parallel dowel has been appended to the framing square, dyadically marked from top to bottom, 0 through 8. Similarly, the top and right hand dowels have also been calibrated, dyadically marked angularly 0 through 8. To enhance precision in determining any $\mathcal{L}$ position, a toothpick is inserted into the center of the left-hand side of $\mathcal{L}$'s left-hand cross tee. With a piece of chalk at the intersection of $\mathcal{L}$ and $\mathcal{R}$, a quadratrix curve is traced as $\mathcal{L}$ and $\mathcal{R}$ are moved uniformly from mark 0 to mark 8.

In particular, with both $\mathcal{L}$ and $\mathcal{R}$ at their respective 0 marks, our performers α and β count down, beating *three, two, one, zero*, whereupon α slides $\mathcal{L}$ downward at a uniform rate while β rotates $\mathcal{R}$ at a uniform rate, reaching their respective mark 1 at beat *one*, and so on, until they culminate at mark 8 on beat *eight*. A little practice helps the synchronization.

4.1 A Quadratrix Legacy?

Here's a second approach to constraining the quadratrix's two degrees of freedom.

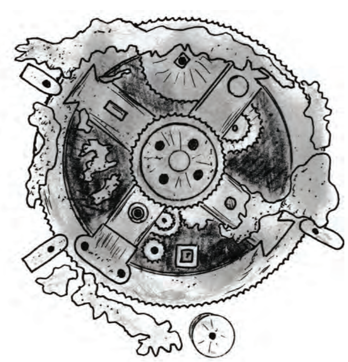


Figure 9: Author sketch of the Antikythera device.

In 1900, sponge divers off the coast of the Greek island Antikythera discovered a wreck sixty meters deep dating back to at least the second century BC. One of the retrieved artifacts, now known as the *Antikythera device*, was a curious blob of melded copper, as shown in Figure 9. Careful inspection of this object revealed it to be an assemblage of over thirty interlocking gear wheels, one of which has nineteen teeth and another of which has 223 teeth, the very numbers making a Metonic cycle, the cycle governing eclipse recurrence: that is, while the earth circles the sun in exactly nineteen years, the moon nearly circles the earth 223 times. Careful reverse engineering has shown the device to be an amazing orrery, a veritable clock of the planets, sun, and moon in their ancient epicyclic synchronous harmony, a tale chronicled in [9] and the basis of the fanciful 2023 Hollywood blockbuster movie *Indiana Jones and The Dial of Destiny*.

So perhaps various Greek artisans, inspired by Hippias's imagined request, went on to synchronize the quadratrix trammel using gears, and then proceeded to fashion orreries, a tradition continued by medieval clock-making guilds, and beyond to all the gadgets of the modern era, to include Pascal's addition machine, Leibniz's multiplication machine, Babbages's difference engine, the mind experiments of Turing's machine, to the computer—perhaps the ultimate trammel.[13]

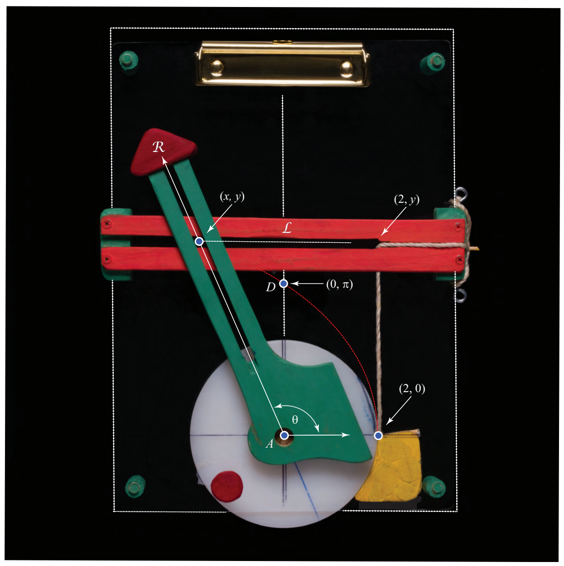


Figure 10: A pulley and string quadratrix trammel.

For a speculative Antikythera device antecedent—using a pulley and string rather than gears—consider the trammel of Figure 10, being a variation of Dinostratus's idea, as Pappus recounted, but in reverse, with $\mathcal{L}$ ascending from AB to DC rather than descending from DC to AB.[14] Onto the back of a plexiglass clipboard $\mathcal{G}$, a pulley $\mathcal{P}$ of radius two units, centered at $A = (0, 0)$, is attached, rigidly coupled via a dowel through $\mathcal{G}$ to a dial $\mathcal{R}$ on $\mathcal{G}$'s front. The ends of a bar $\mathcal{L}$, ever horizontal, can slide along the left and right hand sides of $\mathcal{G}$. With a string $\mathcal{S}$ wrapped counterclockwise about $\mathcal{P}$'s rim, one end affixed into $\mathcal{P}$, and the free string feeding through a small hole in $\mathcal{G}$ at $(2, 0)$ along an imagined x-axis, $\mathcal{S}$'s other end is affixed to $\mathcal{L}$ at $(2, y)$, for $y \geq 0$. Let θ be the counterclockwise radian angle between the positive x-axis and dial $\mathcal{R}$. At $\theta = 0$, $\mathcal{R}$ is pointing right; $\mathcal{L}$'s horizontal midline is aligned with the x-axis; a central portion of $\mathcal{L}$'s body lies tucked within a slot in the base region of $\mathcal{R}$ about A; and $\mathcal{S}$ is ever taut from $\mathcal{P}$ at $(2, 0)$ to $\mathcal{L}$ at $(2, y)$. Sliding $\mathcal{L}$ upwards along $\mathcal{G}$'s front face causes $\mathcal{P}$ and $\mathcal{R}$ to rotate, and $\mathcal{S}$ to unwind from $\mathcal{P}$. For each $0 < \theta < \pi$, let (x, y) be the intersection point of $\mathcal{L}$ and $\mathcal{R}$ (both of which may need to be arbitrarily long as θ grows to π). Observe that $y = 2\theta$ gives the length that $\mathcal{S}$ has unwound! Therefore, the locus of all intersection points (x, y) is a quadratrix curve that intersects the y-axis

[13] To balance Plutarch's and Hardy's arguments—as outlined at length in [13]—about the splintering of mathematics into pure and applied, here's an admission by Oxford numerical analyst [15, p. 23]: "I have long marveled at how most mathematicians prove their theorems without . . . numerical experiments. Whatever the topic, I use the computer to guide me."

[14] In Figure 10, B and C refer to the vertices of the square $ABCD$ as appearing in Figures 1, 3, and 8.

at $D = (0, \pi)$, namely, when $\theta = \frac{\pi}{2}$, in Figure 10.

As a delightful finale[15]: This last quadratrix trammel resolves Dinostratus's dilemma that Z in Figure 1 is undefined. Remember, at $t = 1$, $\mathcal{L}$ and $\mathcal{R}$ are aligned, so a stylus's position at $\mathcal{R}$ and $\mathcal{L}$'s intersection is indeterminate[16]; moreover, a wheel centered at A with radius $\frac{2}{\pi}$ begs the question of Dinostratus's approach (to constructing π). However by similarity, Z's analog in Figure 10 is simply radius 2, and D's analog is resoundingly defined as $\mathcal{R}$ and $\mathcal{L}$'s unique intersection at $\theta = \frac{\pi}{2}$, yielding the ideal construction length π.

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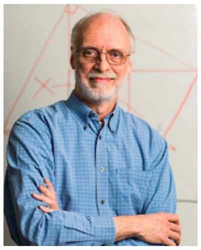
[15] One reviewer of this paper shared their *aha moment* thusly: "Knowing that the quadratrix is capable of constructing π, it seems natural that a trammel for the curve would involve a wheel or circle of some kind."

[16] Of course, the limit notion rescues all.

Logic's Modern British Up-enders, Defenders, and Extenders: Whately's Revitalization of Logic

Calvin Jongsma (Dordt University)



Cal's professional interests include the history and philosophy of mathematics and logic. Springer published his textbook *Introduction to Discrete Mathematics via Logic and Proof* in 2019. An essay recapitulating and updating his dissertation on Whately's revival of logic is the lead chapter in *Aristotle's Syllogism and the Creation of Modern Logic* (Bloomsbury Academic, 2023). He also maintains an annotated database on *Faith and the Mathematical Sciences*.

Abstract

Logic developed dramatically during the last half of the nineteenth century. The baseline for these transformations in Great Britain was the revival of logic by Richard Whately around 1825. Whately successfully defended syllogistic logic as the science of valid reasoning against potent seventeenth- and eighteenth-century detractors—Bacon, Locke, Reid, Campbell, Stewart, and others. In so doing, he made logic an intellectually respectable field of investigation for the next generation of logicians to explore and extend. This included John Stuart Mill (inductive logic), Augustus De Morgan (logic of relations), and George Boole (algebraic logic; propositional logic).

1 Introduction

Logic is not a mainstream concern of practicing mathematicians or computer scientists. Those who express interest in it often come at it with foundations of mathematics or logic circuits in mind. Or perhaps because they've been tasked to teach students how to construct proofs. Few are curious what logic was like before Boole and Frege engineered the mathematical takeover of logic from philosophy. Unless, of course, you're interested in history of mathematics and logic. Like I am.

My Ph. D. thesis topic at the University of Toronto was intended to be Boole's contribution to mathematical logic. That meant exploring: 1) logic before Boole, 2) mathematics at the time of Boole, and 3) Boole's work in the algebra of logic. I went at researching all this systematically.

Starting mostly with the logic background, I found Boole treated Richard Whately as an authority on logic and credited him with resuscitating it about a quarter-century before Boole wrote his *Mathematical Analysis of Logic* (1847) and *Laws of Thought* (1854). Why, I wondered, did logic need reviving, and what was it like before Boole? Curiosity was what got me into a bit of trouble.

Turning to Whately, I discovered that its distinguishing feature was a thorough defense of deductive logic. Why, I mused, did Aristotle need an apologist if his logic was still solidly in vogue two millennia after it originated—as Kant had affirmed just a few decades earlier, claiming that logic had gone neither backward or forward since Aristotle but appeared to be a finished field.

These questions drove me backwards into the 18^{th} and 17^{th} centuries. But to understand the critics' disparagement of logic, I found I first had to delve into the particularities of syllogistic logic so I could assess the criticisms it had provoked.

In the end, I had a dissertation's worth of material just by focusing on Whately and logic prior to Boole. Fortunately, my thesis committee agreed to a changed focus. But since this subject wasn't germane to teaching a range of university-level mathematics courses at Dordt, I let my historical research on Whately lie dormant — for 40 years, until invited to reprise my thesis in a condensed and updated form as the lead chapter in the pricey book *Aristotle's Syllogism and the Creation of Modern Logic* [10] ($120), published in early 2023 by Bloomsbury Academic.

I'd like to share some highlights from this story to explain why a broad-minded mathematical educator might be at least marginally interested in the topic. An expanded version of my chapter (and this talk) can be downloaded for free from *Dordt's Digital Collections*.[1]

2 Syllogistic Logic Fundamentals

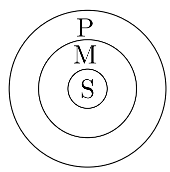
I won't go very far into the weeds on traditional *Aristotelian Logic*, but it may be helpful to mention and then schematize a few central technical details.

Aristotelian Logic recognizes four *categorical* sentence-forms (A, E, I, and O) and focuses on two-premise argument-forms known as *syllogisms*. A type of syllogism or *syllogistic mood* is identified by what sorts of categorical sentences comprise the argument. The most basic positive syllogistic mood is *Barbara*, comprised of three properly linked *universal affirmative* (A) sentences. The most basic negative syllogistic mood is *Celarent* (an *EAE* argument). Simple circle diagrams for the classes S, M, and P nicely illustrate these arguments' validity.

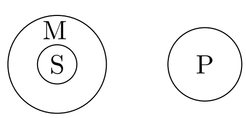
- *Categorical Sentences*
  A: All X are Y   E: No X are Y
  I : Some X are Y   O: Some X are not Y

- *Syllogistic Argument Forms*
  - Three categorical terms, representing classes: S, M, P (Subject, Middle term, and Predicate)
  - Two premises and a conclusion

- *Barbara*
  All M are P
  All S are M
  ∴ All S are P



- *Celarent*
  No M are P
  All S are M
  ∴ No S are P



Barbara instantiates the *positive* (*affirmative*) form of *Aristotle's Dictum*, *Celarent* the *negative* form: *Whatever is universally affirmed or denied of a class can also be asserted of any subclass.* In formulating and diagramming *Barbara* and *Celarent*, M is the full class with S as a subclass, while P is a superclass of M (for *Barbara*) or is disjoint from M (for *Celarent*) and therefore similarly for S. Very simple but potent forms of argumentation.

So much by way of technical background. Now let's turn briefly to consider the criticisms of the British opponents of logic — the attacks by logic's *up-enders*.

[1] Available at https://digitalcollections.dordt.edu/faculty_work/1446/

3 Logic's 17th-Century English Critics

Francis Bacon (1561-1626) was a pioneering English crusader for an inductivist natural philosophy. The scientific method he proposed[2] early in the 17th century in his *Novum Organum* [3] as a competitor to Aristotle's *Organon* (his logical writings) was overly simplistic and lacked an integral role for mathematics or even experimentation, but it resonated with many who felt syllogistic logic and medieval scholastic disputation were unable to arrive at truths about the natural world and were only good for fruitless debates. One needed to examine the things themselves, not expound on what one could say about them. Syllogisms were necessarily confined within the world of words.

The empiricist epistemology John Locke (1632-1704) formulated at the end of the century in his *Essay Concerning Human Understanding* [11] likewise had little use for syllogistic reasoning. In a well-known barb, Locke noted that "God has not been so sparing to Men to make them barely two-legged Creatures and left it to Aristotle to make them Rational."[3] One's native thinking ability — directly comparing ideas to see whether and how they agree — was all one needed, not the artifices of *Aristotelian Logic*.

4 Logic's 18th- and Early 19th-Century Scottish Critics

An even more hostile *ad hominem* attack on *Aristotelian Logic* arose in Scotland. The Common Sense philosopher Thomas Reid (1710-1796) opened the assault with his 1774 *Brief Account of Aristotle's Logic* [13]. Syllogistic logic, he says, is a shallow subject that enables one to artificially deduce conclusions already better known than their premises. Aristotle purposely obscured his analysis by using letters instead of meaningful terms to make his logic seem more profound. *Aristotle's Dictum* may be foundational to logic, but this is a principle "of no great depth." Syllogistic reasoning proceeds from premises to conclusions that are virtually contained in them. To produce genuinely new knowledge, as in natural philosophy, one should use inductive reasoning. Reid thus returns to Bacon's central theme: reasoning useful for expanding knowledge is inductive in nature, not deductive.

Two years later, George Campbell (1719-1796) delivered another broadside in his well-known *Philosophy of Rhetoric* [7]. Echoing Locke, he found syllogistic reasoning artificial and unnatural. Such argument forms are epistemically sterile — of value only for "disputatious wrangling." Syllogisms conclude things better known from things lesser known, instead of the reverse, as needed. In fact, since the conclusion of a valid syllogism is already contained in the premises (as many logics at the time claimed), syllogisms are prime examples of *petitio principii*, the fallacy of begging the question. In establishing the truth of the premises, one will already have shown the truth of the conclusion. To take *Barbara* as an example, to prove that all M are P one must show that all S are P along the way since S is part of M.

Reid's philosophical heir, Dugald Stewart (1753-1828), at first promoted the value of symbolic logical argumentation, perhaps because of his familiarity with mathematics, having taught it for several years at the University of Edinburgh alongside his ailing father. But by 1814 when the second volume of his *Elements of the Philosophy of the Human Mind* [14] was published, he had changed his viewpoint and proceeded to relitigate the deficiencies of *Aristotelian Logic*. Stewart found Campbell's charge of circularity unanswerable, and in a similar vein, he castigated *Aristotelian Logic*'s

[2] This was essentially a process of eliminating all possible causes but one—an extreme form of *disjunctive syllogism*—based on collecting a copious amount of data via observation.

[3] This is in the *Essay*'s fourth, 1700 edition, IV.XVII.4, 671.

practice of reducing valid syllogistic forms to the most basic first figure moods[4] — thereby *demonstrating demonstrations*. This he found inherently circular: the validity of such demonstrations would need to be tested by casting them into a syllogistic format, whose validity was the very thing under investigation.

Like Reid, Stewart also endorsed inductive reasoning for studying nature. He proposed developing a *rational logic*, where induction would play a prominent role. This was a plan he failed to implement, dying in 1828. John Stuart Mill (1806-1873) would carry the project forward in his seminal 1843 work on inductive logic [12].

5 Logic's 19th-Century English Defense: Copleston and Whately

We've now moved into the 19th century. Already at the turn of the century *Aristotelian Logic* was making a comeback at Oxford, which had reformed its university examinations in the subject to make them less perfunctory. One of Oriel College's tutors who took the value of logic to heart was Edward Copleston (1776-1849). Copleston never published a logic text, but his meticulous reworking of the subject for tutoring his students was pivotal. He also became Oxford University's spokesperson, tenaciously defending logic and Oxford's curriculum against contemporaneous criticisms by the Edinburgh mathematician John Playfair and others. He compared their complaints about logic's barrenness to the expectation that a "reading-glass" would produce words for someone unable to read, an analogy Whately would later repeat.

Copleston's ideas found a worthy champion in his student and eventual colleague, Richard Whately (1787-1863). Copleston judged the intellectual climate unripe for a new textbook on logic; Whately did not. So while Whately credits Copleston with being a virtual co-author of his text, in the end it was Whately whose work revived the study of *Aristotelian Logic* in 19th-century England, first as a two-part 1823 article[5] on "Logic" in the *Encyclopaedia Metropolitana* [16] and then, in response to its widespread acclaim, as a textbook (first edition, 1826; ninth edition, 1848).

Whately's *Elements of Logic* [17] opened with an introductory 23-page "Preface" defending the utility of logic. This was bookended by a concluding 23-page "Dissertation", which explored the role of induction vis-à-vis syllogistic logic and upheld the epistemic importance of logic. We'll briefly look at Whately's view of the nature and scope of logic and then summarize his defense of logic against the criticisms of the preceding two centuries.

6 Whately's Conception and Defense of Syllogistic Logic

Logic in Whately's view was both a science and an art. That it was a science was a comparatively new emphasis; at the time, many held that it was the art of thinking in general or that it was the right use of reason in seeking after truth. Stressing logic's scientific character, making the *logical form* of statements and *valid argumentation* central, gave logic a leaner, more focused aim.

Nevertheless, logic was also an art grounded in its scientific character. Whately emphasized that

[4] This was a standard part of traditional Aristotelian logic: reduction showed the primacy and perfection of first figure argument forms, though *reductio ad absurdum* was also required for some reductions.

[5] The publication date of Whately's article managed to elude everyone almost since it was written (even the bibliophile De Morgan was ignorant of the exact date in 1860) until I was able to pin it down definitively in my dissertation research.

any sound argument whatsoever could be resolved into syllogistic forms that could then be tested for validity. For instance, and importantly, as a devout and future churchman, Whately promoted the value of logical reasoning as an instrument for defending the Christian faith.[6] He thought students should be well-versed in logic so they could evaluate theological arguments and supply reasons in support of their beliefs. Whately himself analyzed a number of religious arguments, particularly in his highly respected chapter on "Fallacies." An appendix added to the fourth (1831) edition of his text even provided an extended analysis of Paley's argument for the divine origin of the Christian religion.

Whately's *Elements of Logic* was a well crafted text. It gave students a rationale for studying the subject and an excellent overview, emphasizing the role of *Aristotle's Dictum* as encapsulating all valid reasoning, before taking up the standard specifics about terms, propositions, and syllogistic moods. The enduring success and historical significance of Whately's text, though, was largely due to his vigorous defense of logic against its philosophical detractors, not to its systematic presentation of technical details.[7]

Whately doesn't give a decisive answer to Campbell's caustic criticism that syllogisms harbor *petitio* fallacies since he agreed that every conclusion of a valid argument is a special case of a universal premise: this is more or less the import of *Aristotle's Dictum*. He does note, however, that were the *petitio* charge true, it would vitiate all deductive reasoning, which certainly can't be right, and which would also undermine Campbell's own arguments against the syllogism.

In response to Stewart's trenchant critique that the practice of demonstrating demonstrations was circular, Whately asserts that *Aristotle's Dictum* does not provide a *demonstration* of syllogistic forms but is the "universal principle" on which "all Reasoning *ultimately* depends." This response, too, was less than fully convincing since Whately does seem to use *Aristotle's Dictum* as the deductive foundation for affirming the validity of syllogistic arguments.

With respect to the inutility of syllogistic reasoning generally, a theme revived by Reid, Whately provides a more satisfying rebuttal. In geometry, for example, students obviously learn the truth of new theorems implied by propositions they already know. The truth of the proposition "circles are to one another as the squares on their diameters" wouldn't be known without demonstrating it from previous results.[8] Regarding factual knowledge, Whately grants that one often arrives at new truths using something other than deductive reasoning. An inductive investigation may yield empirical principles that form the deductive basis for a science, but this does not mean that syllogistic reasoning *from* these principles is barren. There's room for both Bacon and Aristotle in pursuing knowledge. As Whately pointedly noted: "A *plough* may be a much more ingenious and valuable instrument than a *flail*, but it can never be substituted for it."[9] Induction may plant a crop in an epistemic field, but syllogistic reasoning reaps the inferential harvest.

Arriving at the conclusion of a syllogism, Whately says, is like finding a vein of buried metal in a field: though the metal is contained therein, this isn't brought to light until it is unearthed and extracted. A result may be virtually contained in a syllogism's premises, but until it is derived from

[6]This was apparent already in his first publication [15] (1819), in which Whately satirized Hume's skepticism about the biblical account of miracles by arguing that the same sort of reasoning would lead one to deny the existence of (the still living) Napoleon.

[7]Whately, following Copleston, doesn't veer very far from Henry Aldrich's readily available and widely used *Artis Logicae Compendium* [1] in his technical presentation, even while streamlining a few parts.

[8]"Essay.2" of [16], 91.

[9]Ibid, page 97. Striking illustrations like this are found throughout Whately's writing.

them it may not be explicitly *known* to the reasoner. In his text's eighth (1844) edition he adds a striking illustration about finding a buried animal skeleton whose identity can only be identified once an off-site naturalist provides the necessary information about such animals. Similarly, while the premises of a syllogism may be known separately, it's only after this knowledge is combined that the conclusion can be drawn. Both premises of a syllogism are needed to infer their consequence.

What is somewhat implicit in Whately's responses but not yet clearly conceptualized is a distinction between epistemology and logic, between a consequence being *logically implied by* or *deduced from* a set of premises and a conclusion being *known* to be true on the basis of the truth of the premises. Whately is moving toward a more formal understanding of logical inference in his logic, but what this means for everyday argumentation still isn't fully recognized.

7 Reception of Whately's Logic and Further Developments

Whately's logic was immediately and universally recognized as authoritative when it came out.[10] It elevated logic from its previous moribund state to being a field deemed worthy of scholarly pursuit. In the first quarter of the 19$^{\text{th}}$ century about a dozen works on logic were published in Britain; in the second quarter, after Whately's publication, a flood of works were produced — more than 50, in addition to the many editions and reprints of Whately's own works on logic.

By mid-century the groundwork was laid for several new logical developments of historic significance. As mentioned above, Mill wrote his famous *System of Logic* [12] in 1843, making induction central (as he thought) to all forms of inference. Other advances in logic were initiated by the English mathematicians George Boole (1815-1864) and Augustus De Morgan (1806-1871). Boole's 1847 *Mathematical Analysis of Logic* [5] was prompted by pondering an acrimonious priority dispute between De Morgan and the Scottish philosopher William Hamilton (1788-1856) over who had first quantified the predicate.[11] Boole expanded his work in his more mature 1854 book *Laws of Thought* [6]. De Morgan's own book, *Formal Logic* [8], appeared on the very same day as Boole's booklet, though his ideas on the logic of relations got worked out in a series of later papers, culminating in 1860 [9]. Later 19$^{\text{th}}$-century logicians such as William Stanley Jevons, John Venn, Charles Saunders Peirce, and Ernst Schröder extended these new perspectives further.

All of these later developments are deserving of talks on their own. But each of them is indebted to Whately for his revival of logic, as they universally acknowledged. Whately's spirited defense of syllogistic logic made the discipline attractive and accessible to a wide audience. Logic thus achieved renewed respectability. While Whately's work didn't itself birth any bold technical novelties, it revitalized British logic and made it possible for others to pursue new paths in a field that had been cleared for active exploration.

[10]The most comprehensive and informative survey of works published in the half-century following Whately's 1823 article is an anonymous article published in 1872 [2], which seems not to be well known.

[11]As it happened, George Bentham (1800-1884) put forward this notion in a book [4] analyzing Whately's logic before either of them 20 years earlier, but his work was almost unknown at the time since it had been remaindered for wastepaper after only 60 books were sold in order to pay creditors affected by the publisher's insolvency.

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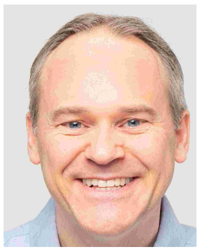
A Status Review of
“The Unreasonable Effectiveness of Mathematics”

Melvin Royer (Indiana Wesleyan University)



Melvin Royer holds B.S.E.E. and M.S.E.E. degrees in electrical engineering and a Ph.D. in mathematics from Purdue University. He has spent most of his career teaching mathematics and related disciplines at Indiana Wesleyan University, where he is currently a professor. He has served as a division chair and as chair of the faculty. His professional interests are mainly in dynamical systems. He and his wife Barb enjoy travel, music, church activities, and healthy food.

Abstract

Eugene Wigner’s 1960 article “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” spurred numerous authors from various academic disciplines to respond with either further support for his position (that the application of mathematics in science is inexplicably and almost “miraculously” effective) or rebuttals to it. It appears that most mathematicians and physicists affirm that applied mathematics is highly effective, while economists and biologists are less inclined to do so. Opinion on whether the effectiveness is actually unreasonable or not seems more divided. Support for Wigner’s position has somewhat declined as the complexity of new problems becomes apparent. Many authors believe that for applied mathematics to remain effective, it will need to shift focus from analytical models and formal proofs to using data memorization, computation, stochastic and heuristic modeling, and artificial intelligence. Philosophers have also studied the issue with some arguing that the effectiveness poses a serious threat to the belief of naturalism.

1 Introduction

In his influential 1960 article “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [1], the physicist Eugene Wigner cites multiple situations in physics where previously existing but seemingly unrelated mathematical truths were found to “unreasonably” provide an effective model. For example, during the early development of quantum mechanics, Born, Jordan, and Heisenberg found that after replacing the position and momentum variables by matrices in certain equations of classical mechanics, the resulting theory predicted spectral properties of the helium atom even though some required symmetry conditions for using the matrices were not satisfied. Wigner comments that “...we ‘got something out’ of the equations that we didn’t put in.” As explained later by Mark Steiner [4], Wigner’s point is more than a series of individually impressive examples. Rather, Steiner elaborates, mathematics has been effective as an overall research paradigm in physics; physicists routinely and successfully use ideas from mathematics to form scientific hypotheses that can then be tested by experimentation.

Wigner uses the term “empirical law of epistemology” to refer to this claim that mathematics developed for beauty and computational manipulability often later turns out to accurately formulate the laws of nature. He notes that just as the principle of invariance has provided physics with a foundation of facts, so has the empirical law provided physicists with the enthusiasm and reassurance to continue research using a mathematical paradigm. He expresses hope that the success of mathematics in the physical sciences would be followed by its success in other academic disciplines and concludes with a famous paragraph expressing optimism for the future of this law:

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

Multiple authors from various academic disciplines have subsequently provided arguments and further examples that such effectiveness exists and is unreasonable, while others have written rebuttals and interesting alternatives. The titles of their articles include provocative variations such as "reasonable effectiveness" and "unreasonable ineffectiveness." It is apparent, however, that authors are attaching somewhat different meanings to the relevant terms. There are actually at least three related issues involved:

1. What content would Wigner consider to be mathematics?
2. Is the application of mathematics actually effective?
3. If the application is effective, is that effectiveness unreasonable or can it be explained?

In this paper we summarize relevant portions of a selection of articles written by authors, beginning with Wigner, working in physics, philosophy, natural language processing, economics, and biology. Many of these articles cover additional topics, sometimes extensively; we only treat portions of the articles that we believe have the strongest bearing on Wigner's claim. The inclusion of articles is far from exhaustive; they were selected mainly on the basis of provocative titles or mutual reference. Finally, we provide a few conclusions that may be drawn from the totality of these articles. In an appendix, we briefly discuss several philosophical issues regarding what it means to "apply" mathematics that are not central to Wigner's idea.

For the reader's convenience, we preview here the articles in the order in which they will be treated. Generally speaking, we first survey articles sympathetic to Wigner's claim, then turn to authors who are less supportive.

1. Wigner (physicist, 1960): Gives examples that mathematics is unreasonably effective when applied to physics.
2. Hamming (mathematician, 1980): Agrees with Wigner that mathematics is effective when applied to physics, then argues that this effectiveness is actually reasonable but finally admits defeat and ends up agreeing with Wigner.
3. Steiner (philosopher, 1999): Defines the philosophical issues, agrees with Wigner, and proceeds to argue that applying mathematics to science is an inherently "anthropocentric" process (one in which humans have a special place in the universe).
4. Colyvan (philosopher, 2001): Agrees with Steiner and claims that multiple philosophies of mathematics fail to explain the unreasonable effectiveness.
5. Grattan-Guiness (historian, 2008): Agrees with Wigner that mathematics is effective when applied to physics, but argues that Wigner's thoughts were incomplete and that the effectiveness is actually reasonable.
6. Focardi and Fabozzi (economists, 2010): Provide evidence that mathematics is moderately effective in economics and explain why it is not as effective as it is in physics.

7. Abbott (engineer, 2013): Argues that math is becoming less effective in engineering and that Hamming's reasons and others show that any effectiveness is actually reasonable.
8. Halevy, Norvig, and Pereira (data scientists, 2010): Argue that mathematical theories have not been very successful in natural language processing and advocate instead for the more successful strategy of machine learning patterns in big data.
9. Velupillai (economist, 2005): Argues that the apparent effectiveness of mathematics applied to economics is due to using inappropriate methodology and advocates for mathematics itself to change to a more computational approach.
10. Lesk (biologist, 2001): Claims that mathematics has been effective in narrow subdisciplines of biology and explains why the effectiveness is not broader.
11. Borovik (mathematician, 2021): Agrees with Lesk on the limited effectiveness of mathematics in biology and makes similar recommendations as Velupillai for the future direction of mathematics.

2 Wigner

Wigner spends time in his article explaining what he means by mathematics and asserting that mathematicians do their work guided by respect for ingenuity and beauty. One point made by both Wigner and Hamming [2] is that to better appreciate the true effectiveness of mathematics, one must move beyond examples taken from simple arithmetic and geometry. For example, the fact that 6 sheep + 7 sheep = 13 sheep and also 6 stones + 7 stones = 13 stones makes the integers seem remarkably effective. Similarly, the fact that the simplest algebraic equations (linear and quadratic) correspond to the simplest geometric entities (lines, circles, and conics) illustrates the effectiveness of analytic geometry. Enrico Fermi was able to estimate the yield of the first atomic bomb test by dropping scraps of paper while the blast wave was passing him, measuring the horizontal deflection of the scraps, and doing a mental calculation.

Such examples might lead one to believe that the effectiveness of mathematics in the physical sciences is inevitable because mathematics is developed using the simplest possible concepts and these were bound to occur in any formalism. In other words, the effectiveness is essentially just an illustration of Occam's Razor. But Wigner (and later Hamming) maintain that simplicity is actually *not* the answer. They both argue that mathematical concepts are actually developed for aesthetic reasons, namely beauty and calculational convenience, rather than for their conceptual simplicity. For example, the complex numbers were developed and are used not because they are "obvious" but rather because they exhibit remarkable properties when used in computation. As evidence against the explanation of simplicity, Wigner emphasizes examples using more complicated mathematics. The example of quantum mechanics, discussed in the introduction and used by Wigner, is far from elementary, involving an infinite dimensional complex Hilbert space with a Hermitian inner product.

Another example used by Wigner is gravitation. From observing falling objects on earth and from imprecise astronomical data, Newton deduced his universal law of gravitation $F = G\frac{m_1 m_2}{r^2}$. Except for special situations requiring the use of general relativity, the law of gravitation remains truly universal and correctly predicts the gravitational force between two objects to within the precision of even modern scientific experiments. Wigner comments "[The law] has proved accurate beyond all reasonable expectations." Yet the law is not simple in the sense that it implicitly involves the use of calculus in defining acceleration.

As a final example, Maxwell's equations use vector calculus in a nontrivial system of partial differential equations. Maxwell is thought to have included the displacement current term $\frac{\partial \mathbf{E}}{\partial t}$ in his equation (written using modern notation) $\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$ at least partly to preserve symmetry with the equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$; this led him to "unreasonably" predict the existence of electromagnetic radiation.

The previous examples illustrate effectiveness in sophisticated situations where there seems to be no simple "obvious" explanation. Wigner believed the complexity involved made the success of the applications "unreasonable."

3 Hamming

The article by the mathematician R.W. Hamming has many similarities to that of Wigner. Hamming also emphasizes the aesthetic development of mathematics and provides additional examples of its effectiveness in the physical sciences. Striking a new direction, he then gives four possible explanations why this effectiveness should actually be expected rather than considered unreasonable.

1. We see what we look for.
2. We select the kind of mathematics we wish to use.
3. Science actually answers comparatively few questions.
4. The evolution of man provided a model.

Hamming's first argument is that we impose the effectiveness by our own methodology. For example, he argues that Galileo likely developed his laws of motion mostly from reasoning (perhaps imagining falling objects of unequal mass tied together) rather than experimentation. Hamming notes that he later learned George Polya also had the same suspicion [3]. Similarly, once Fourier analysis became a central feature of quantum mechanics, certain mathematical inequalities in Fourier theory inevitably lead to an uncertainty principle. Hamming cites a quote by Arthur Eddington capturing this thought: "Some men went fishing in the sea with a net, and upon examining what they caught they concluded that there was a minimum size to the fish in the sea."

Hamming's second explanation is that we forget that mathematics does not always work. For example, when scalars were unable to explain the addition of non-collinear forces, physicists instead used vectors with great success. The tendency then is to remember the success of vectors and forget the failure of scalars.

Thirdly, Hamming notes that science actually answers comparatively few questions about life. For example, science has contributed little toward fundamental explanations of the nature of truth, beauty, and justice. Furthermore, Gödel's theorem shows limitations on formal logic itself, thus barring logical answers to some well-posed mathematical questions.

Finally, Hamming proposes that the biological evolution of humans may explain the effectiveness of mathematics. He suggests that early life forms evolved the seeds of our current ability to form the long chains of reasoning needed in scientific research. Similarly, Darwinian evolution would select forms of life that had the best models of reality. Thus, humans have been "hardwired" to effectively use mathematics. However, Hamming himself openly states that he does not find this argument convincing. Other authors have also skeptically viewed evolutionary explanations; some of these arguments and rebuttal are given by Russell Howell and James Bradley [5] [6].

In the end, Hamming abruptly concludes by acknowledging that these explanations are inadequate. "I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set out to account for...The logical side of the nature of the universe requires further exploration." It is interesting to note that some later authors such as Abbott[10] find Hamming's arguments more convincing than did Hamming himself.

4 Steiner

4.1 The role of analogies

Approaching the topic from a different discipline, Mark Steiner's book *The Applicability of Mathematics as a Philosophical Problem* [4] focuses heavily on the methodology used by the early developers of quantum mechanics. In particular, Steiner spends considerable effort tracing analogies physicists used from classical mechanics to form hypotheses in the new quantum model (the pattern of similar forms between the models is called the "Correspondence Principle"). Wigner's example in the introduction is one such analogy. Another cited by Steiner is that when attempting to extend the Klein-Gordon field equation to the electron, Dirac invented new "number-like" objects to allow factorization of the equation. In this new number system, the equation had additional solutions which suggested the existence of the particle now known as the positron. Subsequently the existence of positrons was experimentally confirmed. It was also realized that Dirac had actually reinvented concepts of Clifford algebra developed by mathematicians in the nineteenth century.

Steiner notes that many of these analogies were what he calls "Pythagorean". That is, at least at the time of their development, the analogies were expressible in no other way than the language of pure mathematics. Steiner agrees with Wigner and Hamming that the development of pure mathematics today is guided by beauty and calculational convenience. Beauty seems to naturally be a human-centered judgment. And calculational convenience is based on the limited capabilities of our brain. So when the developers of quantum mechanics looked to pure mathematics for analogies, they were actually studying humans more than the physical universe. Anthropocentrism is nothing new in science. For example, in the Middle Ages, events were divided into heavenly and terrestrial and the geocentric model of the solar system was defended by some because it gave mankind a special place. But in the development of quantum mechanics, anthropocentrism became central to a modern research paradigm.

4.2 Mathematics as magic

Steiner notes that the claim that human constructions supply insights into reality belongs to the realm of what has often been called magic. Furthermore, mathematics seems unique in this regard; in no other area does manipulating human constructions according to human convenience provide insight into the structure of the world. For example, the rules of chess are also a system of logic, but these rules say nothing about the nature of atoms. The applicability of mathematics to physics is thus the last fortress of magic, but one whose standing seems secure.

Not only has mathematics had isolated successes in explaining problems in the natural sciences, but it has been a successful global research strategy guiding progress. The obvious success of the use of mathematical analogies is then evidence for Steiner's claim that there is a correspondence between the human brain and the physical world. He summarizes the situation as living in a "user friendly" world, which certainly includes the claim that mathematics is unreasonably effective. We will return to Steiner's implications of this assertion in the conclusion of this paper. In addition, the appendix provides a glimpse of Steiner's view of what it means to "apply" mathematics.

5 Colyvan

5.1 Mathematical realism

Just two years after the publication of Steiner's book, the mathematical philosopher Mark Colyvan published a paper entitled "The Miracle of Applied Mathematics" in 2001 [7]. He firmly supports Steiner in asserting that Wigner's paper poses a significant philosophical problem. Colyvan's goal is to show that the problem is broadly relevant in the sense that it cannot be resolved by simply adopting one of the competing currently popular philosophies of mathematics. Specifically he attempts to show that neither the philosophy of Quinian realism nor the anti-realist philosophy of fictionalism resolve the unreasonable effectiveness of mathematics.

Mathematical realism is the idea that mathematical truths exist independently of human knowledge. Some well-known forms of realism include Platonism, which is the idea that mathematical objects exist in some abstract realm, and Aristotelian realism, which asserts that mathematical concepts such as symmetry and continuity can be literally realized in the physical world. The form of realism that Colyvan chooses to study is due to the Harvard philosophers Willard Quine and Hilary Putnam and is referred to by Colyvan as Quinian Realism. A main tenet of Quinian Realism is that if we commit ourselves to believing a scientific theory, then we are equally committed to believing in the existence of entities that are indispensable in that theory.

Quinian realists explain the effectiveness of scientific theories by causality. For example, the theory of electrons explains the phenomenon of electromagnetism because electrons objectively exist and cause this phenomenon. So then such a philosopher could attempt to explain the unreasonable effectiveness of mathematics by claiming that objective mathematical concepts cause the phenomena they explain. However, Quinian realists tend to be Platonists (believing mathematical objects exist abstractly) and also reject the idea that abstract objects can cause physical phenomena. Colyvan asserts that therefore Quinian realism cannot explain Wigner's problem.

5.2 Mathematical fictionalism

Mathematical anti-realism, of which Hartry Field's fictionalism is a main school, is the belief that mathematical objects only exist as constructions of the human mind. Fictionalism holds that mathematics is actually literally false but can still be used because it is helpful in the explanation of theories. To show that mathematics is not indispensable to science, Field (at least partly successfully) shows how Newtonian gravitation can be explained without reference to mathematics at all. But according to Colyvan, fictionalism is powerless to explain how mathematics, which is supposedly not essential to science at all, can guide the development of new science in the way Wigner describes.

Since both realism and anti-realism fail to explain mathematical effectiveness, Colyvan concludes the effectiveness is established as an unreasonable puzzle, perhaps a "miracle." But we next move to authors who begin to question first the unreasonableness then the effectiveness of the application of mathematics.

6 Grattan-Guinness

The mathematical historian Ivor Grattan-Guinness published a rebuttal to Wigner entitled "Solving Wigner's Mystery: The Reasonable (Though Perhaps Limited) Effectiveness of Mathematics in the Natural Sciences" in 2008 [8]. He completely accepts Wigner's claim of effectiveness; his dispute is with its characterization as unreasonable. Grattan-Guinness argues for the "reasonableness" in two ways. First, the natural sciences have actually been effective in the development of mathematics. In other words, mathematics (including advanced topics) frequently borrows concepts from science, just as the reverse is true. (He notes that pure mathematicians sometimes haughtily try to ignore this fact). When two disciplines exchange concepts in parallel, links should be expected rather than regarded as mysterious. Second, both mathematical and scientific theories are built by observing regularities and by drawing analogies from other theories. Furthermore, this theory-building often involves ubiquitous notions such as conservation, duality, continuity, and induction, leading to reasonable sharing of concepts between theories.

Grattan-Guinness gives names to seven variations by which one theory can borrow from another: reduction, emulation, corroboration, importation, revolution, innovation, and convolution. He then spends much of his paper giving historical examples how these variations led to reasonable influences of mathematics and the physical sciences upon each other. As an example of emulation, Joseph Fourier was motivated by problems in heat conduction to develop the representation of functions as trigonometric series. Several decades later, Georg Ohm used these series to give a quite successful mathematical treatment of the science of acoustics. One view would be that the application of Fourier series to acoustics was unreasonably effective. But Grattan-Guiness argues that here the effectiveness is instead quite reasonable. He notes that the physics of heat and of acoustics have commonalities, so when heat (a concept of science) contributed to Fourier series (a concept of math) which in turn contributed to acoustics (a concept of science), it is only natural that the latter application was effective.

In short, Grattan-Guiness claims that Wigner's article is philosophically unsound because Wigner neglected to consider numerous cases where mathematics and physics had clear links in historical development that makes their mutual application quite reasonable. He also points out that Wigner did not fully appreciate the ubiquitous notions that guide the theoretic development of both mathematics and the physical sciences.

7 Focardi and Fabozzi

7.1 Mathematics is effective in economics

The economists Sergio Focardi and Frank Fabozzi are the first of several authors we consider who work in fields outside the physical and mathematical sciences. In their article "The Reasonable Effectiveness of Mathematics in Economics" [9], they assert that mathematics is effective in economics but only moderately so (here the word "reasonable" is used more in the sense of moderation rather than the way Wigner used it). As evidence of effectiveness, they note that mathematics has made some successful economic forecasts. For example, our understanding of risk has been improved by the ability to identify appropriate probability distributions for various economic phenomena. Cyclical behavior and aggregate decision making can also be somewhat explained by modern time series and clustering models. Furthermore, mathematics can also provide information about when it is not possible to make reliable forecasts. Finally, it is possible to outline a research agenda for the possible improvement of the science of economics. These characteristics are among the standards of mathematical success in other fields and should, the authors say, be sufficient to consider mathematics effective in economics.

7.2 Why math is only *reasonably* effective in economics

The vast majority of Focardi's and Fabozzi's paper is devoted to a discussion of why mathematics has not been effective in economics in the same way it has been in the physical sciences. The most fundamental reason is that the subject matter of economics is large scale economies and markets which were deliberately designed to encourage innovation. Economic growth is dependent upon innovation, so total predictability of economic systems is considered stifling and to be avoided. As a result, economies are consciously self-adaptive; any knowledge gained soon changes the nature of the system. While self-adaptation may be good for public policy, it obviously complicates the mathematical analysis of the system.

There are also other reasons why the success of mathematics in economics seems more modest than in the physical sciences. First, small-scale models or experiments are only rarely possible. Second, because of the previously mentioned adaptation of economies, there is actually a shortage of relevant data at any given time. By the time a large pool of data can be accumulated, much of it is no longer relevant. Finally, at least until recently, the necessary computing power to perform realistic simulations was not available.

Focaradi and Fabozzi make the analogy that economic prediction is comparable in scale not to mathematically predicting systems such as an atom, but rather to predicting earthquakes and weather. In these areas, physical science is taking longer to achieve successful mathematical predictions. Furthermore, the mathematical approach taken in fields such as geology, meteorology, and biology is rather different. Rather than relying mainly on axiomatic models with deterministic parameters which are then estimated from experimentation, the relevant mathematics includes complexity, artificial learning, computation, and stochastic methods. The economist Velupillai [12] later argues that mathematics itself must move in these directions to remain an applicable discipline.

8 Abbott

In his article "The Reasonable Ineffectiveness of Mathematics" [10], the physicist and electrical engineer Derek Abbott attempts to counter Wigner's and Hamming's conclusions. He argues both that mathematics is not as effective as supposed and that a non-Platonist view of mathematics helps provide a reasonable explanation for the effectiveness it does have. Platonists hold that mathematics has an abstract existence outside of the physical universe. Abbott believes that most mathematicians are Platonists but nearly all engineers are not. His explanation for this difference is that engineers are thoroughly aware of the limitations, approximations, and contrivances of their own designs, and so do not expect to find ideal structures anywhere else.

As evidence that mathematics is less effective than sometimes claimed, Abbott notes that modern electronics engineers have increasingly abandoned the use of traditional analytical calculations. For example, transistors at the micrometer scale could be described by equations derived from first principles. At the nanoscale, however, complicated side effects overwhelm these equations and necessitate the use of empirically designed computer models. Abbott also notes that the property of invariance is dependent upon the time scale of the observer. We find mathematical descriptions of biological and economic systems challenging in part because such systems change while we are trying to observe them. On the other hand, in physics we are, for example, able to assume the sun is a constant energy source because its fluctuation is not apparent on our time scale.

Regarding whether any effectiveness is "unreasonable," Abbott believes that Hamming's explanations for the supposed mathematical "miracles" are more convincing than Hamming himself judged them to be. He argues in various ways that Wigner and others cherry picked examples in which mathematics was effective while ignoring other cases. He notes, for example, that until recently mathematics has mainly focused on linear systems for reasons of tractability; effectiveness is then already somewhat "baked in" due to the ability to use superposition.

Returning to the idea of mathematical Platonism, Abbott argues for rejecting this viewpoint. He notes that a hypothetical alien society may have mathematics in which idealization and Occam's Razor are not valued, all physical equations are stochastic, and reasoning is done by mental numerical simulation. Freeing ourselves from Platonism allows us to understand how we as humans have oversimplified physical properties by inventing mathematics to fit our own mental capabilities. A non-Platonist view of mathematics as simply a formalist mental tool would, he argues, encourage the acceptance of different types of analyses, thus speeding the development of scientific models.

9 Halevy, Norvig, and Pereira

The remaining authors all to some extent call for mathematics to broaden in scope. The field of natural language processing has obviously developed rapidly in recent years. Halevy, Norvig, and Pereira wrote an article titled "The Unreasonable Effectiveness of Data" [11] while working as research scientists at Google in 2009. Though written before the widespread use of large language AI models, the conclusions of their article seem consistent with current models in AI. In the title, the word "unreasonable" is used in Wigner's sense but applied to a different methodology.

The authors note that sciences involving human behavior have proven more resistant than physics to analysis by sophisticated mathematics. Scientists have developed complex general models for natural language processing tasks, such as speech recognition and machine translation, with some success. However, Halevy, Norvig, and Pereira claim that simple models based on stored probabilities of short consecutive word sequences inevitably outperform general models (assuming massive natural datasets are used to calculate the model probabilities).

The authors' viewpoint is that theories for natural language processing may always be complex and never have the elegance of physics equations. If so, they advise developers to abandon the goal of elegant theories and instead focus on machine learning to "make use of the best ally we have: the unreasonable effectiveness of data." While the development of language models seems impressive and ongoing, the methodology of simplistic models using "brute force" on massive data sets seems rather out of the spirit of what Wigner envisioned as the unreasonable effectiveness of mathematics. In particular, the idea of analogy (using old mathematical ideas to generate new scientific models) is absent, though an unconscious analogy is perhaps being used by patterning the development of artificial intelligence systems somewhat after the human brain.

10 Velupillai

10.1 A contrarian view to mathematical effectiveness

In his article "The Unreasonable Ineffectiveness of Mathematics in Economics," the economist Velupillai presents a decidedly pessimistic view of the effectiveness of mathematics [12]. He uses the word "unreasonable" for his viewpoint that the assumptions of most economists are unwarranted and the word "ineffective" for the fact that his alternately preferred assumptions lead to mathematical intractability.

Early in his paper, Velupillai explains that mathematics has been applied to different branches of economics with varying degrees of success. He chooses in this paper to focus on General Equilibrium Theory (GET), which is a major area of economics research focusing on supply, demand, and prices of an entire economy. A central premise of GET is that prices jointly move toward a global equilibrium; GET therefore is deeply intertwined with the mathematical theory of fixed point analysis.

Fixed points are generally proven using theory from classical real analysis, and Velupillai has two concerns with this procedure. First, many such related theorems (e.g. the Bolzano-Weierstrass Theorem) are proven using an infinite sequence of non-constructive choices, which is a paradigm disallowed by most quantitative disciplines other than economics. Velupillai argues that by accepting such proofs, economists are engaging in meaningless (hence unreasonable) theory. Second, he claims that economics problems actually deal with integer rather than real quantities (e.g. the number of shares of a stock) and should therefore be described by Diophantine equations rather than equations over the real numbers. But these Diophantine equations require very different analysis procedures, and are usually far more difficult to solve (sometime provably so) than the corresponding equations over the real numbers. Velupillai therefore characterizes the application of mathematics to economics as ineffective.

10.2 Redirection of mathematics

Velupillai does not advocate abandoning the use of mathematics in economics, but rather argues that mathematics itself needs to change focus and modality. Linking to visionary ideas from Stanislaw Ulam and later from Stephen Wolfram [13], Velupillai argues that mathematics must and will change in nature from proving precise theorems to addressing general concepts involving computation. He notes that at any given time, mathematics tends to be defined by the questions that its methods can successfully address.

In the language of Wolfram, the mathematical laws of traditional science are analytically solvable and therefore "computationally reducible." The success of mathematics in economics can therefore be viewed as only a subterfuge resulting from overemphasizing these analytically solvable relationships. Applications with laws that are not analytically solvable will be "computationally irreducible" and will therefore, according to Wolfram, imply unpredictability. Velupillai thus explains the effectiveness of mathematics in economics as simply a result of an unhealthy focus on unrealistic and simplistic theories.

11 Lesk

11.1 The situation in biology

The final two articles we consider are placed together since they both address the discipline of biology. However, the first was written before, and referenced by, the work of Velupillai. The molecular biologist Arthur Lesk gave a symposium lecture in 1998 entitled "The Unreasonable Effectiveness of Mathematics in Molecular Biology" and later published an article with the same title [14]. Lesk later revealed, however, that he had wanted his title to read "The Unreasonable Ineffectiveness of Mathematics in Molecular Biology" but was convinced by the symposium organizers to make the change [15]. In essence, Lesk claimed to be pessimistic in the short term but optimistic in the longer term. As was the case with Focardi and Fabozzi, Lesk uses the word "unreasonable" in the sense of moderation rather than in Wigner's sense.

Lesk begins by pointing out that many biologists do not share the foundational optimism of physicists that mathematics will be successful at explaining the processes involved their field of study. A fundamental problem is that the complexity of biological systems obscures the relationships between system inputs and outputs that biologists could potentially express mathematically.

11.2 Computational molecular biology

Lesk then specializes his discussion to molecular biology, which has a goal to progressively map information from a DNA sequence to the resulting protein sequence, then to the protein three dimensional structure, and finally to the protein functionality. Presumably to be able to share successful examples, he further focuses on computational aspects of molecular biology. Though much progress has been made in the subfields of sequence alignment and structure superposition (and so mathematics is partly effective), he notes that the gap to predicting full protein structure remains large (and so mathematics is still partly ineffective).

Lesk notes that a conceptual approach to many such structure problems is clear – write and minimize a thermodynamic energy. The required mathematics, however, is hindered by the level of complexity that resists reduction to smaller problems. The forces known to us that stabilize protein shape act only over short distances and cannot explain how proteins fold starting from an arbitrary position. Further, energy landscapes often have local minima which complicate the search for a global minimum. The mathematical form of the energy of a folded protein is therefore difficult to minimize, even numerically.

Such difficulties have led some computational molecular biologists away from predicting protein structure using theoretical principles and toward recognizing patterns by comparing to previously constructed databases. As with the situation of Halevy, Norvig, and Pereira in natural language processing, this trend toward pattern recognition and machine learning calls into question what qualifies as "mathematics." Memorization and distributed computing seem conceptually different from Wigner's idea.

12 Borovik

12.1 The role of mathematics in biology

The final viewpoint considered here, Alexandre Borovik's article "A Mathematician's View of the Unreasonable Ineffectiveness of Mathematics in Biology" [16], has much in common with the article by Velupillai. Here, the word "unreasonable" is used more in the sense of lament ("it's not fair that mathematics is so much harder to use in biology than it is in physics"). Borovik first confirms the accuracy of a quotation attributed to the mathematician Israel Gelfand:

> Eugene Wigner wrote a famous essay on the unreasonable effectiveness of mathematics in natural sciences. He meant physics, of course. There is only one thing which is more unreasonable than the unreasonable effectiveness of mathematics in physics, and this is the unreasonable ineffectiveness of mathematics in biology.

Borovik's viewpoint is "… for the time being: mathematics is still too weak for playing in biology the role it ought to play." He acknowledges the success of mathematics in the modern subfield of genomics but notes that such work tends to involve computations rather than the complex theories and analogies envisioned by Wigner.

Borovik argues that biology studies *algorithms*, a concept that does not exist in physics. In molecular biology, a crucial algorithm is the determination of protein functionality from DNA sequences, as mentioned earlier by Lesk. And molecular biology has long had a principle known as the Central Dogma which says that information only flows from DNA to protein structure, never in the other direction. As an analogy of the Central Dogma, Borovik notes the mathematical concept of a one-way function. Multiplication of large integers is thought to be an example; finding the product given the factors is easy, but finding the factors given the product is presumably forever hard. He points out that in a similar way, our inability to trace information backward from the protein shape to DNA is one of the main obstacles to the formation of good mathematical models in molecular biology. This inability may be permanent since evolution had no need to develop reversibility in the information flow.

There are several other reasons why mathematics may be less effective in biology than in physics. First, the science of biology involves the study of evolution. Evolution works bottom up, while human logic generally works top down. Evolution is blind and need not produce an optimal organism, merely a survivable one. These features make it difficult to understand a process when all we see is its outcome. Also, biology does not have scale invariance – molecular biology, cellular biology, and organismal biology all involve very different processes at different scales, making it difficult to divide problems into manageable subparts.

12.2 The future of effective mathematics

Borovik argues that mathematics itself needs a paradigm change for several reasons. First, as noted above, mathematics is being called upon to answer new questions to which it is not well suited in its current form. Second, mathematics has become so specialized and complicated that it has become increasingly difficult to find peers to critique one's work. Finally, mathematics education no longer teaches content needed by practicing applied mathematicians.

In order for mathematics to be effective in biology, Borovik recommends multiple paradigm changes. Most importantly, he believes mathematics must become a computer-based discipline in which artificial intelligence will assist in proofs and check their correctness. Second, he recommends an increased emphasis on stochastic approaches. Third, he cites a need for the development of new multidimensional combinatorics. And finally, he laments the need for a dramatic reform in mathematics education. Like Lesk, Borovik seems optimistic in the long term about all these reforms except the one involving mathematics education.

13 Conclusions

From the articles reviewed here, there is a clear lack of consensus on both the degree of effectiveness of applied mathematics and on whether this effectiveness is unreasonable. It does seem that agreement with Wigner's claim of effectiveness has somewhat declined in the years since his article was published as quantitative research from fields outside of mathematics and physics becomes more widespread. Nearly everyone agrees that mathematics is highly effective in physics; opinions and predictions of the situation in biology and economics are more mixed. A commonly cited reason for this difference is that biological systems and economies exhibit systems of higher and irreducible complexity which often can also significantly adapt themselves in observable time scales.

Wigner's claim has become strongly linked with the development of future mathematics. The nature of many current applications is stimulating the development of mathematics in directions that are more computational, stochastic, and heuristic. It is unclear to what extent these efforts will succeed in the way hoped by Wigner. Even if comparable effectiveness is achieved, it seems possible that mathematics will have changed so much that Wigner's original concept of "formal mathematical analogy as a paradigm" will no longer be recognizable.

It is disputed whether any effectiveness is "unreasonable", but the case that it is has broad support among mathematicians and physicists. Consensus among mathematical philosophers on the issue seems unlikely in the near future. If one accepts the conclusions of Grattan-Guinness, then for purposes of Wigner's claim, mathematics is simply a tool which is more effective in some disciplines than in others. However, if one believes the arguments of Steiner, then issues of worldview are raised. Steiner argues that the anthropocentric world he describes poses a problem for belief in naturalism (the universe is a closed system with no guiding mind), which he identifies as the reigning "ideology" of the natural sciences. Such a system should definitely not be anthropocentric, and therefore the world should not be "user-friendly" for investigations by the human mind. The fact that it is, according to Steiner, therefore refutes naturalism. Similar arguments against naturalism by writers such as C.S. Lewis and Alvin Plantinga are discussed by Bradley and Howell [6].

Steiner does not propose an alternative view to naturalism, though he notes that physicists seem to have largely abandoned pure naturalism in favor of a working theory that relies heavily on aesthetic principles of beauty and symmetry. But Bradley and Howell note that theism is a compelling explanation for an anthropocentric universe. If humans are created in the image of God, it seems reasonable that this image includes an inherent rational and aesthetic capability for understanding His creation.

Appendix: Philosophical Issues of Application

Steiner[4] defines the application of mathematics as using mathematical theorems as premises in logical deductions. For example, if we accept the premises that $7 + 5 = 12$, we have seven apples, we have five pears, we have no fruits other than apples and pears, and no fruit is both an apple and a pear, then we conclude we have twelve fruits. In particular, this definition excludes direct application to the physical world, which would require stances on certain philosophical issues of mathematical realism. The deductions reached by this "application" of mathematics may be ones that make predictions about the physical world, but that is a separate issue.

There is actually a semantic problem with the fruit example. In the premise $7+5=12$, the numeral '7' refers to an object (in this case the number seven), while in the premise "we have seven apples," 'seven' is used as a characterization of the apples. So in fact, the argument as presented needs more justification. The logician Gottlob Frege gave a satisfactory resolution to this problem [4, p. 15-19]; mathematics can thus effectively be applied in a semantic sense.

There is no trace of semantic concerns in Wigner's article; he and subsequent responding authors were not concerned with the question of applicability in this sense. But there are other philosophical issues to consider, which Steiner calls "metaphysical" problems, stemming from a possible gap between mathematics and the physical world. Regardless of whether mathematical objects actually exist (a debate in the philosophy of mathematics that the author Colyvan [7] discusses), mathematics is not claimed to be directly causal in the physical world. But in physics, nearly every relation is taken to be causal, so how can mathematics be applied to the world at all? Steiner asserts that Frege solved this problem as well, applying mathematical entities to "concepts" (whether these concepts themselves then describe the physical world is irrelevant).

The metaphysical issue of applicability is closer to Wigner's claim than the semantic one, but still not the issue that fascinated him. For Wigner and most of the other authors considered here, the possibility that mathematics can be applied to scientific theories about the physical world is taken as a given; the issue is the degree to which such application is helpful to the development of these theories. This degree of helpfulness is what Steiner calls "descriptive" applicability and is the topic of Wigner's article and of this paper.

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Embracing the Mystery: How Mysterious Mathematics Bolsters My Faith

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Abstract

Mathematical modes of thinking provide a helpful framework for engaging mysteries of faith. In mathematics, we routinely wrestle with ideas that are challenging to wrap our minds around, results that challenge our notions of certainty, reveal our finite human limitations, and shift our perspectives. Experiencing these challenges in the realm of mathematics enables us to expect and embrace challenges in matters of faith and truths in God's Word that are difficult to comprehend. This paper will explore mysterious mathematics that bolsters my Christian faith and fosters awe and wonder for God, the ultimate mathematician.

1 Introduction

The connection between mathematics and Christian faith can often seem elusive to those outside the discipline (and to many inside the discipline as well). After all, many people falsely view mathematics as "a set of facts, rules, and procedures" [2, p 3][1] and determine that there is no decidedly Christian way to utilize them. Mathematics, however, involves much more than computation; it is a study of patterns and an exploration of dimension, shape, and quantity, as well as change and uncertainty. Mathematics reveals structure in God's creation and much more—it reveals mysteries, counter-intuitive results or apparent contradictions that remind us that God's ways are above our own. As I wrestle with, and stand in awe of, these difficult-to-understand truths, I am reminded that challenges in mathematics have parallels in my faith walk.

I feel strongly that it is necessary and valuable to wrestle with and revisit difficult questions. When struggling with a mathematical problem or concept, we often need to walk away and return later with fresh eyes. Time and space or a change in perspective may be necessary to understand the concept more deeply or make progress toward the problem's solution. I see parallels between this and reading God's Word. The more we engage God's Word, the more we learn from it. Sometimes we read a familiar passage, and we see it with new eyes; the Spirit gives us new insights. After all, "the word of God is alive and active. Sharper than any double-edged sword. . . ." (Hebrews 4:12) Mathematics can be living and active too. Often the deep connections between mathematical concepts themselves and their spiritual applications or ramifications require time for growth, processing, dialogue, and reflection. Sometimes the clarity and connections only come after we have

[1] Full Quote: "Mathematics – a set of facts, rules, and procedures – is a 'package' to be passively received. . . . Mathematics teachers are supposed to spend time explaining or 'covering' material from the textbook. . . . Teachers verify that students have received knowledge by checking the students' answers to make sure they are correct." Original source: [7]

been exposed to and wrestled with more complex mathematics. We cannot necessarily make sense of some mathematical truths in the moment or the first time we encounter them. There is a level of mathematical maturity that is needed before we are ready to make some of these connections between mathematics and faith. As students experience more mathematics, they are better equipped to process how mathematical principles are inter-weaved throughout God's creation—how God is present in the unexpected connections and cohesion of mathematics. This idea is foundational to my view of math-faith integration. I feel like it can feel forced when the students lack mathematical maturity. If pushed too hard the integration can feel artificial, but as we grow in our mathematical understanding—as we are exposed to more and deeper mathematical complexity—as we begin to realize that certainty is illusory, encounter our human and mathematical limitations, and gain insight from different perspectives, we begin to see how our pursuit of mathematics fosters an acceptance of and appreciation for the mysteries of God and His Sovereignty.

2 Certainty

My view of how Christian faith interacts with mathematics has been shaped by many sources, but two that I especially resonate with are Robert Brabenec's article "The Impact of Three Mathematical Discoveries on Human Knowledge," which I will address throughout my paper, and a novel by Gaurav Suri and Hartosh Singh Bal entitled *A Certain Ambiguity*. One reason I resonate with this novel is that it highlights how faith is essential in mathematics. Faith is expected in Christianity, but, surprisingly, it is also necessary in mathematics. This novel along with *Mathematics Through the Eyes of Faith* is the subject of an article/book review in the Summer 2012 issue of Christian Scholar's Review by Dave Klanderman and Sharon Robbert in which they discuss how both sources are utilized to prompt students to reflect on the interplay between mathematics and a reformed Christian worldview. The novel paints a compelling picture of why we pursue mathematics and how mathematicians think, and it includes story-lines which explore aspects of mathematics related to faith. Characters include Ravi, a college student who is learning about mathematics, about his (now deceased) grandfather, and seeing both with new eyes; Bauji, the grandfather, an Indian mathematician who possessed a purely mathematical worldview; and Nico, Ravi's mathematics professor, who inspires Ravi and his students to see the beauty and wonder inherent in mathematics. The mathematics that these characters explore will form the backdrop and impetus for some of the mathematical ideas that I want to discuss further in this paper.

Among other things, the novel explores the implications of a purely mathematical worldview and the inevitable shortcomings of such a worldview. While studying in America, Ravi learns that his grandfather, Bauji, had been imprisoned in (the fictional town of) Morisette, New Jersey in 1919 for violating the town's blasphemy law by insisting that Christianity is, "a procession of fools who have given up on reason" [14, p 51]. Bauji believes only in that which can be proven mathematically; that is, he believes only in that which can be reasoned deductively. This certainty is illusory. His worldview is eventually crushed by the discovery of non-Euclidean geometries which are based on distinct sets of axioms.[2] (Specifically, each non-Euclidean geometry replaces Euclid's fifth postulate with a statement negating the existence or uniqueness of the line in the parallel postulate.[3]) Bauji believes that all truth can be deduced rationally from a set of initial axioms, but the discovery of non-Euclidean geometries showed that equally valid mathematical systems can be built on different

[2]The belief that Euclidean geometry provided the unique mathematical model for the physical world was universally accepted as absolute truth until the early nineteenth century. This is the first of the three mathematical discoveries Robert Brabenec discusses in his article. [3]

[3]Euclid's fifth postulate can be stated as "given a line and a point not on the line, there is exactly one line through the point that will be parallel to the original line."

(and even contradictory) sets of axioms. Applied to Bauji's belief system, two rational people could use logical and valid deductive reasoning to arrive at contradictory "truths" from each other by starting with different sets of axioms. If there is not only one set of initial assumptions to start with, then there's no hope for using mathematical deduction to arrive at absolute truth. This devastates Bauji's worldview because he relies on reason alone, and he has now seen that while reason is necessary, it is not sufficient to know truth.

The belief that mathematics embodied absolute truth was Bauji's faith, and the discovery that it does not shattered his faith. Instead of shattering my belief/faith, I find it encourages and strengthens me because Christianity and mathematics both require faith. Even in mathematical reasoning, one must start with some basic assumptions (axioms), and those assumptions (axioms) cannot be proven. This theme is also addressed in *Mathematics Through the Eyes of Faith* Chapter 6: "Proof and Truth," which discusses the axiomatic structure of mathematics and how we proceed in mathematics from first principles, which seem obvious but cannot be proven, and thus must be taken on faith.

We are certain to always have some ambiguity in mathematics and in life. In our work, in our relationships, in our belief systems, in our modes of thinking, we desire completeness and consistency. A mathematical system is complete if it is self-contained, and it is consistent if there are no possible contradictions within it. However, as the Brabenec article details, in 1931, Kurt Gödel proved that in mathematics, we cannot achieve both completeness and consistency [3, p 5] and therefore certainty is elusive. Mathematically, this means that no axiomatic system which contains the natural numbers as a model can be proved consistent without going outside the system, yet by relying on propositions outside the system, the system becomes necessarily incomplete. This apparent contradiction means that we (mathematicians) must embrace uncertainty and have faith in the foundations of mathematics.

For the fictional mathematician in the aforementioned novel, the discovery of non-Euclidean geometries shattered his belief in mathematics' ability to provide truth; in actuality, this discovery led many disciplines to reject/deny the existence of absolute truth. This response was highly unnecessary, since this discovery in no way proved that absolute truth did not exist, but rather asserted that its existence could not be proved through mathematics. Instead of viewing this discovery as a "death blow" to absolute truth, I see it as evidence that absolute truth, and thus a full comprehension of Almighty God, transcends human understanding.

I find I can relate this accepted understanding about completeness and consistency to my faith. As a Christian I am limited by my finite comprehension. This parallels the truth that there is something outside of ourselves, namely God, that gives credence and sustenance, awe and mystery to our existence. Isaiah 55:8-9 says "For my thoughts are not your thoughts, neither are your ways my ways, declares the Lord. As the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts."

3 The Limitations and Effectiveness of Mathematics

Another important element of how my faith and discipline interconnect is the juxtaposition of the limitations of mathematics with the surprising effectiveness of mathematics. Mathematics is not something to be worshiped because it does not encapsulate or achieve absolute truth. It is inherently limited and flawed, and yet mathematics, in surprising ways, effectively models seemingly unrelated and unconnected aspects of our universe. I enjoy this because it reminds me of the wonder and mystery of God. For example, many abstract mathematical structures that were pursued as mathematics for mathematics' sake, not at all with an eye toward how they could be useful or helpful in applications, have subsequently proven to be effective in describing natural phenomena and solving real-world problems. Many aspects of the physical world are nicely modeled using Euclidean geometry, yet a non-Euclidean geometry was essential to Einstein's theory of relativity.

In, "Mathematics: Always Important, Never Enough. A Christian Perspective on Mathematics and Mathematics Education," Calvin Jongsma investigates how the themes of Creation, Fall, and Redemption can inform mathematics. He writes that God is the Creator and Sustainer of this world and thus the source of the mathematical structure seen in the world around us. In studying this structure, we can gain insights into the nature of our God, yet mathematics is not absolute. Mathematics is important, but as Jongsma says, "it is never enough." [8] So, while we enjoy discovering the mathematical structure of creation, as Christians we need to be aware that the realm of mathematics does not embody absolute truth. Mathematics describes the order and structure of the creation, but it does not tell the whole story. God's awesome design cannot be constrained or contained by mathematics because mathematics is limited. Mathematics, however, does possess "unreasonable effectiveness," as R.W. Hamming notes in, "The Unreasonable Effectiveness of Mathematics," which was a response to the 1960 (similarly titled) paper by Eugene Wigner, a Nobel Laureate in Physics. This effectiveness of mathematics is also explored by Jongsma and in Chapter 8 of *Mathematics Through the Eyes of Faith*. In his paper, Hamming explores how effectively mathematics provides the framework for exploring the universe that we perceive. Yet, without a Christian worldview, he is left without answers to explain why the logical structure communicated through mathematics so effectively models physical reality. I agree with Jongsma that this effectiveness of mathematics is not unreasonable, but rather, indicative of the nature of the "One who gave them mathematical features" [8, p 5]. So, rather than be puzzled by the unreasonably effective applications of mathematics to physics, economics, etc., we should give glory to God for His creative design.

The beautiful yet unexpected ways that mathematical constructs predict and model the physical world also point to our God and Creator despite mathematics' limitations. In "Does Mathematical Beauty Pose Problems for Naturalism?" Russell Howell considers many such examples. Consider the field of complex analysis. The complex numbers were studied for their usefulness in mathematics, not for their relevance to the physical world, yet now they play a prominent role in quantum mechanics. This is just one of many examples of a mathematical abstraction that was later found to have practical significance.

Let's also consider the human preference for symmetry. This unexpected utility and desire for structural beauty gives evidence of an intelligent design and allows me to marvel at the surprising effectiveness of mathematics. This harkens back to our wrestling with mathematical ideas as we seek to balance or reconcile mathematics' limitations with its surprising effectiveness.

3.1 Higher Dimensions & Infinity

There are aspects of mathematics which are very difficult to grasp, and there are some things that we can prove are unknowable, but I believe that these aspects of mathematics should foster an appreciation for that which is difficult for our limited, finite minds to comprehend about our Lord and Savior. The Lord God Almighty is one God, but He exists in three persons—Father, Son, and Holy Spirit. He is eternal—He has no beginning or end—and He is omniscient, omnipotent, and omnipresent. All these attributes transcend our finite human understanding, and they should, for if we struggle with mathematical understanding, we should be in absolute awe and wonder when faced with the glory and majesty of our Creator. Two concepts related to limitations that I want to explore are higher dimensions and infinity, both of which play roles in several examples to follow.

David Neuhouser, in "Higher Dimensions in the Writings of C.S. Lewis," explores how mathematical constructs provide insights into the nature of our Creator. As a former student of Dr. Neuhouser, I have been greatly influenced by his perspective. He provides many interesting examples of the use of mathematical imagery in the works of various authors, including C.S. Lewis, but I will narrow my focus to include only a few. Restricted by our finite comprehension and capacity, we are incapable of completely describing or comprehending God. The notion of higher dimensions, however, can provide interesting insights. What if time really is 2-dimensional? Our perception is that we are trapped in a sequential existence, unable to change the past while moving toward the future, much like movement along a line. But what if time is really like a plane? Is what we perceive with the finite capabilities of our minds a clear picture of the actual reality? Maybe our perception of reality as 3-dimensional is flawed. Could the contemplation of the fourth or higher spatial dimensions provide insight into how we, as believers, are all parts of one body? Higher dimensional mathematical imagery opens our minds to interesting possibilities.

The first chapter of *Mathematics Through the Eyes of Faith* addresses the question of whether, "mathematical concepts like higher dimensions and infinity, point beyond themselves to a higher reality?" [4, p 6]. Contemplating higher dimensions is spurred by Edwin A. Abbott's 1884 novel, "Flatland: a romance of many dimensions"[1] and the subsequent animated films "Flatland: The Movie" [15] and "Flatland 2: Sphereland" [16]. All of these sources describe the visit of a sphere to a 2-dimensional world in which only its cross-sections are visible.[4] The sphere tries desperately to convince a square that the third dimension exists, but he is unsuccessful until he pulls the square into the third dimension. Excited by the third dimension, the square now wants to see the fourth dimension, but the sphere claims it does not exist. The square, having seen the connection between 2D and 3D is confident that there must be higher dimensions as well. What I find useful in this story (and in my contemplation of higher dimensions) is how this mathematical imagery can give me an interesting perspective on certain Biblical imagery.

Consider what a "Flatlander" would see if your hand encountered Flatland. Your fingertips would appear to them as five separate non-polygonal closed curves. He would be totally incapable of comprehending that they are in fact connected and belong to the same hand. Thinking along these lines, it is possible to imagine how all the members of the Body of Christ could also be connected, just through a higher dimension. Or consider John 20 when Jesus appears to the disciples even though the doors are all locked. Maybe he entered through another dimension. I am not claiming that he did or that we are connected in this way, but this mathematical imagery gives me a richer understanding of these passages than I would otherwise have.

This brings us to notions of infinity. We know that there exist infinitely many natural (i.e., counting) numbers; in fact, there exists a countable infinity of them.[5] It can also be shown that the set of rational numbers can be put into a one-to-one correspondence with the set of natural numbers, yet there exists no one-to-one correspondence between the set of natural numbers and the set of real numbers. Since the infinity of the rational numbers is countable whereas the infinity of the real numbers is not, there are multiple "sizes" of infinity.

[4] As the sphere passes through the 2D world, it appears as circles that are constantly changing in diameter.

[5] We say that this size of infinity is countable because the natural numbers can be arranged in a list whose elements can be counted one by one.

What about eternity, how can we fathom a life that never ends? Everything that we experience has a beginning and an end; we are born, and we die, but God grants us everlasting life. How can there be no end? Eternity, like infinity, is difficult to comprehend; our finite, human limitations cloud our view and inhibit our ability to see from God's perspective.

The professor, Nico, from *A Certain Ambiguity* explains it beautifully: "The story of infinity is a story of how far the human mind can take us. But it is also the story of boundaries that we may not cross, no matter what. We will see amazing facts that must be true but also raise tantalizing questions that seem to be unanswerable. Not because mathematicians just happened not to have found an answer so far, but rather because they couldn't possibly. Our current set of assumptions about infinity are not strong enough to lead to an answer to some questions. *Ever*." [14, p 14] In mathematics and in comprehension of Almighty God, there are things we cannot know, profound mysteries, and it is okay not to know; it is just part of our finite, human limitations.

Building on the mystery of infinity, we will turn again to Professor Nico and experience his joy as he shares a surprising infinite series result with his students. Nico's students have been investigating infinite series, and he has just revealed the amazing result[6] that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}. \tag{1}$$

Nico says: "Sometimes when I look at this equation with fresh eyes, I'm amazed and in awe all over again. Equations like this represent why I fell in love with mathematics. There are so many unexpected connections, so much order when you would expect none, a mostly hidden tapestry into which we get a few limited glimpses through the efforts of our brightest minds. Who made these connections? Why do they exist? I can only say that God must be a mathematician."[14, p 62]

I value how different perspectives enable us to discover connections between seemingly unrelated mathematical concepts. This compels me to worship the Creator and Sustainer responsible for these connections. I believe that there are parallels for this in our faith journey as we wrestle with apparent contradictions and mysteries. Sometimes when we look at matters of theology, our perspective is clouded because we are unable to see from God's perspective. We may encounter a verse or passage of Scripture that seems confusing or paradoxical, and we struggle because our finite limitations inhibit our ability to resolve the conflict, but we know that another perspective exists. Through further study, we may see things from a different perspective which brings understanding, but it is also possible that full comprehension may be beyond our finite capacities. We need to recognize our limitations and rest in the assurance that there is nothing in mathematics or life that God has not thought through.

4 Perspectives and Patterns in Recursive & Closed Sequences

In mathematics, we often find that changing our perspective yields new insights. Similarly, in matters of faith, a change of perspective often brings a lot of clarity. What an incredible reminder that as we move through life, we must recognize that things which we struggle to make sense of in the moment have purpose from God's perspective.

[6] We will revisit this result later in the paper and discuss why it is amazing.

Although Nico was referring to an infinite sum (i.e., series) in my previous example, I want to investigate numerical sequences because they are some of my favorite mathematical structures to explore.[7] These examples are helpful because their exploration mirrors how so much of mathematics is developed—we start with small, concrete examples, look for patterns and properties, and then generalize for all numbers, objects, etc., that exhibit the same traits. In looking for ways to generalize or extend the results to a larger context, changing our perspective allows us to approach ideas in new ways.

These next several sequence examples will move from basic to more challenging and will highlight aspects of mathematical reasoning that have implications for Christian faith. As Nico says "...mathematics is not a spectator sport. You have to do it to appreciate it, and doing it requires patience and persistence. You can love a song without being able to sing, but that doesn't work in mathematics. Nevertheless, the beauty is there for you to find." [14, p 14] I want to expand on this idea. To find the beauty in mathematics, you need to do it. To discover surprising, unexpected connections, you need to work the problems. To find the joy and awe in mathematics and in God's intricate creation, you need to immerse yourself in it. You must interact with it. Similarly, to know God you must abide in Him, you need to be immersed in His Word, you need to wrestle with ideas, much like we wrestle with mathematical concepts and problems. It is in wrestling with difficult-to-accept ideas that awe comes in both our faith journey and in mathematics. We will next explore some numerical sequences by searching for patterns and generalizing them, and in the process discover structure and beauty, inspiring awe for our Creator and Sustainer.

A recursive sequence (or recurrence relation) is one in which we have a fixed starting point and a clear method for how to get from one term to the next. For example, given the sequence

$$1, 4, 7, 10, 13, 16, \ldots,$$

we can see that the first term (or starting point) is 1 and the pattern that allows us to move from one term to the next can be described as "add 3." Mathematically, we will express this recursive representation of the sequence as

$$a_1 = 1 \text{ and } a_n = a_{n-1} + 3 \text{ for } n \geq 2.$$

The recursive definition of a sequence is nice because it expresses how we can progress from one term to the next, but a major drawback is that if we need to know the value of a specific term, say for example, the 100^{th} term, we know that it would be 3 more than the value of the 99^{th} term, which would be 3 more than the value of the 98^{th} term, and so on. To find the 100^{th} term, we would need to find all 99 terms that come before.

However, finding the closed form of a sequence will allow us to determine the value of a particular term in a sequence based solely on its position (term number) in the sequence. Our sequence happens to be arithmetic, so its closed formula has the form $a_n = a + d \cdot (n-1)$, where a is the value of the first term, n is the term number, and d is the constant difference (i.e., the amount we add each time). The closed form for this particular sequence is given by $a_n = 1 + 3 \cdot (n-1)$ which simplifies to $a_n = 3n - 2$. (We can also see how this is derived in Table 1.)

Although this is a simple example, notice that from one perspective, finding the hundredth or thousandth or millionth term is tedious even with technology, but from another perspective the

[7] We might note that series and sequence concepts are related because an infinite series can be rewritten as an infinite sequence of its partial sums.

Term Number (n):	Term Value (a_n):	Pattern
1	1	
2	4	1+3
3	7	$4+3=(1+3)+3=1+3\cdot 2$
4	10	$7+3=(1+3\cdot 2)+3=1+3\cdot 3$
5	13	$10+3=(1+3\cdot 3)+3=1+3\cdot 4$
6	16	$13+3=(1+3\cdot 4)+3=1+3\cdot 5$
$\vdots$		
n	$3n-2$	$1+3\cdot(n-1)$

Table 1: Deriving the closed form for the arithmetic sequence example.

value is easily accessible. Prayer and answers to prayer, can be like that. I may be facing some hugely complicated problem or situation, but sometimes with prayer the answer reveals itself or a burden is lifted. We would love to know the closed form for our existence; we want to know how everything is going to work out, but we must experience life sequentially trusting God to work out the details in His way.

Let's consider another example: the sum of consecutive numbers. Assume that we want to find the sum of the first n natural numbers, where n can be any arbitrary natural number. For example, if $n = 5$, then we are talking about the sum $1+2+3+4+5 = 15$. We can re-contextualize this finite series problem as a sequence problem by defining the n^{th} term in the sequence as the sum of the numbers 1 through n. Then our question can be rephrased as "what is the value of the n^{th} term in this sequence?" and we can look at the pattern of the terms to discover the recursive relationship that it exhibits (as demonstrated in Table 2).

Term Number (n):	Sum	Term Value (a_n):	Recursive
1	1	$a_1 = 1$	
2	$1+2$	$a_2 = 3$	$a_2 = a_1 + 2$
3	$1+2+3$	$a_3 = 6$	$a_3 = a_2 + 3$
4	$1+2+3+4$	$a_4 = 10$	$a_4 = a_3 + 4$
5	$1+2+3+4+5$	$a_5 = 15$	$a_5 = a_4 + 5$
6	$1+2+3+4+5+6$	$a_6 = 21$	$a_6 = a_5 + 6$
7	$1+2+3+4+5+6+7$	$a_7 = 28$	$a_7 = a_6 + 7$
8	$1+2+3+4+5+6+7+8$	$a_8 = 36$	$a_8 = a_7 + 8$
9	$1+2+3+4+5+6+7+8+9$	$a_9 = 45$	$a_9 = a_8 + 9$
10	$1+2+3+4+5+6+7+8+9+10$	$a_{10} = 55$	$a_{10} = a_9 + 10$
$\vdots$			
n	$1+2+3+\cdots+(n-1)+n$	a_n	$a_n = a_{n-1} + n$

Table 2: Finding the recursive pattern for the sum of consecutive numbers.

Since the differences are increasing rather than constant, this sequence is not arithmetic, so we will need to change our perspective to discover its elusive closed form. A variety of methods exist, but we will discover the closed form by exploring the seemingly unrelated staircase problem.

In Figure 1, we have the first staircase which consists of a single block, the second staircase which requires 3 blocks to construct, the third staircase which requires 6 blocks to construct, and so on. We would like to determine how many blocks it would take to make a staircase whose tallest stair

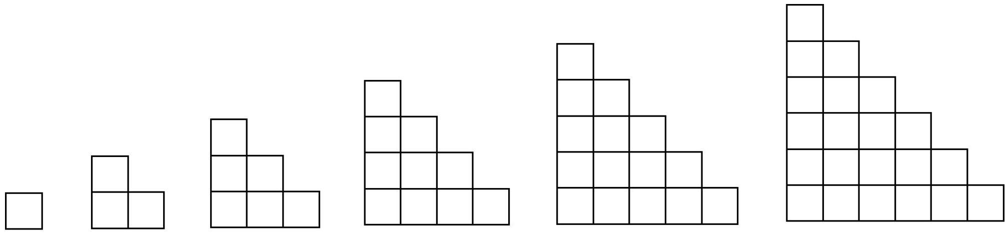

Figure 1: Staircase images for $n = 1$ to $n = 6$.

is made from n blocks, where n can be any natural number. To see the connection between this problem and the prior sequence problem, notice how these staircases are built. To construct the n^{th} staircase we need to make stairs of height $1, 2, 3, \ldots, n$. The total number of blocks needed will be $1 + 2 + 3 + \cdots + (n-1) + n$. This is really the same question in disguise, and viewed from this perspective, the closed form of our sequence would correspond to the number of blocks in the n^{th} staircase. The question now becomes "is there an easy way to find the number of blocks in the n^{th} staircase?" This is where a visual representation provides helpful insights. For example's sake, let's take two copies of the 5^{th} staircase and arrange them as shown in Figure 2.

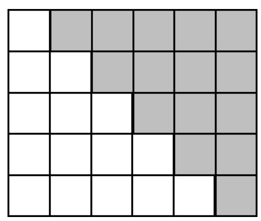

Figure 2: Arrangement of two "staircases".

We can then generalize to obtain the result for the n^{th} staircase. From Figure 2, we see that if the tallest stair was made from 5 blocks, then two copies of the staircase can be arranged to form a 5×6 rectangle, or if we generalize to the n^{th} case they can be arranged to form an $n \times (n+1)$ rectangle. The number of blocks in one staircase would then be half this amount. There will be $\frac{5 \cdot 6}{2} = 15$ blocks in our staircase with tallest stair of height 5 (which matches a_5 in Table 2) and we can generalize this to see that a staircase with tallest stair of height n will consist of $\frac{n(n+1)}{2}$ blocks. This means that the elusive closed form definition of our sequence is just $a_n = \frac{n(n+1)}{2}$. Thus, the recursive definition $a_n = a_{n-1} + n$ for $n \geq 2$, where $a_1 = 1$, and the closed form definition $a_n = \frac{n(n+1)}{2}$ describe the same sequence.

We have seen that viewing our sequence problem from a different perspective allows us to more clearly see how to derive a closed form representation. However, this is not the only perspective which can prove enlightening here. The seemingly unrelated "handshake problem" can lead to this formulation as well. The handshake problem asks "If each person in a group shakes hands with each other person in the group, how many handshakes will occur?" For example, let's consider a group of 6 people. Each of the 6 people shakes 5 hands, so we would have $6 \times 5 = 30$, except that this actually double counts each handshake since a handshake involves two people but should only be counted once. The actual number of handshakes would be $\frac{6 \times 5}{2} = 15$. Generalizing this, we have that in a group of $n + 1$ people[8], the number of handshakes will be $\frac{(n+1) \times n}{2}$ which (because multiplication is commutative) matches the closed form we just derived for our prior sequence.

[8]The group has $n + 1$ people, rather than n because a person does not shake their own hand, so each of the $n + 1$

On the surface, these three problems are seemingly unrelated, but as we have now seen they are intimately connected, and viewing our original problem from one of these other perspectives can provide us with necessary insights to make the transition from the recursive representation to the closed form.

I like how some patterns have a clear structure while others become clearer when viewed from a different perspective, patterns in which decomposing a visual representation gives clarity and insight, because viewing sequences from these different perspectives points us to the wonder and mystery of God. God's ways inspire awe and wonder because they are so different from our own. Our human limitations and fallenness cloud our view. Looking at the life of Jesus, we can see just how different our human and God's divine perspectives are. "The greatest must be the servant of all," "the last shall be first," "gaining life through losing life," etc. (Matthew 23:11, 20:16, & 16:25, respectively). The insights gained through a shift of perspective in these sequence problems are a reminder that some of our questions or struggles in matters of faith are because we are unable to see them from God's perspective.

Consider yet one more numerical sequence example: $5, 11, 19, 29, 41, 55, \ldots$. What is the next term in the sequence? What comes after that? What are the recursive description and the closed form? Once again, exploring the additive ways that we move from one term to the next leads to the recursive representation shown in Table 3.

Term Number (n):	Term Value (a_n):	Pattern: (add next even number)	Recursive:
1	$a_1 = 5$		
2	$a_2 = 11$	$11 = 5 + 6$	$a_2 = a_1 + 6$
3	$a_3 = 19$	$19 = 11 + 8$	$a_3 = a_2 + 8$
4	$a_4 = 29$	$29 = 19 + 10$	$a_4 = a_3 + 10$
5	$a_5 = 41$	$41 = 29 + 12$	$a_5 = a_4 + 12$
6	$a_6 = 55$	$55 = 41 + 14$	$a_6 = a_5 + 14$
7	$a_7 = 71$	$71 = 55 + 16$	$a_7 = a_6 + 16$
8	$a_8 = 89$	$89 = 71 + 18$	$a_8 = a_7 + 18$
9	$a_9 = 109$	$109 = 89 + 20$	$a_9 = a_8 + 20$
10	$a_{10} = 131$	$131 = 109 + 22$	$a_{10} = a_9 + 22$
$\vdots$			
n	a_n		$a_n = a_{n-1} + 2n + 2$

Table 3: Finding the recursive pattern for the sequence $5, 11, 19, 29, 41, 55, \ldots$.

The question then becomes, can we find a closed form? Once again, a visual representation provides helpful insights. The first four numbers in our sequence can be represented by the four block representations in Figure 3.

Each block figure can be decomposed into a square of tiles and a "handle" consisting of a single row of tiles. Exploring this decomposition in Table 4 leads to a closed form representation for the sequence.

So, it appears that the closed form of this sequence can be represented as $a_n = (n+1)^2 + n$ or equivalently as $a_n = n^2 + 3n + 1$, but how do we know that we are correct? The pattern and visual

people will shake n hands. Or if we think of the process sequentially, the first person will shake n hands, the second, $n - 1$ hands, and so on until the last person who will shake 0 hands, so the total number of handshakes will be $n + (n-1) + (n-2) + \cdots + 2 + 1 + 0$ matching our staircase and sequence problems.

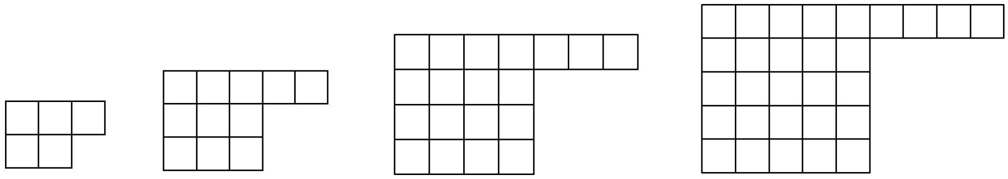

Figure 3: Block representations for the first four terms of $5, 11, 19, 29, 41, 55, \ldots$

Term Number (n):	Term Value (a_n):	Pattern: Square + "handle"	Sum:
1	5	$2^2 + 1$	$4 + 1 = 5$
2	11	$3^2 + 2$	$9 + 2 = 11$
3	19	$4^2 + 3$	$16 + 3 = 19$
4	29	$5^2 + 4$	$25 + 4 = 29$
5	41	$6^2 + 5$	$36 + 5 = 41$
6	55	$7^2 + 6$	$49 + 6 = 55$
$\vdots$			
n	$a_n = n^2 + 3n + 1$	$(n+1)^2 + n$	

Table 4: Finding the closed form for the sequence $5, 11, 19, 29, 41, 55, \ldots$

evidence seem clear, but how can we be certain that our closed form is correct? Maybe the pattern just works for the numbers that we tried, not for all natural numbers. Mathematically, we gain this certainty by proving inductively that our sequence is correct.[9]

Experiencing how different perspectives can increase our level of understanding and discovering unexpected connections while exploring numerical sequences deepens my appreciation of the mathematical structure of God's creation. Which perspective is most meaningful can vary from one individual to the next, but the more views we have and comprehend, the deeper our understanding will be. There are sequences (like the sequence of perfect squares, $1, 4, 9, 16, 25, 36, 49, \ldots$) where the closed form is actually more accessible than the recurrence relation. This reminds me of times in our lives when we are able to see the end goal, but struggle with the details and steps needed to get there. However, for most sequences (as we have seen), finding the recursive representation is fairly doable, while finding the closed form is significantly more challenging. One last sequence example not only explores the importance of perspective, but also highlights the mystery of God.

5 The Truly Mysterious: Seemingly Paradoxical and Counter-intuitive Aspects of Mathematics and Faith

In both mathematics and faith there are all-encompassing, awe-inspiring mysteries. Consider the Fibonacci sequence (the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$) which frequently appears in nature in leaf patterns, flower, pine cone, and pineapple spirals, etc. Its recursive definition is

$$F_1 = 1, F_2 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3,$$

and its closed form is

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

[9] I will spare you the details here, but for those who are interested, a mathematical induction proof is included in the appendix.

This closed formula is not at all obvious, and the fact that it involves the Golden Ratio is quite surprising. This mirrors our Christian walk. As we read God's word and deepen our relationship with Him through prayer, we slowly conform more and more to the image of His Son. Much like the recursively generated sequence, we can see and experience our incremental growth as we faithfully follow Christ on a daily basis, but who we will become eludes us. We do not know what our futures hold; the closed form of our existence is beyond our grasp.

Our existence is sequential much like a recursive sequence. We move from one moment to the next, but in our human limitations, we cannot see the whole picture. As it says in 1 Corinthians 13:12, "For now we see only a reflection as in a mirror; then we shall see face to face. Now I know in part; then I shall know fully, even as I am fully known." In God's transcendence He sees the closed form of our existence.

Let's return to the novel, *A Certain Ambiguity*, and explore why the infinite series result Nico presented is so amazing. Recall equation (1):

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots=\sum_{i=1}^{\infty}\frac{1}{i^2}=\frac{\pi^2}{6}.$$

Notice that the numbers that we are adding on the left are just rational numbers, but all of a sudden, π appears on the right. The number π is irrational. It is the ratio of the circumference to the diameter of a circle. How can a sum of rational numbers be equal to an irrational number? This is indeed a mystery.

Determining the value of $\sum_{i=1}^{\infty}\frac{1}{i^2}$ was so difficult that it garnered a name: the Basel Problem. In 1735, after the Basel Problem had challenged mathematicians for the better part of the previous century, Euler discovered its surprising solution. [6, p xxii]. Since that time, many other proofs of this astounding result have been published using a wide array of mathematical methods. Although Euler's initial argument is less than rigorous, we will explore it here because of its use of Taylor series which students investigate in Calculus II and because it highlights once again one of the many unexpected connections inherent in mathematics.[10] We, like Euler, will use the fact that an n^{th} degree polynomial, $P(x)$, with $P(0)=1$ can be factored as

$$P(x)=\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\left(1-\frac{x}{a_3}\right)\cdots\left(1-\frac{x}{a_n}\right), \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are the roots of $P(x)$ which may or may not be real, and we will assume that we can utilize this factorization for infinite polynomials as well.[11]

Let's begin by considering the seemingly unrelated function $f(x)=\sin x$. The Taylor series,[12] for $\sin x$ is given by

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-+\cdots, \tag{3}$$

so for $x \neq 0$, the Taylor series for $\frac{\sin x}{x}$ is

[10] The proof sketch included is similar to that given in [6] pp. 46-47.

[11] The assumption that this factorization applies for infinite polynomials is a big Eulerian leap of faith, but this can be rigorously shown using the Weierstrass factorization theorem.

[12] Technically this is a Maclaurin series since the series is centered at $x=0$.

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - + \cdots . \tag{4}$$

Let's define a function

$$P(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & x \neq 0 \end{cases} . \tag{5}$$

It follows that $P(x)$ is the infinite polynomial

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - + \cdots . \tag{6}$$

Since $P(0) = 1$, we can factor this polynomial using equation (2) if we know the roots of the polynomial. Since $\frac{\sin x}{x}$ has roots at $x = k\pi$ where $k \in \mathbb{Z}$ the function $P(x)$ will similarly have roots at $x = k\pi$ where $k \in \mathbb{Z}$ as long as $x \neq 0$. So, $x = \pm k\pi$ for $k = 1, 2, 3, \ldots$ will be the roots of $P(x)$. Factoring $P(x)$ using these roots and equation (2) gives us

$$P(x) = \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots \tag{7}$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\cdots . \tag{8}$$

Expanding the right side of this equation yields an infinite polynomial with only even degree terms. Let's focus on the second-degree terms in the expansion which will be obtained using the $-\left(\frac{x}{k\pi}\right)^2$ from one factor and 1's from the remaining factors. The coefficient of x^2 in the expansion will be

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = -\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}. \tag{9}$$

But the coefficient of x^2 in the Taylor series representation of $P(x)$ is $-\frac{1}{3!}$. Matching up like coefficients, we obtain $-\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = -\frac{1}{3!}$ and thus

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{3!} = \frac{\pi^2}{6}. \tag{10}$$

As I mentioned when we first encountered equation (1), this result (and the ingenuity of this "proof") is awe inspiring. First of all, the idea of a sum of infinitely many non-zero terms having a finite value challenges our finite comprehension yet results like this are common throughout mathematics. Secondly, that the finite value of this infinite sum of rational numbers is the irrational number $\frac{\pi^2}{6}$ defies reason, yet we have just looked at a sketch of Euler's argument for its truthfulness. This leads me to echo Nico's claim that "God must be a mathematician!" [14, p 62]

Mathematics is filled with these surprising, fascinating, counter-intuitive results. In mathematics, we are continually confronted with ideas that are hard to wrap our minds around. Similarly, when

I read Scripture and encounter ideas that seem contradictory or hard to understand, rather than being discouraged, I appreciate the mystery and worship the God in whom all things hold together.

What I would like to do in the remainder of this paper is to discuss a couple of faith concepts that are mysterious—things that can be challenging to wrap our minds around—and explore just a few mysterious and surprising examples from my mathematics classes that foster my sense of awe, wonder, and appreciation for that which is clear and yet mysterious about the great I AM. [13]

Consider the mystery of the incarnation. John 1:14 says "The Word became flesh and made His dwelling among us. We have seen His glory, the glory of the one and only Son, who came from the Father, full of grace and truth." Jesus Christ is one distinct person with two natures, each possessed completely and inseparably. He is fully God and fully man. The Heidelberg Catechism on Lord's Day 6 in the responses to questions 16 and 17 explains that "human nature, which has sinned, must pay for its sin; but a sinner could never pay for others;" [18, p 18] thus Christ must be truly human and truly righteous, but that He must also be truly God in order to bear the weight of God's anger. He is of one substance with the Father, yet at the same time, He is of one substance with man. He was always God and He became man through being conceived of the Holy Spirit and born of a virgin. His claims to be God, His miracles, and His sinless life led the disciples to believe He is divine, and His ascribing titles of divinity to Himself (e.g., John 10:30, "I and the Father are one.") prove His full deity.

Another common struggle is with the profound mystery of God's Sovereignty and man's free will. Romans 8:29 says "For those God foreknew He also predestined to be conformed to the image of His Son, that He might be the firstborn among many brothers and sisters." I believe that we are predestined and that we can only come to saving faith if we are chosen and called by God, but we are responsible for our actions. In our daily lives, we choose to sin. Romans 3:23-24 says, "for all have sinned and fall short of the glory of God, and all are justified freely by His grace through the redemption that came by Christ Jesus." We are born with a sin nature, although through being born again in Christ and indwelt by the Holy Spirit, sin no longer has any power over us, yet we still choose to sin sometimes. We have some freedom, for God is not responsible for our sins, yet He has ordained all the days of our lives. I do not fully understand this, but I do not think that I am supposed to. I think that some of our theological disagreements result because of the limitations of our finite comprehension. Maybe the true answer to some of our theological disagreements is more of a "both/and" rather than an "either/or." I think that Hebrews 6 offers challenges to both Calvinists and Arminians. The Calvinist position is that those who "fall away" were never "true" believers, but I'm not fully satisfied with that explanation. Even the Heidelberg Catechism (in Lord's Day 23, question 60 [18, p 38]) uses the language of acceptance in discussing how we are made right with God. Acceptance requires action which seems to indicate some human responsibility even though the believer can only come to Christ if he or she is drawn (John 6:44). Salvation is a profound mystery. All I can say is, "Thanks be to God for His indescribable gift." (2 Corinthians 9:15).

I think the mysteries of mathematics can help students who wrestle with these concepts and questions about God. There are aspects of faith that are difficult to understand. I want to meet students there, to listen, to go with them through the complexities that lead us to a place of awe-inspiring wonder of God's majestic creation. In mathematics, we encounter results that are counter-intuitive which stop us in our tracks. So many mathematical concepts draw us to the brink of our human comprehension and yet God holds it all together. Studying mathematics helps me sit with things

[13] Exodus 3:14

about God that I struggle to wrap my mind around and equips me to be okay with not knowing, not having to demand answers. I believe it may help students too.

Next, we will consider several mathematical examples. None of them are light and fluffy. They are all fairly heavy (at least to those outside of mathematics), but they illustrate ways in which my students encounter mysteries of mathematics that can motivate awe for God the perfect mathematician. One example of a mysterious result from Calculus involves improper integrals. For a continuous function $f(x)$ on a closed interval $[a,b]$ the integral from a to b denoted by $\int_a^b f(x)dx$ calculates the (signed) area between the curve $f(x)$ and the x-axis on the interval from $x=a$ to $x=b$.[14] Improper integrals then are those that involve an infinite limit of integration[15] or those that are undefined at some value within their interval of integration causing what we can think of as an infinite spike. We will look at a couple of surprising results arising from improper integrals corresponding to regions of infinite width and note that similar surprising results would occur if we explored improper integrals corresponding to regions with infinite height.

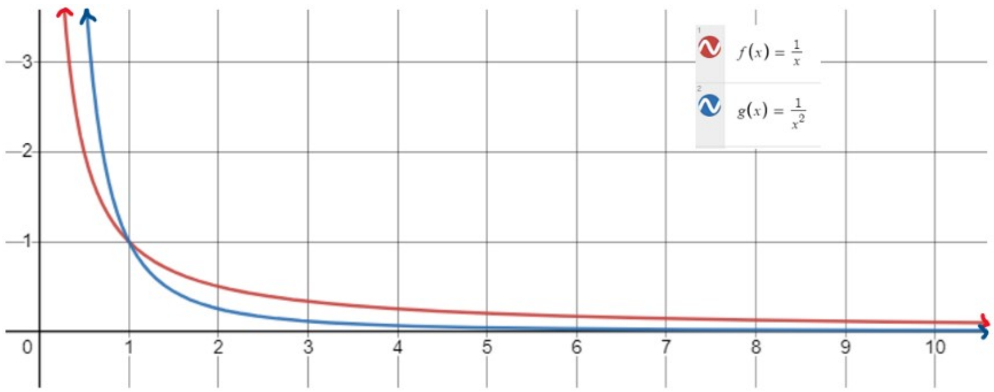

Figure 4: Graphs of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ whose graphs (in the first quadrant) are shown in Figure 4 and consider the improper integrals from 1 to infinity of these functions.[16] We can visualize the values of $\int_1^\infty \frac{1}{x}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ by looking at the graphs in figures 5 and 6, respectively, where the shaded area denotes the value of the integral.[17]

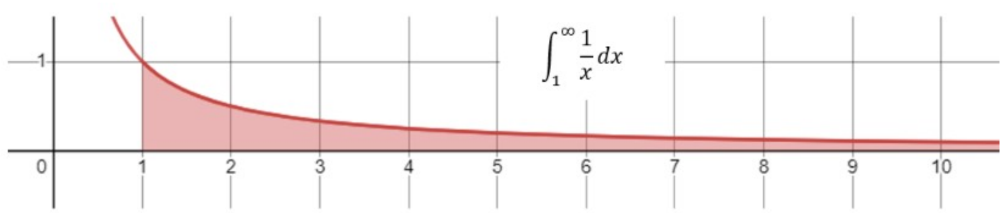

Figure 5: Graph of $\int_1^\infty \frac{1}{x}dx$ with (truncated) area shaded.

Looking at these graphs, we do not see much difference, so we might expect that the values of these improper integrals would be similar, but if we shift our perspective and investigate their values numerically (shown in Table 5), some interesting differences emerge. The value of $\int_1^b \frac{1}{x}dx$ continues

[14]This area is signed because areas between the curve and the x-axis that lie above the x-axis give positive area and those between the curve and the x-axis that lie below the x-axis count as negative area.

[15]We can think of the regions defined by these integrals as having infinite width.

[16]Graphs constructed using Desmos.

[17]Notice that these images are necessarily truncated since we cannot actually show a region with infinite width.

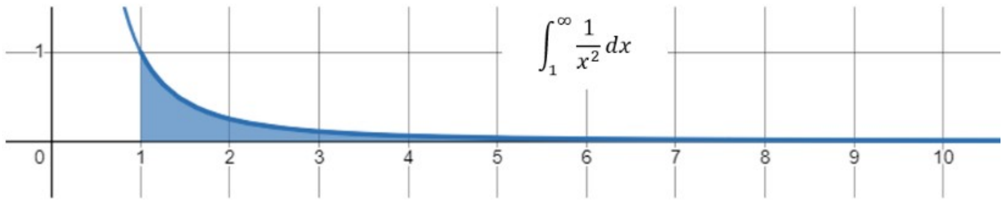

Figure 6: Graph of $\int_1^\infty \frac{1}{x^2}dx$ with (truncated) area shaded.

to grow as we increase the upper limit of integration,[18] whereas $\int_1^b \frac{1}{x^2}dx$ appears to be approaching 1 as $b \to \infty$. This is a surprising result. How can these seemingly similar integrals have such different behavior?

$$\int_1^{10} \frac{1}{x}dx \approx 2.302585 \qquad \int_1^{10} \frac{1}{x^2}dx = 0.9$$
$$\int_1^{1000} \frac{1}{x}dx \approx 6.907755 \qquad \int_1^{1000} \frac{1}{x^2}dx = 0.999$$
$$\int_1^{100{,}000} \frac{1}{x}dx \approx 11.512925 \qquad \int_1^{100{,}000} \frac{1}{x^2}dx = 0.99999$$
$$\int_1^{10{,}000{,}000} \frac{1}{x}dx \approx 16.118096 \qquad \int_1^{10{,}000{,}000} \frac{1}{x^2}dx = 0.9999999$$

Table 5: Estimating $\int_1^b \frac{1}{x}dx$ & $\int_1^b \frac{1}{x^2}dx$ for increasing values of b.

The algebraic perspective will allow us to formalize what we have seen from the graphical and numerical perspectives and show that $\int_1^\infty \frac{1}{x}dx$ diverges[19] while $\int_1^\infty \frac{1}{x^2}dx = 1$ giving us certainty about our conclusions. We will use the idea of limits to show these results. Using Calculus, we see that

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} \ln x|_1^b = \lim_{b\to\infty} \ln b - \ln 1 = \lim_{b\to\infty} \ln b = \infty,$$

so $\int_1^\infty \frac{1}{x}dx$ diverges, but

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} -\frac{1}{x}\bigg|_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = 1.$$

We have shown that the area between $\frac{1}{x}$ and the interval $(1, \infty)$ on the x-axis is infinite, while the area between $\frac{1}{x^2}$ and $(1, \infty)$ on the x-axis is exactly 1. This can be challenging for students (and the rest of us too) to wrap our minds around. This is clear and yet mysterious. The mathematical theories of limits and integrals can give us certainty that these results are correct, but from our finite perspective, something just does not seem right. How can these integrals that seem so similar, in fact, be so different?

Integrals explore areas and behaviors on a continuous scale, but similar challenges emerge when we explore discrete phenomena. Let's consider the infinite series that mirror our improper integral

[18] In other words, $\int_1^b \frac{1}{x}dx \to \infty$ as $b \to \infty$.

[19] In other words, the integral does not have finite value.

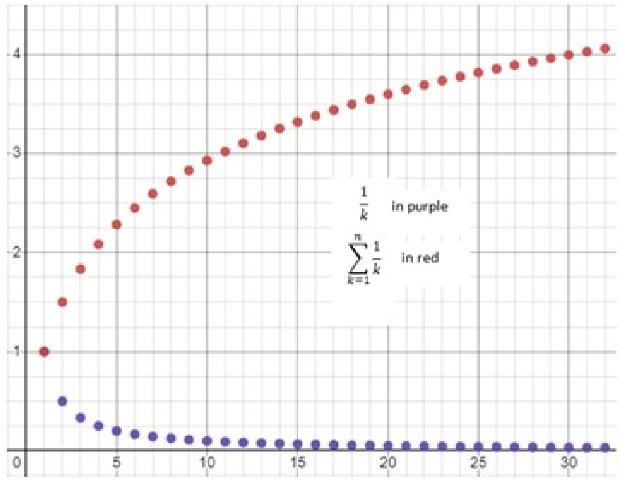

Figure 7: Graph of $\sum_{k=1}^{\infty} \frac{1}{k}$.

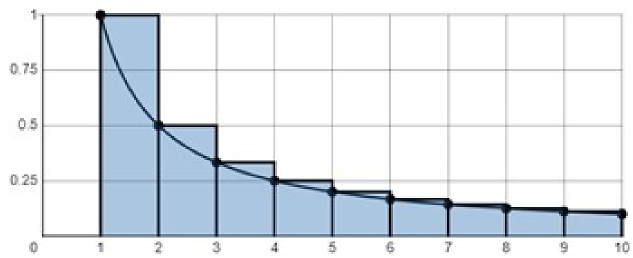

Figure 8: Picture of the 9^{th} partial sum of the harmonic series as a sum of the areas of rectangles.

examples. The series $\sum_{k=1}^{\infty} \frac{1}{k}$, the harmonic series, diverges, which can be shown by connecting this series to the corresponding improper integral. Figure 7 shows the individual terms from the sequence in purple and the sum of the terms, (i.e., the series values) in red.[20] Our graphical evidence seems to indicate that even though the individual terms in the sequence decrease to 0 quite quickly, the sum of the terms continues to increase to infinity, and thus the series diverges. We can (and do in Calculus II) prove this by thinking about this series as a left Riemann sum approximating the integral $\int_1^{\infty} \frac{1}{x} dx$. Figure 8 gives a finite visual of this representation. If we think of approximating the integral $\int_1^{\infty} \frac{1}{x} dx$ using rectangle sums where each rectangle has a width of 1 unit and each rectangle's height is given by $\frac{1}{k}$ where k is the left endpoint of the k^{th} subinterval, then each rectangle has area $\frac{1}{k}$ and the sum of the areas of the rectangles[21] is given by $\sum_{k=1}^{\infty} \frac{1}{k}$. Looking at Figure 8, we can see that because $f(x) = \frac{1}{x}$ is a decreasing function on $[1, \infty)$, the Riemann sum will overestimate the area under the curve and thus

$$\sum_{k=1}^{\infty} \frac{1}{k} > \int_1^{\infty} \frac{1}{x} dx.$$

But we have already shown that $\int_1^{\infty} \frac{1}{x} dx$ diverges, so $\sum_{k=1}^{\infty} \frac{1}{k}$ (i.e., the harmonic series) must also diverge.[22]

[20] Graph constructed using Desmos series plotting module: www.desmos.com/calculator/8op6xpfuks.

[21] The sum of the areas of the rectangles approximates the area under the curve, i.e., the integral value.

[22] Since the series value is larger than the integral value and the integral diverges, the series must also diverge.

Turning now to $\sum_{k=1}^{\infty} \frac{1}{k^2}$, we realize that we have already found the sum for this infinite series. We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to $\frac{\pi^2}{6}$ because this is the amazing result that Nico introduced in *A Certain Ambiguity*, and we explored in Equation (1). Looking at the graphical representation of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ in Figure 9, we can see that the convergence of the series is clear although finding its exact value requires some sophisticated mathematics (as we saw earlier). In the realm of series, we also encounter mystery—series whose terms seem very similar can have drastically different behavior. We might note that since both of these series have the form $\sum \frac{1}{k^p}$, they are actually examples of what we call p-series and using the connection between infinite series and improper integrals, it can be proven that a p-series will converge (i.e., have a finite value) when $p > 1$ and diverge for $p \leq 1$.[23]

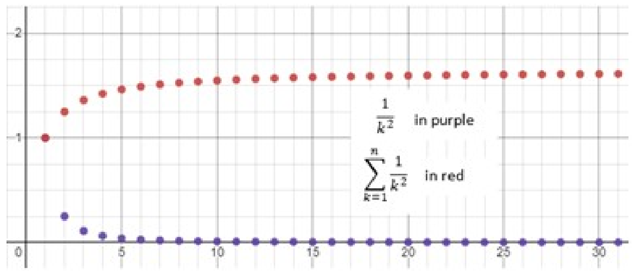

Figure 9: Graph of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

We have seen how in Calculus, wrestling with infinite objects like improper integrals and infinite series leads to mysterious, counter-intuitive results, but the mystery extends far beyond Calculus into every branch of mathematics. Let's look at one last example from Mathematical Statistics which yields a mysterious expected value result connected to our series exploration [5, p 111]. In this example, we will let X be the number of interviews a student has prior to getting a job with probability mass function (i.e., pmf) given by

$$p(x) = \begin{cases} \frac{k}{x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \text{where } k = \frac{6}{\pi^2}.$$

First, let's unpack what this means and verify that this is a reasonable pmf for this scenario:

$$\begin{aligned} p(1) &= \frac{6}{\pi^2} \approx 0.60709 \\ p(2) &= \frac{1}{4} \cdot \frac{6}{\pi^2} = \frac{3}{2\pi^2} \approx 0.1520 \\ p(3) &= \frac{1}{9} \cdot \frac{6}{\pi^2} = \frac{2}{3\pi^2} \approx 0.0675 \\ &\vdots \end{aligned}$$

These values seem to be reasonable probabilities for the likelihood that the student gets a job on their k^{th} interview. All these values are between 0 and 1 and

$$\sum p(x) = \sum_{x=1}^{\infty} \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x^2} = 1 \text{ since } \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6},$$

[23] This result allows us to immediately know the behavior of infinitely many series. We know, for example, that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ will diverge while $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}}$ will have a finite value, moreover, we can determine the behavior of a whole multitude of other similarly structured series through comparisons to these known p-series.

so this is a valid probability mass function. Now let's consider the expected value of X, that is, let's calculate the mean or expected number of interviews a student will have prior to getting a job. The mean, $\mu = \mathrm{E}[X] = \sum xp(x) = \sum \frac{xk}{x^2} = k \sum \frac{1}{x}$. Therefore, $\mathrm{E}[X] = \frac{6}{\pi^2} \sum \frac{1}{x}$, but $\sum \frac{1}{x}$ is the harmonic series which we know diverges from our earlier example, so we get the surprising result that the expected number of interviews before getting a job is infinite. How can this be? This seems counter-intuitive. The mathematics is clear (to mathematicians), yet the result is mysterious. This is yet one more example of that which is clear and yet mysterious in mathematics. It fosters my expectation that the more I study the Word of God and follow the Son, the more clear and yet mysterious aspects of His Word, work, and nature I will encounter.

6 Conclusion

The study of mathematics is filled with mystery. It is the whole breadth of mathematics—the modes of thinking, the things we wrestle with, the strange connections we make—that enable us to be okay with uncertainty, to be aware of our limitations, to explore different perspectives, and to embrace mystery. It is through the wrestling and revisiting, the experiencing and reflecting, that we are able to see how the challenges in mathematics can reflect and, if we choose, foster our Christian faith.

We have explored notions of certainty, how in mathematics, we must start with axioms that we necessarily accept without proof. Thus, like Christianity, mathematics requires faith in its foundations. As humans, we have limitations, but God does not; He is the great I AM! The Lord God Almighty is omniscient, omnipotent, omnipresent, and eternal; how can we make sense of that? Still, mathematical mysteries can spark our imagination and inspire awe for the God who ordained and sustains all things. Contemplating the mysteries of infinity and higher dimensional objects within mathematics stretches our minds, providing imagery that, though limited, may give glimpses of a fuller and deeper reality that transcends our existence and points toward God's majesty. Viewing sequences in their recursive and closed forms and from different perspectives help us to appreciate the patterns and structures of God's creation. Thinking about our experience as being caught in the recursive while God possesses the closed form gives insight into how different our human and God's divine perspectives are.

When I encounter clear and yet mysterious concepts in mathematics—things that are difficult to wrap my mind around, they lead me to praise God who has imbued mathematics with these attributes. These examples also enable me to be content not knowing or fully grasping truths about God's attributes. We can and should be continually learning and growing, but our finite limitations will always preclude us from a full understanding of some aspects of God, for "As the heavens are higher than the earth so are [God's] ways higher than [our] ways and [God's] thoughts than [our] thoughts." (Isaiah 55:9). There are so many truths in mathematics and Christian faith that are hard to comprehend—aspects of God that transcend our finite understanding—so in both mathematics and faith I choose to embrace the mystery.

Appendix

Mathematical Induction proof that the recursive and closed forms agree for the sequence $5, 11, 19, 29, 41, 55, \ldots$.

The general idea of mathematical induction involves two components: Show explicitly that the closed formula generated is correct for the first couple of terms and then show that if we assume that our closed form is correct for the $(n-1)^{\text{th}}$ term, it has to be correct for the next term, i.e., the n^{th} term.

To make this clearer, consider our example. If we substitute 1 into our closed formula $a_n = n^2+3n+1$, we get $a_1 = 1^2+3(1)+1 = 1+3+1 = 5$, which matches the first term in our sequence and if we plug in 2, we get $a_2 = 2^2+3(2)+1 = 4+6+1 = 11$, which matches the second term in our sequence. Thus, we know that our closed formula is correct for the first two terms. Now what our induction hypothesis says is that, if we know that our closed formula is correct for a particular term, it must be correct for the next term. Therefore, if the closed formula is correct for $n = 2$, it must be correct for $n = 3$. But then since it is correct for $n = 3$, it must be correct for $n = 4$, and so on, so if we know that our induction hypothesis is true, we know that our closed form is correct for every value of n.

Proposition 1. *The sequence defined by $a_n = a_{n-1}+2n+2$ for $n \geq 2$, where $a_1 = 5$ is equivalently described by the closed formula $a_n = n^2+3n+1$.*

Proof. Let $n \in \mathbb{Z}^+$ be given and let a_n be the sequence recursively defined by $a_n = a_{n-1}+2n+2$ for $n \geq 2$, where $a_1 = 5$ We must show that $a_n = n^2+3n+1$.

Base case(s): We verify that the recursive and closed forms match for $n = 1, 2, 3, 4$ in Table 6.

Table 6: Showing the recursive and closed forms match for $n = 1, 2, 3, 4$.

n	Recursive form: $a_n = a_{n-1}+2n+2$	Closed form: $a_n = n^2+3n+1$	Match?
1	$a_1 = 5$	$a_1 = a+3+1 = 5$	yes
2	$a_2 = a_1+2(2)+2 = 5+6 = 11$	$a_2 = 4+6+1 = 11$	yes
3	$a_3 = a_2+2(3)+2 = 11+8 = 19$	$a_3 = 9+9+1 = 19$	yes
4	$a_4 = a_3+2(4)+2 = 19+10 = 29$	$a_4 = 16+12+1 = 29$	yes
$\vdots$			
$m-1$	$a_{m-1} = a_{m-2}+2(m-1)+2$	$a_{m-1} = (m-1)^2+3(m-1)+1$	assume
m	$a_m = a_{m-1}+2m+2$	$a_m = m^2+3m+1$	show

Induction Step: Let $m \in \mathbb{Z}^+$, $m \geq 2$ be given and assume that we have verified that the recursive and closed forms of the sequence agree for $n = 1, 2, 3, \ldots, m-1$.

We must show that the recursive and closed forms agree for $n = m$. That is, we must show that $a_m = m^2+3m+1$, where a_m is recursively defined to be $a_m = a_{m-1}+2m+2$.

We know that the recursive and closed forms agree for $n = m-1$, so

$$\begin{aligned} a_{m-1} &= (m-1)^2+3(m-1)+1 \\ &= m^2-2m+1+3m-3+1 \\ &= m^2+m-1. \end{aligned}$$

Now consider a_m. From our recursive definition, we know that $a_m = a_{m-1} + 2m + 2$. But from our induction hypothesis, we know that $a_{m-1} = m^2 + m - 1$, so by substitution, we obtain:

$$\begin{aligned} a_m &= a_{m-1} + 2m + 2 \\ &= m^2 + m - 1 + 2m + 2 \\ &= m^2 + 3m + 1. \end{aligned}$$

Therefore, if the recursive and closed forms match for $n = m - 1$, they must match for $n = m$.

Therefore, for all $n \in \mathbb{Z}^+$, the sequence recursively defined by $a_n = a_{n-1} + 2n + 2$ for $n \geq 2$, where $a_1 = 5$ is equivalently described by the closed formula $a_n = n^2 + 3n + 1$. □

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Heavy Metal, Super Mario, C.S. Lewis, and Calculus II

Derek Thompson (Taylor University)

Derek Thompson received a Bachelor's of Science in Mathematics from Indiana Wesleyan University, a Master of Arts in Mathematics from Ball State University, and a Ph.D. in Mathematics from Indiana University-Purdue University Indianapolis. He received Taylor's Distinguished Lecturer award in 2017 and the Joe Burnworth Teaching Award in 2019. But playing with his three daughters Ellie, Hannah, and Lydia has taught him a lot more about teaching since then.

Abstract

In the fall of 2023, I had several experiences that I viewed in retrospect as "transcendent," and that caused me to reflect on other past experiences in my life. How do these experiences shape our teaching, for better or for worse?

1 Transcendent Experiences

Like many of you, I have spent my entire life in academia. I went straight from being a student, to being a student, to being a student again, a graduate assistant, and then a professor. I have endured countless conversations about how to enforce attendance in math classes, as we all have. Last fall, I had a revelation about teaching that seems so obvious in retrospect, yet seems worth sharing here. But, like any good math paper, let's start with a definition.

According to some dictionary somewhere on the internet, the word *transcendent* means "beyond or above the range of normal or merely physical human experience." For the purpose of this article, I will say that a transcendent experience is something beyond normal, everyday life. I hope that each of us experienced something akin to this when we accepted Christ as Lord and Savior, but here are some other examples from my own life.

- (1991) At six years old, I was playing at the neighbors' house and slipped while running by their pool and fell in the deep end. No one else was outside. After I-don't-know-how-long, their golden retriever, Seamus, pulled me out by biting the collar of my shirt. I never learned to swim after that, and still panic when my head is underwater.

- (1993) I was eight years old, and my mom and her sisters went on a shopping trip to Shipshewana, a touristy town in northern Indiana full of mom-and-pop stores. They left me in the car all day (these were different times). I read the entirety of the book *Dragons of Autumn Twilight* by Margaret Weis and Tracy Hickman. I had never been so thoroughly sucked into an imaginary setting, and my mind has been captivated by medieval fantasy settings (*The Legend of Zelda, Final Fantasy, Magic: the Gathering, Harry Potter*) ever since.

- (2004) I saw *Napoleon Dynamite* in the movie theater. I suppose this one needs context! I went with some friends on a whim, and had never heard of the movie nor seen a preview. Keep in mind that YouTube did not exist until 2007. The everyday experience of wasting an embarrassing amount of time watching people doing stupid things on small screens was not commonplace in 2004, and seeing it happen on a big screen was confusing, hilarious, and transcendent. It changed my definition of "movie."

- (2005) I did a secondary education practicum at Marion High School, and the teachers had a carry-in lunch. Someone brought homemade bread pudding, and it was the best dessert I have ever enjoyed in my entire life, to this day. I have since ordered bread pudding at any and every restaurant that serves it, and only met disappointment. I wouldn't give just anything, but I'd give a lot of things, to identify the cook and have that dish recreated for me.
- (2006) I took Modern Abstract Algebra at Indiana Wesleyan University with Dr. Melvin Royer. When he proved to us that $\mathbb{Z}$ was isomorphic to $2\mathbb{Z}$, at first I didn't believe him, and then it felt like my third eye had been opened.

I could give many more examples: the births of my daughters, my grandmother's passing, the Spirit-filled decision to adopt a child, a dungeon from *The Legend of Zelda: Twilight Princess,* driving for the first time. But I think the examples above are sufficient to illustrate properties of transcendent experiences:

- They are surprising or shocking;
- They stick permanently in your memory;
- They permanently change your worldview in some way.

The third item is the most difficult. While *cognitive dissonance* is a relatively recent term, it is a common phenomenon. We all too easily see it in others, but how often do we fail to let a transcendent experience shape us, or to recognize how it applies to our teaching?

2 Connections, Integrity, and Metacognition

When I joined the faculty at Taylor University, I was joining the ranks of five full professors, all of whom were fonts of wisdom. In particular, Mark Colgan has said many times that mathematics is really just about making connections between things. After hearing that many times, I finally understood the Henri Poincaré quote, "Mathematics is the art of the giving the same name to different things." Many times I've had to explain to a student that a bounded set is different from a bounded function, which is different from a bounded operator, which is different from a bounded sequence, but it's all really the same thing. Our Discrete Mathematics book has a great diagram when it introduces direct proof, which basically says that all Direct Proofs follow the same pattern:

State definition $\Rightarrow$ Change the setting $\Rightarrow$ Work with the new, easier setting $\Rightarrow$ Change back.

"Changing the setting" is a great problem-solving skill for life. How often have I had to solve parenting issues by reimagining the scenario in a different light, with variables removed or changed? I firmly believe that the skills I've developed in mathematical problem-solving have improved my ability as a parent, a citizen of my country, even as a Christian, but I have finally learned to import *back into* mathematics the experiences I learn elsewhere.

Another important lesson from my elders was "read Parker Palmer." In *The Courage to Teach*, Palmer begins by stating "This book builds on a simple premise: good teaching cannot be reduced to technique; good teaching comes from the identity and integrity of the teacher." Palmer then defines *integrity* this way:

"By integrity I mean whatever wholeness I am able to find within that nexus [of identity] as its vectors form and re-form the pattern of my life. Integrity requires that I discern what is integral to my selfhood, what fits and what does not—and that I choose life-giving ways of relating to the forces that converge within me: do I welcome them or fear them, embrace them or reject them, move with them or against them? By choosing integrity, I become more whole, but wholeness does not mean perfection. It means becoming more real by acknowledging the whole of who I am." [2]

Palmer then discusses various aspects of self that teachers integrated (or didn't) into their identity: mentors who evoked them, subjects that chose them, their upbringing and their family's attitude towards academics. Twenty-five years later, his work is as important as ever, and in fact, I am going one step further along the same line.

In my secondary education courses at Indiana Wesleyan University, we had to write "Metacognitive" papers. Despite writing all of those papers, I graduated college still not knowing what that word truly meant—but now I do. I agree wholeheartedly with Palmer's premise, that teaching with integrity is what makes the whole thing work. My new revelation is the metacognition required to realize integrity more fully. Before I can welcome or reject forces that converge with in me, I have to *recognize* all the forces that are actually present, no matter how far away from teaching they may seem.

3 October of 2023

I said that I could give more examples of transcendent experiences in my own life, and now I will. During the Fall 2023 semester, the way I approached teaching (and mathematics as a whole) was radically reshaped by three seemingly unrelated experiences, all in the month of October.

3.1 *Coward* by Haste the Day

Prior to the pandemic, I frequently attended concerts, including a half-dozen trips to Cornerstone Festival in Bushnell, Illinois (may it rest in peace). When I became a Christian, I discovered a love for heavy metal and hardcore music, in part because other contemporary Christian music felt like it was 5-10 years behind its counterparts, while the metal scene was fresh and innovative. Those festivals did much to form my faith in those early years (I converted in high school), showing me that there were more ways to worship than the narrow (and often boring) music I experienced as I began to attend church regularly.

In October 2023, I had not attended a concert for over four years, but was convinced by a friend to overpay for secondhand tickets to see Haste the Day and Oh Sleeper, old favorites of ours. When the opening act started playing, I was quite excited to be hearing live music again!

But I also felt old, and tired, and weak, and out of place. In my youth, I was no stranger to the mosh pit. But now I found myself thinking "Why am I here? I am too old for this. I can't go back. It would be embarrassing. This experience is closed off for me." While I was thinking this, Haste the Day played their song *Coward:*

I am looking into darkness
To hide my face
Don't show me who You are
'Cause I am a coward

And in that moment I realized that my remedial mathematics students (and sometimes my calculus students) feel exactly the same way. Math is a closed door for them. It's been too long, and it's embarrassing to try and get back into it. If I can't enjoy worshipping God listening to my favorite music, how can I ask my students to do something much harder, and much more public?

So I did indeed join the mosh pit, shout, scream, push, and generally act stupid. I went home covered in other people's sweat and stench, with worshipful words written on my heart:

All this time I've been running
From what You've spoken over me
But my legs are weary
And my will is tired
But it's where You'll find me
Find me on my knees

I'm told that I wasn't very useful in class the next day; apparently students had to shout to get my attention and I was plenty sore. But it reminded me that the path for my students is *through* and then *past* their fear, that my class is part of where God is calling them, and to help them stop running.

3.2 Super Mario Bros. Wonder

In 1991, when I wasn't drowning in my neighbor's pool, I was playing my brand new *Super Nintendo Entertainment System.* While I had enjoyed the original Mario games on NES, *Super Mario World* was full of surprises, secret exits, hidden levels and bonus worlds, and it had me in its grip in a way that makes parents panic and read articles about gaming addiction.

Many Mario games came and went in the next 33 years, but in October 2023 *Super Mario Bros. Wonder* released. The trailers looked stupid, even for a game about an Italian plumber who rides a dinosaur to save a princess of mushroom people from an evil turtle: now the plants could talk, and Mario could turn into an elephant and spit water. My first thought was that the developers of the game were probably taking illegal drugs. But the most powerful drug is actually nostalgia.

Each stage of *Wonder* has a "Wonder Effect." You touch a large bluish flower and the stage transforms: pipes start crawling like worms, or the stage turns upside down, or time speeds up and slows down. Almost every stage has a unique effect. For three weeks straight, I could not wait to get back into the game just to see what surprises were in store. I don't have a better way to say it than the old cliche: I felt like a kid again. Every moment was wonderful in the sense of being *filled with wonder*, just as the name promised.

This video game also got me thinking about my mathematics courses. So often, students who are struggling feel like they are trying to find their way through a long, dark, dank tunnel and they cannot see the light at the end. But what if the trip is long and dark because they're blindfolded and being led to a surprise party? That's what playing this game felt like—a joyous pursuit into darkness, seeking the wonderful discoveries that awaited, rather than a fearful plunge into a drowning sensation, like the terror I felt in that pool so long ago.

3.3 *Out of the Silent Planet* by C.S. Lewis

Before I attended the Haste the Day concert or played Super Mario Bros. Wonder, I finally read *Out of the Silent Planet* by C.S. Lewis. I say "finally" because it seems my entire department had read it, as had most Taylor students—it had been required in our first-semester Foundations of Christian Liberal Arts course in years past.

I have always enjoyed science fiction and fantasy, but nothing I have read accomplished the things that C.S. Lewis did with this trilogy. First, in a modern era where Christians seem to draw lines over Biblical interpretation and orthodoxy, it was a joy to read Lewis simply playing a theological "what if" game. But secondly, he had a way of weaving philosophical insights into the text, and then demonstrating them with the behavior of characters in the story.

In one scene, Ransom has arrived on the planet of Malacandra and escaped his captors, and then encounters an alien for the first time. He resists the self-protective instinct to run away, because he is too fascinated by the creature to do so. While setting the scene, Lewis writes, "Love of knowledge is a kind of madness." [1] How many mathematicians have you read about for whom this is true? I recall waking up at 4 a.m. in college to resolve a number theory problem I had apparently been doing in my sleep, then going back to bed.

While *Out of the Silent Planet* was an excellent book in its own right, the way it changed me is that I finally learned how to make connections like those described in Section 2. Seeing Lewis illustrate the philosophical principles he espoused with his characters got me finally to see the ways in which transcendent experiences of my own could shape my teaching, if only I could make the connection.

It is only because I read this book and finally understood this concept that I was able to see how other experiences in my life, like the Haste the Day concert and playing Super Mario Bros. Wonder, could inform my teaching. The next section describes how they did so.

4 Joy, Wonder, and Playfulness

After having these experiences, my teaching in Calculus II and in my remedial mathematics course began to shift. I also teach our freshman Problem Solving course for majors, and a key component in that class is teaching students to loosen up and look at problems from different angles—to play with them. I began to see a new definition for *play*:

> Play is participating with joy and wonder . . . and without fear.[1]

I see students on a spectrum of playfulness. At the "bad" end, students are terrified to be in your classroom. They don't want to speak or be called upon, they don't understand what's going on and are just trying to survive the experience. Some students tolerate math class, but are just there to get their grade and move on. Others eagerly participate, but are just hoping to memorize the right answers, get a good grade and carry on. But the best students are the best not because they know the right answer, but because they enjoy questioning it: "but what if we did this instead?" They are eager to *play*.

At the beginning of this article, I mentioned the classic difficulty of getting students to come to class. What if class was so joyful, so inviting, so wonderful, that skipping was never really something

[1] While I *believe* I constructed this definition, I am sure aspects of it (and this entire paper) are owed to *Mathematics of Human Flourishing* by Francis Su [3], as reading that book was a transcendent experience as well.

a student considered? A tall order, for sure. While I hope all of us enjoy our profession and go to class happy, I began to purposefully infuse my class with joy and wonder.

If a student asked, “Do we need to know this for the test?” I would respond with, “Forget about the test, how lucky are we to discover these aspects of God’s creation?” And in Calculus II in particular, I have centered each week of content as a build-up to a joyous “Wonder Effect” for the week: Gabriel’s Horn, Euler’s identity, the concept of Taylor series, even the idea that some integrals cannot be computed by hand (as Paul wrote the Corinthians, “For now we see only a reflection as in a mirror”).

I doubt everyone who reads this paper loves heavy metal or plays video games. But my question is really this: what transcendent experiences have shaped your worldview, for better or for worse? How can you transfer the good to your classroom, and avoid the bad? How can you make your classroom joyful, playful, even transcendent, by making these connections?

I have read many articles about effective teaching, but the truth is that many of my Calculus II students are engineers that I never see again. They graduate (for the most part) but I don’t really know how they do in Calculus III, Differential Equations, or afterwards. What I do know is that after eagerly, excitedly, playfully, joyfully showing that $e^{\pi i} + 1 = 0$ via Taylor series, one of my students said softly, but loudly enough for the whole room: “Oh, wow.” They may not remember everything from my class, but they’ll remember the joy and wonder.

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Mentored Research Projects in Applied Mathematics: Encouraging Creativity while Providing Structure

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Abstract

Mentoring undergraduate students in research can be a daunting task for any faculty member. Furthermore, the students often need help with how to concentrate their efforts productively when working on an open-ended research project. Here, we present a framework to approach a mentored research project in applied mathematics as a semester-long "course" structured around three milestones: project proposal, progress report, and final results. Students receive a syllabus, complete with instructions, deadlines, and rubrics for the assessments related to the progress of their project. This approach leaves space for creativity while providing a structure and opportunities for faculty guidance and feedback. Along the way, students learn how to frame their research question, collect data, learn from primary literature, generate a poster, and present their results orally and in writing.

1 Introduction

When I started as a mathematics professor at Wheaton College 12 years ago, I was informed that I was expected to include my students in my research activities. I did not know precisely how to do this: my only research until that point was related to my Ph.D. dissertation topic in nonlinear dynamical systems modeling of cellular processes, a narrow and specialized field inaccessible to most undergraduate students. At that time, I did not know if I could continue any research in that field beyond my dissertation on my own, let alone mentor students in the process. Fortunately, several faculty members at Wheaton College who had developed their own research mentoring methods reached out to me and offered to mentor me in mentoring my students. Dr. Paul Isihara, in particular, had developed a thriving mentored research program in applied mathematics. His methods largely inspired me to develop my own approach.

At Wheaton College, about 20% of the math majors become involved in some form of research experience in mathematics during their undergraduate studies. Research experience is not a requirement for graduation, but many high-achieving students seek this type of experience, especially those considering graduate school. Students gain research experience through one of the following avenues:

1. **A *Research Experience for Undergraduates* (REU) program.** Large universities offer these 8-10 week long full-time summer programs nationwide. The National Science Foundation funds most of them and provides travel and lodging benefits, as well as a generous stipend.
2. **Summer Research at Wheaton College.** Wheaton College offers to fund a small number of professors and students to collaborate on an 8-10 week long full-time research project during the summer. Like a REU program, the college offers free lodging and a stipend.

3. **Semester-long Mentored Research course at Wheaton College.** Every semester, one of the math professors teaches a *Mentored Research* course, which counts as a 2-hour course release for the professor and grants two credit hours to the enrolled students. Every professor is free to structure this course as they see fit.

In this paper I will present my version of the Mentored Research course, which I have developed and taught multiple times since 2012. After modifying, refining, and restructuring the course repeatedly to respond to new challenges every year, I want to share some of my experience. As I explain below, my most successful mentored research experiences have been those that provide the structure that undergraduate students expect from any course while giving sufficient freedom for students to experience the open-ended nature of original research, take ownership of the process, and allowing them to create and experiment with their own approaches.

Because of my background as an applied mathematician, my Mentored Research course focuses exclusively on applied mathematics research, i.e., the type of research that supports research in another field usually conducted according to the scientific method. However, the general course format and structure can be adapted to other types of mathematical research.

2 Mentored Research Course Format and Structure

2.1 Course Structure

Mentoring undergraduate students in research presents many unique challenges:

- **Undergraduate students have minimal mathematical background.** Finding a topic accessible to students who have only been exposed to lower-division mathematics (calculus, linear algebra, etc.) is a significant obstacle.

- **Undergraduate students often struggle with open-ended questions.** Lower-division students have only been exposed to problems for which there is a unique, correct answer and for which there is often a unique approach. Many of these students are perplexed by the open-ended nature of research questions, for which not even the professor knows the answer or the best way to tackle the problem.

- **Undergraduate students often lack the discipline and motivation for independent study.** Students usually relegate research to the lowest priority, especially when juggling research demands with the strict deadlines enforced by their other classes and activities.

As a result of these challenges, students may become frustrated and feel like they are spinning their wheels, investing large amounts of time and effort with little reward. The professor may also become frustrated and decide to take the research into their own hands and do everything themselves while the students merely watch. To avoid falling into these traps, the professor should structure a Mentored Research course like any other undergraduate course, including:

- a clear and detailed syllabus,

- regularly scheduled in-person meetings with the professor,

- weekly deadlines and assessments,

- clear milestones throughout the semester with corresponding deadlines.

This familiar course structure (syllabus, deadlines, etc.) is comforting and encouraging for undergraduate students, who expect this type of structure for all their courses. The structured format urges them to approach a mentored research project as any other course, dedicating time and effort every week alongside their other classes and activities.

2.2 Course Format, Prerequisites, and Learning Outcomes

The Mentored Research course usually tackles 2 or 3 different projects concurrently, with each project assigned to 1 or 2 students. Registration is by invitation only (i.e., not open enrollment): the professor approaches up to 6 capable and motivated students who have previously expressed interest in conducting a mentored research project and offer them the opportunity to enroll.

The formal prerequisites for the Mentored Research course are Calculus (I-III) and Linear Algebra. Experience in differential equations, statistics, and programming is preferred, but not absolutely necessary, as it is usually possible to craft a research topic accessible to students without this knowledge.

The learning outcomes stated on the syllabus are as follow. By the end of the course, students will have learned to:

1. organize and plan all phases of an original research project,
2. identify a research gap and define the topic and scope of a research project,
3. conduct literature search,
4. collect data & build a mathematical model,
5. learn analytical and computational tools to solve a model & analyze results,
6. present a project proposal, progress report, and final results.

2.3 Project Topics

A valid project topic for the Mentored Research course must satisfy three requirements:

1. **It must apply mathematics to another field.** The project must use a mathematical approach to answer a research question in an outside field (i.e., not proving a "pure" math theorem).
2. **It must contain some form of quantitative mathematical modeling.** The project cannot be merely qualitative or simply computing a dataset's descriptive statistics.
3. **It must be original.** The project cannot be purely expository; it must seek to answer an original research question.

Outside these three requirements, students can choose any topic that aligns with their interests. The professor may have to help the students define a valid research question, define the scope, and propose the approach. To this end, the first assignment is to generate a *mind map* to help identify the possible research avenues and define the topic and scope. Of course, the proposed topic and scope can change substantially throughout the project as students learn about the field, generate results, and encounter roadblocks. The professor can also provide a list of ideas to help students choose a topic.[1]

Here are some examples of recent research projects conducted at Wheaton College:

- *The Syrian Refugee Crisis: Modeling the Dynamical Mechanisms of Refugee Flow.* This project applied a quantitative approach to study migration flows in space and time. Students built a dynamical systems model that replicated the patterns in the data collected from the United Nations High Commissioner for Refugees website. Students read and learned about the factors driving refugee flow described in qualitative studies (e.g., family reunification, cultural familiarity, political stability, and societal violence) and quantified them to determine the model's numerical parameter values.

- *A Mathematical Model for Gang Membership in North Lawndale Accounting for Rehabilitation and Fringe Crime Imprisonment.* This project was based on the work of Sooknanan *et al.* [1] on the mathematical modeling gang membership in Trinidad. The students modified the model and applied it to the case of North Lawndale, a Chicago neighborhood where Wheaton College partners with a non-for-profit organization that provided qualitative data. The main difficulty for the students was to understand the published model and replicate its results before extending it.

- *Volatility and Loss of Information in the Irish Dance Scoring and Judging System.* This project was proposed by a student who was also a competitive Irish dancer at the national level. She had access to large amounts of score data from previous competitions where the public had criticized the results as being unfair. The project compared the scoring systems of various sports whose scoring system includes a subjective component (such as figure skating, gymnastics, and diving) to identify the sources of bias and risk of error propagation.

- *Sales Performance Analysis of Regional Frito-Lay Sales Representatives Based on Social Network Analysis.* This project was based on a dataset obtained from a business contact of a faculty member in the Business department. Although the students had not progressed beyond linear algebra in their coursework, they were able to build a social network model based on digraphs and matrix algebra. They also learned about principal component analysis and applied it to their model.

2.4 Workflow

The workflow for an applied mathematics research project progresses according to the following phases in sequence:

Phase 1: Learning, Literature Search, Data Collection

After determining the general topic of the research project, the first step is to learn about the field of application, beginning with searching the internet with a standard search engine and consulting books and magazines from the library. Concurrently, the professor and students should search the

[1] Personally, I keep on my desk a folder of project idea "seeds" that I collect as I read articles, come across interesting datasets or talk with friends and colleagues. At any given time, my folder contains 10-20 "seeds" (not necessarily all viable and fruitful) that I share with undecided students to awaken their creativity.

scholarly literature with a specialized search engine (such as *Web of Science*) to learn about the latest research in the field and to determine which questions remain open. The ultimate goal is to find a *research gap*, i.e., a question or problem that the existing research has not answered. Identifying a research gap may prompt the students to refine or modify the title and scope of their proposed research project. Students should carefully document every source they read, as they will need this information to build the *literature review* section in their project reports.

In addition to learning about the field of application, the literature search phase is the time for students and professor to learn (or review) the mathematical and computational methods they plan to use. Depending on their background, students and professor may have to learn (or brush up on), for example, the basic principles of dynamical systems modeling, statistical data analysis techniques, or linear programming.

Since mathematical models are based on qualitative and quantitative observations of the world, the researchers must determine early in the project what data they will need and how they will obtain it. Sometimes, the professor or a student may already have a raw dataset, which can determine the topic and research question. Sources of data include:

1. **Free databases on the internet.** Governments and nongovernmental organizations post large amounts of data on the internet. Examples include: the United Nations Statistics Division [2], the World Bank [3], the U.S. Federal Government [4], the Federal Reserve Economic Data [5], and the Bureau of Labor Statistics [6].

2. **Scholarly literature.** Many journals require that their authors make their datasets available to readers.

3. **Personal research contacts.** The professor may know colleagues or contacts at other institutions willing to share their datasets. Natural or social science professors often have raw datasets amenable to original mathematical modeling projects. Similarly, local businesses and nonprofits may be willing to provide data and partner with the college for a research project that benefits them.

4. **Self-collected.** Collecting one's own data for a mathematical research project is discouraged because data collection can be time-consuming and expensive. Furthermore, data collection efforts can distract from the primary pedagogical goal of focusing on the mathematical modeling aspect of the research.

Phase 2: Model Building

In this phase, the students decide on a mathematical modeling approach that leverages their data to answer their research question. The process usually begins with establishing simplifying modeling assumptions and defining the model's variables (dependent and independent) before writing the equations. The resulting models usually fall into one of the following categories:

1. **Dynamical Systems model.** This type of model typically has the form of an initial-value problem based on differential equations that describe a system's evolution mechanisms to predict its possible future outcomes. An important goal is to estimate the parameter values so that the model's solution replicates the data. Instead of a continuous time variable, this type of model can also proceed by discrete time steps, leading to difference equations or recurrence relations that can be solved without resorting to calculus. Possible variations may include

modeling spatial variations or distributions using continuous space variables, leading to partial differential equations, or discretizing space into a network of nodes and vertices to obtain a dynamical graph theory model.

2. **Empirical model.** This type of model typically relies on statistical tools to reveal patterns in the data, starting with descriptive statistics and extending to regression analysis, including linear regression, logistic regression, and multiple regression. An essential task consists in quantifying the significance of the model's results via significance tests and confidence intervals.

3. **Operations Research model**. This type of model often takes the form of a constrained optimization problem, which identifies the decision variables, an objective function, and a set of constraints. This can lead to a linear programming, integer programming, or nonlinear programming model, depending on the form of the variables and equations.

Phase 3: Generating Results

Once the model is defined, the researchers can run simulations, analyze scenarios, find solutions, and perform analyses (sensitivity analyses, stability analyses, etc.). This phase usually requires that the researchers familiarize themselves with a computational platform to perform these tasks, generate results, and represent them graphically.

The phases described above progress iteratively. For example, the first round of results may reveal information that leads to adjusting the research question, collecting more data, adjusting the model, considering a different approach, etc. The researchers cycle through these phases throughout the semester, except before one of the three milestones (see below), when they pause their research efforts to prepare a project report and an oral presentation.

2.5 Semester Structure and Weekly Cycle

As a two-credit course, the class meets 100 minutes every week of the semester (15 class sessions). Here is the schedule of weekly activities:

Week #	Activity	Meeting
1	Introduction and assigning teams	All together
2-3	Defining topics, scope, data, tools	Individual teams
4	**Presentation of Project Proposals ***	**All together**
5-8	Collecting data, building model, preliminary results	Individual teams
9	**Presentation of Progress Report ***	**All together**
10-14	Refining model, analyzing results	Individual teams
15	**Presentation of Final Results ***	**All together**
(post-semester)	Optional oral presentation(s) and/or publication	-

All students meet together with the professor on the first week. During this first class session, the professor explains the expectations and structure of the course. Then, students and professor discuss possible project topics, data, methods, and each student's mathematical background and interests. By the end of the first session, professor and students decide on the number of projects (2-3) and students per team (1-2) for the semester. By the following week, each team should establish a shared working folder containing all research materials for everyone to access throughout the semester.

Every week after that, excluding the three *Milestone weeks* (indicated by a * on the table), the professor sets up a classroom to conduct a weekly *advising session* with each team. Each team meets individually with the professor for 30 or 45 minutes (depending on whether there are two or three teams), while the other team(s) work separately in another room. Here are the objectives of the advising sessions:

- The professor gives feedback to the students on the previous week's work and discusses their progress, if necessary. Students share what they have learned, report on the roadblocks they encountered, and ask questions.
- Together, the professor and students brainstorm ideas for the next step(s). The professor guides the students in their learning, data collection, model building, or result-generating efforts. If necessary, the professor teaches and demonstrates new approaches or computational tools.
- The professor assigns tasks for the students to complete and posts them on the LMS (e.g., Canvas, Backboard) by the end of the day to ensure that the students understand what the professor expects them to complete before the next session. The students have five days (i.e., two days before the next session) to complete the assigned tasks and submit their work. Depending on the progress stage and model type, these tasks could be, for example:
  - "Find, download, and read article X in journal Y, make notes, and send me your questions and comments."
  - "Find data for X on the internet, organize it in a spreadsheet, and send it to me."
  - "Conduct a correlation analysis for the 16 variables. Send me the results and your decision (based on your results) on which five to retain for the regression model."
  - "Run 25 time-course simulations for various combinations of parameters α and β, each ranging from 0 to 0.1 in increments of 0.02. Send me the results as a set of 25 time-course graphs."
  - "Write and send me a revised version of the progress report with the updated Discussion section and the completed bibliography."
- After the students submit their work for the week, the professor has one or two days to prepare for the next session. During this time, the professor views the students' work and plans the next steps. At this time, the professor also assigns the *weekly progress grade*, which reflects the degree of completion of the assigned tasks and the students' overall proactiveness and work ethic. This grade is assigned on a 4 point scale according to the following rubric:
  - 4 points: All tasks completed
  - 3 points: Tasks attempted and 75% completed
  - 2 points: Tasks attempted and 50% completed
  - 1 point: Tasks attempted and 25% completed
  - 0 points: Tasks not attempted.

  The primary purpose of the progress grade is to keep the students accountable and on task. Because of the unknown nature of open-ended research, students may run into roadblocks that prevent the completion of an assigned task. In this case, the student can earn full credit by explaining the attempt and the roadblock encountered. The weekly progress grades are averaged and factored into the overall semester grade (see 2.7).

On the three *Milestone* weeks, instead of the advising sessions, all students in the class meet for the entire 100-minute session. During these meetings, each team presents their project to the other class members as an oral presentation. In the remaining time, the professor gives instructions concerning the next milestone (see 2.6).

2.6 Milestones

Each research project is divided into three *milestones*: (1) Project Proposal, (2) Progress Report, and (3) Final Results. Each milestone culminates with each team submitting a formal paper and delivering its content as an oral presentation on a *milestone week* (see the schedule table in 2.5).

The papers and corresponding presentations, graded according to rubrics posted beforehand, weigh heavily in the overall course grade (see 2.7). The papers are formal documents written in LaTeX, with proper mathematical syntax and formatting. For the oral presentations, each team delivers a 16-20 minute presentation with visual support (PowerPoint, Google Slides, or Beamer) to the rest of the class. The oral presentations can be announced on campus and made open to the public if feasible. Students should view these presentations as formal events and imagine themselves presenting to a broad audience. The milestone weeks are also collaborative opportunities where students are encouraged to ask questions about each other's projects, celebrate each other's achievements, brainstorm together, and help each other overcome roadblocks.

The three milestones outlined below are inspired by the periodic reporting events followed by project managers in the industry. After a project proposal, the researchers periodically report on their progress to the stakeholders (customers, superiors, funders) via progress reports before ultimately submitting their final results.

Milestone #1: Project Proposal (week 4 of 15)

The project proposal represents the research project's public launching event. It sets the project's vision, scope, goals, and requirements regarding time, resources, and data. Ideally, the proposal should sell the project, highlighting its relevance and practical importance, as if it were appealing to donors to fund the research. The content of the project proposal should include:

- An introduction of the team members, along with their roles, expertise, and qualifications,
- A proposed title with a description of the project's topic, scope, relevance, and motivation,
- The research question(s) and the desired or anticipated outcomes,
- A literature review with identification of a research gap,
- A description of necessary data (either available or to be obtained) to complete the project,
- The proposed mathematical modeling approach and computational solution/analysis/simulation methods, including software and hardware requirements,
- Preliminary results (if available),
- The timeline of anticipated future milestones (presentations and publications).

Milestone #2: Progress Report (week 9 of 15)

The progress report occurs at the halfway point of the project timeline. It should provide some results and describe how the project may have changed due to roadblocks encountered or unexpected discoveries. The report should include:

- A summary of the research topic and research question(s),
- All significant changes since the project proposal, such as the title, topic, scope, research question(s), research gap, data, methods, team members, or timeline. For each changed item, the report should explain the reason for the change,
- Data and mathematical methods, including modeling assumptions, variables, equations, etc.,
- Preliminary solutions, simulations, and analyses, presented graphically or in table form,
- A preliminary discussion, highlighting exciting, surprising, or counterintuitive results,
- Expected future results.

Milestone #3: Final Results (week 15 of 15)

The final paper should follow the standard scientific structure of a research article in applied mathematics: Abstract, Introduction (including the literature review, research gap, and research question), Methods (model, data, and solution methods), Results, Discussion, and Conclusion.

In addition to the final paper and oral presentation, each team must generate a research poster at the end of their project. Although the students are not required to present their project as a poster presentation during the semester, the poster is ready if they decide to participate in a poster presentation at a later date.

2.7 Assessment

The course grade is weighted as follows:

- **Weekly progress (25%).** Average of the weekly progress grades earned every week.
- **Project proposal (15%), Progress report (20%), Final results (30%).** Each milestone is graded according to a rubric (see below) posted beforehand.
- **Poster (10%).** Graded based on its content, structure, and visual appearance.

Each of the three milestones is graded according to a rubric (see Appendix), which includes:

- the content (40%),
- the formatting of the paper or report (25%), including its organization, sectioning, writing style, and the formatting of tables, figures, mathematical syntax, and bibliography,
- the format and delivery of the oral presentation (25%), including the slide design, timing, pacing, and interaction with the audience.

In a team of two (or more) students, the last 10% of each milestone grade is awarded to each student for completing a confidential *collaboration survey* in which each student reports on the dynamics of their team's collaboration. This information allows the professor to regularly detect conflict or work inequity between team members throughout the semester and promptly address any issues. (If a student is working alone, he/she is automatically receives these points.)

3 Post-Semester Opportunities

A project conducted in a Mentored Research class can provide opportunities beyond the end of the semester, such as publication in a journal or presentations at a conference.

Publication

The final paper of a mentored research project is an excellent foundation for a manuscript suitable for publication in a peer-reviewed journal.[2] The process usually requires one or two semesters of additional work, but motivated students are often interested in continuing to collaborate and make progress beyond the semester's end. Alternatively, the professor may hand off the project as a work-in-progress to a different team of students. Possible venues for the publication of applied mathematics research include:

- Mathematics journals, such as *Mathematical Gazette*, *Simulation*, *PRIMUS*, and *UMAP*.
- Non-mathematics journals, such as the *Journal of Migration Studies* or the *Journal of Humanitarian Logistics*.
- Student journals, such as *Pi Mu Epsilon*.

Presentations

By the end of the semester, each team has a ready-made and rehearsed oral presentation and a poster, allowing them to present their research project at a research fair or poster presentation on campus or at a conference with minimal extra work.

[2] At Wheaton College, 20-25% of projects initiated in a Mentored Research class become the subject of a future peer-reviewed publication.

Appendix: Grading Rubric for Milestones

	4-Excellent	3-Good	2-Fair	1-Insufficient
Content (Paper and Oral Presentation) 40%	All sections are complete, compelling, and correct.	Sections mostly complete, but some important information missing in one or more sections.	Two or more sections are incomplete or contain incorrect information.	Esential information missing in multiple sections.
Paper Formatting 25%	Report is well organized, clear and logical structure, excellent flow, tables and figures are complete, clear math syntax, complete bibliography.	Report is clear and organized, writing style is adequate, tables and figures OK but missing some essential information, math syntax is adequate, complete bibliography.	Sectioning is at times unclear or incorrect, tables and figures are unclear, math syntax is unclear or incorrect, incomplete bibliography.	Report is unclear or sectioning is incorrect, tables and figures missing essential information, math syntax is incorrect, incomplete bibliography.
Oral Presentation 25%	Excellent slide design, legible, clear, and appealing. Excellent delivery, pacing, and interaction with audience. Finishing within 1-2 minutes of allotted time.	Slides are legible and complete. Delivery and pacing is adequate. Finishing within 1-2 minutes of allotted time.	Some slides are incomplete or unclear. Delivery is too fast or too slow. Finishing more than 2 minutes before or later than allotted time.	Slides often illegible or unclear. Insufficient interaction with audience. Finishing more than 2 minutes before or later than allotted time.
Participation Survey 10%	Full credit for completing a confidential participation survey, where each student reports on the dynamics of their team's collaboration.			

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Insight and Illumination in Mathematical Learning: Exemplars of Transitions to Understanding in the Classroom from Elementary to Graduate School

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Abstract

Jacques Hadamard describes a four-stage epistemological process: preparation, incubation, illumination, and verification. Hadamard developed this scheme based upon Henri Poincaré's reflections on the process that led to his discovery of Fuchsian functions (i.e., automorphic functions) which he described in his wonderful paper *La Genese de la Creation Mathematique* (*Mathematical Discovery* in English). Philosopher and theologian Bernard Lonergan builds on the writings of Aquinas to develop his theory of insight which elaborates the transition from incubation to illumination. Insight and illumination provide an analytical lens for transitional moments as students learn ever more abstract mathematical concepts. We offer exemplars of these transitions at the elementary, middle, high, undergraduate, and graduate school levels. We provide connections to theorists in mathematics education such as Jean Piaget (stages of development of understanding functions), Pierre van Hiele and Dina van Hiele-Geldof (levels of understanding geometry), and Dietmar Küchemann (levels of understanding letters or variables). We also provide pedagogical strategies which align with Christian faith commitments and promote growth in understanding.

1 Introduction

As professors of mathematics and mathematics education at the university level, we often observe students experiencing an "aha" moment that marks the progression from cognitive struggle (productive or otherwise) to conceptual understanding. Richard Skemp contrasts this more interconnected relational or conceptual understanding with a more disconnected instrumental or procedural understanding [14]. Rather than merely computing correct answers by following a proscribed algorithm, students must make critical connections among different mathematical concepts to form an essential idea or conjecture.

This article places these "aha" or discovery moments in a broader context. We discuss learning theories espoused by Jacques Hadamard and Bernard Lonergan. Each offers terminology to align with this "aha" moment, and they situate the critical moment of discovery within a larger process that begins with a new problem and culminates with the verification of a proposed solution. Next, we document a series of exemplars of this type of student thinking at all levels of learning mathematics, from elementary school to graduate school and beyond. Some of these exemplars, or vignettes, arise from a classroom of students taught by one or more of the authors while others are representative of elementary, middle, and high school mathematics classrooms. Where appropriate, we connect pedagogical practices that align with our own faith commitments with these vignettes.

In a final section, we offer a few conclusions that emerge from this discussion. We seek to remind the reader of pedagogical choices that can promote these "aha" moments in our students and that some of these choices also align with our Christian faith commitments.

2 Insight and Illumination

Henri Poincaré articulated his own reflections on the process of mathematical discovery in his paper *Mathematical Discovery* (in French *La Genese de la Creation Mathematique*[11]). The account he gave was in his work focusing on, what he called, Fuchsian functions (automorphic functions in modern parlance). These are functions that satisfy certain differential equations that have a form similar to those equations that have elliptic functions as their solutions. Thus Poincaré sought a theory of Fuchsian functions which possessed a theory analogous to that of elliptic functions. In that manner, he was working from the standpoint of being an expert in complex function theory in general, and on elliptic functions in particular. In drawing upon this expertise, his initial foray to determine the possibility of the existence of Fuchsian functions was an attempt to produce them as ratios of infinite series in a way analogous to how elliptic functions can be expressed as ratios of infinite series, called theta functions. In Poincaré's account, he spent an all night, caffeine-fueled exploration whereby he experimented in creating Fuchsian functions using hypergeometric functions. It is during this wild bout of experimentation that he hit upon one type that satisfied the criteria he gave for such a function to be Fuchsian. Subsequently, he was able to describe his new function as a ratio of infinite series in the spirit of elliptic functions (he called the functions in the ratios theta-Fuchsian).

Here we see an initial example of mathematical discovery and the structure the underlies it. Poincaré initiates this project by formulating a problem within his area of expertise, complex function theory, that seeks to develop a class of functions that are solutions to a type of differential equation, as studied by Fuchs, in a way analogous to elliptic functions. This analogy provided a heuristic which guided Poincaré to how these functions, if they existed, should come to be and then, subsequently, be classified. Through his deep understanding and knowledge of complex functions and elliptic functions in particular, Poincaré experimented and explored what was tacitly proximal and distal, in his conscience (à la Michael Polanyi), the aspects of the theory that could be brought to bear to bring those functions from unbeing to being. It is this latter act, one which considers and combines the various aspects of known or newly understood aspects of the theory, which suddenly brings together into an intelligible unity, a function that satisfies the required solution to the problem. This moment of illumination can be viewed, à la Bernard Lonergan, as a supervening act of understanding which, furthermore, is a movement from analysis to synthesis. This act resolves the problem by a further reflective act of verification in which the conditions, designed to insure the proposed solution is valid, are checked by showing that the components of the solution do in fact intelligibly unite into a proper solution. In one further illustration of this process of discovery and the role illumination plays in it, Poincaré further relates how he next develops and classifies Fuchsian functions by carrying further the analogy with the theory of elliptic functions. Elliptic functions are understood by the period lattices associated to them in the complex plane. The transformations that preserve them are Euclidean in type. Poincaré was able to identify the transformations for defining the Fuchsian functions he thus found, but they were not Euclidean in type. To fully develop a theory Fuchsian functions, the classification of their transformations was a necessary step to take. Poincaré, of course, achieved it, here is his account:

> Just at this time I left Caen, where I was living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with my conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience's sake I verified the result at my leisure.[11, p. 392]

We can see here that Poincaré was prepared because of previous work he conducted in plane geometry that he possessed a tacit knowledge, in a distal manner, of the theory of transformations. Through a process of incubation in which the theory of complex functions and plane transformations were tacitly understood together in his sub-conscience, suddenly he understood in a synthesis of intelligible unity that the transformations of Fuchsian functions were those of a non-Euclidean plane. This then led to a fuller classification of all Fuchsian functions.

Inspired by Henri Poincaré's reflections, Jacques Hadamard describes the following four-stage epistemological process to achieving invention and discovery in mathematics: (1) preparation, (2) incubation, (3) illumination, and (4) verification [3]. Bernard Lonergan builds on the writings of Aquinas [1] to develop his theory of insight which elaborates the transition from incubation to illumination and then to judgment via rational reflection [8]. Hadamard's four stage cycle itself can be articulated within Lonergan's unified framework of empirical, intelligent, and rational consciousness. First, through the empirical consciousness, there is need for *preparation* during which one becomes familiar with the problem setting, definitions, axioms, and key results logically deduced. Next, via the intelligent consciousness, there is a time of *incubation* during which the individual reflects on the growing knowledge of the problem to develop a heuristic from which potential solution strategies may be formulated. At the third stage, by *illumination* within the intellect, a person grasps the essence of the solution to the problem. Finally, by the rational consciousness, a person must *verify* the solution, look for possible generalizations, and explore new questions raised by this solved problem.

Learning is achieved at a particular stage through the oscillating Hadamard cycle eventually resulting in illuminations and verifications, resolving all lingering questions through judgments. Lonergan, building on the theology of Aquinas, develops his own theory of insight, and he describes conditions, like Hadamard's incubation phase, leading to insight which aligns with Hadamard's illumination. According to Aquinas, our capacity to come to knowledge of truths, both potential and actual, in creation, is due to our nature as image bearers of God. Our intellect possesses a light derived from our participation in the True Light, that is the Word of God from whom all things are made and have their being. From this light, what is at first seen as individual and separate terms in the data become seen in a holistic unity in which the terms and relations are understood from their participation in the whole.

3 Mathematical Illumination Vignettes from Elementary School to Graduate School

We next offer a series of short vignettes or exemplars involving both students and teachers of mathematics at a wide variety of levels of sophistication. Some of these vignettes are drawn from the classroom of one of the three authors, while others describe a common experience in elementary, middle, or high school mathematics classrooms. In each vignette, you will observe the need for a person to progress through the four phases articulated by Hadamard, with particular emphasis on the transition from incubation to illumination through learning activities carefully selected and implemented by the classroom teacher. Where appropriate, we link pedagogical practices that align with Christian faith commitments to the learning environments designed to provide opportunities for learners to achieve these “aha” moments of illumination.

3.1 Elementary School Exemplars

In the elementary grades, students engage geometric concepts at increasingly sophisticated levels. Pierre van Hiele and Dina van Hiele-Geldof identified different levels of understanding and engagement with geometric concepts, shown in Figure 1 [18]. At the earliest stage, students reason holistically. A student labels a figure a “triangle” because it has the general shape as other objects previously identified as triangles. At the next level, students begin to analyze and sort different objects based upon properties such as the number of sides. The subsequent level requires an abstraction to recognize underlying relationships among different shapes. So, for example, a student can now classify a square as a special type of rectangle due to its four right angles but that also has four congruent sides, developing the identification of patterns through analogy and generalization. The transition from informal deduction to more formal deduction occurs at the following level. Teachers must structure the learning environment so that the *preparation* and *incubation* phases lead to appropriate *illumination* and ultimately to *verification*.

Figure 1: The five van Hiele levels of geometric understanding.

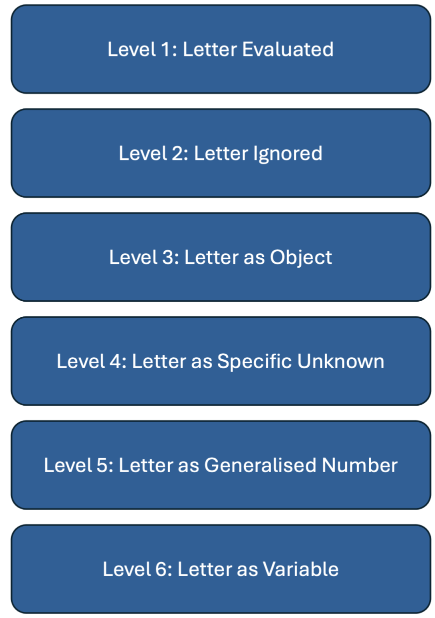

Figure 2: The six Küchemann levels of understanding letters.

As children progress through elementary school, they must make a transition from whole numbers to rational numbers. After mastering addition, subtraction, multiplication, and division of whole numbers, students in grades 3 through 5 encounter fractions and must adjust their thinking to correctly operate with these fractions. Initially, most students will exhibit the misconception of overgeneralization. For example, they will compute the sum of two fractions by adding the numerators and adding the denominators. Other students may ignore the denominators entirely, focusing solely on the numerators. Additional difficulties may surface when finding common denominators or when converting between mixed numbers and improper fractions [17]. Teachers can create learning activities designed to allow students to explore the meaning of fractions through manipulatives such as fraction wedges, fraction tiles, or Cuisenaire rods. Using clusters of 3-4 students can promote a learning community in which reasoning can be explained and conjectures made. Meanwhile teachers can monitor these learning clusters and provide question prompts that foster student explanations and productive math struggle [13]. These learning clusters also help to foster a community of hospitality, where each learner serves as a peer guide for others in the group [15]. This notion of community and hospitality aligns with a teacher's Christian beliefs that each student bears God's image and should seek to serve their peers as colearners [6].

3.2 Middle School Exemplars

A similar model of growth can be found in the transition from arithmetic to algebra. Dietmar Küchemann describes a total of six different levels of understanding of the concept of a variable, shown in Figure 2. When middle school students first encounter letters in a mathematical context, they seek to evaluate them for a specific number, operate with them as objects, or simply ignore the letters entirely [7]. However, for these same students to succeed in learning algebra, they must be prompted to understand these letters as specific unknowns, as generalized numbers, and ultimately as variables. Küchemann noted that instruction in the first algebra course often focuses on these three most advanced levels of understanding of letters, whereas most of the students in these courses enter the year operating at the three more basic levels of understanding. Teachers must help these students to view algebra as a generalization of arithmetic and to see algebra as the movement from number as understood via manipulation of operations and equality to understanding of number subjected to and defined by operations subject to rules and axioms. Note that the first three of these levels are more procedural whereas the latter three levels are more conceptual [4].

Students typically move through these different levels of understanding variables prior to moving through similar understanding levels for functions, shown in Figure 3 [10]. Jean Piaget and his colleagues identified a total of four stages or levels of understanding of the concept of mathematical function. At the first level, the student is unable to make any meaningful connection between the input and output variable. At the second level, the student may connect specific input values to corresponding output values, but these connections are haphazard rather than systematic. The systematic connection between input and output values characterizes the third level. Students operating at this level can detect an underlying pattern and extend it sequentially. For example, a student may notice that the output value increases by 3 for every unit increase in the input value. When asked to find the output value corresponding to an input value following a significant gap in the input values, the student must compute output values for all the intervening input values following a sequential reasoning process. Finally, at the fourth level, students can describe a generalized relationship between input and output values, commonly as an explicit formula. It should be noted that extended time may be needed for some students to discover the generalized relationship for some of these functions. Allowing "soaking time" during in-class learning activities or an extra day to complete an assignment may prove useful. Some teachers also remove the time pressure on unit tests by allowing students to start a bit earlier or stay a bit later. These allowances, though sometimes requiring flexibility and perhaps an alternate test location, reflect a teacher's Christian commitment to allow students to demonstrate the fullest extent of their learning based up on God-given abilities and a willingness to use them fully [6].

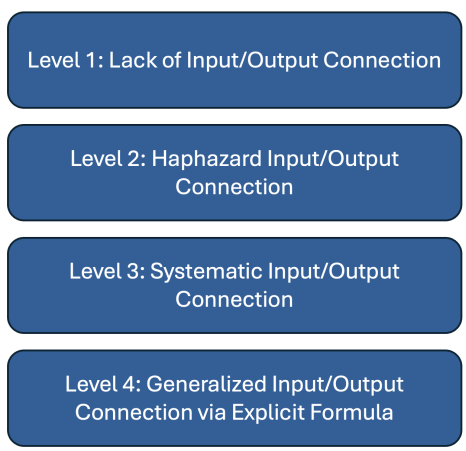

Figure 3: Piaget's four stages of understanding functions.

3.3 High School Exemplars

In the second year of algebra, students learn multiple methods to solve quadratic equations, including factoring, using the quadratic formula, and completing the square. For most students, each of these methods are entirely algebraic. However, from a historical perspective, the completion of the square method is primarily geometric. When preservice teachers encounter this geometric aspect of the method it necessitates a shift in their thinking to accommodate this new context or approach. Once again, such students must have ample time to prepare and to allow incubation prior to experiencing the illumination of the integration of the spatial and numerical or algebraic features of this method [9]. As Christian teachers, we embrace the notion of God as creator of a well-ordered and interconnected world. Making the effort to demonstrate connections among history, mathematics, science, and other subjects helps students to grasp the ways in which their experiences of God's world crosses arbitrary disciplinary boundaries.

As students transition from algebra to calculus, another illumination phase occurs. Whereas linear functions have a constant slope, higher degree polynomial functions and other families of functions, such as trigonometric, exponential, and logarithmic, have non-constant slopes. As students face the challenge of describing the slope of one of these functions at any point, teachers introduce the concepts of the slope of a tangent line that leads to the notion of a derivative, defined as the limit of the slope of related secant lines. Although students quickly acknowledge that the slope of the curve is changing, they often struggle with the concept of a limit and its application to the formal definition of a derivative.

3.4 College and University Exemplars

When undergraduate mathematics students take their first course in abstract algebra or group theory, prior mathematical experiences have familiarized these students with properties such as associativity and commutativity. However, students must now apply these concepts to algebraic groups. Later in the course, these same students often struggle when they initially encounter factor or quotient groups. Since the groups and their subgroups now function as elements, there is an added abstraction required in the mind of the student. Professors must offer plenty of examples to encourage students to pass through *preparation* and *incubation* to *illumination* and ultimately *verification.*

A second exemplar occurs in the advanced course in geometry offered at most universities and taken by mathematics majors, especially those students preparing to teach mathematics at the high school level. After an in-depth exploration of Euclidean Geometry, students are introduced to geometries in which the Parallel Postulate is not provided as an axiom. Students must expand their viewpoints to consider non-Euclidean geometries in which there may be zero or multiple lines through a given point which are parallel to a given line. Once students have time to think deeply (incubate) about potential implications of this major change in axiomatic assumptions, they discover (via illumination or insight) the world of triangles whose angular sums are more than (or less than) 180 degrees.

The distinction between stating a fact by rote and having a deeper understanding of it is one that is measured by the difference found in mere memory of stated definition and actual meaning. Such a distinction is connected in unity in a divine grounding in God as the Word (*Logos* or *Verbum*). The Christian faith theologically identifies Word as the second person of the Triune nature of God, as the prologue of the Gospel of John indicates:

> In the beginning was the Word, and the Word was with God, and the Word was God. The same was in the beginning with God. All things were made by him; and without him was not any thing made that was made. In him was life; and the life was the light of men. And the light shineth in darkness; and the darkness comprehended it not. (John 1:1-5, KJV)

The further reference to light here can be understood as the source of illumination within each person from which insight arises, as indicated further along in this chapter: “That was the true Light, which lighteth every man that cometh into the world” (John 1:9, KJV).

Beyond this Gospel, other passages that connect Christ as the source of illumination from which truth comes to be known include the following:

> But all things that are reproved are made manifest by the light: for whatsoever doth make manifest is light. Wherefore he saith, Awake thou that sleepest, and arise from the dead, and Christ shall give thee light. (Ephesians 5:13-14, KJV)

Furthermore, scripture helps distinguish between a knowing that is merely rote memorization through a deeper knowing that comes from understanding by the notion of wisdom:

> For the Lord giveth wisdom: out of his mouth cometh knowledge and understanding ... Wisdom is the principal thing; therefore get wisdom: and with all thy getting get understanding. (Proverbs 2:6; 4:7, KJV)

Lonergan identifies the aspect of human nature by which a person arrives at a deeper knowledge as being centered within the conscience, which he defines as “awareness immanent in cognitive acts” and further parses that activity in terms of the empirical (via the senses), intelligent (via grasping, insight), and rational (via judgment) consciences. Seeing this as the activity of arriving at truth via wisdom, scripture understands this activity as grounded in the spirit:

> For what man knoweth the things of a man, save the spirit of man which is in him? Even so the things of God knoweth no man, but the Spirit of God. Now we have received, not the spirit of the world, but the spirit which is of God; that we might know the things that are freely given to us of God. Which things also we speak, not in the words which man’s wisdom teacheth, but which the Holy Ghost teacheth; comparing spiritual things with spiritual. (1 Corinthians 2:11-13, KJV)

3.5 Exemplars from Graduate School and Beyond

Insight and illumination continue to impact the development of mathematical ideas, including doctoral dissertations and ongoing mathematical research. We first offer an example from the first author’s doctoral dissertation within the discipline of mathematics education [5]. In a study including twenty-five preservice teachers, nineteen elementary and six secondary, each participant completed written instruments designed to assess the level of understanding of the concepts of a variable and a function. Videotaped interviews were also completed, and each participant was asked to explain some answers from these instruments as well as to solve additional mathematical problems. Based on these data, each participant was assigned to one of four levels of understanding of variable based upon assessment indicators provided from a published instrument [2]. The first level corresponds with the *letter evaluated* stage, the second level corresponds to the *letter ignored* and *letter as object* stages, the third level corresponds to the *letter as specific unknown* stage, and the fourth level corresponds to the *letter as variable* stage. Similarly, each participant was assigned to one of the four levels of understanding functions, defined by Piaget and discussed earlier (progressing from no link between input and output variables to haphazard links, to sequential links, and finally to generalized links). The results are summarized in Table 1.

Function → Variable ↓	Level 2 (with some Level 3)	Level 3	Level 3 (with some Level 4)	Level 4
Level 1	1	0	0	0
Level 2	0	2	0	0
Level 3	1	2	2	0
Level 4	0	2	9	6

Table 1: This table indicates the number of preservice teachers classified based on their understanding of Functions (at Levels 2-4) and Variables (at Levels 1-4).

While this was a small-scale study, it is striking that the upper triangular portion of the data matrix is empty. After some time to allow these data to incubate in the researcher's mind, a key insight or illumination was reached. Namely, that a sophisticated understanding of the concept of a variable is a necessary but not sufficient condition for a sophisticated understanding of a function.

During graduate school, the third author, in working on his doctoral thesis, needed to make a key calculation of a map on the cohomology of the dihedral group of the square acting on certain vector spaces over the field of two elements. After nearly a year of being stymied, a discussion one day with his advisor suggested a possibility of relating that map to a certain map in group cohomology called induction. That night, while contemplating how induction works, it suddenly occurred to him (illumination) that the map in question was almost related to the composite of the induction through a certain symmetric group and then followed by the restriction back to the putative dihedral group. The difference between the two maps was the simple identity map. This allowed the implementation of a tool in group cohomology called the double coset formula. After a weekend of intense calculations, a key relation was established in the homotopy groups of simplicial abelian Hopf algebras over the field of two elements. Six months later, the final set of relations were produced establishing the keystone of his doctoral dissertation, which was subsequently published [16].

In a relatively recent example from ongoing research in mathematics, Andrew Wiles worked for nearly a decade to produce a proof of Fermat's Last Theorem. After a lengthy incubation stage, he encountered a critical conjecture by Gerhard Frey and later proven by Kenneth Ribet that a counterexample to Fermat's Last Theorem provides the data for forming an elliptic curve that is not modular. Inspired by this insight, Wiles began a tour-de-force effort to prove the Shimura-Taniyama Conjecture which states that all elliptic curves are modular. After several years, Wiles completed the proof and submitted a completed manuscript to the Inventiones Mathematica. Unfortunately, a fatal gap was found and he spent another year attempting to close the gap. At the brink of admitting defeat, Wiles attempted to isolate the source in his proof that led to the critical gap. In a stunning "aha" moment, Wiles suddenly saw (illumination) that the method he was using to establish a key formula could be replaced by a different method he abandoned years prior. This enabled the gap to be closed, the proof to be completed, and subsequently published. Nova Online later produced *The Proof*, a video that documented this process from *preparation* to *incubation* to *illumination* and finally to *verification* [12].

4 Conclusion

In each of the vignettes from elementary school through university level, teachers structure a learning community that emphases collaboration, assists peers through productive struggle, and values each learner as a bearer of God's image [6]. Students possess the power of illumination and insight only by participation in the Word who is the "True Light that illuminates everyone that comes into the cosmos" (John 1:9). Vignettes from graduate school and ongoing mathematical research also acknowledge the role of dissertation advisors, research colleagues, and the broader mathematical community. Each "aha" moment for students and eventually for mathematicians can be a celebration of growth in understanding using their God-given abilities to reason ever more abstractly while also applying their newfound knowledge to discovering connections within God's created world.

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Self-Graphing Equations


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Abstract

Can you find an xy-equation that, when graphed, writes itself on the plane? This idea became internet-famous when a Wikipedia article on Tupper's self-referential formula went viral in 2012. Under scrutiny, the question has two flaws: it is meaningless (it depends on typography) and it is trivial (for reasons we will explain). We fix these flaws by formalizing the problem, and we give a very general solution using techniques from computability theory.

1 Introduction

Suppose your friend sends you an xy-equation and you start graphing it. After graphing for a few minutes, you notice that what you've graphed so far looks like the letter x. You continue graphing, and you notice that you've just plotted the letter y on your graphing paper. After some more work, you notice that you've just added the symbol $+$ in between the x and the y. You continue this way for many hours until the equation is completely graphed. Then you step back and realize that what you've written on your graphing paper is the very equation your friend sent you!

A self-graphing equation is an equation such that, when you graph it, you get the equation itself back, written on your graphing paper. This idea received a lot of attention when a Wikipedia article on Tupper's self-referential formula went viral in 2012[1].

The problem of finding a self-graphing equation is meaningless because it depends on typography. It's also trivial, if no limit is placed on what functions one can use in the equation and how they are written. Indeed, fix a particular image $I \subseteq \mathbb{R}^2$ of the equation "$F(x, y) = 1$" on the plane (for example, the letter F could be the union of a vertical line segment and two horizontal line segments; the parentheses could be fragments of Bézier curves; and so on), and define $F : \mathbb{R} \to \{0, 1\}$ by

$$F(x,y) = \begin{cases} 1 & \text{if } (x,y) \in I, \\ 0 & \text{otherwise.} \end{cases}$$

By construction the graph of the equation $F(x, y) = 1$ is I, an image of the equation $F(x, y) = 1$ on the plane. So $F(x, y) = 1$ is trivially a self-graphing equation. Clearly, in order to make the problem nontrivial, it is necessary to specify which functions are allowed, and how they are written!

We will address the two problems of meaninglessness and triviality by formalizing the problem. Then, rather than focusing on any particular typography or any particular choice of what functions

[1] Tupper's so-called self-referential formula is not actually self-referential at all (nor did he himself call it self-referential [6]). Rather, it's a formula whose graph contains every possible 106×17-pixel bitmap. Tupper later posted an actually self-referential formula on his website [7], but it received less attention. Tupper's original formula has been generalized by Somu and Mishra [4]. Trávník has also published a self-graphing formula [5].

are allowed, we will instead give sufficient conditions thereon. Any typography, and any choice of functions, which satisfies these sufficient conditions, will be guaranteed to yield a self-graphing equation. To do this, we will invoke the so-called *recursion theorem* from computability theory (appropriately, the same theorem which was classically used to prove the existence of self-printing computer programs, also known as *Quines*).

2 Formalization

"What was a compelling proof in 1810 may well not be now; what is a fine closed form in 2010 may have been anathema a century ago" [2]

Definition 1. If $A \neq \emptyset$ is a finite alphabet, write A^* for the set of finite strings from A. By a *notion of equations* we mean a finite alphabet A together with a function $\mathrm{Gr} : A^* \to \mathscr{P}(\mathbb{R}^2)$ assigning to every string $\sigma \in A^*$ a subset $\mathrm{Gr}(\sigma)$ of $\mathbb{R}^2$ called the *graph* of σ.

For example, if A contains symbols x, y, $+$, ^{2}, $=$ and 1, and if $\sigma \in A^*$ is the string "$x^2 + y^2 = 1$", then $\mathrm{Gr}(\sigma)$ might be (but we do not require it to be!) the unit circle centered at the origin. Or, if A contains symbols r, θ, cos, $=$ and 1, and if σ is the string "$r = 1 + \cos\theta$", then $\mathrm{Gr}(\sigma)$ might be the graph of a cardioid. Or if σ is the string "$+ =$" (or if σ is the blank string), then $\mathrm{Gr}(\sigma)$ might be an error message, "Error: Invalid equation", written on the plane (as, say, a union of points, line segments, and Bézier curve fragments).

Definition 2. By a *glyphed notion of equations* we mean a triple $(A, \mathrm{Gr}, \mathrm{Gl})$ where (A, Gr) is a notion of equations and $\mathrm{Gl} : A \to \mathscr{P}(\mathbb{R}^2)$ is a function assigning to each $x \in A$ a set $\mathrm{Gl}(x) \subseteq \mathbb{R}^2$ called the *glyph* of x.

If A contains the symbol 0, then $\mathrm{Gl}(0)$ might be, for example, the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, \frac{1}{2})$ (so as to nicely fit in the 1×1 unit square $[0,1]^2$, lending itself to a monospace font where each character is 1 unit wide). But it does not have to be. If A contains the symbol X, then $\mathrm{Gl}(X)$ might be, for example, the union of the line segment from $(0,0)$ to $(1,1)$ and the line segment from $(0,1)$ to $(1,0)$ (again nicely lending itself to a monospace font where each character is 1 unit wide). But it does not have to be.

Definition 3. (Extending glyphs to strings)

1. For all $S \subseteq \mathbb{R}^2$ and all $r \in \mathbb{R}$, let $S^{\to r} = \{(x + r, y) : (x, y) \in S\}$, the result of translating S to the right by r units.
2. Whenever $(A, \mathrm{Gr}, \mathrm{Gl})$ is a glyphed notion of equations, we will extend Gl to a function on A^*, also written Gl (this will cause no confusion), as follows. Let $\sigma \in A^*$.
   - If σ is the empty string, let $\mathrm{Gl}(\sigma) = \emptyset$.
   - If σ is the string of length 1, whose first (and only) character is $x \in A$, let $\mathrm{Gl}(\sigma) = \mathrm{Gl}(x)$.
   - Otherwise, σ is the string $x_0 \ldots x_k$ where each $x_i \in A$. Let

$$\mathrm{Gl}(\sigma) = \mathrm{Gl}(x_0)^{\to 0} \cup \cdots \cup \mathrm{Gl}(x_k)^{\to k}.$$

Thus, $\mathrm{Gl}(\sigma)$ is the result of writing σ on the plane, from left to right, translating the glyph of each ith character to the right by i units. The resulting union is particularly easy to visualize if we assume that for every $x \in A$, $\mathrm{Gl}(x) \subseteq [0,1]^2$. In that case, the glyphs of A comprise a monospace font where every character has width 1, and the glyph of a string in A^* is the result of writing the glyphs of the individual characters from left to right in the usual way. This assumption will make the results in this paper more intuitive, but, interestingly, the whole paper will work just fine without this assumption.

Definition 4. (Self-graphing equations) Let $\mathcal{A} = (A, \mathrm{Gr}, \mathrm{Gl})$ be a glyphed notion of equations. By a *self-graphing equation in* $\mathcal{A}$ we mean a string $\sigma \in A^*$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gl}(\sigma)$.

3 Computability theory preliminaries

Definition 5. 1. For any sets X and Y, we write $f : \subseteq X \to Y$ to indicate that f is a function whose codomain is Y and whose domain is some subset of X.

2. For all $n \in \mathbb{N}$, let $\varphi_n : \subseteq \mathbb{N} \to \mathbb{N}$ be the nth computable function (assuming some fixed enumeration, possibly with repetition, of the computable functions).

3. A function $f : \subseteq \mathbb{N} \to \mathbb{N}$ is *total computable* if $\mathrm{dom}(f)$ (the domain of f) is all of $\mathbb{N}$.

We state the following celebrated result from computability theory without proof.

Theorem 6. (The Recursion Theorem) For every total computable $f : \mathbb{N} \to \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{f(n)}$.

4 Self-constraint: a sufficient condition for the existence of self-graphing equations

In this section we fix a glyphed notion of equations $\mathcal{A} = (A, \mathrm{Gr}, \mathrm{Gl})$ (where A is a finite nonempty alphabet).

Definition 7. By a *Gödel numbering of* A^* we mean a bijection[2] $\ulcorner \bullet \urcorner : A^* \to \mathbb{N}$ such that there is some algorithm for computing $\ulcorner \sigma \urcorner$ (for $\sigma \in A^*$) as a function of σ. We refer to $\ulcorner \sigma \urcorner$ as the *Gödel number* of σ (we think of $\ulcorner \sigma \urcorner$ as a numerical encoding of σ).

Definition 8. The glyphed notion of equations $\mathcal{A}$ is *self-constrained* if there exists a Gödel numbering $\ulcorner \bullet \urcorner$ of A^* and a total computable $f : \mathbb{N} \to \mathbb{N}$ such that:

- For all $n \in \mathbb{N}$, if $\varphi_n(0) = \ulcorner \tau \urcorner$ for some $\tau \in A^*$, then $f(n) = \ulcorner \sigma \urcorner$ for some $\sigma \in A^*$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gl}(\tau)$.

If f is as in Definition 8, then f should intuitively be thought of as being computed by an algorithm which takes an input $n \in \mathbb{N}$ and outputs an equation whose graph is the output of $\varphi_n(0)$ (if any), written on the plane. The strings in question are encoded by Gödel numbers to standardize the functions in question and allow the usage of standard computability theory, but intuitively one should think of f and φ_n as outputting strings from A^*. If $0 \notin \mathrm{dom}(\varphi_n)$ then it does not matter what $f(n)$ is, only that $f(n)$ be defined.

[2]One could change this definition to require only that $\ulcorner \bullet \urcorner$ be an injection instead of a bijection, which would be more typical of Gödel numberings. We chose to require the Gödel numbering function to be bijective in order to avoid technical complications.

Remark 2. It is not required, in the algorithm which computes $f(n)$, for $\varphi_n(0)$ to actually be computed as a preliminary step. It is not even required that the algorithm computing $f(n)$ determine whether or not $\varphi_n(0)$ exists (and indeed, this would be impossible, as it would require solving the Halting Problem). The work of computing $\varphi_n(0)$, or even of determining whether $\varphi_n(0)$ exists, can be delegated to whoever has to *graph* the output of $f(n)$.

We can illustrate Remark 2 with the following analogy. Say that $k \in \mathbb{N}$ is an *FLT-counterexample* (here FLT stands for "Fermat's Last Theorem") if $k > 2$ and there exist positive integers a, b, c such that $a^k + b^k = c^k$. For every $x \in \mathbb{R}$, let $\psi(x)$ be the number of FLT-counterexamples $\leq x$. A teacher does not need to know Fermat's Last Theorem in order to assign a student the task of graphing the equation $y = \psi(x)$. Without knowing Fermat's Last Theorem is true, a teacher can even, with some tedious mechanical effort, rewrite $y = \psi(x)$ in "closed form" (at least if the closed form is allowed to include infinite sums—see [1]). Knowledge of Fermat's Last Theorem is required in order to *graph* the equation, not to *state* it.

We will now show that self-constraint is a sufficient condition for existence of a self-graphing equation. At first glance, self-constraint might seem like such a strong requirement as to leave one in doubt whether any reasonable notions of equations actually satisfy it. We will give an example in Section 5 of a notion of equations which is self-constrained and therefore has a self-graphing equation, and the example should help the reader to better understand how self-constraint can be satisfied. Basically, the key is that infinite products or infinite sums can be used to encode quantifiers $\exists$ and $\forall$.

Theorem 9. If $\mathcal{A}$ is self-constrained then there exists a self-graphing equation in $\mathcal{A}$.

Proof. Let $\ulcorner\bullet\urcorner$ and $f : \mathbb{N} \to \mathbb{N}$ be as in Definition 8. Subclaim: There is a total computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\varphi_{g(n)}(0) = f(n)$.

This Subclaim is actually a special case of a theorem from computability theory called the "*Smn* theorem", but we will sketch a direct proof here. Let $g : \mathbb{N} \to \mathbb{N}$ be the function computed by the following algorithm:

1. Take input $n \in \mathbb{N}$.
2. Let $X = f(n)$.
3. Let P be the following algorithm:
   (a) Take input $m \in \mathbb{N}$.
   (b) Output X (ignoring the value of m).
4. Output an encoding of P (a number k such that φ_k is the function computed by P).

For any $n \in \mathbb{N}$, by construction $\varphi_{g(n)}$ is the function computed by the above algorithm P (for the given n). Thus $\varphi_{g(n)}(0)$ is computed by ignoring the input $m = 0$ and outputting $X = f(n)$. Thus $\varphi_{g(n)}(0) = f(n)$. Since we have provided an algorithm for g, g is computable. Clearly $\mathrm{dom}(g) = \mathbb{N}$, so g is total computable. This proves the Subclaim.

Let g be as in the Subclaim. By the Recursion Theorem (Theorem 6) there is some $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{g(n)}$. In particular,

$$\varphi_n(0) = \varphi_{g(n)}(0) = f(n) \text{ is defined. } (*)$$

Let $\sigma, \tau \in A^*$ be such that $f(n) = \ulcorner\sigma\urcorner$ and $\varphi_n(0) = \ulcorner\tau\urcorner$. We claim σ is a self-graphing equation in $\mathcal{A}$. To see this, compute:

$$\begin{aligned} \mathrm{Gr}(\sigma) &= \mathrm{Gl}(\tau) && \text{(Definition 8)} \\ &= \mathrm{Gl}(\sigma), && \text{(By } *, \sigma = \tau) \end{aligned}$$

as desired. □

5 A Concrete Context for a Self-Graphing Equation

To conclude, we will give an example of a particular glyphed notion of equations $\mathcal{A} = (A, \mathrm{Gr}, \mathrm{Gl})$ not too unlike how we write and graph equations in practice. We will argue that this particular $\mathcal{A}$ is self-constrained. Thus, Theorem 9 guarantees the existence of a self-graphing equation in $\mathcal{A}$.

For an alphabet, let

$$\begin{aligned} A = &\{a, b, c, \ldots, z\} && \text{(Letters)} \\ &\cup \{0, 1, 2, \ldots, 9\} && \text{(Digits)} \\ &\cup \{(\} \cup \{)\} && \text{(Left and right parentheses)} \\ &\cup \{+, \cdot, -, /, \hat{}, =\} && \text{(Plus, times, minus, division, exponentiation, equality)} \\ &\cup \{\Pi, \_, \infty\} && \text{(Infinite product machinery)} \end{aligned}$$

(for concreteness, A can be taken to be a subset of $\mathbb{N}$ of cardinality $26 + 10 + 2 + 6 + 3 = 47$). The reader should think of ^ as an exponentiation operator, as in the equation 2^3 = 8 (read: "2 to the power 3 equals 8"). The character Π should be thought of as an infinite product symbol, to be used (in combination with ^, _, =, ∞, and parentheses) as in the equation: $\Pi\_(n = 0)\hat{}\infty(1\hat{}n) = 1$ (read: "The product, as n goes from 0 to ∞, of 1^n, equals 1").

Choose glyphs $\mathrm{Gl} : A \to \mathcal{P}(\mathbb{R}^2)$ for writing A such that each such glyph is written inside the square $[0, 1] \times [0, 1]$ using pixels of dimension $\frac{1}{100} \times \frac{1}{100}$, each such pixel being a translation, by an integer multiple of $\frac{1}{100}$ horizontally and an integer multiple of $\frac{1}{100}$ vertically, of the square $[0, \frac{1}{100}] \times [0, \frac{1}{100}]$. For example, Gl(+), the glyph of the + sign, might be $([\frac{50}{100}, \frac{51}{100}] \times [0, 1]) \cup ([0, 1] \times [\frac{50}{100}, \frac{51}{100}])$ (the first argument to $\cup$ being a rectangle of height 1 and width 1/100 and the second argument to $\cup$ being a rectangle of height 1/100 and width 1), which can clearly be formed by such pixels.

Define $\mathrm{Gr} : A^* \to \mathscr{P}(\mathbb{R}^2)$ so that for every $\sigma \in A^*$, if σ is a valid equation, then $\mathrm{Gr}(\sigma)$ is the graph of σ. If σ is not a valid equation, then let Gr be some arbitrary nonempty subset of $\mathbb{R}^2$, for example, an error message written on the plane (we only require it to be nonempty so as not to inadvertently make the empty string a trivial self-graphing equation). For example, if σ is the string "$x\hat{}2 + y\hat{}2 = 1$", then $\mathrm{Gr}(\sigma)$ is the unit circle; if σ is the string "$x\hat{}2 = -1$" then $\mathrm{Gr}(\sigma)$ is the empty set.

In this way, we obtain a glyphed notion of equations $\mathcal{A} = (A, \mathrm{Gr}, \mathrm{Gl})$. We will argue that $\mathcal{A}$ is self-constrained and thus (by Theorem 9) admits a self-graphing equation. In other words, we will argue (Definition 8) that there is a Gödel numbering $\ulcorner\bullet\urcorner$ of A^* and a total computable $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $\varphi_n(0) = \ulcorner\tau\urcorner$ then $f(n) = \ulcorner\sigma\urcorner$ for some $\sigma \in A^*$ such that $\mathrm{Gr}(\sigma) = \mathrm{Gl}(\tau)$.

Let $\ulcorner\bullet\urcorner : A^* \to \mathbb{N}$ assign numbers bijectively to strings from A^* in some way that could be written out as an algorithm. There are many ways to do this and it does not matter which way it is done.

As one example, we could linearly order A and then enumerate A^* by listing all the length-0 strings in A^* (in alphabetical order), followed by all the length-1 strings in A^* (in alphabetical order), followed by all the length-2 strings in A^* (in alphabetical order) and so on, and let each $\ulcorner \sigma \urcorner$ be the position in which σ occurs in the resulting list.

We want $f(n)$ to output $\ulcorner \sigma \urcorner$ for some $\sigma \in A^*$ such that the graph of σ is $\mathrm{Gl}(\tau)$, where $\tau \in A^*$ is the string whose code is output by $\varphi_n(0)$ (if $0 \in \mathrm{dom}(\varphi_n)$). For such τ, what does it mean for a pair $(x, y) \in \mathbb{R}^2$ to be in $\mathrm{Gl}(\tau)$? It means that

$$\exists\ a, b, c, d, e \in \mathbb{N} \text{ s.t. } P(n, a, b, c, d, e), \tag{*}$$

where $P(n, a, b, c, d, e)$ is the statement:

> The nth Turing machine (i.e., the Turing machine which computes φ_n), when run with input 0, halts after exactly a steps, with output b, and when b is interpreted as a string τ (using $\ulcorner \bullet \urcorner$), τ has length at least $c+1$—let the cth symbol of τ (counting from 0) be called τ_c—and the $\frac{1}{100} \times \frac{1}{100}$ pixel with bottom-left coordinates $(d/100, e/100)$ is an element of $\mathrm{Gl}(\tau_c)$, and $(x - c, y)$ (the result of translating (x, y) to the left by c units) is in said pixel (so that (x, y) is in the translation of said pixel by c units to the right, which is said pixel's representation in $\mathrm{Gl}(\tau)$ by Definition 3).

Let's examine the subclauses of $(*)$.

- The subclause "The nth Turing machine, when run on input 0, halts after exactly a steps, with output b", can be expanded out into a complicated statement in the language of arithmetic ("There exists k such that k encodes a sequence $C_0, C_1, \ldots, C_a$ of Turing machine snapshots such that...").
- The subclause "the $\frac{1}{100} \times \frac{1}{100}$ pixel with bottom-left coordinates $(d/100, e/100)$ is an element of $\mathrm{Gl}(\tau_c)$", can be written as a finite disjunction of quantifier-free statements about individual pixels, namely, at most $100 \cdot 100 \cdot |A|$ such disjuncts: one per $\frac{1}{100} \times \frac{1}{100}$ pixel in $[0, 1]^2$ per symbol in A. For example, if the glyph of symbol "+" includes pixel $[50/100, 51/100] \times [0, 1/100]$, then this pixel-symbol pair contributes the quantifier-free disjunct: $(\tau_c = \text{"+"}) \wedge (d = 50) \wedge (e = 0)$.
- The subclause "$(x - c, y)$ is in said pixel" can be rephrased as "$d/100 \le x - c \le (d+1)/100$ and $e/100 \le y \le (e + 1)/100$".

We claim that all subclauses of $(*)$ can be written as equations of the form $E = 0$ using only symbols from A; to see this, we reason inductively:

- Atomic subclauses like "$d = 50$" can be written as $d - 50 = 0$.
- Atomic subclauses like "$e/100 \le y$" can be rewritten as "$y - e/100 - |y - e/100| = 0$", and the absolute values can be replaced with symbols from A by using the fact that $|x| = (x^2)^{1/2}$.
- (Disjunction) If two subclauses can be written in the form $E_1 = 0$ and $E_2 = 0$ using only symbols from A, then so can their disjunction, because "$E_1 = 0$ or $E_2 = 0$" is equivalent to "$(E_1) \cdot (E_2) = 0$".

- (Negation) If a subclause can be written in the form $E = 0$ using only symbols from A, then so can its negation, because "not($E = 0$)" is equivalent to $0^{E^2} = 0$ (since $0^0 = 1$ [3] but $0^x = 0$ for all positive x).

- (Existential Quantifiers) If a subclause $E = 0$ can be written using only symbols from A (where E may involve a variable v), then so can the clause $\exists v(E = 0)$ for any variable v, because $\exists v(E = 0)$ is equivalent to $\Pi_{v=0}^{\infty}(1 - 0^{E^2}) = 0$.

- (Conjunction, Universal Quantifiers) Closure under conjunction and universal quantification follow because "$E_1 = 0$ and $E_2 = 0$" is equivalent to "not(not($E_1 = 0$) or not($E_2 = 0$))" and "$\forall v(E = 0)$" is equivalent to "not($\exists v$ not($E = 0$))".

Thus, $(*)$ itself can be written as an equation $E = 0$ using only symbols from A. Fix such an E. For every $n \in \mathbb{N}$, let $\overline{n}$ be the string of n's decimal digits (for example if $n = 311$ then $\overline{n}$ is the length-3 string "311"). For every $n \in \mathbb{N}$, let $E(\overline{n}) = 0$ be the equation obtained by replacing all unquantified occurrences of n in $E = 0$ by $\overline{n}$. Define $f : \mathbb{N} \to \mathbb{N}$ so that for all $n \in \mathbb{N}$, $f(n) = \ulcorner E(\overline{n}) = 0 \urcorner$.

By construction, $f(n)$ outputs (the code of) the equation $E(\overline{n}) = 0$ whose graph is the set of all points (x, y) satisfying $(*)$, i.e., the set of all points in $\mathrm{Gl}(\tau)$ where $\ulcorner \tau \urcorner = \varphi_n(0)$ if such a τ exists.

Thus, f witnesses that $\mathcal{A}$ is self-constrained. By Theorem 9, there is a self-graphing equation in $\mathcal{A}$.

In some sense, the crucial key in this example is that the infinite product allows for the expression of the unbounded logical quantifier $\exists$. Together with the propositional logical connectives (AND, OR, NOT), unbounded quantification enables expression of anything that can be expressed in first-order logic.

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Divine Odds:
Examining the Probability of Christ Fulfilling Prophecies

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Abstract

In 1952, Peter W. Stoner published his seminal work *Science Speaks*, which discusses, among other things, Bible prophecies in relation to probability estimates and calculations. In this tribute to the original work, we consider the prophecies in Isaiah 53, often attributed as the gospel of the Old Testament. The Great Isaiah Scroll, discovered in Qumran Cave 1 in 1946, makes these prophecies especially important to the Christian faith as the scroll has been dated over one hundred years prior to Christ's birth. The probability of eight of the prophecies in Isaiah 53 is calculated, and a discussion follows centered around the fulfillment of the prophecies in the life and ministry of Jesus Christ.

1 Introduction

The field of apologetics can be traced back to the early days of Christianity. The term "apologetics" comes from the Greek word apologia, which means a formal defense or justification. Apologetics is the task of presenting a well-reasoned, intellectual defense of the truth claims of the Christian faith [30]. The idea of apologetics is rooted in 1 Peter 3:15–16, which says, "[B]ut in your hearts honor Christ the Lord as holy, always being prepared to make a defense to anyone who asks you for a reason for the hope that is in you; yet do it with gentleness and respect, having a good conscience, so that, when you are slandered, those who revile your good behavior in Christ may be put to shame" [4]. In the early church, persecuted Christians defended the faith against paganism, polytheism, and political threats, the greatest example being the apostle Paul throughout the book of Acts. These early apologists laid the foundation for the field, which has continued to evolve over the centuries, addressing new challenges and criticisms as they arise. New techniques and strategies have also emerged as these challenges have arisen, leading to a large and diverse field of differing apologetic approaches.

It is common for one to integrate their field of expertise into their apologetic arguments. St. Augustine of Hippo combined philosophy and theology [2], for example, while a modern-day example is C.S. Lewis, who combined philology, philosophy, and theology in his writings such as *The Chronicles of Narnia* and his World War II BBC broadcasts [66]. While it was common that many early theologians were also well trained in mathematics, this is not as common today. That does not mean, however, that there are no modern-day mathematicians who utilize their faith for the purpose of apologetics. One modern-day apologist from the twentieth century is Peter Stoner. Stoner, chairman of the Department of Mathematics and Astronomy at Pasadena City College until 1953, is the coauthor of *Science Speaks*, an evaluation of certain Christian evidences [50]. The book, among other things, examines biblical Old Testament prophecies in relation to probability estimates and calculations.

In order to explore the relationship, Stoner utilized a unique class he offered at Pasadena City College titled Christian Evidence, which was sponsored by the Inter-Varsity Christian Fellowship. For part of this course, students took eight Old Testament prophecies from various books and attempted to answer the question, "One man in how many men has fulfilled this prophecy?" The class discussed each prophecy at length and agreed on a reasonable and conservative estimate, with an emphasis on the conservative nature of the estimate. In this context, a conservative estimate means erring toward values that make the final probability larger rather than smaller. Stoner taught this course twelve times, and the final estimates resulted from the work of over 600 students. The prophecies and the estimates can be found in Table 1.

Scripture	**Prophecy**	**Probability**
Micah 5:2	Born in Bethlehem	1 in 280,000
Malachi 3:1	Had a forerunner to prepare the way	1 in 1,000
Zechariah 9:9	Entered Jerusalem as a king riding on a colt	1 in 10,000
Zechariah 13:6	Betrayed by a friend then wounded in hands	1 in 1,000
Zechariah 11:12	Betrayed for 30 pieces of silver	1 in 10,000
Zechariah 11:13	30 pieces – returned, thrown, used in purchase	1 in 100,000
Isaiah 53:7	Did not defend himself at trial	1 in 10,000
Psalms 22:16	Was crucified	1 in 10,000

Table 1: Scripture, prophecies, and estimated probability from *Science Speaks*.

Stoner concluded by utilizing the independence rule in probability to multiply all of the probabilities together and then divided the result by an estimate of the number of people to have ever lived (as of 1952), resulting in a final probability of 1 in 10^{17}. The visual he provided was to cover the state of Texas two feet deep in silver dollars, where one of the dollars was marked on the bottom. Then, blindfold a man and send him out to select a single silver dollar. This is the likelihood that any one person could fulfill just these eight prophecies, the probability only decreasing if additional prophecies are added.

While *Science Speaks* was well received at the time of its publication, in recent years it has been met with mixed criticism—to the point that Don W. Stoner, a grandson of Peter Stoner, has said, "At this point in time, *Science Speaks* is more likely to be weighed for its historical significance than its usefulness as an evangelical tool" [51]. While the book is still highly regarded for its innovative and pioneering approach, much of the criticisms from theologians are with regard to the interpretation of the prophecies used. In this paper we will take a similar approach to Stoner by combining Old Testament prophecies with probability, but narrow the focus to prophecies that are less debated and are more easily quantifiable. That being said, all of the prophecies we selected are still extremely difficult to estimate and may be incorrect. We will discuss our methods of attempting to get the best, most "conservative" estimate we can in Section 4 and the limitations in Section 6.

Before we begin, a quick disclaimer. This paper involves statistical analysis of historical events, including topics such as poverty, capital punishment, and crucifixion. While these are discussed within a probabilistic framework, we acknowledge the weight of these subjects and their historical and ethical significance. Our intention is to approach these discussions with both analytical rigor and respect for their broader implications; however, the reader should be aware of the weight of the following argument. If there is one thing all humans can agree on, it is the brokenness of this world, and we do not want to treat this lightly.

Throughout this paper, we analyze biblical prophecy through a probabilistic framework. While some readers may hold the assumption that the Bible is the inerrant word of God, our approach does not require this presupposition. Instead, we evaluate prophecies and historical events using external corroborations where available, ensuring that the analysis remains accessible regardless of theological stance.

In Section 2, we provide background on the Great Isaiah Scroll and the importance of the prophecies in Isaiah 53, from which we will select the prophecies to analyze. Section 3 contains the Scripture of Isaiah 53. In Section 4, we discuss the methodology for calculating the probabilities of the specific prophecies selected. Section 5 contains the results, Section 6 presents a brief discussion, Section 7 contains an illustration, and Section 8 offers our conclusion and steps for future research.

2 Background

In 1947, seven ancient scrolls were discovered in Qumran Cave 1—the first discovery among nearly one thousand scrolls that would be uncovered over the next decade in various other caves in Qumran [14]. These scrolls are commonly referred to as the Dead Sea Scrolls, as they have all been located in the canyon of the Wadi Qumran, along the northwest coast of the Dead Sea. One of the original scrolls came to be known as the Great Isaiah Scroll (1QIsa^a), the largest and best-preserved scroll discovered in the region. The scroll is nearly complete, containing all sixty-six chapters across fifty-four columns [26]. In addition, the scroll has been dated to c. 125 B.C., making it one of the oldest discovered scrolls—some one thousand years older than the oldest manuscripts of the Hebrew Bible known to us before the scrolls' discovery (i.e., the Aleppo Codex, c. 920 A.D., and the Leningrad Codex, c.1008 A.D.)[36].

The Great Isaiah Scroll belongs to the Masoretic Text family, the Hebrew text of the Old Testament. The consensus among the first generation of scholars who analyzed this text was that 1QIsa^a preserved a popular version of the Masoretic Text. While it does contain differences, they are often minor. Emmanuel Tov, for example, indicates that 1QIsa^a is similar to other copies we have, with the exception of minor details such as a different approach to spelling [53]. Biblical scholar Gleason L. Archer wrote:

> Even though the two copies of Isaiah discovered in Qumran Cave 1 near the Dead Sea in 1947 were a thousand years earlier than the oldest dated manuscript previously known (A.D. 980), they proved to be word for word identical with our standard Hebrew Bible in more than 95 percent of the text. The 5 percent of variation consisted chiefly of obvious slips of the pen and variations in spelling [1].

The closeness of 1QIsa^a to the Masoretic Text, in addition to being dated at least 100 years prior to Christ's birth, is of special importance. By the time of the apostles, the prophecies contained in Isaiah were already well known and established. For example, Matthew 3:1–3 states [4]:

> In those days John the Baptist came preaching in the wilderness of Judea, "Repent, for the kingdom of heaven is at hand." For this is he who was spoken of by the prophet Isaiah when he said, "The voice of one crying in the wilderness: Prepare the way of the Lord; make his paths straight."

Other examples include Acts 7:49, Acts 3:13, and Matthew 4:15–16. As such, the prophecies of Isaiah concerning the Messiah of the Old Testament are considered accurate to their original writing, now over 2,500 years ago.

Now that we have established the text is reliable, we begin to narrow our focus on the specific prophecy contained within. In addition to other prophecy, Isaiah contains four Servant Songs, which are prophetic songs about the Messiah. The fourth and final Servant Song—which is the focus of this paper—reveals precise details of the servant's mission that could not have been known to anyone but God. Some biblical commentators consider it the most important chapter in the entire Old Testament, as it contains a prophecy of the Atonement of Jesus Christ. During a sermon, Charles Spurgeon said:

> Mr. Moody was once asked whether his creed was in print. In his own prompt way, he replied, "Yes, sir; you will find it in the fifty-third of Isaiah." A condensed Bible is in this chapter. You have the whole gospel here [49].

Martin Luther declared that every Christian "must memorize this passage" [31]. John Calvin titled his sermons on Isaiah 53 *The Gospel According to Isaiah* [6]. John MacArthur wrote an entire book on Isaiah 53 titled *The Gospel According to God* [32]. Franz Delitzsch wrote, "[It] is the most central, the deepest, and the loftiest thing that the Old Testament prophecy, outstripping itself, has ever achieved" [9]. Given the overwhelming support for Isaiah 53's importance, we too shall focus on this remarkable text.

3 Scripture

The scripture we are studying actually begins at the end of the fifty-second chapter of Isaiah. Due to chapters and verse numbers being added at a later date, the section we are going to confine ourselves to is Isaiah 52:13 - 53:12, which is the final three verses in chapter fifty-two and all of chapter fifty-three. We shall use the English Standard Version (ESV), a common Bible translation for biblical study and scholarship [10]. Here is the entire passage, structured to convey the poetic verse style characteristic of the book of Isaiah [4]:

13 Behold, my servant shall act wisely;
he shall be high and lifted up,
and shall be exalted.
14 As many were astonished at you—
his appearance was so marred, beyond human semblance,
and his form beyond that of the children of mankind—
15 so shall he sprinkle many nations.
Kings shall shut their mouths because of him,
for that which has not been told them they see,
and that which they have not heard they understand.

53 Who has believed what he has heard from us?
And to whom has the arm of the Lord been revealed?
2 For he grew up before him like a young plant,
and like a root out of dry ground;
he had no form or majesty that we should look at him,
and no beauty that we should desire him.
3 He was despised and rejected by men,
a man of sorrows and acquainted with grief;
and as one from whom men hide their faces
he was despised, and we esteemed him not.

4 Surely he has borne our griefs
and carried our sorrows;
yet we esteemed him stricken,
smitten by God, and afflicted.
5 But he was pierced for our transgressions;
he was crushed for our iniquities;
upon him was the chastisement that brought us peace,
and with his wounds we are healed.
6 All we like sheep have gone astray;
we have turned—every one—to his own way;
and the Lord has laid on him
the iniquity of us all.

7 He was oppressed, and he was afflicted,
yet he opened not his mouth;
like a lamb that is led to the slaughter,
and like a sheep that before its shearers is silent,
so he opened not his mouth.
8 By oppression and judgment he was taken away;
and as for his generation, who considered
that he was cut off out of the land of the living,
stricken for the transgression of my people?
9 And they made his grave with the wicked
and with a rich man in his death,
although he had done no violence,
and there was no deceit in his mouth.

10 Yet it was the will of the Lord to crush him;
he has put him to grief;
when his soul makes an offering for guilt,
he shall see his offspring; he shall prolong his days;
the will of the Lord shall prosper in his hand.
11 Out of the anguish of his soul he shall see and be satisfied;
by his knowledge shall the righteous one, my servant,
make many to be accounted righteous,
and he shall bear their iniquities.
12 Therefore I will divide him a portion with the many,
and he shall divide the spoil with the strong,
because he poured out his soul to death

and was numbered with the transgressors;
yet he bore the sin of many,
and makes intercession for the transgressors.

4 Probability Methodology

There are many prophecies contained in the few verses of Isaiah 53—prophecies concerning the person of Christ, the atonement of Christ, Christ's reward, and more. For the purpose of this study, we will focus on prophecies that concern the person of Christ, as these are quantifiable. In total, we will examine eight prophecies from this section. For each prophecy, we attempt to quantify the probability—or the odds—of any one person fulfilling it, using a data-driven approach. The specific prophecies and their associated probabilities are detailed in this section.

Following the terminology used by Stoner, we refer to "conservative estimates" when assigning probabilities. In this context, a conservative estimate refers to selecting probability values that are more likely to overestimate rather than underestimate the true value. This approach aims to ensure that the final computed probability is, if anything, higher than it should be—making the conclusion more cautious. As a result, the actual probability is likely even smaller than our estimate. This method is essential because estimating these probabilities is inherently difficult, and all estimates are open to debate. Using a conservative estimate helps minimize uncertainty by favoring an approach least likely to support the argument we are making—that a single person could satisfy all eight prophecies. As discussed, we will consistently use the most conservative estimates for the purposes of this argument. This idea will be explored further in Section 6, where several limiting factors are addressed.

4.1 Fill The Earth

Before discussing the specific prophecies and their related probabilities, it is helpful to first determine how many people have ever lived. This is a topic of fierce debate in both theological and secular circles. Beginning in 1995, the Population Reference Bureau has been estimating the total number of people who have ever lived on Earth. As of 2022, the estimate was 117 billion [27]. Regardless of one's view on creation, this estimate is widely regarded as accurate within the scientific community and will be utilized for the purposes of this paper. It is clear that humanity as a whole has taken to heart the Genesis 2:1 command: "be fruitful and multiply, and fill the earth."

The focus of Stoner's work was to determine, "What is the chance that any man might have lived from the day of these prophecies down to the present time and fulfilled all of the eight prophecies?" To answer this question, we need an estimate for the number of people who have lived since the writing of the prophecies in Isaiah, not just the total number of people ever to have lived. The dates for the book of Isaiah are debated among scholars; however, most agree that the book was completed around the fifth century B.C.E.[10]. The Population Reference Bureau estimates that in 8,000 B.C.E., 8,993,889,771 people had ever lived, and in 1 C.E., 55,019,222,125 people had ever lived[27]. There are no estimates available for the years in between, so we will use the estimate from 1 C.E., as it is much closer to the 5th century B.C.E. and very close to when the Great Isaiah Scroll is estimated to have been written. This means there were approximately 6.2×10^{10} people who had ever lived since roughly the time the prophecies were written.

4.2 Isaiah 52:13

Isaiah 52:13 says, "He shall be high and lifted up, and shall be exalted." This raises the question: how many people have been high and lifted up, and exalted?

Merriam-Webster defines exalted as: "(1) elevated in rank, power, or character; or (2) held in high estimation, glorified, or praised" [12]. The Hebrew word for exalted means to be high, to exalt [20]. The Hebrew phrase "high and lifted up," used only three times in the Old Testament—all in Isaiah—describes God's majesty and transcendence, specifically His exalted status [10]. The first usage, in Isaiah 6:1, details a vision Isaiah had where he saw into the heavens. He saw the King "high and lifted up," and heard the seraphim declare, Holy, holy, holy is the Lord of hosts; the whole earth is full of His glory!" This suggests someone being lifted up and exalted to the same level as the heavens [20]. As such, this would describe one of the most important people to ever have lived, beyond even a celebrity—someone remembered throughout all of time.

There have been multiple attempts to list the most important or exalted people to ever live, such as those in [18] and [43]. These lists tend to feature no more than 100 people, many containing fewer. Therefore, we propose that it is reasonable to assume that 100 people have been raised up and exalted to be remembered throughout time, out of everyone who has ever lived. Since this applies to all people, not just those born after the writing of Isaiah, we will use 117 billion in our calculation. Thus, we will estimate that 100 out of 117,000,000,000, or one person in 1.17×10^9, has been exalted.

4.3 Isaiah 52:14

Isaiah 52:14 says, "As many were astonished at you—his appearance was so marred, beyond human semblance, and his form beyond that of the children of mankind." This raises the question: how many people have been marred beyond human semblance?

In 2003, the United Nations began gathering data from countries regarding violent and sexual crimes. This included national figures on offenses and victims of selected crimes recorded by the police or other law enforcement agencies. These data are submitted by Member States through the United Nations Survey of Crime Trends and Operations of Criminal Justice Systems (UN-CTS) or other means. Included in this data are metrics for serious assault, which is defined as the intentional or reckless application of serious physical force inflicted upon a person, resulting in serious bodily injury. In 2022, a total of 85 countries reported the number of serious assaults. The average rate was 132 serious assaults per 100,000 people in the population [58].

There are several issues with using this metric as an estimate for the Isaiah 52:14 prophecy. First, these assaults may not necessarily result in victims being marred beyond human recognition. This is further complicated by the fact that if one is beaten to the point of being beyond human recognition, an argument can be made that the chances of survival beyond twelve hours of the beating are low, which would likely result in the assault being classified as a homicide rather than a serious assault [21]. Second, many countries that did not report their data may have a higher-than-average rate of serious assaults, which would make this estimate an understatement. Finally, this metric does not reflect all of history, where the frequency of serious assaults may have varied significantly. Using the 2022 data, the odds are 1 in 758. However, given the aforementioned difficulties, we have decided to use a rougher estimate of 1 in 1,000. Given the violent nature of human history, we believe this is a conservative estimate [62].

4.4 Isaiah 53:2

Isaiah 53:2 says, "For he grew up before him like a young plant, and like a root out of dry ground." Franz Delitzsch comments that this "depict[s] the lowly and unattractive character of the small, though vigorous, beginning" [9]. Here, "small" refers to the plant, symbolizing the future Messiah. This leads to the question: how many grew up in a poor family or without a family altogether?

In each society throughout history, the measure of who is considered poor changes based on current circumstances. For most of human history, all people were hunter-gatherers, and only with the emergence of farming did livelihood conditions change [3]. François Bourguignon and Christian Morrisson, leading economists in estimating world poverty levels over time, define the poverty line as consumption per capita of $2, expressed in 1985 Purchasing Power Parity [5]. As recently as two hundred years ago, roughly 80% of the world was considered to be living in poverty [45][54]. The World Bank estimates that it was not until the late 1960s that the global poverty rate dropped below 50% [64]. In the past century, the rate has steadily dropped to around 10% in 2019 [39].

For a broad approach to defining poverty that works across history, we consider a conservative approach and define anyone living under the median income as poor. This implies that 50% of society is considered poor, so there is a 1 in 2 chance that the person referenced in the prophecy was poor.

4.5 Isaiah 53:3

Isaiah 53:3 says, "He was despised and rejected by men." This raises the question: how many have been despised and rejected?

Psychologists define rejection as usually an explicit verbal or physical action that declares that the individual is not wanted as a member within a relationship or group" [63]. One study found that 67% of a representative U.S. sample admitted using the silent treatment (deliberately not speaking to a person in their presence) on a loved one, and 75% indicated that they had been a target of the silent treatment by a loved one"[13]. Another study found that "Ostracism in the workplace is a common experience. For example, in a study conducted by O'Reilly (2015), 70% of employees stated that they were exposed to ostracism in the past... In a study conducted with 2,000 managers/employees in the USA, 67% of the participants stated that they did not talk to someone else deliberately, and 75% stated that they were exposed to such behavior at least once (cited in Harvey et al., 2018). All these data support the idea that organizational ostracism is common"[22].

If we say that 67% of people have been despised and rejected, the odds would be 1 in 1.5. However, we propose using 50% as an estimate for the percentage of people who have been despised and rejected, as this simplifies the math with odds of 1 in 2. Given that these studies focus solely on the workplace and do not involve more intimate relationships, this estimate is almost certainly a conservative overestimate.

4.6 Isaiah 53:5

Isaiah 53:5 says, "He was pierced for our transgressions; he was crushed for our iniquities." Most scholars, dating back to Thomas Aquinas and Isidore of Seville, agree that this describes crucifixion [61][25]. Crucifixion is defined as the act of crucifying, which means putting to death by nailing or binding the wrists or hands and feet to a cross [8]. This clearly satisfies the description of piercing, but there are many ways to be pierced that do not involve being crushed. The main cause of death in crucifixion is asphyxiation, where the body is no longer able to support itself and the lungs are crushed under the weight of one's own body.

Normally, to breathe in, the diaphragm must move downward. This enlarges the chest cavity, and air automatically moves into the lungs (inhalation). To exhale, the diaphragm rises, which compresses the air in the lungs and forces the air out (exhalation). For a victim hanging on a cross, the weight of the body pulls downward on the diaphragm, causing the air to remain trapped in the lungs. To exhale, the victim must push up on their nailed feet. The difficulty in exhalation leads to slow suffocation. Carbon dioxide builds up in the blood, resulting in a high level of carbonic acid in the blood. The body responds instinctively, triggering the desire to breathe. At the same time, the heart beats faster to circulate the available oxygen. The decreased oxygen causes damage to the tissues, and the capillaries begin leaking watery fluid from the blood into the tissues. This results in a buildup of fluid around the heart and lungs. The collapsing lungs, failing heart, dehydration, and the inability to supply sufficient oxygen to the tissues ultimately suffocate the victim. The victim's own body crushes them to death [47].

While crucifixion is certainly not the only way to be both pierced and crushed, estimating the number of people who have experienced both is not feasible. Therefore, we will instead attempt to estimate the probability of someone being crucified. This raises the question: how many people have been crucified?

Crucifixion has been employed in different parts of the world and across various historical periods. However, it is most notably remembered as a brutal form of control and punishment in the Roman Empire approximately 2,000 years ago. Probably originating with the Assyrians and Babylonians, crucifixion was first used systematically by the Persians. In the 4th century BC, Alexander the Great adopted crucifixion and brought it to the Mediterranean shores, and during the Punic Wars, the Romans learned and then perfected the practice [19][44]. The practice was abolished in the Roman Empire by Constantine I after AD 320, but it has been used sporadically, most recently by the terrorist group ISIS in 2016[11].

Despite the fact that crucifixions are well-documented in various ancient sources, one of the best-known examples being Jesus of Nazareth in the Bible, there have only been two archaeological discoveries of crucifixion remains [55][23]. As such, it is difficult to determine how many people have been crucified throughout history. The scholar who discovered the first crucifixion remains in 1968 simply remarked, "From ancient literary sources, we know that tens of thousands of people were crucified in the Roman Empire. In Palestine alone, the figure ran into the thousands"[56]. We propose using 1 in 1,000,000 as the probability of any one person being crucified as a conservative estimate, which would suggest that roughly 117,000 people have been crucified in total.

4.7 Isaiah 53:7-8

Isaiah 53:7 says, "He was oppressed, and he was afflicted, yet he opened not his mouth; like a lamb that is led to the slaughter, and like a sheep that before its shearers is silent, so he opened not his mouth. By oppression and judgment he was taken away; and as for his generation, who considered that he was cut off out of the land of the living, stricken for the transgression of my people?" There are two different themes here that we will attempt to address as a whole. The first is that when oppressed and afflicted, a man would not open his mouth to complain. The second, building on the imagery of a lamb being led to slaughter, is that he would be cut off from the land of the living. Combining the two themes describes someone who remains silent throughout oppression, affliction, and death, not once opening their mouth to complain. In modern terms, this leads to the question: how many did not try to defend themselves during trial when the penalty was death?

We begin by estimating the number of executions per year. Worldwide, there were an average of 805 executions per year from 1985 through 2022 [16]. However, this data excludes China, as it does not publish its execution statistics. While China does not report its data, in August 1983, the central government began its first "strike hard" campaign against crime, resulting in an estimated 5,000 executions during the first three months [37]. One study estimates that the number of executions in China likely represents about 90% of the global total [65]. If we take the data from 1985 through 2022 and assume that 805 executions represent 10% of global executions, the total number of executions per year worldwide can be estimated at approximately 8,050. We'll assume around 8,000 executions per year. This leads to the question: of the 8,000 projected executions per year, how many people waived the appeals process?

The only reputable source that has reliably tracked this data over the last fifty years is the United States. Since the reinstatement of the death penalty in 1976, there have been 1,589 executions in the U.S., for an average of 33 executions per year [35]. This makes up only 0.4% (33/8000) of the total global estimated executions annually. According to the U.S. Death Penalty Information Center, as of May 2024, "since the reinstatement of the death penalty in 1976, at least 150 defendants have been volunteers — approximately ten percent of all executions" [34]. A "volunteer" is defined as an individual who waived at least part of their ordinary appeals process or who terminated proceedings that would have entitled them to additional legal process prior to their execution. The Death Penalty Information Center has compiled a list of all volunteers, including details of when along the process the individual volunteered. Of these, only 11 of the 152 volunteers pleaded guilty and volunteered from the beginning of the legal process. This suggests that only 0.7% (11/1589) of American executions involved individuals who did not attempt to defend themselves at any point during their trial.

Given that this data represents only one country (which makes up less than 1% of yearly executions) over the last fifty years, and that country's justice system is founded on individual rights [59], it is reasonable to assume that throughout history, the percentage of people who waived their defense was significantly smaller [48]. This period in the United States, for example, does not consider the time before the Civil Rights movement, let alone before the abolition of slavery. Globally, the Universal Declaration of Human Rights was not passed by the United Nations until 1948, and it is still not recognized by all countries [57]. Japan, for example, has recently come under scrutiny for not adhering to the presumption of innocence, similar to time periods in past centuries [52]. As such, using the limited evidence we have, we propose a conservative estimate of 1 in 10,000 for the odds of someone not defending themselves at trial when the penalty was death.

4.8 Isaiah 53:9a

Isaiah 53:9 says, "And they made his grave with the wicked and with a rich man in his death." This leads to the question: how many people were buried with, near, or in a rich man's tomb? We must first determine how many people are classified as rich.

Pierre Concialdi argues, "The top 1 percent has become, in this respect, a widely popular, if not totally accepted, definition of 'rich' people... [The] relative thresholds usually mimic the definition of standard monetary poverty lines by setting a threshold at a given distance from the median income: people are considered rich if their income is above x times the median income. For instance, Franzini et al. (2016) define affluent, rich, and super-rich as people whose incomes are, respectively, above 3 times, 5 times, or 10 times the median income" [7].

The definition of 'rich' varies by society depending on the amount of disposable income available for spending. For our purposes, we will define 'rich' as anyone in the top 1% to 10% of income earners. Now, the question is: how many men who were not rich were buried near, with, or in a rich person's tomb? According to the National Funeral Directors Association's (NFDA) 2023 Cremation and Burial Report, the U.S. cremation rate is expected to increase from 60.5% in 2023 to 81.4% by 2045 [60]. This means there are fewer and fewer graves for bodies to be placed in, as traditional burials are on the decline. While the number of traditional burials is decreasing, there are still many people who have been buried throughout history, a number that cannot be easily estimated.

Historically, there are many instances of people being buried near or with the rich, such as the pharaohs of Egypt, who were buried with servants for the afterlife [38]. As such, we will generously estimate that 1 in 1,000 individuals have been buried in or near a rich person's tomb.

4.9 Isaiah 53:9b

Isaiah 53:9 says, "Although he had done no violence, and there was no deceit in his mouth." While there are various interpretations of these verses, the Hebrew word for "although" refers back to the earlier part of the verse and passage. Sections 4.7 and 4.8 discuss the burial of someone after a silent death. Taking this interpretation into account, Isaiah 53:9b leads to the question: how many people have never committed violence or lied, yet were still put to death?

Throughout history, many innocent individuals have met their deaths. The first recorded genocide dates back to at least Carthage in 146 BC [29], and there has been an ongoing history of genocides, abuses of power, and numerous other instances where innocent people were put to death. To estimate this, we will focus on data that we do have: wrongful convictions.

Multiple studies have attempted to estimate the frequency of wrongful convictions, a topic that remains highly debated. A 2001 study in the state of New York found that 1% of convictions were false convictions [40]. A 2007 study involving 798 Ohio criminal justice professionals found that wrongful felony convictions in their jurisdictions occurred between 0.5% and 1% of the time [41]. Another study found that estimates vary from less than 1% up to 5%[15], while a study by the National Academy of Sciences found that death penalty convictions are incorrect 4.1% of the time[17]. The Innocence Project claims that the rate is around 1% [42].

All of these studies were conducted in the United States and only consider the last fifty years. No international organization tracks wrongful convictions or attempts to estimate them globally, making the U.S. studies our best available estimates. Given that the United States houses 22% of the current global prison population [42], estimates based on U.S. data may be reasonably extrapolated. Due to the conservative nature of the Innocence Project's estimate, we will use their figure of 1%, or 1 in 100, as our estimate, acknowledging the difficulty of obtaining a true estimate, especially across all of history.

5 Results

A summary of the calculated probabilities from Section 4 can be found in Table 2. We can now calculate the probability that any one person fulfills all eight of these prophecies. In *Science Speaks*, Stoner assumed the probabilities he estimated were independent. In probability theory, two events are considered independent if the occurrence of one event does not affect the probability of the occurrence of the other. However, the probabilities we selected do not satisfy this assumption. Given that the independence assumption has been violated, we will adjust our estimates to account for the potential interdependencies among the events, thereby recalculating the probability that one person fulfills all eight of these prophecies.

Scripture	**Prophecy**	**Probability**
Isaiah 52:13	High and lifted up, exalted	1 in 1.17×10^9
Isaiah 52:14	Marred beyond human semblance	1 in 1,000
Isaiah 53:2	From a poor background	1 in 2
Isaiah 53:3	Despised and rejected	1 in 2
Isaiah 53:5	Crucified	1 in 1,000,000
Isaiah 53:7	Did not defend themselves while at trial	1 in 10,000
Isaiah 53:9	Buried in or near a rich man's tomb	1 in 1,000
Isaiah 53:9	Innocent yet put to death	1 in 100

Table 2: Scripture, prophecies, and estimated probability from Isaiah 53.

Specifically, we will utilize conditional probability to estimate the likelihood that a single person fulfills all eight prophecies. Additionally, we will calculate the probability under the assumption of independence for comparison purposes. In the following sections, let A_1 represent the event that a person is "high and lifted up and exalted," A_2 represent the event that a person is "marred beyond human semblance," A_3 represent the event that a person is from a poor background, A_4 represent the event that a person is "despised and rejected," A_5 represent the event that a person is crucified, A_6 represent the event that a person does not defend themselves at trial, A_7 represent the event that a person is buried in a rich man's tomb, and A_8 represent the event that a person is innocent yet put to death.

5.1 Conditional Probability

When independence cannot be assumed, the following theorem may be used.

Theorem 3. *For events $A_1, \ldots, A_n$ whose intersection has not probability zero, the chain rule states:*

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{k=1}^{n} P(A_k | A_1 \cap \cdots \cap A_{k-1}).$$

As we have eight prophecies, by Theorem 1 our equation would be

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8) = {} & P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_1 \cap A_2) \\ & \times P(A_4 \mid A_1 \cap A_2 \cap A_3) \times P(A_5 \mid A_1 \cap A_2 \cap A_3 \cap A_4) \\ & \times P(A_6 \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \\ & \times P(A_7 \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6) \\ & \times P(A_8 \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7). \end{aligned}$$

We will assume that someone being high and lifted up and exalted and someone being marred beyond human semblance are independent, as such $P(A_2 \mid A_1) = P(A_2) = \frac{1}{1{,}000}$.

We will assume that being marred beyond human semblance increases the probability that someone is from a poor background [33]. We will assume this increases the probability from 1 in 2 to 1 in 1.5, meaning they are 33% more likely to be marred beyond human semblance. Thus, $P(A_3 \mid A_2 \cap A_1) = \frac{1}{1.5}$.

We will assume being high and lifted up and exalted increases someone's probability of being despised and rejected [46]. In addition, we will assume being from a poor background also increases the probability of being despised and rejected [24]. We will assume this increases the probability from 1 in 2 to 1 in 1.25, meaning they are 60% more likely to be despised and rejected. Thus, $P(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{1}{1.25}$.

We will assume that being marred beyond human semblance and being from a poor background increases someone's probability of being crucified [47]. We will assume this increases the probability from 1 in 1,000,000 to 1 in 100,000, a 1000% increase. Thus, $P(A_5 \mid A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{100{,}000}$.

We will assume that someone not defending themselves at trial has a higher probability if they have been marred beyond recognition [47]. We will assume this increases the probability from 1 in 10,000 to 1 in 1,000.

We will assume being high and lifted up and exalted increases the chance of one being buried in or near a rich man's tomb [38]. We will assume this increases the probability from 1 in 1,000 to 1 in 500. Thus, $P(A_7 \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6) = \frac{1}{500}$.

Finally, we will assume that someone not defending themselves while at trial increases the probability of them being innocent yet put to death. We will assume this increases the probability from 1 in 100 to 1 in 50. Thus, $P(A_8 \mid A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7) = \frac{1}{50}$.

These adjustments can be found in Table 3. The final probability would be 1 in 1.82×10^{25} that someone would fulfill all eight prophecies. If we divide this by the number of people who have ever lived since the writing of the prophecies from Section 4.1, this results in a probability of 1 in 2.94×10^{14}.

Scripture	**Prophecy**	**Probability**
Isaiah 52:13	High and lifted up, exalted	1 in 1.17×10^9
Isaiah 52:14	Marred beyond human semblance	1 in 1,000
Isaiah 53:2	From a poor background	1 in 1.5
Isaiah 53:3	Despised and rejected	1 in 1.25
Isaiah 53:5	Crucified	1 in 100,000
Isaiah 53:7	Did not defend themselves while at trial	1 in 1,000
Isaiah 53:9	Buried in or near a rich man's tomb	1 in 500
Isaiah 53:9	Innocent yet put to death	1 in 50

Table 3: Scripture, prophecies, and estimated probability from Isaiah 53, adjusted for dependence.

5.2 Independence

If we were to assume independence, which we discussed in 4 as being violated, we would simply multiply each probability together form Table 2 without any adjustments. As we have eight prophecies, our equation would be

$$P(A_1 \cap A_2 \cdots \cap A_7 \cap A_8) = P(A_1) \times P(A_2) \cdots \times P(A_7) \times P(A_8).$$

The resulting probability would be 1 in 2.14×10^{28}. If we divide this by the number of people to have ever lived, the resulting probability would be 1 in 3.88×10^{17}. This value is extremely close to Stoner's original estimate in *Science Speaks.*

6 Discussion

An argument can be made that the estimates we used are incorrect. The eight prophecies selected are certainly difficult to estimate, despite our best efforts to do so. As the purpose of this study was to produce a conservative estimate, as was the goal of *Science Speaks*, we acknowledge that many, if not all, of these values may be incorrect. In fact, they may be too conservative at times. If this is the case, then the probabilities would be even smaller.

An additional argument could be made that the adjustments made in Section 5 were not correct, especially since they were subjective. Once again, the goal was to be conservative in nature, and we firmly believe we made the most conservative estimate we could. Given the evidence we do have, we believe our adjustments are all reasonable estimates for the equation; however, further adjustments may always be made, which would change the final probability. Our estimate is 200% larger than if independence is assumed, resulting in a much larger (albeit still small) probability.

No apologetic tool, outside of the gospel itself, is perfect. Picking eight prophecies, all of which would have been well known to the Jewish audience around the time of Jesus of Nazareth, raises the argument that the New Testament authors and even Jesus himself could have intentionally fulfilled these prophecies. This argument, however, was something the first-century Jews showed to be invalid. Acts 5:34-40 says [4]:

> But a Pharisee in the council named Gamaliel, a teacher of the law held in honor by all the people, stood up and gave orders to put the men outside for a little while. And he said to them, 'Men of Israel, take care what you are about to do with these men. For before these days Theudas rose up, claiming to be somebody, and a number of men, about four hundred, joined him. He was killed, and all who followed him were dispersed and came to nothing. After him Judas the Galilean rose up in the days of the census and drew away some of the people after him. He too perished, and all who followed him were scattered. So in the present case I tell you, keep away from these men and let them alone, for if this plan or this undertaking is of man, it will fail; but if it is of God, you will not be able to overthrow them. You might even be found opposing God!' So they took his advice.

As it is safe to say the followers of Jesus have not scattered, the argument that Jesus purposely fulfilled prophecies falls apart.

7 Illustration

One of the most well-known illustrations of *Science Speaks* is the famed Silver Dollar illustration. The illustration is that, if you take 1×10^{17} silver dollars and cover the State of Texas, the resulting stacks would be roughly 2 feet deep. If you mark the bottom of a single one of them, and then blindfold someone and send them out to pick out a single silver dollar, the probability that the silver dollar selected is the marked one is 1 in 1×10^{17}.

While a useful image for years, the illustration is a little outdated. Don Stoner has recently said, "Silver dollars are now collector's items rather than what a person might carry in their pockets" [51]. As such, a similar illustration can be done with the well-known shape of an Oreo cookie. A classic Oreo cookie is, on average, 4.44 cm in diameter and 0.798 cm in thickness [28]. We shall use our conditional probability from 5.1. As our probability is smaller that Stoners, this would only cover slightly over 65% of the surface area of Texas. Instead, we shall consider the state of California, the third largest state. The surface area of California is 163,695 square miles. If we take 2.94×10^{14} Oreos, this would cover the state of California with a single layer of Oreos with enough left over to cover 7% of the state with a second Oreo. Given this image, if a single Oreo is a vanilla Oreo and you send someone who is blindfolded out into the state of California and have them select a single Oreo, the probability that it is the vanilla Oreo is 1 in 2.94×10^{14}.

8 Conclusion

This study was limited to only eight prophecies in one chapter of the book of Isaiah, some of which are hard to get precise probabilistic estimates for. Even if some may argue that we were too generous with our estimates, there are roughly 300 Messianic prophecies in the Old Testament that scholars believe Jesus Christ of Nazareth has already fulfilled. If we were to consider additional prophecies, the probability that any one person would fulfill all the prophecies would become even smaller. As such, to the extent that we know this blindfolded man cannot pick out the marked Oreo, we know that Jesus Christ is the Messiah. While this may not be a perfect apologetic argument, we still believe this work, in addition to Stoner's original work, is a valuable tool and argument for the case for Christ. Given the finding of the Dead Sea Scrolls, the reliability of the original text, and the significance of the fifty-third chapter of Isaiah, we believe the argument we present is a strong addition to the many other arguments for Christ. As Romans 1:20 says, "For his invisible attributes, namely, his eternal power and divine nature, have been clearly perceived, ever since the creation of the world, in the things that have been made."

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The Crooked Made Straight:
The Geometric Work of Alfonso of Valladolid

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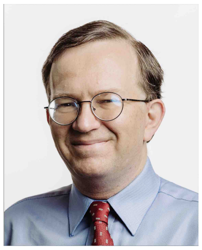

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Abstract

Because of its appearance in Handel's *Messiah*, the prophecy of Isaiah 40:4 is very familiar—that "every valley shall be exalted . . . the crooked [made] straight, and the rough places plain." The New Living Translation, however, renders this, "Straighten the curves," and this is exactly how the 14th-century Jewish convert to Catholicism Alfonso of Valladolid thought of it in his *Sefer Meyasher 'Aqov*, a Hebrew treatise on the nature of geometry.

This paper provides a non-technical introduction to Alfonso's context, Alfonso himself, his approach to geometry, and his connection of geometric ideas to Scripture and faith. I will lean heavily on the impressive critical work of Ruth Glasner, Avinoam Baraness, and others in translating and placing Alfonso's work in context.

1 Introduction

One of the positive recent trends in teaching mathematics, particularly its history, is being open about its cultural embedding. A notable effort is the TRIUMPHS project ([22]), which emphasizes using primary sources to understand concepts; as a banal example, it is difficult not to learn about ancient Chinese reliance on various grains when that is so prominent in examples of what we would now call row-reduction! A deeper example is the importance of spherical trigonometry in calculating the *qibla*, or direction to face Mecca for prayer, in Islamic mathematics (see [23]).

Cultural embedding does not mean giving up on the importance of logic, or on an abstraction of mathematical objects. Rather, it emphasizes the universality of mathematics, in that *real* people cared about the same things, and came to the same mathematical conclusions, despite their varied cultures. Further, it means that the role of religion in the lives of billions of humans does not have to be discounted in favor of mathematics living outside of that aspect of human life. (It would be hard to discount it, given the explicitly religious motivation of so many thinkers since the Pythagoreans, but many have tried.)

A fascinating example of this embedding is the mathematical work of the man variously known as Abner of Burgos (as a rabbi) or Alfonso of Valladolid (after his conversion to Catholicism), who lived roughly 1260–1347 in the medieval Spanish kingdom of Castile. Although he is primarily known for his role in the polemics between Jews and Christians in medieval Castile, he also cared very much about Euclidean geometry. His one extant mathematical work, the *Sefer Meyasher 'Aqov*, clearly arises from his cultural context, while also attempting to elucidate timeless truths.

Indeed, he is explicitly trying to go beyond known truths. As we will see, Alfonso directly connects his attempts to reform geometry, measure curves, and refute an actual infinite to the Hebrew Bible and his faith, while engaging philosophers and geometers from throughout the Greek and Islamic traditions. We will begin by introducing Alfonso's times in Section 2, and summarize his non-mathematical life and work in Section 3. Section 4 will go into the highlights of his mathematics, while in Section 5 we conclude with his theological connections.

It is important to point out that this organization is solely for the convenience of the reader and author, and is *not* one Alfonso would have agreed with! For him, the theology and mathematics were deeply interconnected. We hope that this is evident by the end of the paper, and that it will prove useful to anyone looking for a good example of that complex cultural embedding we really all inhabit.

2 Medieval Iberia

Even though the reader may know Valladolid and Burgos as medieval tourist destinations, medium-sized cities in modern Spain[1], for the purposes of Alfonso's life we should not think of them in this way. The Roman province of Hispania was long gone, as was the Germanic Visigothic kingdom which replaced it. Instead, for centuries the Iberian peninsula (modern-day Spain and Portugal) had been divided among a number of petty Muslim kingdoms and several consolidating Catholic kingdoms, inhabited by a broad mix of the children of Arabs, Berbers, Jews, Hispano-Romans, and Goths—a true melting pot.

This was firstly a legacy of the Muslim conquest of nearly the entire peninsula by a North African contingent led by Arabs. By about 750 only Alfonso I of Asturias in the far north remained to raid what in the meantime had become a unified state controlled by the Umayyad family, the former rulers of the entire Islamic world.

Much ink has been spilled on the wonders of this realm, known as al-Andalus[2]. And though it had the same restrictions for its Christian and Jewish residents as other Muslim regimes, and sometimes over the centuries it had rulers just as inimical to Jews as the later Catholic rulers of Spain would be, it was in general a place where multicultural academic learning flourished, from medicine to theology—in Arabic, the language of the ruling court.

Particularly relevant for Abner/Alfonso is the mathematical and philosophical content. As a snapshot, by around 1000 AD the works of Aristotle, along with many commentaries, notably the most recent ones from Ibn Sina (Avicenna in the West) were available in the capital of Cordoba—and so were Euclid's *Elements* and Ptolemy's *Almagest* (itself an Arabic title). Cordoba became even more important philosophically in the twelfth century with the births of the Aristotelian jurist and doctor[3] Ibn Rushd (Averroes) and the towering Jewish philosopher Moses Maimonides. While Ibn Rushd was mostly in the good graces of a new dynasty, Maimonides fled forced conversion to other parts of the Islamic world, where he continued to write in Arabic (using Hebrew letters).

Further, continued mathematical developments were pouring in. On the Arab side, al-Khwarizmi's algebra compendium, the number theoretic and optics work of al-Haytham (Alhazen), and also geometric work on things like the parallel postulate by people such as Thabit ibn-Qurra, were all

[1] In fact, Columbus died in Valladolid, and El Cid is buried in Burgos.
[2] Ironically, after the Germanic Vandal tribe who had tried to take over Iberia, and then been pushed out.
[3] Nearly all philosophers were also doctors at this time, it seems.

available. The Jewish community was also well invested; for example, Rabbi Ben Ezra wrote several works at this time on the then-new Indian numeration system. This tradition continued in what is now southern France after the expulsion of the Jews in 1148, with the famous Levi ben Gerson (Gersonides) and the ben Tibbon family, who even translated Euclid into Hebrew (see [18]).

Abner/Alfonso's family was not expelled, though, because of the second main historical factor in medieval Iberia—the Reconquista, or reconquest, of Spain by Christians. From the days of Alfonso I, the small Catholic kingdoms of the far north (such as Castile, Leon, and Navarre) first made some areas of Iberia not worth holding on to by the emirate, then later outright conquered them from the rulers of al-Andalus. By 1100 not just Burgos and Valladolid, but Madrid and, above all, the famous archbishop's seat, center of learning, and Visigoth capital of Toledo were all firmly in the hands of the kings of Castile. By the time Abner was born, nearly all of modern Spain and Portugal, including Cordoba, was part of some Christian kingdom, though the area of Gibraltar and Granada held out several centuries longer as Muslim client states.

Cultural change is usually slower than conquest, though, and the new Spanish/Portuguese rulers did not force Muslims (or Jews) to convert until the end of the 1400s. So in many cities there was continued interaction between the rich intellectual heritage of al-Andalus, the Jewish doctors and rabbis, and the new Latin-writing Catholic scholastics. Abner had access to a huge amount of Arab literature, his Jewish heritage and philosophers, and (for him, to a lesser extent) the Western Christian tradition as well.

3 From Abner to Alfonso

Abner was most likely born about 1260 in Burgos, which had a good-sized Jewish community. Abner, according to autobiographical comments in books such as "Teacher of Righteousness" (*Mostrador de Justicia*, only extant in his Castilian version), was a rabbi, led a yeshiva, and engaged in other typical tasks for a man of learning of the time (probably as a doctor, for instance). It is clear that Burgos was a thriving center (see [4]) and its scholars "should be seen as part of the foundation of the Jewish scholarship in Toledo," so Abner certainly would have had ample opportunity to absorb his tradition.

As we will see, this must have included a sizable amount of mathematics. However, it is in the realm of philosophy and the Torah that he wrote the majority of his works. At the time Abner seems to have been an orthodox Aristotelian in the tradition of the Jewish/Muslim philosophers mentioned above, with standard beliefs for a rabbi of the time and place.

Abner's own story of his conversion to Christianity starts with an experience in Avila (closer to Madrid), where many Jews had gone to see a potential Messiah to be announced on a specific date in 1295. Instead, crosses appeared on their white robes of prayer ([2]), and Abner was not the only one who wondered whether this was a sign from the true Messiah. After many years of uncertainty and several relevant dreams, he finally officially converted around 1320 (so, in late middle age), and from then on devoted most of his time to persuading his former coreligionists that "conversion to Christianity was the only way to be saved from the Exile" ([20]). By all accounts this conversion was quite sincere, unlike many to either Islam or Christianity in the medieval world which were motivated by force, economic opportunity, or other reasons.

What is particularly interesting about Abner/Alfonso is that he became, in a sense, *doubly* apostate. This is obvious in the sense that he became a Christian, but his critique of much of Aristotelian science was cutting-edge, given Aristotle's dominant position in Islamic and Jewish philosophical circles. There certainly had been various critiques of Aristotle before, and ones Abner would have been familiar with[4], like al-Ghazali's concerns over creation (see [1]); Abner similarly was concerned with "extreme Aristotelianism within contemporary Judaism" which "read the Torah to justify [heretical] philosophical beliefs" [11]. A particularly pointed critique was that of dimension, which is relevant below. For those interested in the Neoplatonic aspect of Alfonso's philosophy and theology, see [19].

We should not see these things as being opposed to each other. Even if one declines to go as far as [11] and say that "his conversion and acceptance of Christian dogma ... provided him [with] a solution to [philosophical and existential] conflicts" within Judaism, nonetheless it is clear from his writings that both his philosophical and religious opinions were carefully considered and integrated. In fact, many of Alfonso's Christian theological positions seem to have been little influenced by orthodox teaching, and his philosophy likewise is mostly devoid of contact with the relevant Latin-writing debates[5]. As an example, Alfonso suggests that "what made Jesus unique among human beings was that he was born of the highest matter, enabling him to unite with the highest degree of divine essence that a human can attain" ([19]).

Alfonso's primary influence was on the contemporary Jewish situation in Castile. First, he was called on to give advice to the ruling authorities regarding Jewish prayers against apostates such as himself—a matter considered to be of some importance in a period where the position of Jews was slowly becoming more and more restricted. More importantly, his polemical pro-conversion writings spawned a virtual cottage industry of rebuttals, many of which themselves were influenced by his philosophy (such as on free will, see [19]). Moreover, the discussion was not trivial. His main dialogue ends with an unconverted Jew, and in another work he specifically critiques Jewish philosophers who (in his opinion) are actually irreligious, as opposed to their simply erring coreligionists ([20]). Because his work almost exclusively used the Hebrew Bible and Talmudic commentaries, seeing "his former faith as revealing the inherent truth of Christianity" ([11]), it influenced Jewish commentators for several centuries (in addition to providing quotes from Jewish authors to Christian writers in a Romance tongue).

Finally, it is worth noting that the identification of the well-known Abner/Alfonso with the author of the *Meyasher 'Aqov* was not straightforward. The author himself does not give any self-indication other than the extremely common name Alfonso. The original translation to Russian in [10] claims their identity, and another initially skeptical researcher confirmed it in [6]. In [7] numerous examples of idiosyncratic language and thought are given which make even the layperson feel confident!

4 Straightening the Curves

4.1 The General Goal

After these extended background sections, we now come to the heart of the matter. We let Alfonso/Abner speak for himself[6] (emphasis added):

[4] Probably in Arabic, not the also-widespread Hebrew translations; see [8].

[5] While Alfonso translated much of his work into Castilian, his direct knowledge of Latin seems to have been limited to pieces of the Psalter "included from memory according to an oral familiarity," [21].

[6] All quotes from *Sefer Meyasher 'Aqov* are taken from the English translation in [7].

> [M]y original intention in this book [is] to *find a constructible straight line* that equals ... the infinitely many curves of different species, and to find a constructible ... area *equal to the area of a circle* ... to find a constructible body ... *equal to the body of a sphere* ... [and] to find a way to *divide any rectilinear angle*[.]

Even a reader unfamiliar with the history of mathematics should recognize that this is tantamount to creating a significant part of calculus. To summarize the first three of these goals, he wants to:

- Find the arc length of an arbitrary curve, defined either geometrically or via motion,
- Find the rectangular area of a circle, and
- Find the volume of solid curved objects.

An ambitious list to be sure! This is particularly so since even at that time those familiar with the Greek tradition would have known that it was unlikely one could 'square the circle' with conventional Euclidean tools.

However, the first question of 'straightening the curves' is likewise famous—notably for the circle, of which Archimedes' approximation of π is only the most well-known in a long line across many cultures. In fact, it was only in the early modern era that Torricelli did the first such calculation, by *exactly* computing the arc length of any given piece of the equiangular spiral. Even at that time the philosophical notion of which types of curves could even *in principle* be thus 'linearized' was a matter of controversy to, among others, Descartes and Galileo[7].

So it is quite appropriate that Alfonso/Abner decided to call his work the 'Straightening of the Curved' (or 'Rectifying the Curved' in [7]), as the one-dimensional version of this general problem:

> Therefore, I call this work Meyasher 'Aqov, as this is its main goal and it is very useful in mathematics.

Let us see how he proposed to achieve these goals. Any references to Alfonso/Abner's work follows the English translation and commentary in [7].

4.2 Improving Euclid

It was already pretty well known since antiquity that Euclid's framework was not sufficient to answer Alfonso's questions. Euclid was missing some necessary axioms, such as an axiom of continuity, even for non-controversial propositions. Moreover, he had left himself open to attack on some less obvious issues, notably regarding Postulate 5—the so-called Parallel Postulate, on which see below.

In the first section of his text, Alfonso introduces new postulates, but not simply to repair Euclid. In his view, his new axioms patch up several holes, defend the philosophical underpinnings of geometry, *and* allow it to achieve the various rectification/quadrature goals he has in mind.

His first complaint is understood most easily by recalling Euclid's Proposition I.4, which is essentially the side-angle-side congruence theorem for triangles[8]. Nearly every college geometry text

[7] The latter in his unpublished *De Motu*; I'm grateful to user `Andrew R.` on the History of Science and Mathematics StackExchange site for this reference; see `https://hsm.stackexchange.com/a/18164/9624`.

[8] Two useful resources for students on Euclid and these issues are [13] and [5].

focusing on axiomatics (such as [24]) spends significant time critiquing Euclid's use of Common Notion 4 (*Things which coincide with one another equal one another*) along with 'superposition' or 'application' (depending on your translation of Euclid) in proving this proposition. Can you really just move a triangle onto another triangle? Usually this is then followed by a discussion of Hilbert's or Birkhoff's modern axiom schemata, which both (in some fashion) take the SAS congruence theorem as a fundamental postulate.

Alfonso begins by improving "Euclidean superposition" to clarify that it must be instantaneous, which (for him) solves the problem of I.4. In addition, he allows a "mental" version of this superposition, which he claims is necessary for understanding results such as the generalization of the Pythagorean Theorem to areas of shapes other than squares (Euclid VI.31, which Alfonso implicitly extends to curved figures). So far, so good.

But Alfonso is just getting warmed up. Two issues which college geometry classes do not typically discuss, but which were well-known at his time, were the implicit use of motion in the *definition* of a circle and the definitions of curved solid objects such as a cone. Toward the end of the first section[9], Alfonso introduces two new postulates to address this problem.

The first new postulate is the allowability of *imagining* continuous motion. His main stated reason for pursuing this is to allow the Parallel Postulate to become a proposition (again, see below), but it clearly can subsume the mental superposition, and could be quite useful in measuring areas. The commentary in [7] points out this idea had been discussed for some time by Arabic-writing geometers, so Alfonso's idea is not as novel here as are his goals in doing so.

He then doubles down on this with his second "Postulate of Measurement." This is a postulate by analogy. When we measure a finite continuous quantity such as length (Alfonso says), we can divide it into pieces with the same measure, using a lower-dimensional object (such as a line segment dividing a rectangle). Alfonso claims we can use this as an axiom for comparing measures as well, as long as the ratio is constant, even if that comparison extends indefinitely. That this is self-evident enough to be an axiom is shown by, among other things, the constant ratio of the length of the month to that of the (sidereal) year continuing as long as one desires. We will see thoughts even more reminiscent of Cavalieri's work shortly.

4.3 Philosophy and Infinity

The first new axiom, as [7] points out, can be thought of as anticipating axiom schemata for geometry based on transformations, not Euclid's axioms. But the second one is more difficult for us to understand, because we no longer worry about dimensionality so much in measuring! However, much of the controversy over the infinitesimally small over the centuries was because Aristotle disallowed comparison of quantities of strictly different dimension, so this notion is actually very important. To wit, can we really talk about a rectangle as being composed of line segments? In the Greek mindset, such non-homogeneous comparisons were not allowed[10].

Along these lines, this paper does not pretend to have any but the most superficial overview of the then-contemporary philosophical debates. However, it is very important to recognize that Alfonso engages at length with the entire tradition (including Aristotle, ibn Sina and ibn Rushd)

[9] Sections I.5.1 to I.5.3 in the subdivisions used by [7].

[10] We can see this even in the 1600s when algebraic notation was initially always homogeneous, like $aab + ccc$ rather than $a^2 + c^3$.

on these topics, because motion of geometric objects was directly connected to motion of other objects (part of what was then called 'physics')—even if we do not ordinarily associate Aristotle with mathematics. A nice example is at the end of the second section, where Alfonso includes a significant discussion of the so-called Aristotle's wheel paradox[11].

As another example, when motion occurs, does it create a void left behind? This was another significant question, even if the motion involved was purely geometric. Before unveiling his axioms, Alfonso makes sure to give equal time to those who have (in his view) an incorrect notion of motion, and then refute them. The commentary in [7] is enlightening for those wishing to learn more, though as a research monograph it presupposes substantial background knowledge.

The best example of this philosophical orientation for the audience of the current article may be when he begins explaining various errors in geometric proofs (by both Islamic and Jewish writers, at various points in the second section of Alfonso's text). One main issue here is that some proofs can be read as implying the existence of an actual infinity, which, like the void, was a topic of serious philosophical importance. Interestingly, even though Alfonso decidedly does not believe there is an actual infinity, he first critiques illegitimate (in his eyes) uses of motion to demonstrate this nonexistence.

Instead, he uses his new 'proof' of the Parallel Postulate and motion of *finite* magnitudes to demonstrate the impossibility of an actual infinite line; see Figure 1. Suppose we have an infinite straight line AB; then we use perpendiculars to construct a parallel line CD through any point C not on AB. Crucially, now that Alfonso's new postulates give access to the Parallel Postulate, CD and AB are *everywhere* equidistant (Euclid I.33). Now Alfonso imagines (!) rotating a common perpendicular AC through C (on the new line); this defines more lines which will always meet the original, supposedly infinite, line AB somewhere (labeled A' here). But these meeting points will still be that same distance from the line CD, which means (according to Alfonso's view of motion) that when the rotation finally meets CD, it could only have done so "instantaneously," which Alfonso then shows is not possible either. Note the imagined continuous motion of the rotating line.

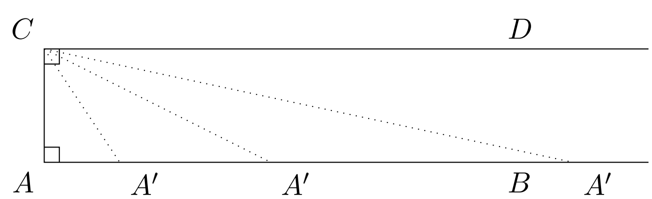

Figure 1: Parallel lines

So infinity is impossible. How else did Alfonso use his hard-won new postulates?

4.4 The Parallel Postulate

The modern reader will not be surprised that, despite his enthusiasm, Alfonso did not prove the Parallel Postulate, did not develop hyperbolic geometry, and did not square the circle. However, he did so in creative ways which exemplify just how large a variety of mathematicians were pursuing these questions long before their solutions arise in the 'standard' story.

[11] On comparing the circumference of two unequal circles with the same center. See `https://www.gocomics.com/calvinandhobbes/2024/06/25` for an entertaining closely related example, one which students will enjoy.

After dispensing with a number of purported proofs of Euclid's Postulate 5, Alfonso attacks it himself[12]. The main device he uses is closely related to the Khayyam-Saccheri quadrilaterals (already anticipated by ibn Qurra) which typically occur in exposition of neutral geometry in college geometry courses. Such quadrilaterals have two adjacent right angles with equal sides. Apparently inspired by al-Haytham's school, he takes a right triangle and 'doubles' it along the hypotenuse to create a quadrilateral with equal *opposite* right angles (and equal sides). Proving that this figure is actually a rectangle is his path to proving Euclid's Parallel Postulate.

His overall strategy is interesting; refer to Figure 2. Suppose $\triangle ABC$ to have a right angle at A, generating quadrilateral $ABCD$ with another right angle at D. One can double the length of side CA to CG, and then make a right angle at G with side length AB to construct another right $\triangle CGH$ which can be 'doubled' in the same way to a quadrilateral $CGHK$. In Euclidean geometry, the fourth vertices D, K must coincide; do they in neutral geometry? Alfonso carefully considers a number of subcases to (supposedly) show this[13]. Finally, he then returns to superposition and demonstrates that this new quadrilateral has two parts, one of which can be applied to the original quadrilateral. Then he uses standard angle measure arguments to see that all of the above are rectangles.

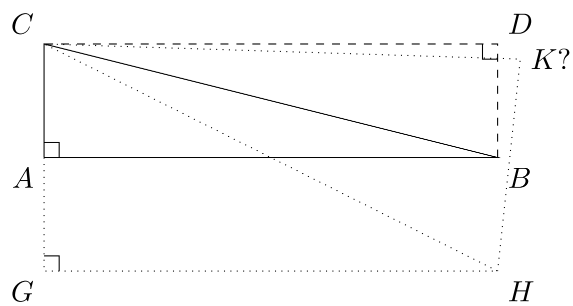

Figure 2: Quadrilaterals for neutral geometry

Today we know that the existence of *any* rectangle is logically equivalent (given the other postulates) to the Parallel Postulate, so his task was fatally flawed. It would be an interesting exercise to ask students to construct Alfonso's quadrilateral in a hyperbolic model, to come up with all possible cases of where the points on the two quadrilaterals might be, or to compare his strategy with the different rectangles of Khayyam, Saccheri, and Legendre.

4.5 Integration?

As far as we know, Alfonso intended to use the measurement postulate and the idea of allowing "division of magnitudes into lower-dimensional magnitudes" ([7], Commentary to Section 4) to lay foundations for the quadrature of some entities. Unfortunately, the only manuscript has just a few lines of his ideas before breaking off, and it seems unlikely he achieved his intention fully. There is nothing at all suggesting that the goal of measuring curves and the quadrature of curved bodies was achieved, which was to have been the fifth and final part of his treatise.

Alfonso's assertion in what we *do* have is that two rectangles with a common side have areas in ratio

[12] In Section II.3 by the editors' numbering.

[13] For full details, as well as discussion of a host of textual issues indicating we may not have Alfonso's final redaction of his thoughts, see the commentary to II.3.2 in [7].

to "that which falls inside the first [rectangle] of the infinitely many heights drawn on the basis ... to all that which falls inside the second [rectangle]" ([7] Part IV). That is to say, for this extremely simple example Alfonso proposes to use the motion of a (lower-dimensional) point on partly shared sides of the two rectangles to define "infinitely many [lower-dimensional] perpendicular [slice]s" of each rectangle—and then to compare their ratios. It should be clear that this relies heavily on his earlier principles, including his explicit call to Plato for justification in the outline of the entire work. Glasner and Baraness stress that Alfonso does *not* use the usual Eudoxian or Archimedean arguments by exhaustion in addressing area.

By the time Alfonso alludes to the same principle working for "similar cylinders and similar prisms with the same height," it is quite evident his argument is analogous to Cavalieri's work. Cavalieri is more careful about his dimensionality arguments, which used indivisible 'all lines' to actually fulfill the promise of comparing volumes of cylinders and prisms. Note also the similarity to the work of Liu Hui and the Zu family in medieval China (see [15]), Kepler's wine barrels, and even to Oresme's proto-integration of functions (see for instance [3]). Alfonso's musings seem to be yet another independent appearance of this principle *before* the more rigorous work of Fermat and others in France (and, eventually, Newton and Leibniz).

4.6 More Geometry

Although Alfonso did not (to our knowledge) succeed in his overall mathematical goals, he still matters more generally (outside of the sheer curiosity expressed in this paper) by exemplifying the huge lacunae we have in the *transmission* of mathematical, or indeed scientific, knowledge between cultures. This is demonstrated by Part III of the text, which we have not yet discussed.

Here, Alfonso includes a wide variety of geometric propositions beyond Euclid from various areas, including ones less familiar to moderns like proportions and spherical trigonometry. What interests us here is not that he was learned enough to want to disseminate others' propositions (for that matter, [8] suggests that Alfonso was not just versed in Archimedes, but actually translated Archimedes into Hebrew for his own purposes). Instead, it is that he apparently had access to mathematical traditions we otherwise have no evidence would have been available in Iberia at that time period.

First, Alfonso provides proofs of Hippocrates' famous quadrature of not the circle, but portions of circles called lunes. Not only that, he explicitly refers to this personage as a model for his notions of superposition in the first part of the text. There is a long discussion in [16] (and [10] which I do not have access to) of his possible sources, noting that there simply are no sources known to have been available at the time, whether in Arabic or not, which identify *Hippocrates* as the originator.

A second example is the construction of the 'conchoid of Nicomedes', an elegant locus (see e.g. *Nicomedes*, [18]) from around the time of Archimedes. Although ibn-Qurra mentions it in regards to trisection (see [12]), Alfonso's use of it in trisection *and* solving the cube-doubling problem is not known elsewhere ("unique", per [14]). As Glasner says in her book review [9], "how a Jewish scholar who lived in Castile in the fourteenth century learned [of] Nicomedes' conchoid is still far from clear."

Finally, the last proposition in the third part shows Alfonso's knowledge of a particular construction with two circles known today as an al-Tusi couple. In this mechanism, a small circle rotates within a larger one of twice the radius, and a specific point on the smaller circle then generates linear motion. This accords well with his imagined motion, since "linear motion and circular motion are not two distinct types" ([7]), and in a non-Aristotelian manner. Glasner and Baraness point out that "Alfonso was somehow informed about some mathematical ideas that were raised in this school [of the new observatory al-Tusi built after Baghdad's capture by the Mongols]" ([7], p. 42), and this is just one of several indications he had access to their work.

If Alfonso knew of these results, who else might have known about otherwise 'lost' theorems? Copernicus used either al-Tusi couples or something very similar to them in his heliocentric model. Some researchers (e.g. [17]) now suggest that Copernicus became aware of them from either *Sefer Meyasher 'Aqov*, or a common source also available in Italy (where Copernicus studied on several occasions), though there are other proposed solutions. Whether this is true or not, it is wise to have humility in our assessment of the flow of the history of ideas—not Eurocentric, to be sure, but also not prioritizing any other civilization. People often like sharing ideas, and we should be clear about that.

5 Alfonso the Author

We've seen how Alfonso/Abner had long been rethinking Euclid. We conclude this paper with what clearly came first for the author, whether as Abner or Alfonso—the glorification of the creator of the universe in his studies.

Let us return to the title 'Straightening of the Curved'. We remarked on its appropriateness earlier due to content. But it is doubly appropriate for his theology, because in the Hebrew any reader even somewhat familiar with Isaiah's prophecies would note the clear allusion to the second half of Isaiah 40:4, here in English from the New Living translation[14]:

> Fill in the valleys, and level the mountains and hills.
> Straighten the curves, and smooth out the rough places.

While nearly the entirety of Abner's *oeuvre* uses primarily Hebrew and Arabic sources, most of the writing we have is that of Alfonso. Though from his other work we know he believes he has found the Messiah of this prophecy, his Biblical connections are firmly rooted in his Jewish heritage, and that is one reason it seems reasonable that he had been working on this book for many years before his conversion.

For the first example, recall the demonstration of the non-existence of an actually infinite line. Immediately after finishing this, he continues as follows:

> From here one can (if one wishes) come near to the understanding of the existence of God, as he sets actual specific limits on distances after having been potentially unlimited.

He follows this with several quotes from Job 38 directly claiming the "dimensions", "measur[e]," and "halt[ing]" which God placed upon the earth itself. This does not seem to be a haphazard proof-text, designed to ingratiate himself with a religious local potentate, as one might interpret

[14] As a famous Messianic prophecy, this made it into Handel's Messiah from the King James translation as 'the crooked made straight.'

the introductions to various medieval Islamic and Christian mathematical texts. After all, Alfonso XI of Castile probably was conversant in neither Hebrew nor mathematics! Instead it seems to be a theological confirmation of his philosophical commitment to the non-existence of infinity—or perhaps the other way around.

That is not the only place theology seeps into the philosophical ideas behind his mathematics. We regret that our understanding of Alfonso's theology is limited to the sketches in sources like [20] and [7] (Section 4 of the Introduction). Nonetheless, it should already be clear that his use of different dimensions is due to a strong philosophical stance, and it would be surprising in his milieu if that were *not* connected to a strong theological stance as well. Citing Sadik's thesis, Glasner and Baraness in [7] summarize Alfonso's view as being, "it is God, not matter, that endows bodies with dimensionality." This is in connection with his theology of incarnation, but should also apply to any other bodies, including geometric ones. Certainly notions of atomism, infinity, and indivisibility were discussed in (among other things) the nature of the Eucharist for centuries to come, even though they largely came from theoretical areas such as mathematics.

Finally (for us, but first for Abner/Alfonso), what is the justification for even studying this at all? Then and now, many ask the question of whether the religious life is conducive to the study of seemingly unrelated matters like abstract mathematics, but Alfonso clearly rejects this point of view. His introduction alludes to the Talmud, Ecclesiastes, Isaiah, Numbers, Deuteronomy, and Psalms[15] while praising this kind of general learning. He has "pursued learning consistently without distraction," he has "not exert[ed] himself for nothing in vain," and the knowledge has "emanated from the holy spirit." God has "created everything for his glory," and "the whole earth is full of his glory."

And part of that glory is the goal Abner had, and Alfonso has, in writing *Sefer Meyasher 'Aqov.*

> [T]o glorify [the Lord's] name and greatness by the mensuration of the circle [was] from my youth to my old age the one thing I desired from the Lord.

To Abner/Alfonso, nothing could reveal God's glory more than to unveil the hidden truths of geometry, and he believed he had done so. 'Straighten the curves,' indeed.

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Finally, even though it should be obvious, it is important to acknowledge that this paper is not primary scholarship. It is merely an attempt at bringing the work of several scholars (particularly Glasner and Baraness) to a more general audience, and synthesizing it differently for that audience. Any mistakes or errors in understanding or interpretation of his, or their, work is solely mine. Thanks are also due to all who have worked to preserve Alfonso/Abner's work, especially Mordekhai Finzi, the 15th-century owner of the only known manuscript (now held at the British Library).

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Using the Statistical Reasoning Framework as a Tool in Theology

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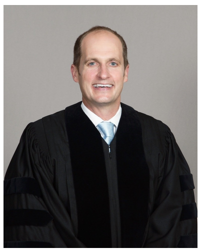

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Abstract

The Statistical Reasoning Framework (SRF) provides a useful tool for Statisticians and Christians to analyze the world and God Himself. This paper describes the use of the SRF in statistical classrooms in addition to providing an example of how it may be used to help answer a theological question.

The Statistical Reasoning Framework (SRF) initially described in 2007 by Franklin et. al. was generalizations of work by statisticians for decades in addition to scientists and philosophers for centuries helping to deal with the omnipresence of variability [6]. The SRF was influenced from years of research in statistics education and more formally in the Guidelines for Assessment and Instruction in Statistics Education (GAISE) College report [1] and the Cobb report [4]. The four major processes of the statistician in the SRF were described as formulating questions, collecting data, analyzing data, and interpreting results. The GAISE II College report [3] specifically addressed the SRF as the statistical investigative cycle as part of its first goal to promote statistical thinking. The GAISE II Report [2] continued to expand on the SRF calling in the statistical problem-solving process.

1 The Statistical Reasoning Framework

1.1 Formulate Statistical Investigative Questions

By developing questions that anticipate variability, statisticians are very specific when beginning the process of statistical reasoning. The untrained researcher often begins with questions that have deterministic answers, such as, "Is there a statistical difference between. . . " In which follows a solution of yes or no. Children begin with such questions like, "What is my favorite color?" With slight re-wording, students in early grades begin to create questions that require the collection of data and analysis to draw inference about a certain population. Their language becomes less inwardly focused and looking to understand the world around them, "What's the favorite color of 8 year olds?"

As K-12 students develop the ability to formulate questions, teachers initially play a key role, and questions are targeted at the population of students within the classroom [6]. These skills are further expanded by students beginning to create their own questions of interest and seeking generalizations by expanding to populations larger than the classroom. Instructors should also consider opportunities to ensure specific questions brought to the classroom exhibit unique sampling needs, box plots, histograms, scatter plots, etc. to lend themselves to deeper discussions in later phases of the SRF. As students move to the college setting, emphasis on questioning should expand to include impacts of data collection from design decisions and potential needs of Institutional Review Boards [1].

1.2 Collecting Data

The key role in the data collection process is the acknowledgment of variability and accounting for this variability. The role of random sampling, sample size, experimental designs, random assignment, sampling designs, sampling variability, and pairing data are some of the key standards for students to develop in an early statistics course [6]. Many of these standards may be acquired by students in the classroom through active learning and projects affording students opportunities to see potential bias that occurs and other impacts of poor and strong designs [1]. Additionally, teachers may want to provide data from another source outside the classroom to connect with statistical questions that may not be explored quickly from a classroom setting and emphasize the development of other standards within the classroom setting. However, it is important that students develop the habits of mind of data interrogation when using data acquired from external sources. They should understand what the data represent, identify the variable type, recognize the matched or unmatched state of the observations, and compose a general context for how the data were collected [2].

K-12 students initially begin the data collection process by conducting census of the classroom with little to no designs for differences. Data collection designs start with simple experiments and progress towards comparative experimentation by incorporating random allocation. Students should progress from classroom census to sample surveys to draw inference about a population and advance towards the use of random selection with different design processes. As students begin experimental designs, attention should be placed on induced variability to help design for differences in experiments. As students move to the upper grades and college level, instructors should include an emphasis on ethical considerations to situate legal and ethical data collection and experimentation [1].

1.3 Analyzing Data

The general theme of all data analysis is to account for variability and represent it well [2, 6]. This analysis often conveys different numerical statistics such as percents, means, standard deviations, confidence intervals, etc. Analysis also provides visual depictions of data that help a consumer of the analysis understand the variability such as histograms, box plots, dot plots, scatter plots, etc. Students develop the use of these distributions initially as examples, then as tools for analysis and developing the distribution as a global concept.

The progression of students' ability to analyze data in the K-12 setting is dependent on their mathematical experiences and abilities. Students in K-12 settings are traditionally and initially exposed to bar graphs with categorical data sets and questions that are within their systems of counting fluency. Students' experiences are advanced as they learn about rational numbers, univariate analysis, and move towards graphing on number lines. Often in upper middle school, students are introduced to the coordinate system and begin opportunities to work with paired data in scatter plots and two-way frequency tables.

As students mature through mathematical proficiency, statistical proficiency is much different and paramount for development. Students move from recognizing and representing variability within a group to quantifying that variability. They should further develop this ability in measurement variability to compare between groups. Using this development from visual to quantifiable variability, students should begin acknowledging sampling error and develop ways to describe and quantify it.

1.4 Interpreting Results

A key difference between mathematics and statistics is the interpretation of the problem in the presence of variability [2, 6]. In statistical problem solving, the end result is not certainty but interpretations in the presence of variability that often involve estimates. Statisticians are encouraged to look beyond the data to make generalizations while in the presence of variability. In which, the use of probability becomes a key mathematical process that should be included to help students develop the ability to interpret results effectively.

Initially, students at the K-12 level do this very little. Their focus is at the classroom level, recognizing differences using additive and multiplicative reasoning between two to four categorical variables. Students spend little time focusing on generalizations of their statistical questions to other classrooms or schools. However, teachers must begin to push students to begin to acknowledge the similarities and differences between populations and the use of representative samples to draw inference. Teachers must provide students with opportunities to experience the differences between observation and experimental studies and their impact on inference. Probability should be utilized as a tool to quantify the strength of models in interpretation. Students should have opportunities to become aware of the distinction between correlation/association and causation.

2 Theoretical Framework

Before moving further in this paper, it would be useful for the reader to understand the context of the authors' Biblical hermeneutics. According to Corley et al. [5], the authors of this report take a Modern Reformed hermeneutic. Corley et al. described the hermeneutics of Protestant Reformation leaders as inductive and faith-oriented. They emphasized four principles that are the focus of Biblical interpretation:

- The focus of Scripture is on Christ, not the church nor man.
- The ultimate purpose of the Bible is salvation, not knowledge.
- The basis for Christian doctrine and practice is the Bible.
- The authority for interpreting the Bible is in the individual.

Modern approaches to Biblical interpretation were developed through the Westminster Theological Seminary and described by the Princeton method [5]:

> Princeton's methods of scriptural interpretation relied heavily on the principles of Scottish common sense philosophy. Their defense of scriptural authority was based upon the notion that empirical induction is the primary source of truth and that all reasonable people intuit moral absolutes. They defined theology as a science, mining the Scriptures for facts much like a scientist gathers data from nature.

The authors contend that individual interpretation of the Bible does not mean that individual interpretation is favored at the expense of rejecting a collective interpretation of the church through creeds and historical theology. Similarly, holding to the 5 *solas* of the reformation, the authors do not believe that *sola scriptura* means "solo" *scriptura*. In particular, 19th-century sectarian movements and theological liberalism that extol the individual conscience over the boundaries of theology worked out by the church within the bounds of biblical authority and historic creeds/confessions are not what is meant. We believe that the only "infallible rule of interpretation of Scripture is the Scripture itself; and therefore, when there is a question about the true and full sense of any scripture, it must be searched and known by other places that speak more clearly" [7]. Generally, the authoritative interpretation proceeding from the individual "can be no other but the Holy Spirit speaking in the Scripture" such that "not only the learned, but the unlearned, in a due use of the ordinary means, may attain unto a sufficient understanding of them" [7].

Hermeneutical approaches vary widely depending on the philosophical, cultural, and theological context in which they are applied, with notable differences between reformed and Wesleyan methods of biblical interpretation. The modern reformed hermeneutics utilized by the authors of this paper tend to prioritize the authority of Scripture and emphasize a doctrinally fixed, historical-grammatical approach that seeks to uncover the objective meaning of the text as intended by the original author, with particular attention to God's sovereignty—a method that may make it more difficult for modern readers to apply to their lives. In contrast, Wesleyan hermeneutics incorporate a more dynamic, pragmatic approach that values the text's application to the contemporary believer's life, with an emphasis on personal holiness and grace. While both traditions affirm the centrality of Scripture, the Wesleyan method places greater emphasis on the lived experience of the Christian community and its ability to interpret Scripture in light of reason, tradition, and experience—elements of the Wesleyan quadrilateral. This model's emphasis on experience and reason can lead to subjectivity and inadvertently elevate human authority to an equal level with divine revelation. Thus, it was not used to reduce the variability that would be induced from human reason and experience. The hermeneutical approach utilized in this paper sought to use scripture to interpret scripture.

3 The Consistent God

Theological questions arise throughout humanity's journey here on earth not because of variability of Gods (1 Corinthians 8:6) or any variation within God himself (James 1:17). Theological questions arise because of our limited understanding and ability to know the God of the Bible through any medium in the flesh (Ecclesiastes 8:16-17; Romans 8:9) without special revelation (2 Timothy 3:15-17; John 1:12-13,18). We have but shadows of God moving through our midst and in ways which He was pleased make himself known (Exodus 33:18-20; Hebrews 1:2) to us through scripture (2 Peter 1:21). The Lord Jesus himself cited scripture as coming from the Creator himself (Matthew 19:4-5, 12:36; Romans 9:17; Galatians 3:8; Hebrews 3:7).

Christians today may look to God Himself to help reduce variability. His Spirit leading us in paths of righteousness for His name's sake (1 Samuel 12:22; 2 Kings 19:34; Isaiah 37:35, 43:35, 48:9-11;

Ezekiel 20:44, 36:32, 39:35; Psalm 25:11, 31:3, 79:9, 106:8; Joshua 7:9; Philippians 1:29; Romans 8:36; 2 Corinthians 4:5) is a wonderful place to begin. But God's operation across only one person, say ourselves, is a much too limited approach to understanding the God of the universe who knows the very hairs on each person's head (Luke 12:7) and provides food and water for every animal on the planet (Psalm 147:9). On this side of heaven, we are left with evidences of His hand across the lives of His children, the Church, the Israel of God (Galatians). Unlike Protestant Wesleyan Theologians who often choose to look at Sacred Scripture, the primary source of understanding God, through tradition, reason, and experience of men outside the Bible (Wesleyan Quadrilateral), we have the Bible itself to help cast light on God Himself (Psalm 119:105,130; 19:7-8), Theology. For this reason, the question formulation, data collection, and inferential reasoning proceeding through examples will rely solely on scripture.

4 Theology and Statistical Reasoning Framework

4.1 Developing Theological Questions

As we come to experience God, we are often left in wonder. We are left with questions about the works of His hands. Though we may pray for our physical eyes to be opened to see the spiritual world around us (2 Kings 6:17-20), God may see fit to leave us spiritually and/or physically blinded (Deuteronomy 29:29). We may wonder things like, why towers fall and kill people (Luke 13:4). In fact, one of the largest questioners in the Bible was Jesus. He asked Simon Peter, "But who do you say that I am" (Luke 9:20). Not because He did not know who He was, but to allow for Peter to glorify Him in his response, "You are the Christ, the Son of the living God" (Matthew 16:16). Jesus response praises his answer to the question by a solution that was not from Peter or Jesus, "For flesh and blood has not revealed this to you, but my Father who is in heaven" (Matthew 16:17).

Statistical questions understand that variability is present with a general aim that there is an underlying truth or fact contributing to an experiment. Questions are posed in a way to extract more than one solution. In theology, the underlying truth is God. We must shape our questions around the consistency of God across the varying experiences of His people. Though these people vary in nationality, ethnicity, culture, experiences, race, sex, and socio-economic status, the God that shapes and directs their lives is unchanging.

Thus in parallel nature, a good statistical question is one that anticipates variability and a good theological question is one that anticipates variability with a presupposition of a constant God. It seeks to understand the ultimate purpose of the Bible, salvation: How was Pharaoh's heart hardened?

This particular question is a traditional question that many Biblical readers are challenged with as they read Romans 9:17-18.

> For the Scripture says to Pharaoh, "For this very purpose I have raised you up, that I might show my power in you, and that my name might be proclaimed in all the earth." So then he has mercy on whomever he wills, and he hardens whomever he wills (Romans 9:17-18).

Pharaoh appears to be lifted as an example of the Lord's hardening of a heart to exhibit mercy to whom he wills and harden whom he wills. To understand this theological question using the SRF, readers will need to begin collecting data.

4.2 Collecting Data to Answer Theological Questions

As with a good statistical question that anticipates variability, the data collection process must acknowledge variability in the collection process. In statistics, different collection techniques are used to reduce the variability that may be present when sampling while other techniques induce variability to test treatments or treatments between controls. Random assignment is used in experiments to reduce for the differences that may exist between groups subjected to different treatments or controls in addition to controlling for other factors that may have not been controlled for during the methodology.

Following the Modern Reformed Hermeneutic, data collection should largely flow from the Scripture itself. Using this process, collecting data from the Bible should provide less variability. Using the most historically accurate renderings of the Biblical text should also reduce some of the interpretations from other sources across the span of years of Biblical recitation and transcription. For this reason, verses used in this analysis were drawn from the English Standard Version though other translations such as the New American Standard Bible also place strong emphasis on historical accuracy.

To help ensure data collection that directly relates to Pharaoh and his heart being hardened, data collection was focused on the events and interactions in Exodus. Thus, data collection to answer the question of the hardening of Pharaoh's heart is largely found between Exodus chapters 4 and 14. These chapters contain 19 different verses that describe the hardening of Pharaoh's heart and directly correspond to the events that transpired for the exodus of the Israelites from Egypt.

4.3 Analyzing Data to Answer Theological Questions

The central role of analyzing data in statistics is to understand the variability that was found within the data that was collected. This process in statistics requires reasoning about distributions and statistics. Statisticians use graphical displays, two-way tables, and numerical summaries to explore, describe, and compare the variability within the data.

In addition to context, as described above, analysis should also include semantics and syntax [5]. Semantics focuses on the original language background and use of words in those contexts. It does not necessarily require analyzers to be fluent readers of Hebrew, Greek, or Latin [5]. It does require analyzers to dive directly into the use of words to identify the particular meaning to a passage. When considering syntax, the analyzer should look at the grammar of a verse or passage of the Bible, focusing on the rules of how words are used and relate with one another. Syntax is about the relationship between words rather than just the words themselves.

Combining these two ideas of analysis from statistics and hermeneutics, the 19 verses from Exodus can be split into three different themes. The first theme are verses that directly say that Jehovah, God, hardened the heart of Pharaoh. The second theme using syntax and semantics conveys a general hardening, in which neither Pharaoh nor Jehovah is accredited to the hardening of Pharaoh's heart. The final theme are verses that convey Pharaoh as the direct operator of the hardening of his heart. These verses are shown in Table 1. The parentheses indicate the order of verse appearance.

God's Hardening	Being Hardened	Self Hardening
4:21 (1)	7:13 (3)	8:15 (6)
7:3 (2)	7:14 (4)	8:32 (8)
9:12 (10)	7:22 (5)	9:34 (11)
10:1 (13)	8:19 (7)	
10:20 (14)	9:7 (9)	
10:27 (15)	9:35 (12)	
11:10 (16)		
14:4 (17)		
14:8 (18)		
14:17 (19)		

Table 1: The Hardening of Pharaoh's Heart (Exodus)

Upon initial inspection of the vast number of verses showcasing God's hardening of Pharaoh's heart, one may jump to quick conclusions that God must have hardened Pharaoh's heart. However, further analysis may be helpful in understanding the variability from the data collected. A number of theologians have argued that Pharaoh first hardened his own heart then God hardened Pharaoh's heart. To see how much this claim corresponds to the text, a time series plot was created. To visualize this possibility, verses depicting God's hardening were categorized as 3, a general hardening was classified as a 2, and a self-hardening was categorized as a 1. Figure 1 displays the hardening of Pharaoh's heart chronologically.

After visualizing the line plot, it is evident that God's hardening was depicted first followed by a general hardening in which neither is provided as the cause to hardening. What follows is a self-hardening in the later parts of the plagues. A clear pattern of God's hardening could be seen in the later portions of the story from Exodus. To dive into the possibility of an initial self-hardening followed by God's hardening, a closer look at verses that describe Pharaohs's heart being hardened and the verses of general hardening could be useful in reducing the variability.

In the first verse in which Pharaoh is depicted as hardening his own heart, we read the following: "But when Pharaoh saw that there was a respite, he hardened his heart and would not listen to them, as the Lord had said" (Exodus 8:15). This particular emphasis on "as the Lord had said" was provided to help bring clarity to the process of hardening at an earlier stage of the story. Going back to the immediately three previous hardening depictions in chapter 7, we find similar wording in verses 13 and 22, "as the Lord had said". Thus, it appears that the general hardening and even the self-hardening verses push the analyzer to move back to the event of the burning bush in which Jehovah God spoke to Moses, "And the Lord said to Moses, "When you go back to Egypt, see that you do before Pharaoh all the miracles that I have put in your power. But I will harden his heart, so that he will not let the people go" (Exodus 4:21). Deeper review of other general hardening verses that also include the wording, "as the Lord had said" include 8:19 and 9:35.

4.4 Interpreting Analysis to Answer Theological Questions

The analysis phase is really focused on attempting to understand the variability present from the data. In the interpretation phase of the SRF, statisticians seek to make conclusions while in the presence of variability with particular attention toward making inferences while taking variability into account. An essential component during the interpretation is to stay within the context of the

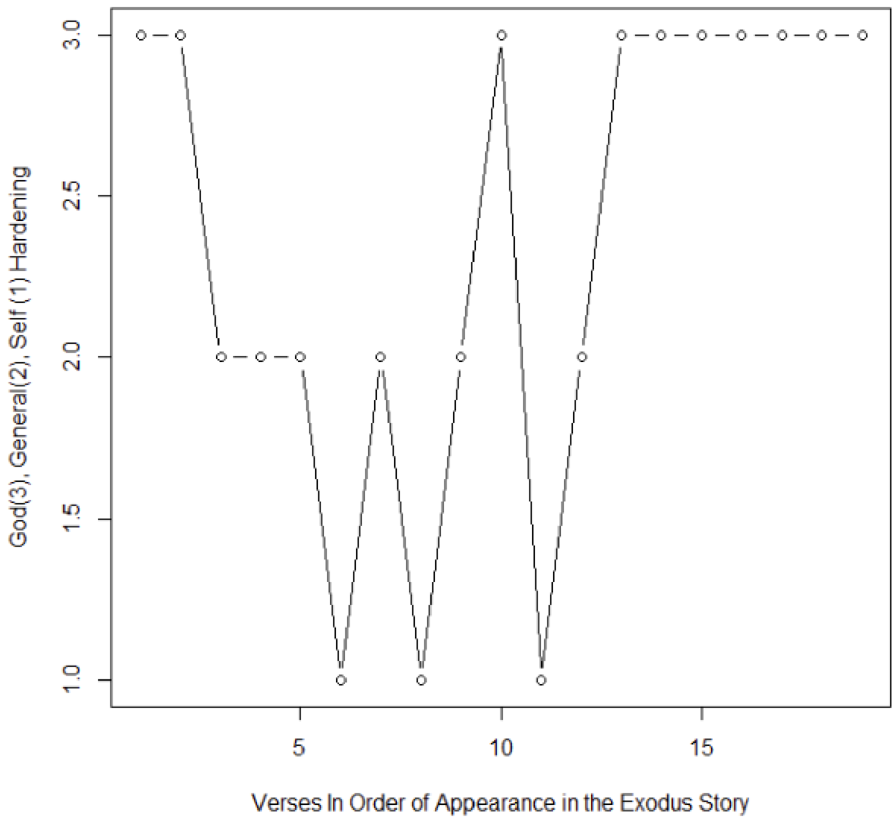

Figure 1: Time Lapse of Verses on the Hardening of Pharaoh's Heart (Exodus)

data and methods of analysis. This is the reason why statisticians speak with uncertainty often in forms of type I and II error using confidence levels.

As part of the inference from answering a theological question, there should be an impact on the believer. When our eyes have been opened to see and love our Lord more fully and His salvation for His people, we should be drawn to deeper affections. After analyzing any theological question, the interpreter should walk away both with a deeper knowledge and love for God (Hosea 6:6).

Upon deeper review of the hardening of Pharaoh's heart, we find strong evidence to support what was originally read in Romans 9:17-18, that God hardened Pharaoh's heart. Of 19 verses, there are ten verses that directly state this from the story within Exodus while only three that push against this notion. Of the 6 verses from the story within Exodus that cast shadows on the act of hardening, four of these draw us to reference the initial act of God's confession of hardening of Pharaoh's heart along with one verse of self-hardening. Though all variability within the verses of the Exodus story may not be reduced of variability, a total of 15/19 or 79 percent of verses indicate Jehovah as the first cause of Pharaoh's hardened heart with little to no pattern of chronological distinction.

Given such a strong indicator of knowledge that it is very likely the Lord who hardens hearts and shows mercy to whom He wills, what is their to gain in love from such knowledge? Just as Elijah prays before calling fire from heaven that the people not only "know" God, but that they know that God turned their hearts to God (1 Kings 18:37). There must be some added benefit in love to knowing God controls the hardening and softening of the heart. Romans 9:23 helps us with this insight, "in order to make known the riches of his glory for vessels of mercy, which he has prepared beforehand for glory." Knowing that God has softened the Christian's heart, should bring overwhelming joy with astonishment that he would love us despite our sin. We, like Jesus, should thank God for both the opening and blinding of each person's eyes (Matthew 11:25). In light of our offenses and sinful nature, God is fully justified in hardening every human heart. The question from Jacob, the deceiver, is not why did you harden the heart of Esau, but why did you choose to provide me a heart of repentance and faith?

5 Conclusion

In a quickly developing age of Artificial Intelligence, we must begin to develop clear and concise ways of creating questions, collecting data, analyzing data, and interpreting analysis. The SRF provides a clear and precise method for those attempting to problem solve about questions that arise from the real world using data. In fact, this process parallels how we may also understand the God of the universe through scripture. We know that God wants us to seek Him (Hebrews 11:6). Creating questions to understand God, theological questions that believe that the Lord of the universe as unchanging from age to age (Psalm 55:19, 102:27, Malachi 3:6, etc.) is a wonderful start. Using sacred texts, the Bible, to answer these questions has spanned the process of answering theological questions from generation to generation and was a common practice of our Lord Jesus (Wilmington, 2017). Analyzing these data from the Bible, making meaning from verses, and synthesizing what is being taught within one verse and among the different verses is essential to begin answering the question about our Lord that began the investigation and is encouraged by our Lord (Psalm 119:65-72). Finally, those investigating theological questions should walk away from analysis with interpretations or inferences that reduce variance from data collection to know bring both a deeper knowledge of God and a deeper love for Him (James 3:17, John 14:21, 2 Corinthians 4:6, etc.). Some interesting theological questions that readers may investigate themselves using the SRF as a model may include:

- For whose sake does God save His people?
- What baptism types occurred within the Bible?
- For whom did Christ die?
- How is saving grace dispensed to men?

As students explore these theological questions and collect data, they may consider multiple ways to analyze their data. For example, a student may consider a 2x2 table for baptisms that count the frequency of family/individual baptisms as one factor and the mode of baptism as the other. They may find there are many more family baptisms in the Bible than they had ever considered before and begin associating individual baptisms to the context of the persons being baptized (the unwed Paul and the Ethiopian Eunuch in the desert). Whatever paths students take, the SRF provides a wonderful opportunity for students to begin making sense of God and the Bible, Biblical theology.

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Craig, Cardinality, and the Possibility of an Infinite Past

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Abstract

We analyze two arguments popularized by William Lane Craig for temporal finitism. Although these arguments are often advanced to support the Kalam cosmological argument for a beginning of the universe, they have broader implications for how one understands God's relationship to time. The first argument appeals to Hilbert's hotel to argue that if there existed an actual infinity, absurdities would follow. The second argument alleges that an infinite collection could not be formed by additive succession. Although we ultimately find the arguments to be uncompelling, we believe they serve as a meaningful point of contact between mathematics, philosophy, and theology.

Introduction

Reflection on the infinite has a long history in philosophy and theology as well as mathematics, thus it serves as a meaningful point of contact between these disciplines. Since Aristotle distinguished the notions of the potential and actual infinite – an ever-growing yet always finite collection versus one that actually contains infinitely many elements – for most of Western history, the actual infinite was regarded as impossible. This distinction also exists in mathematics, where infinity is often used as a convention to denote unbounded growth, as reflected in Gauss' declaration, "I protest above all against the use of an infinite quantity as a completed one, which in mathematics is never allowed. The infinite is only a *façon de parler*: one has in mind limits approached" (quoted in [5, p. 120]). In the 19th century, however, Georg Cantor introduced ways of thinking about completed (that is, actual not potential) infinite sets in mathematics. Cantor's handling of an actual infinite was initially rejected by the academic community, running against deeply ingrained philosophical and theological intuitions. Nevertheless, Cantor appealed to theology to defend the notion of an actual infinite [6, pp. 167-186]. In particular, he found justification for the actual infinite in Augustine's discussion of God's exhaustive foreknowledge of the eternal future: "All infinity is in some ineffable way made finite to God, for it is comprehended by his knowledge" (*City of God*, XII.18). While Cantor's work came to be respected and his handling of the actual infinite is today accepted by most mathematicians as essential to modern set theory, the possibility of an actual infinite existing in the real world continues to be debated.

This debate over the possibility of an actual infinite features in arguments for God's existence. Notably, William Lane Craig has popularized the Kalam cosmological argument which is expressed by the following syllogism from [3]:

1. Everything that begins to exist has a cause.

2. The universe began to exist.
3. Therefore, the universe has a cause.

Craig supports the second premise of the argument by arguing for temporal finitism: the view that time extends only finitely far into the past. To defend temporal finitism he appeals to both contemporary cosmological discoveries and philosophical arguments against the possibility of an actual infinite number of moments (or events). If one accepts Craig's arguments for temporal finitism, it has significant implications for one's doctrine of God. In particular, if successful, the arguments demonstrate not only that the universe has a beginning, but also that time (physical or metaphysical) must have a beginning, and therefore God has not experienced an infinite series of past moments (or events). Although this may not pose a challenge for classical theists, who conceive of God as timeless, it may raise problems for those theists who understand God to be temporal, such as in neoclassical or open theists, for if temporal finitism is true, then such theists must affirm that there was a first temporal moment in the life of God.

Indeed, Craig himself advocates for this position. He understands God to be temporal with creation yet atemporal apart form creation, rejecting that God experienced an infinite series of moments prior to creation. This view faces challenges, though, as "many contemporary philosophers find themselves at a loss for words as to how God's timeless phase of existence could be related to God's temporal phase of existence" [13, p. 37]. Moreover, some Biblical scholars and theologians read in Scripture an affirmation of God as having a temporal everlasting past. Commenting on the Biblical description of God as existing "from everlasting to everlasting" (Psalm 90:2), for instance, R.T. Mullins notes that this Hebrew construction "quite literally means from perpetual duration in the indefinite past to perpetual duration in the indefinite future. This is a deeply temporal portrayal of God. Psalm 90 not only portrays God in temporal terms, it also speaks of God existing alone before creation" [11, p. 33] Indeed, it appears natural to read "from everlasting" as indicative of God having enjoyed an infinite number of moments prior to the creation of the universe. Under such a reading, one may still affirm with Craig that the physical universe is past finite while maintaining that (metaphysical or divine) time is past infinite. In fact, Craig often qualifies his own statements of temporal finitism to focus on physical (rather than metaphysical) time. For instance, he asserts, "the number of past events must be finite in number and, hence, the temporal series of past, physical events is not beginningless" [2, p. 200]. However, if successful, Craig's argument against an actual infinite would seemly rule out a metaphysical, as well as physical, infinite past, as nothing in his formulation of the argument (discussed below) limits its scope to only the physical.

Given the implications for Biblical interpretation and the doctrine of God, the purpose of this paper is to assess if Craig's philosophical arguments for temporal finitism are successful or not. We focus on Craig's formulations of two arguments which he gives in several of his books including *Time and Eternity*, as well as *Theism, Atheism, and Big Bang Cosmology*, which he coauthored with Quentin Smith, and *Creation out of Nothing*, which he coauthored with Paul Copan. As Craig's arguments invoke notions from Cantorian set theory, particularly cardinality, this paper serves to highlight how mathematical notions feature in contemporary theological debate. Although we ultimately judge Craig's arguments to be unconvincing, we believe they exhibit a meaningful interaction between mathematics and theology and can serve to inspire meaningful theological reflection from students of mathematics as well as interest in mathematics from students of philosophy and theology.

The Impossibility of an Actual Infinite

Craig formalizes his first argument against an infinite past as follows:

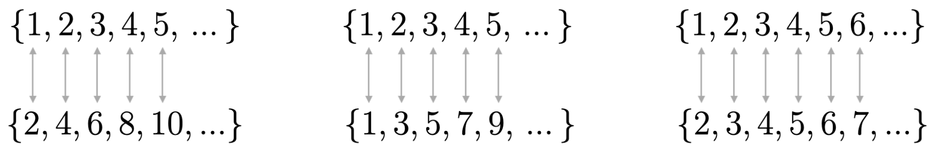

Figure 1: One-to-one correspondences between infinite sets.

1. An actual infinite cannot exist.
2. An infinite temporal regress of events is an actual infinite.
3. Therefore, an infinite temporal regress of events cannot exist.

Craig further specifies that existence in Premises 1 and 3 refers to extramental existence, that is, being instantiated outside of the mind. This is a necessary distinction since Craig does not want to challenge the existence of the actual infinite in the mathematical realm, but this move exposes Craig to a critique by Hedrick who points out that Craig is a presentist and as such does not in fact believe that past moments exist [7, p. 27-46]. Craig's position has also been critiqued by Malpass who avers that if successful, Craig's argument would also demonstrate that the future cannot be infinite [8, p. 786-804]. While interesting, we will not press either of these critiques or assess Craig's responses to them here.

In defense of the first premise, Craig raises the thought experiment of Hilbert's hotel, introduced by the German mathematician David Hilbert to highlight the paradoxical nature of the infinite. The thought experiment involves imagining a hotel with an infinite number of rooms, one corresponding to each positive integer 1, 2, 3, 4, ..., and such that there is a guest in each room. What happens if another guest visits and requests a room? One might think there is no way to accommodate the new guest as all of the rooms are occupied, but by exploiting the infinite nature of the hotel, it is possible to make room for the new arrival. In particular, the hotel manager needs simply to ask the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, and so on, such that each guest moves one room over, freeing up room 1 for the new guest.

Things get even more counterintuitive. Suppose an infinite bus arrives with an infinite number of passengers: passenger 1, passenger 2, passenger 3, and so forth. And suppose the passengers would each like a room at the fully occupied hotel. Incredibly, the manager can accommodate all of the new arrivals! To do so, the manager asks the guest in room 1 to move to room 2, the guest in room 2 to move to room 4, the guest in room 3 to move to room 6, and so forth, so that each guest in room n moves to the even-numbered room $2n$. This then frees up all of the odd rooms – 1, 3, 5, 7, ... – allowing the manager to assign passenger 1 to room 1, passenger 2 to room 3, passenger 3 to room 5, and so forth, sending passenger m to the odd-numbered room $2m - 1$.

Powering the thought experiment is the Cantorian property of an infinite set, namely, that one can exhibit a one-to-one correspondence between the entire set and a proper subset of it. For instance, there is a one-to-one correspondence between the entire set of natural numbers $\{1, 2, 3, 4, ...\}$ and the proper subset of evens $\{2, 4, 6, 8, ...\}$ that pairs 1 with 2, 2 with 4, 3 with 6, and so forth. Similarly, there is a one-to-one correspondence between the natural numbers and the subset of the odds. See Figure 1. Such correspondences make possible the above moves by the manager to accommodate more guests. Craig maintains that these counterintuitive features of the infinite is what makes the existence of an actual infinite in the real world impossible:

> If an actual infinite were to exist, then the principle of correspondence would be valid with respect to it. And if an actual infinite were to exist and the principle of correspondence were to be valid with respect to it, then the various counter-intuitive situations would result. But because these situations are absurd and so really impossible, it follows that the existence of an actual infinite is impossible [2, p. 203].

However, just because the scenarios are counterintuitive does not imply that they are, in fact, absurd. As Pruss notes, "while Hilbert's hotel is indubitably *strange*, the strange and the absurd (or impossible) are different, as is proved by the strangeness of the platypus" [14, p. 12]. What exactly is the absurdity Craig has in view? He expresses it using the idea of cardinality from set theory. For a finite set, the cardinality is the number of elements contained in the set. For example, the cardinality of $\{a, b, c\}$ is 3. Cantor showed that there is also a coherent notion of cardinality for infinite sets. In particular, the cardinality of any set that exhibits a one-to-one correspondence with the set of natural numbers, $\{1, 2, 3, 4, ...\}$, is denoted by $\aleph_0$, which utilizes the first letter of the Hebrew alphabet. Cantor showed that there exist infinite sets, such as the set of real numbers, that do not exhibit a one-to-one correspondence with the natural numbers, and hence are regarded as having a larger cardinality.

Note that since there is a one-to-one correspondence between the natural numbers $\{1, 2, 3, 4, ...\}$ and the evens $\{2, 4, 6, 8, ...\}$ the cardinality of the evens is $\aleph_0$. Similarly, the cardinality of the sets $\{1, 3, 5, 7, ...\}$ and $\{2, 3, 4, 5, ...\}$ are also $\aleph_0$. This leads to Craig's perceived absurdity. In a fully occupied Hilbert's hotel, if all of the guests in the even rooms would check out, this would leave the guests in the odd rooms. Thus, one finds that subtracting infinity (even-numbered rooms) from infinity (all rooms) gives infinity (odd-numbered rooms), which Craig denotes with the expression

$$\aleph_0 - \aleph_0 = \aleph_0. \tag{1}$$

However, if instead in the fully occupied hotel, all of the guests except for the guest in room 1 check out, then one has subtracted infinity (rooms 2,3,4,. . .) from infinity (all rooms) and found a remainder of one (room 1), which Craig denotes by

$$\aleph_0 - \aleph_0 = 1. \tag{2}$$

Together, equations 1 and 2 produce Craig's absurdity. He explains, "the contradiction lies in the fact that one can subtract equal quantities from equal quantities and arrive at different answers" and again he stresses "we subtracted the *identical number* of numbers from the *identical number* of numbers and yet did not arrive at an identical result" [2, p. 206].

One might attack Craig's argument by averring that an infinite series of moments is unlike an infinite collection of objects (or hotel rooms) since one cannot remove past moments the way that one can remove objects from a collection (or have people check out from a hotel). Therefore, while his argument may succeed against an actual infinite physical collection of objects, it does not succeed against an actual infinite series of past events. Aristotle held this position, rejecting the existence of an actual infinite while believing time was past eternal. Or one might seek to distinguish physical moments comprising the history of the universe from metaphysical moments comprising the life of God to avoid the theological implications discussed in the introduction. However, even granting to Craig that there is a way to remove moments (if only mentally), physical or metaphysical, from an infinite series to attempt to recreate this contradiction, his argument is unsuccessful, for his claimed contradiction is based on a confused understanding of cardinal numbers.

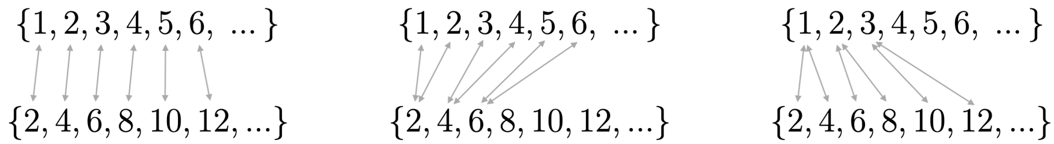

Figure 2: One-to-one, two-to-one, and one-to-two correspondences between the natural numbers and evens. Do these sets have equal, half, or double the elements of one another?

Wes Morriston identifies this problem in his critique of Craig's argument. In their exchange, they discuss a variation on Hilbert's hotel involving checking out books from an infinite library containing books numbered 1, 2, 3, 4, and so on. Morriston points out that when one checks out a subcollection of books, "we are not actually subtracting *numbers* at all" but are removing a subset from a set "and then determining the cardinality of the subset that remains" [9, p. 151]. Yet Craig maintains,

> In arithmetic, subtraction consists of removing the diminution of a given quantity (the minued) by another quantity (the subtrahend) to yield another quantity (the difference or remainder). This appears to be precisely what is going on when one removes various quantities of books from the quantity of books shelved in the library [2, p. 209].

Here Craig has revealed that he is equivocating cardinality with quantity. While it is true that the cardinality of a finite set indicates the number of elements in the set, this is not how the cardinality of an infinite set is defined. Rather, as indicated above, the cardinality of an infinite set is defined in terms of one-to-one correspondence and does not invoke the ill-defined notion of the "number" (or "quantity") of elements in an infinite set. Indeed, there is good reason to resist speaking of the "number" of elements in an infinite set. Consider the sets $\{1, 2, 3, 4, ...\}$ and $\{2, 4, 6, 8, ...\}$. As discussed above, there is a one-to-one correspondence between these sets. Yet, as shown in Figure 2, there is also a two-to-one correspondence between the sets: 1 and 2 in the first set pair with 2 in the second set, 3 and 4 pair with 4, and so on. If one maintains, as Craig seems to, that since there is one-to-one correspondence between the two sets they must have the "same number" of elements, then one should also maintain that since there is a two-to-one correspondence between the sets there are "twice as many" natural numbers as evens. Going further, there is also a one-to-two correspondence, pairing 1 from the set of naturals with 2 and 4 from the set of evens, 2 with 6 and 8, and so on. Thus, one would conclude that there are "twice as many" evens as there are natural numbers! Indeed, this way of talking produces absurdities, but the absurdity arises not from the existence of infinite sets or the well-defined notion of cardinality. Rather, it arises from mistaking cardinality as a count of the "number" or "quantity" of elements in an infinite set.

It is an abuse of notation to treat $\aleph_0$ as denoting the number of elements in an infinite set, the number of books in an infinite library, or the number of rooms in Hilbert's hotel, yet this is precisely how Craig utilizes it. In doing so, he misapplies a concept true for finite cardinality (that it represents the number of elements in a set) to infinite cardinality. Almeida offers a similar critique, noting that Craig applies a "principle of arithmetic [that] is true for domains of finite numbers but false [for] transfinite" domains [1, p. 50]. Nevertheless Craig maintains, "In transfinite arithmetic, inverse operations of subtraction and division are prohibited because they lead to contradictions; but in reality, one cannot stop people from checking out of the hotel if they so desire!" [2, p. 204] But the problem is not that one cannot remove (physically or mentally) an infinite subset of books from an infinite set of books, rather it is that removing those books does not correspond to subtraction of *quantities* but to the subtraction of *sets*. Removing the even books from the collection of all books can be represented by the set operation

$$\{1,2,3,4...\} - \{2,4,6,8,...\} = \{1,3,5,7,...\}. \quad (3)$$

And removing all books numbered 2 and larger from the collection of all books is represented by

$$\{1,2,3,4...\} - \{2,3,4,5,...\} = \{1\}. \quad (4)$$

Now Craig wishes to move from the subtraction of sets in equation 3 to $\aleph_0 - \aleph_0 = \aleph_0$ (equation 1) and from the subtraction of sets in equation 4 to $\aleph_0 - \aleph_0 = 1$ (equation 2) to derive an absurdity. But what is his justification for this? Craig merely asserts, "Equivalent sets [in one-to-one correspondence] are regarded as having the same number of members,"[2, p. 200] without defense or citation.

Craig may respond that having the same cardinality is precisely what he means when he asserts that two infinite sets have the same quantity of elements. But then his objection that "we have subtracted identical quantities from identical quantities and found non-identical remainders" becomes "we have subtracted two sets with the same cardinality from two sets with the same cardinality and found remainders with differing cardinality." And where is the absurdity in this? Rather, this is precisely the case in equations 3 and 4 and it is hard to see why Craig would consider these statements contradictory.

For Craig, affirming that there is a one-to-one correspondence between the full set of hotel rooms (or books) $\{1,2,3,...\}$ and the subset of even rooms $\{2,4,6,...\}$ undermines Euclid's maxim that the whole is greater than the part.

> One maintains that the whole is greater than a part, while the other maintains that the whole is not greater than a part. But which principle is to be scarified? Both seem to be intuitively obvious principles in themselves, and both result in counter-intuitive situations when either is applied to the actual infinite. The most reasonable approach to the matter seems to be to regard both principles as valid in reality and the existence of an actual infinite as impossible [4, p. 24].

But there is only tension between Euclid's maxim and a one-to-one correspondence between infinite sets if one equivocates on the meaning of "greater." The set $\{1,2,3,...\}$ is greater than the set $\{2,4,6,...\}$ in the sense that the latter is a proper subset of the former. Meanwhile, the set $\{1,2,3,...\}$ is not greater than the set $\{2,4,6,...\}$ in the sense that one can find a one-to-one correspondence between them. From these, Craig wants to deduce that the first observation indicates that there are fewer evens then there are natural numbers and from the second observation that there are the same number of both, but neither statement says anything about the relative number (or quantity) of elements in the sets as, again, these are ill-defined notions for infinite sets.

Yet Craig insists that "while the actual infinite may be a fruitful and consistent concept within the postulated universe of [mathematical] discourse, it cannot be transposed into the spatio-temporal world, for this would involve counterintuitive absurdities" [2, p. 210]. This notion of "transposing" mathematics unto the real world might explain Craig's instance on treating the cardinality of infinite sets as denoting the "number" or "quantity" of elements in the sets. In particular, he could maintain that there is some truth about the number of elements in any real-world collection. Then, he could argue that one can produce the contradiction between Euclid's maxim and the Cantorian property that an infinite set has a subset in one-to-one correspondence with it. But it is incumbent on Craig to defend this position and, in doing so, to clearly define what he means by the number of elements in an infinite set. Otherwise, Craig has merely shown that the same counterintuitive features of the infinite that feature in mathematical discourse also feature in a real-world (physical or metaphysical) infinite collection, but this is not reason to declare it an "absurdity" or treat it as a contradiction. Certainly, there are many counterintuitive and paradoxical features of the world.

Craig traces his argument against an infinite past to the major 11th century Persian Muslim theologian Al-Ghazali. Instead of an infinite hotel or library, Al-Ghazali considered the orbits of Earth and Saturn around the Sun. He argued in *Al-Iqtisad Fi Al-I'tiqad* that if the past is infinite and both had been orbiting the Sun for eternity, then both would have orbited an infinite number of times. However, Al-Ghazali noted that astronomers had discovered that the Earth completes approximately 30 orbits in the time it takes Saturn to complete one. Thus, he reasoned, both the Earth and Saturn have orbited the Sun the same number of times (infinite), yet the Earth has rotated the Sun 30 times as often as Saturn has—an absurdity! While Craig's use of Hilbert's hotel is more sophisticated, his argument suffers from the same error as Al-Ghazali's: both treat infinity (or $\aleph_0$) as a number and therefore claim a contradiction when comparing two infinite sets of equal cardinality.

Impossibility of forming an actual infinite via additive succession

Craig offers a second philosophical argument against an infinite past which he formalizes as follows:

1. The series of events in time is a collection formed by successive addition.
2. A collection formed by successive addition cannot be actually infinite.
3. Therefore, the series of events in time cannot be actually infinite.

A version of this argument also traces back to Al-Ghazali and was embraced by Immanuel Kant. Craig describes Premise 1 as obvious, arguing, "The past did not spring into being whole and entire but was formed sequentially, one event occurring after another" [2, p. 211]. Elsewhere Craig states:

> It is important to understand exactly *why* it is impossible to form an actual infinite by successive addition. The reason is that for every element one adds, one can always add one more. Therefore, one can never arrive at infinity [4, p. 31].

However, the proponent of an infinite past (physical or metaphysical) does not affirm that there was a moment in the past when the collection of prior moments "arrived" as an infinite set. Rather, if the past were infinite, then at every past moment, the collection of prior moments would be infinite. Now Craig maintains that "the only way in which an actual infinite could come to exist in the real world would be to be instantiated in reality all at once" [2, p. 212]. But here by "come to exist" Craig seems to beg the question, for the proponent of the infinte past does not believe that there was any moment when the collection of prior moments became infinite, rather it always was infinite.

Nevertheless, Craig offers a more sophisticated argument against an infinite past. He maintains that the view "that the infinite past could have been formed by adding one member after another is like saying that someone has just succeeded in writing down all the negative numbers, ending at -1" [2, p. 214]. That is, Craig argues that if there have been countably infinite hours (or events, moments, etc.), then there would exist a one-to-one correspondence between past hours and negative numbers, so one could imagine such a countdown. Craig then concludes that such a countdown ending at this hour is absurd for it could have just as well been completed at a previous hour, for then it was also the case that there exists a one-to-one correspondence between past hours and negative numbers. "We could ask, why did he not finish counting yesterday or the day before or the year before? By then an infinite time had already elapsed, so that he should already have finished by then."

One can avoid the implied absurdity in Craig's question by pushing back on his claim that a one-to-one correspondence between an infinite past and the negative numbers constitutes the existence of such a countdown. But even if one grants Craig that there could exist such a countdown of past hours (or moments), if only as a mental imagination and not in the real world, one can still resolve Craig's question. To this end, Morrison incisively notes:

> The Principle of Correspondence entails at most that all the numbers could have been counted by now, not that they would have been. The proper response is therefore to say, "Yes, there could have been a counter who wouldn't be finished until next year or a hundred years from now. But there could also be one who is finishing now" [10, pp. 70-85].

That is, we can grant Craig that it *could have been* possible for an eternal counter to count down through all the negatives throughout eternity until this hour if this was the hour that the counter had been counting down until. And it *could have been* possible for a counter to count down to some other hour, if that were the hour the counter had been counting down to. But there is no contradiction in simply observing that a counter *could* have counted down to this or some other hour.

Craig responds by advancing his argument one step further. He notes that there is a one-to-one correspondence between the hours counting down to now and those counting down to some prior hour to argue that there does not appear to be a *sufficient reason* why the countdown ended now rather than at some other hour. That is, if the counter could have counted down to another hour, what explains why it counted to this hour rather than the other? But it is unclear how simply pointing to the existence of a one-to-one correspondence is enough to produce an absurdity. Indeed, one could make a similar objection to a finite countdown, say of 24 hours, terminating at this hour: there is a one-to-one correspondence between the 24 hours leading to this hour and the 24 hours counting down to any other hour. Now Craig may respond that the difference between the finite and infinite cases is that in the finite case, what explains the difference in when the countdowns terminate is the difference in when the countdowns started whereas in the infinite case there is no such difference as there is no starting moment. Indeed, the infinite countdowns have no starting moment, but they are readily distinguished by the fact that at every past moment they have different counts remaining. That is, one infinite countdown ends at this hour and another at another hour precisely because at all prior moments the one countdown was counting down to this moment whereas the other was counting down to the other moment. To revive his argument, Craig must argue why this obvious difference between the two imagined infinite countdowns is not sufficient to account for why one ends at one moment rather than another.

There is a theological formulation of the infinite countdown. In particular, if God has an infinite past and exhaustive foreknowledge, then one might regard Craig's eternal countdown thought experiment as occurring in the mind of God, for throughout the everlasting past God would have perfectly anticipated the arrival of this moment, and every other one, as "known to God from eternity are all His works" (Acts 15:18). It is not clear, though, if such a divine countdown should be regarded as extending infinitely far into the past. On the one hand, whatever their view of divine foreknowledge (determinism, Molinism, open theism, etc.), Christian theologians generally agree that prior to God's decision to create, the future was open with countless possible timelines that God could potentially act to bring about, including this one which God did create. Under this view, God's foreknowledge of the present would only extend finitely far back to God's decision to create. On the other hand, theologians typically affirm that prior to the decision to create, God had natural (and, for the Molinist, middle) knowledge of what God *could* possibly (and feasibly) create and, thus, if God has an infinite past, there could have been in the mind of God an eternal anticipatory countdown to this (possible) present moment and all other (possible) moments. Whatever view one takes on these questions, in light of the above discussion, there appears to be no absurdity that arises from the possibility of such an eternal anticipation of the present in the mind of God.

Conclusion

To summarize, Craig's two arguments in support of temporal finitism seem to largely break down upon inspection. Firstly, while Hilbert's hotel and the infinite library reveal the surprising, even counterintuitive, nature of the infinite, there is nothing inherently absurd about them. The only absurdity that arises is from Craig's abuse of the cardinal $\aleph_0$, treating it as representing an ill-defined notion of a literal count of the number (or quantity) of elements in an infinite set. Thus these examples fail to support the premise that an actual infinite cannot exist. Secondly, the claimed impossibility of forming an actual infinite through additive succession also appears to be an unconvincing argument for temporal finitism. Therefore, while cosmological evidence may favor a beginning of the physical universe, the possibility of an eternal metaphysical (or divine) past remains open.

Granted, contemplation of God enduring an infinite series of moments prior to creation raises a host of theological questions that we do not take up here. There exists much debate within Christian theology and the philosophy of religion on how to understand God's relationship to time, with some affirming that God is timeless and others affirming that God experiences temporal succession. This disagreement exists even among Christians with a high view of Biblical revelation; certainly, it is possible that some questions about God's relationship to time – especially prior to creation – may be underdetermined by the text of Scripture. Therefore, there may be a role for philosophical argumentation to help articulate and develop one's doctrine of God. In this article we have seen the role that mathematics can play in such philosophical arguments. Nevertheless, we would be wise to confess with Augustine in his reflection of past eternity, "I know not what I know not" (*Confessions*, Book XI Chapter 12).

If in fact God has existed temporally from past eternity – "from everlasting" – what would characterize the infinite divine life prior to creation? Here we might be aided by a social view of the Trinity, with God the Eternal Father, God the Eternal Son, and God the Eternal Spirit relating to one another in acts of giving, receiving, and sharing in love that filled everlasting (metaphysical) time prior to creation. Indeed, Jesus seemly refers to these pre-creation relations of love when he prays to the Father, professing "you loved me before the foundation of the world" (John 17:24). Under such a reading, only God has experienced the fullness of an actual infinite past of intra-Trinitarian love, yet humanity is invited to participate in the ever-growing, potential infinite future of this love. Hence, the Son prayed to the Father, "you sent me and loved them even as you loved me" (John 17:23).

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Ethics as Instruction

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Jeremy Case is a professor of mathematics at Taylor University and holds a Ph.D. from the University of Minnesota and a M.A. from Miami University. He has attended every ACMS conference from 1995 to 2024, and he was president of the ACMS from 2015 to 2017. Recently, he served as chair of the Special Interest Group of the Mathematical Association of America on Statistics and Data Science Education (SIGMAA SDS–ED).

Abstract

As a mathematician who has had to transition to teaching statistics, I have tried to incorporate recommendations from the statistical education literature to make my course more like statistics than like mathematics. This article will share my journey in teaching statistics and the features of an ethics assignment I inherited and adapted. One of the perceived benefits of the ethics assignment was its ability for students to communicate statistical approaches and practices.

1 Introduction

When people hear that I am teaching statistics, I often hear comments such as, "You can make statistics say anything you want," or "75% of the statistics are made up on the spot." Even though I have always heard them, I have become increasingly bothered by the dismissive tone. Perhaps because I identify more with statistics or perhaps because there is a growing mistrust of statistics and science [12], I do not want such statements to go unchallenged.

Do students perceive the discipline of statistics as untrustworthy or a field of falsehoods? Is statistics so arcane and mysterious that denigrating it is a defense mechanism? I have wondered if such views act as disincentives to learn statistics.

This reflection traces my venture in teaching statistics and my ambivalence in teaching ethics with a single assignment. My journey involves this organization between my first ACMS conference to the variety of workshops the ACMS provided. This article is not intended to show how successful I was in terms of leading students toward ethical behavior because there is evidence to show there can be little or no correlation between ethics instruction and ethical practice. I do not have the evidence for future behavior. Instead, I am proposing that a focus on ethics may help motivate students towards understanding, or at least articulating, the statistical process.

Now when people say, "You can make statistics say anything you want," I respond, "Just like words." What principles we employ verbally and statistically matter.

2 Should Mathematicians Teach Statistics?

In the spring of 1995, I was finishing up my mathematics Ph.D. and had accepted a position at Taylor University. I knew my courses for the year and viewed Introductory Statistics as the easiest to teach. I must have thought that determining the statistical test, plugging in formulas, and looking up values in a table were not going to be that difficult. My first ACMS conference would radically change my perspective of statistical instruction.

David S. Moore gave two plenary talks at the 1995 ACMS conference at Taylor University [15, 16]. A statistician at Purdue University, Moore reiterated the challenges he had been making for more than a decade to the mathematical community regarding statistical instruction. As an example, Moore had hosted a forum [17] in the *College Mathematical Journal* with the provocative question, "Should Mathematicians Teach Statistics?" to incite serious discussions related to the teaching of statistics at the undergraduate level. While a mathematical science, statistics is not a subfield of mathematics and so should not be taught as such.

One of Moore's ACMS talks focused on the *content*, or what we want our students to learn. The second was on *pedagogy*, or how we want students to learn. These presentations mirrored the recommendations produced by a joint committee of the American Statistical Association and the Mathematical Association of America:

> Any introductory course should take as its main goal helping students to learn the basics of statistical thinking [9, p. 5].

These include the need for data, the importance of data production, the omnipresence of variability, and the quantification and explanation of variability. The report emphasized the subject matter of statistics is data, and students should experience data with the tools that organize a statistician's approach to data.

Moore advocated for a greater dialogue between models and data. Inappropriate mathematical approaches to statistics focus on the probability–based formal inference theory at the expense of the design of data production and exploratory data analysis. A theory–driven, probability–based presentation of statistics does have a respectable basis in trying to avoid presenting statistics as magic. However, examining the data and its production can undermine proposed models. Moore modified the phrase attributed to George Box, "All models are wrong, but some are useful." Instead, Moore said, "Mathematical theorems are true; statistical models are effective when used with skill." [16, p. 4].

To me, these statements were radical challenges. With greater computing power and the increasing use of statistics in other disciplines and in non-academic professional settings, statistical practice had changed and so should its instruction. As is today, the majority of statistics courses were being taught by mathematicians, and I was going to fall in line by teaching statistics the traditional way. In my defense, the first AP Statistics test would not occur for another two years. Moore's textbook, *The Basic Practice of Statistics* [14], implementing his ideas was just being published that year. The use of graphing calculators was still a novelty and a common job interview question for mathematical hires. Statistical software such as SAS and SPSS were command based and difficult to use, let alone teach.

I was overwhelmed with how to teach my introductory statistics courses. What I had heard at that ACMS conference was "Mathematicians should not teach statistics." Perhaps what I should have heard was "Mathematicians should not teach statistics like mathematics."

3 Should Mathematicians teach Ethics?

The answer to this question seems to be obviously "yes," but the pedagogy to teach ethics is not so clear. Done inappropriately, ethical instruction may have little or even negative effects on ethical behavior.

My ambivalence towards ethical instruction arises from anecdotal evidence. My friend's teenager took a freshman business ethics course at another faith-based institution. According to her, while the course was quite helpful in presenting ethical dilemmas, the course inadvertently provided ways in how to get away with illegal practices. She also witnessed several students cheating on the Business Ethics final exam. As another example, a social work colleague told me of post-test results from a class wrestling with diversity issues showed that some students had greater stereotypes than before the class. (They did not publish these results.)

Christian Smith and Patricia Snell's *Souls in Transition: The Religious and Spiritual Lives of Emerging Adults* [18] outlines another challenge in teaching ethics at the college level. Exploring the spiritual lives of American teenagers as they enter their twenties, the book reports that nearly all emerging adult respondents said that knowing right or wrong in life is easy. Just pay attention to their feelings when such a situation arises:

> The vast majority are moral intuitionists —that is, they believe that they know what is right and wrong by attending to the subjective feelings or intuitions that they sense within themselves when they find themselves in various situations or facing ethical questions [18, p. 46].

In a complex world, ethical decisions are not always so clear cut to be trusted to what feels right. Some emerging adults say difficult moral problems can be averted by structuring their life in a way to avoid them in the first place [18, p. 46].

Furthermore, studies on ethical instruction are mixed. For example, a meta-analysis of ethics instruction in the sciences by Alison L. Antes *et al.* [2, pp. 2–3] identified three frameworks commonly used to teach ethics.

1. **Ethical Sensitivity** focuses on an awareness of the ethical implications and an empathetic understanding of how others might be affected by a situation.

2. **Developmental Nature of Ethical Behavior** emphasizes the philosophical nature of moral dilemmas with the idea that a higher level of moral development will translate into improved moral reasoning and ethical behavior.

3. **The Cognitive Nature of Moral Reasoning** places greater emphasis on the need to think through and analyze complex ethical problems before responding. Moral reasoning is a function of how one thinks through an ethical problem, and ethical behavior improves as ethical problem–solving and decision–making skills are enhanced.

The report concluded ethics instruction in the sciences is "at best moderately effective as it is currently conducted:"

> [A]lthough there appears to be a general consensus about the importance of ethics education for researchers and scientists, there is little agreement about the most effective approach to instruction, or even the most appropriate goals for these programs. . . . Some ethics courses

have been shown to induce the desired effects, whereas others indicate little or no effects of ethics instruction on learning outcomes [2, p. 2].

There is evidence ethical instruction does have the potential to be effective if carefully designed and evaluated. How can one assignment overcome great challenges to lead towards ethical behavior? How does someone in the mathematical sciences design such instruction?

4 Retooling and the Promising Syllabus

After teaching Introductory Statistics for a handful of years, I had a break from teaching statistics until my colleague Ken Constantine retired in 2015. After failing to hire a statistician, the department assigned me to the calculus–based regression course Advanced Statistics. Advanced Statistics is a second semester course with the prerequisite being either Introductory Statistics or Mathematical Statistics. There are typically 12–25 students in the course with a large percentage of majors in business or finance but also with majors in math and computer science. The textbook is *Stat2: Modeling with Regression and ANOVA* [6].

With the continual development of statistical education, I recognized I needed more training. I spent part of a Sabbatical looking through the statistics education literature such as the GAISE report [5]. From there I tried to change my instincts as a mathematician to think more like a statistician by emphasizing the context of the data, experimental design, and what conclusions can be drawn from studies. Following David Moore, I strove to present an intellectual framework that made sense of the statistical tools in analyzing and describing data, producing data, and inference from data. I wanted to emphasize statistical thinking, conceptual understanding, integrate real data within its context and purpose, and foster active learning.

For the syllabi in several of my classes, I prepare what Ken Bain and James M. Lang call the "promising syllabus" [4, p. 74], [13]. While the syllabus includes all the procedures and regulations of a typical syllabus, a major part of the promising syllabus is to lay out the promises or opportunities the course offers to students. What kind of questions will the course help them answer, and what abilities would the course help them develop? The focus shifts from what will be covered to what students will take away from the course.

My Advanced Statistics syllabus frames statistics as a course in epistemology, rhetoric, and ethics. That is, how do we know? What can we trust when we have imperfect knowledge? I try to make the distinction between mathematics and statistics myself: "... [T]his course tries to determine what variation can be attributed to chance and what correlations can be made between phenomena. Unlike many other mathematical courses, statistics does not rely on deductive reasoning. Instead, statistics employs inductive reasoning from certain assumptions and requires an ability to read, interpret, and write" [8, p. 1].

Statistics is a means of rhetoric and a language to communicate experimental or scientific results which are not always straightforward. "There are ways to represent data which can favor one point of view over another. There are shades of uncertainty that must be taken into account before making life changing decisions. We must examine the assumptions and context the data explore." [8, p. 1]. Due to inherent certainty, this course is also about integrity and ethics. How can we be faithful in graphical and statistical representations when we are tempted to circumvent undesirable interpretations?

Ken Constantine was the one who gave me the insight to recast ethics as a foundation of statistics. I quote him in my syllabus:

> Statistics presupposes that there is a reality to be discovered but that it is disclosed to us only via limited, imperfect, variation-laden data. In the face of such randomness, we seek to make inferences which correspond to true reality. Thus we don't know if what we're saying is true or false, but our goal is correspondence with reality and an explicit understanding of the extent of our uncertainty. ...Statistics culture has a deeply ingrained sense of ethics, primarily in letting data speak for themselves. Any statistician who appears to be using methods as a persuasive weapon would be frowned upon. The motivation for this sense of ethics is usually articulated as both professional integrity and concern for the implications of conclusions we help researchers to draw. For some statisticians, the basis for these ethical concerns may be religious; for others it may not be religious. In any case, the ethical concern is pervasive and deep in the Statistics community [10, p. 4].

The promising syllabus also proposes what knowledge of skills students will gain by the end of the semester. My syllabus mentions/talks about the applications of the various statistical topics, but I also include that students will have opportunities to deepen the connections of integrity with the Christian faith as we explore the tenuous nature of our knowledge. With the power of statistics and data analysis, the students and I must become a team, and learning is our shared responsibility. "From everyone who has been given much, much will be demanded; and from the one who has been entrusted with much, much more will be asked." Luke 12:48b (New International Version).

Finally, Bain posits that the promising syllabus begins a dialogue in which students understand the nature and progress of their learning [4, p. 75]. A syllabus can become a powerful influence on setting high standards and encouraging people to achieve them.

My purpose for including these statements in the syllabus was for students to avoid cheating, to do their best work, and to not change the p-value to suit their needs. It did not occur to me that ethics would help in presenting the practice of statistics.

5 Ethical Challenges in Statistics

There is no shortage of evidence of the intentional misuse of statistics. Data manipulation, p-hacking, and outright falsifying data are too common. Michael Stob at the 2017 ACMS conference at Charleston Southern, presented "P-values Considered Harmful." He included the story of the study regarding the effectiveness of power posing [7]. The calculations of p-values were suspicious, and others had trouble replicating the study.

FiveThirtyEight, a website known for its analysis of opinion polls and sports, used to provide within its story "Science Isn't Broken" a tool in which one could tweak variables until it demonstrated that Democrats were better than Republicans for the economy [3]. One could also tweak the variables until the p-value was low enough to show Republicans were better.

In the same article, they referenced a study where 20 research groups were given the same data, but came up with different conclusions. The variability was not due to unethical or sloppy work. The researchers were highly competent. Researchers must make subjective choices.

Since Advanced Statistics involves model–building, students are exposed to the idea of confounding variables masquerading as an effect of interest. Regression lends itself to control for latent variability in order to more clearly discover the effect or differences that really do seem present. The hope is that they would understand and be able to discover interaction effects which are present in general public policy discussions. Since there are judgment calls to be made, ethics enters the picture.

6 The Ethics Assignment

The Advanced Statistics course included an ethics assignment which I had inherited from Ken Constantine. The assignment consisted of reading the American Statistical Association's "Ethical Guidelines for Statistical Practice" [1] and an article by Vardeman and Morris, "Statistics and Ethics: Some Advice for Young Statisticians" [19]. I adapted the assignment to include a reflection paper and set up several checkpoints. In order to prepare for the class discussion, students would need to post on our learning management system and come to class with the following summary statements in hand: two summary sentences regarding the ASA Code, two summary sentences regarding the Vardeman/Morris article, two insights or questions, and two connections to their discipline. One of the benefits to this approach is that students have something to contribute from their summary statements. Comments connected to their discipline provided more personal incentives. After the class discussion, their statements should help in writing the reflection.

The ASA ethical guidelines recommend statisticians to act in good faith. They should employ appropriate sampling methods, disclose conflicts of interest, and accept full responsibility for professional performance. They should acknowledge their assumptions, the limits of the statistics, and the financial sponsor. With regard to potential misconduct of others, statisticians should recognize that differences of opinion and honest error do not constitute unethical behavior. More broadly, statisticians should treat research subjects, colleagues, and other statisticians appropriately—in the vein of what some Christians could call "a right relationship with others."

The main point of the Vardeman article is their proposition that statistics is about integrity:

> The vital point is that this discipline provides tools, patterns of thought, and habits of heart that will allow you to deal with data with integrity. At its core statistics is not about cleverness and technique, but rather about honesty [19].

Vardeman and Morris propose that statistics provides a framework "for dealing transparently and consistently with empirical information from *all* fields" [19, p. 21]. Statistics also has "ways of avoiding being fooled by both the ill intent (or ignorance) of others" [19, p. 21].

The article provides other advice for statisticians summarized here.

1. Never take advantage of your peers.
2. Do not whine.
3. Work on weaknesses rather than absolve them.
4. Do not denigrate the strengths of others.
5. If you submit work, it will be complete and represent your best effort.
6. To have integrity in your field, you must be knowledgeable about the system of study.

7. Recognize the limitations of [statistics].
8. Perform analyses which allow interpretations which are tenable but not popular in your organization.
9. Ethical [statistical] practice requires that you take responsibility for acquiring substantive understanding.
10. You cannot do this unless you have strength of character and integrity which takes the ethic of self-reliance, thoroughness, and hard work.
11. Understand fully what your assumptions say.

To write their responses, students are given leading questions such as how the ethical issues in statistics relate to their own discipline, values which seem to be most crucial, and examples of statistical sins of commission and omission.

7 Surprising Results

I expected comments regarding ethical decisions and how ethical students plan to be. What took me by surprise was that students were making the connections of the ethics assignments to the *practice* of statistics. Thinking of statistics as an ethical practice is a novel idea to students. They have heard how you can make statistics say anything you want and how to mislead with graphs and diagrams. Many are skeptical of a discipline which can be mysterious and arcane. Based upon their homework, I wonder many times if they are developing the dispositions and habits of statistical thinking.

Based upon what they wrote in the ethics assignment, I see evidence that they are able to *articulate* the importance of statistical approaches and procedures. I am not sure when they began the course they fully grasped the larger purposes of statistical procedures as a means of curtailing bias. Seen in the light of ethical behavior, the checks and balances in statistics serve a purpose.

Here was one response regarding the ethics assignment:

> A lot of the thoughts that we discussed and wrote about were not revolutionary new ideas, but rather logical thoughts that I had never connected with ethical statistical practice before. I now have come to realize the importance of including all relevant information when reporting statistical conclusions and how intentional one must be in pursuing what the data reveal.

A few reported a greater appreciation for statistics:

> Something I have never thought about during my time in this class was the deeper meaning of what it is to be a statistician. At its core, statistics is not about how you can cleverly present data or the techniques you employ to come up with your conclusions. To be a statistician is to be a pursuer of truth.

Another put a positive light on statisticians:

> Reading these articles really gave me a deeper appreciation for people that pursue statistics as a field of study and career. One quote that really struck me was, "Integrity is a pattern

> of life, not an incident." You don't become a person of integrity just by going through life. You have to actively pursue it and make it who you are at your core. Integrity does not just apply to people who work in statistics.

Many students reported "surprise" and "new thinking" in either their ethics report on their reflection at the end of the term. Others mentioned the importance of data in context. A student talked about the ethical role of the person doing statistics and the context of the data:

> The advice given in the Vardeman and Morris article was interesting and surprising. I used to view statistics as a profession that is amoral, removed from ethical interpretation... By saying a statistician has a moral role, we have a responsibility much greater than just the data. The data tells a story, and we must convey all aspects of the data and its story well in order to be a moral statistician. Numbers are not amoral because numbers correspond to people and their creativity which is inherently moral.

7.1 Data in Context

A common theme in distinguishing statistics from mathematics is the emphasis of data in context. A computer science major wrote:

> This is why I think it is important that programmers understand that there are limitations to statistics and technology and to be open about the categories of data that is being used. Furthermore, data does not exist in a vacuum. Data has a context and a programmer is likely to at some point work in a field where they do not understand the context.

A political science/philosophy major discussed interpreting the results echoing fragments from David S. Moore.

> In no scenario are numbers just numbers. A set of numbers or an equation must always be interpreted by someone to make them have any value. It is this interpretation and then reporting of the conclusions that the ethics of statistics is derived out of. When I encounter problems with statistics (especially probabilities or numerical data that so many politicians often present as fact), I now look at it with healthy skepticism. I ask things like "How was this sampled?", "Where did it come from?", or "What statistical process was used to obtain this number?"

There is also a recognition that truth is not as clear cut in statistics as in mathematics.

> There are measures in place that provide a guideline in statistics of what is generally important and what isn't generally important but there is a lot of "grey" area in statistics.

Another student cited ethics as a motivational tool:

> I have learned mainly from the readings and the discussions that ethics in Statistics ... takes a much more abundant look at the why of statistics. Without looking up any kind of definition, I am willing to claim that statistics is all about using quantifiable knowledge to interpret truth and use it to benefit those in society. A Christian worldview will say that it is to glorify God. It really gave me an eye-opening paradigm shift about the purpose of statistics, because it's not about avoiding the wrong thing as that could be considered depraved indifference

> if statistical knowledge could have been used for good. I am now using my new views on ethics in stats and in life to be diligent in doing the right thing even if I know I'm not doing the wrong thing. It has given me a sense of duty in my intentions and in my prevention of ignorance. I hope to find much more purpose in my future job and life.

Here are some of the strengths I found from the ethics assignment: Multiple interactions with the ideas, connection to student's own field of study, student articulation of their perspectives with a personal response, and a response to claims rather than generating claims.

The statistics education literature promotes real world data with purpose as a motivation for learning statistics. Ethics may be another motivator for some:

> If I'm being completely honest, I have one huge problem with this article. And that problem is that we didn't read this at the beginning of the year. The way I do work is a very holistic approach, where if my heart and passion is behind a project or work, I am really able to put forth a lot of effort and not think too much about getting tired! This happens because I'm genuinely invested in the work I'm doing at that point and the work becomes fun and purposeful. However, if the work that I'm doing is not connected to a larger purpose I lose interest and motivation quite quickly. When I read articles like this one it establishes a clear goal that shows that the work that I am doing is bigger than just turning in homework in Blackboard.

7.2 Being Knowledgeable is a Key to Ethics

The 2022 Higher Education Research Institute at UCLA study reported that 59% of Taylor University and first-time freshmen reported "Becoming an authority in my field" was "Not Important" or "Somewhat Important" [11]. According to Vardeman and the ASA, to be ethical requires being knowledgeable and an expert in your field. Just as a tax accountant needs to have a deep understanding of the tax code to determine what is appropriate, a statistician must be dedicated to being the best statistician he or she can be and to know and understand the assumptions and procedure in order to make right choices.

Several students articulated the importance of expertise with regards to ethics. One student connected this idea to both statistics and his field of study:

> The course has taught me the immense power that exists within statistics. Along with this is the power that statistics can hold in swaying opinion both personally and societally. The ethics discussion and the finance projects helped me truly see how important statistics can be… Moving forward in my career, I see the importance that having a solid understanding of statistics has. I also see the importance of analyzing statistics in an ethical and moral way that is not meant to deceive.

Other students summarized the need for expertise in their own fields:

> My favorite piece of advice was, "to have integrity in your field, you must be knowledgeable." This is absolutely true, and it definitely applies to my field of finance.

> The Ethical Guidelines from the American Statistical Association state in almost every list of responsibilities for the ethical statistician that they must know the procedures and know the moral hazards. This means that acting ethically in this line of work is not just about making the right decisions, but also knowing what the ethical guidelines and boundaries are.

While there still may be a question of the will to act rightly, these students were able to articulate the need for being knowledgeable in order to make right decisions.

8 Observations and Caveats

Unfortunately, the evidence I am providing is at best a retrospective study. Over the years teaching Advanced Statistics, I did not notice these patterns until my last year of teaching the course. Rereading the responses of earlier years, I found similar responses. Since my teaching assignment has changed, I no longer am teaching the class. I have been unable to design a better observational study such as having giving a pre-test of students attitudes towards statistics.

My main claim in this paper is that an ethics assignment within a statistics class encourages a better understanding of statistical concepts, procedures, and dispositions. It can also motivate students to study the discipline.

I am not willing to conclude this assignment will lead all students to ethical behavior in the future—particularly when facing complex issues involving artificial intelligence, big data, and dwindling resources. A carefully planned curriculum with strong pedagogical practices would have a better chance of influence long term behavior. Just because someone says they will be moral does not make it so. I am reminded of Matthew 21:28–31 and the story of the son who says he will work in the vineyard but does not. I am not sure how one tests for future behavior.

On the other hand, my claim is less lofty. Students were able to provide positive articulations in areas of statistics and ethics. They were able to report the importance of data, data collection, and the accounting of variability. They conveyed how statistical integrity impacts their personal and professional interests. In the larger context of other ethical, theological, and spiritual imprints on their lives through other classes and experiences, the assignment, however small, may be a contributing factor towards ethical and godly behavior.

With their potential pitfalls, statistics and ethics require purposeful and planned instruction. Connecting the two in terms of content and pedagogy may not be easy or guaranteed. Even so, there may be unintended benefits.

8.1 Acknowledgment

I would like to thank Ken Constantine for the original assignment and his help in pointing me towards emphasizing integrity as a primary feature of statistical practice.

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Spiritual Benefits of Mathematical Practice

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Abstract

In the moment-to-moment experience, what is it really like to do mathematics? Given the abstraction of its goals, the mental exertions of its methods, and the deep intangibility of its objects, one might be tempted to say that doing mathematics is meditative. As a meditative activity, maybe doing mathematics can be of benefit for the spirit. In this article, I will investigate two examples of the spiritual benefits of mathematics. I will focus on mathematics as a practice—as a human activity—and argue that this focus on practice is helpful both for my present argument and generally for the integration of faith and learning.

1 Introduction

Major inspiration for this work comes from the writing of Simone Weil. Unusual for a religious thinker, she was deeply familiar with mathematics both through her own study and through frequent discussions with her brother, André Weil. Though she mentions mathematics only infrequently in her writing, those mentions point to deep insights for anyone interested in the theology of mathematics. (Vance G. Morgan's book [9] provides a convenient and detailed account of Weil's thought on mathematics.) I can even point to a specific passage as the direct motivation for this article.

> [O]ne does double harm to mathematics when one regards it only as a rational and abstract speculation. It is that, but it is also the very science of nature, a science totally concrete, and it is also a mysticism, those three together and inseparably. [12]

In curriculum documents and in pitches to students, mathematics tends to be motivated in two ways. The first is direct: mathematics is a set of skills and techniques necessary for some future situation, either just the arithmetic required to be a good citizen or the technical numeracy needed for some particular kind of activity in a student's anticipated future. The second is indirect: mathematics has benefits outside of its actual content knowledge. It teaches abstraction, problem solving, logical thinking, possibly even creativity within rigid systems. To find such arguments, you need look no farther than the curriculum guides for high-school mathematics in your own jurisdiction; I know the curriculum guides in my home province of Alberta make these arguments clearly.

Somewhat less frequently, the secondary benefits of mathematics are extended from mental skills such as problem solving to the building of moral character: teaching humility concerning one's own opinions, reliance on evidence in place of personal authority, care about definitions and precision of language, and other similar lessons. Though many have made these arguments, let me point to the recent book Mathematics for Human Flourishing by Francis Su [11] as a very fine example.

Surely, mathematics has all these benefits: to study mathematics is, indeed, to improve the mind and strengthen the character. To borrow Dr. Su's title, I do believe that mathematics can be a path to human flourishing. But beyond mental skills and moral foundations, I'm interested in something still more. In addition to how mathematics improves the mind and the character, I want to know how, if at all, mathematics improves the spirit or nourishes the soul. Though I know some philosophers and theologians make distinctions between the terms 'spirit' and 'soul', I'll use them synonymously in this paper. What is important to me for this discussion is the spiritual reality of being human, a reality gifted to us in our creation which cannot be reduced to the physical or the mental aspects of our being.

So here's the project: to investigate what mathematics does for the spirit. I anticipate such a project holds interest to my readers and to fellow ACMS members. Philosopher Adam Drozdek gives a description that serves as a good starting point.

> Knowing God is the ultimate goal. However, the cognitive apparatus can be exercised with respect to other objects, to geometrical objects and numbers, for example, since the difference between knowing God and knowing numbers lies not in the cognition itself but in the nature of cognitive objects. Hence, analysis of how and what we know about objects of mathematics is a stepping stone to analysis of our knowledge of God. Mathematical sciences make us more sensitive to more subtle kinds of knowledge, and without the preparatory exercise of mathematics, "our mind could not bear great light" of this knowledge. [5]

Activities that we undertake for the betterment of our souls can be called spiritual practices. Even traditionally understood, these are many: a wide variety of types of prayer, meditation, reading of scripture and other worthy spiritual books, confession, pilgrimage, worship, acts of devotion, acts of service and ministry, and so on. In the spirit of "prayer without ceasing", I would argue that almost all human activities (at least, outside those which are simply destructive) can have an aspect of spiritual practice: they can do something for the spirit. Mathematics is not unique in having spiritual benefit, but the particulars of how mathematics benefits the spirit are, I believe, fairly specific to the discipline.

Finally, I believe the way to interogate mathematics as a spiritual practice is to focus on mathematics *as a practice*. That is, so treat mathematics as a human activity instead of, say, a body of truths. To move in that direction, let me talk about a bit of my own journey understanding faith integration in mathematics.

2 Pedagogical Contexts

This consideration of mathematics as a spiritual practice represents a shift for my own understanding of faith integration, particularly in the classroom. I teach at The King's University in Edmonton, Alberta, Canada. This is an institution that very explicitly requires and evaluates the work of faith integration. During my first few years at King's, my focus was on the objects—the outputs—of mathematics. I imagine that many of my readers who have worked on faith integration likely shared this same initial focus: talking about mathematical things seems like the reasonable starting place. The history of the philosophy and theology of mathematics has also focused on objects. What are mathematical things? How do they exist? How are they part of the created order, the fall, redemption? How can we know them? What mathematical truths does God know?

Inspired by the writing of Imre Lakatos and Reuben Hersch, I went through a change of mind a few years ago: instead of thinking about what mathematical things are, I instead started to focus more and more on what mathematical activity is. And I found this a much more fruitful approach. Instead of mathematics conceptually starting with sets and numbers and equations, I reconceptualized mathematic by thinking primarily about human beings gathered around a chalk board: puzzling together, solving problems, making definitions, writing proofs.

This also changed my perspective regarding possible connections between mathematics and the divine. If I focused on objects, I was drawn to the Augustinian idea of mathematics as ideas in the mind of God. In this scheme, there is a divine gift: access to mathematical objects as valuable tools and fascinating playthings for humans. However, when I focused on the activity of mathematics, then the gift changes. Instead of a gift of objects, it is a gift of capacity: the ability to think mathematically. This seems a rich gift, and a gift that I prize immensely. Through this gift, I am led back to the question of this article: what does the capacity for mathematical thinking, graciously accorded to humans by God for their benefit, do for the spirit? How does mathematical thinking and puzzling and struggling benefit the soul of a human being?

I found this framing allowed the whole question, the whole project, to make sense to me. It's not that easy to inquire what spiritual benefit, if any, results from the mere existence of a Noetherian scheme over the complex numbers. It's much more approachable to investigate what spiritual good results from the process of learning how to define, construct and understand Noetherian schemes over the complex numbers. This learning and mental constructing is a thing that a human being does. By going through this mental process (a lengthy and taxing mental process, if my memory of graduate school retains any accuracy), something happens to the mind of said human being. And, as I will argue, something can also happen to the spirit.

3 Intangibility

My question and my project are broad; many possible avenues of investigation are possible. In this article, let me present two ideas, two fruitful avenues of inquiry.

First, I would argue that mathematics is good practice in reaching the intangible. The intangibility of mathematics things is immediately reminiscent of the intangibility of spiritual pursuits. The theologian Volker Kessler puts it this way.

> [M]athematics is not an empirical science. Mathematical objects are not part of our sensual world; they cannot be touched, and mathematical formulae cannot be tested empirically. ... Mathematics shares these properties with the discipline of spirituality. Spiritual knowledge is not part of our sensual world, nor can spiritual statements be tested empirically. Spiritual knowledge is often given by inner visions. [8]

I have very vivid memories, particularly from senior undegraduate and graduate school experiences, of deep frustration with the intangibility of mathematics. I'm supposed to be studying some manifold, but its points are somehow whole lines in projective space. Then they are further removed by a series of blow-ups along some submanifolds. Then Grothendieck comes along and tells me that there are no points at all, just morphisms over some base scheme making the whole construction relative and categorical. I wanted to study some geometry, but the actual tangible objects of my geometry fall so desparately out of reach. My experience here is from algebraic geometry, but I know from talking with colleagues that the same frustration exists for topological loop spaces, Lie groups, vertex operator algebras, and almost all other glorious fragments of mathematics.

To study mathematics is, eventually, to get over this frustration. To realize that whatever access you manage to build for your mathematics, however indirect and tenuous, is simply *what you have to work with.* You come to realize that you'll never have any kind of real visualiation of that multi-dimensional variety you are trying to understand, that the sheaf of functions you are using to try to understand its properties is all that you get. Mathematics is training—is practice—in dealing with the intangible. And since there is a resonance with the intangibility of spiritual things, I argue that this is a spiritually valuable experience. Here is how Douglas Hofstadter puts it, while talking about the proof of Gödel's Incompleteness Theorem.

> All of this [Gödel's numbering scheme] may seem terribly perverse, but if so, it is a wonderful kind of perversity, in that it reveals the profundity of humanity's age-old goals in mathematics. Our collective quest for mathematical truth is shown to be a quest for something indescribably subtle and therefore, in a sense, sacred. [7]

If faith is being sure of what we hope for and confident in what we have not seen, then there is necessarily a strange intangibility in the activity of faith. The spiritual realm is an invisible one. This is not to discount the very tangible experiences of the Christian life: there are times and moments when the spirit is experienced directly and powerfully. And perhaps there is nothing quite as immediate as spiritual experience itself. But much of the life of faith is reflection on that experience. Particularly during this reflection on our spiritual experiences, it's not difficult to fall into the frustration of feeling out of touch, literally and metaphorically. Insights of the spirit are not nice, clear words from out the cloud: they require interpretation and discernement, whether they are drawn from scripture, conversation, reflections, reading, or many other indirect sources. Spiritual growth and understanding are hard to grasp, hard to pin down, hard to point at. The experience of doing mathematics, through sharing this frustation of grasping at something always out of reach, can help us to be more comfortable with all this intangibility.

Karen Olsson wrote a book on both André and Simone Weil titled *The Weil Conjectures.* In that book, she summarized Simone Weil's ideas about the spiritual value of mathematics in a way resonates with my arguments.

> Simone conceives of a civilization in which mathematical reasoning, mystical belief, and existential loneliness formed an energetic triangle. The Greeks, she writes, experienced intensely the feeling that the soul is in exile: exiled in time and space. Mathematics could bring some ease to the exiled soul, she says. Doing math could free you from the effects of time, and your soul could come to feel almost at home in its place of exile. [10]

I think there is something quite poignant here. This isn't about solving the problem of exile, whether it is a spiritual exile of the human condition during the fall or a mathematical exile of never being able to directly experience or visualize the objects of our study. This is about dealing with the existential situation of exile, of living in the midst of it.

I feel I've only touched the surface of this idea in the present article. Taking this farther, I believe that the internal mental focus on intangible objects makes mathematics something like meditation. To meditate, as a religious practice, is to contemplate a spiritual idea, mantra, prayer or other touchstone. Indeed, I imagine that many of my readers may agree that their own mathematical work may, at times, feel meditative. I'm very interested in the history of Christian mysticism, both ancient, medieval and contemporary. I feel strongly there there is something to make of this parallel, some way in which mathematics, as a kind of meditation, acts as a spiritual practice. I will not proceed farther in this direction in the present article, but I do believe there is something quite promising and fruitful here for future work.

4 Infinity and Paradox

In addition to dealing with the intangible, mathematics more specifically reckons with infinity. Indeed, many authors and thinkers have spent much breath and ink wondering about this aspect of mathemtics. Reflection on infinity is something that I used very early in my career to talk with students about what mathematics might have to say about God. As I said before, my initial attempts at faith-integration were focused on the objects. There is a lot of value asking about infinite objects: particularly since Cantor, the mathematics of infinity is a wonderfully rich subject. But I found new insights in my shift from thinking about objects to thinking about practice. It's not just that mathematics defines infinite things; the great value is really that mathematics gives practice in thinking infinite thoughts, at least as far as human are capable of such thinking.

I believe a very valuable way to talk about mathematics and infinity is to focus on the creation and resolutions of paradoxes. Both mathematics and Christianity deal in paradox, directly and indirectly. Though mathematics is an activity of reason, it bumps into the limits of that reason. Infinity is a paradox. In an infinite set, a proper subset can match up perfectly with the whole set. Therefore, the subset is both smaller than the whole and the same size as the whole, all at the same time. This is a paradox. Some of us have spent so long thinking about infinite sets that this paradox might no longer arrest our thinking, but it's worthwhile to forget what we have learned and reconsider the strangeness and incongruity of the situation. It really is a paradox: something that shouldn't make sense. And, for much of history, this paradox and others (the nature of the continuuum, infinite divisibility, Zeno paradoxes, etc.) lead to a negative perspective of infinity: it is the domain of chaos, unrelated to any kind of positive logical order. But, in a long process that culmulated with Cantor, something changed. I really like how French philosopher Alain Badiou describes this situation.

> What happened was that Cantor had the brilliant idea of treating positively the remarks of Galileo and Pascal—and those of the Portuguese Jesuit school before them—in which these authors had concluded in the impossibility of infinite number. As often happens, the invention consisted in turning a paradox into a concept. [1]

The breakthrough to understand infinity is to treat a paradox as a concept. Moreover, mathematics gives an environment for practicing this method: it's a place where paradoxes can be synthesized into real ideas. I believe there is a spiritual wisdom here. Many Christian traditions talk about the mysteries of the faith, which include, of course, creation, incarnation, resurrection, trinity, and others. These are called the mysteries of the faith, not the reason of the faith, because they involve something that transcends reason, something that is often paradoxical. I like how Chesterton describes this concept.

> Mysticism keeps men sane. As long as you have mystery you have health; when you destroy mystery you create morbidity. The ordinary man has always been sane because the ordinary man has always been a mystic. He has permitted the twilight. He has always had one foot

> in earth and the other in fairyland. He has always left himself free to doubt his gods; but (unlike the agnostic of today) free also to believe in them. He has always cared more for truth than consistency. If he saw two truths that seemed to contradict each other, he would take the two truths and the contradiction along with them. ... He has always believed that there was such a thing as fate, but such a thing as free will also. [3]

Following Chesterton, maybe it is true that mature Christianity requires the contemplation of the mysteries of the faith, of going beyond what simple reason can conclude about the world, of having this sanity-keeping mysticism that Chesterton describes. If so, then any discipline which gives practice for this paradoxical insight is a discipline which can bear spiritual fruit. Mathematics does this. Mathematics shows how a paradox can become a concept, and that the fruits of this paradox-concept are rich and beautiful and mesmerizing.

Still talking about infinity, it's valuable here to point out that infinity is an axiom in all modern treatments of set theory. The paradox of infinity requires not a proof, but the adoption of an axiom. Again, I like how Badiou puts it:

> Now, just like the empty set, or zero, the infinite will not be deduced; we have to decide its existence axiomatically, which comes down to admitting that one takes this existence, not for a construction of thought, but for a fact of Being. [2]

Now infinity is accepted as an axiom, but what is an axiom if not a leap of faith? To decide that an infinite set is an idea worth thinking is to give oneself *permission* to think such as thing. To handle paradox requires a movement of faith. The substance of these leaps of faith differ greatly, of course, between mathematics and religion, to say nothing of their relative importance. But the movement is similar, and the practice of doing mathematics can help one understand and become comfortable with the choice, with the leap of faith.

5 Two Cautions and Conclusion

Let me finish with two cautions to myself over this particular project. First, I'm not trying to claim that my analysis is entirely unique to mathematics. You could likewise wonder how the activity of studying economics, or forestry, or Russian literature is a spiritual discipline. I'm sure you could find some interesting insights there. This is always the case with the project of integration of faith and learning: much of what can be said applies to many disciplines, and the same question can be brought to bear in many places. That said, I think that mathematics has some unique and specific benefits. There are attributes of the activity of mathematics that it shares with few other modes of human thought: its abstraction, its intangibility, its handling of paradox, its seeming universality, its seemingly eternal nature. These aspects of mathematics seem to point to spiritual questions, and we should not cower away from those implications.

Second and more importantly, any such project like this must always be careful with idolatry. Because of its spiritual resonances, many practicioners of mathematics have wanted to go all the way with equating mathematics and spirituality. (Among many sources, I could point to the writing of David Hilbert or G. H. Hardy in the early twentieth century). Since mathematics feels like studying something eternal and something infinite, there is always the temptation to conclude that mathematics is, uniquely, the eternal and the infinite discipline. If I think that mathematics is meditative, it's easy to slide into the idea that mathematics is prayer, that it can replace prayer.

It's easy to make mathematics itself into God. Therefore, it is always important to maintain some distance between whatever spiritual benefit mathematics may produce and spirit itself.

For examples of this tendency, Daniel Cohen has an excellent historical study of the apotheosis of mathematics in Victorian England in his book *Equations from God*.

> One obvious corollary to this philosophy [of John Norris] is that the pursuit of mathematical knowledge could be the basis of a rich spiritual life removed from cathedrals and priests. For Victorian mathematicians who felt uneasy with the Church yet who wanted to remain active in their faith, this association of religious devotion with the discipline of mathematics proved extremely appealing. [4]

Infinity is a good example of a location for this tendency toward idolatry. I've argued that mathematics, through studying infinity, is good practice for navigating the paradoxes of the mysteries of the faith. But it should never be assumed that mathematical infinity is the same thing as theological infinity. I would argue that God is infinite in a way that is fundamentally different from how the real numbers are infinite or how the entire von Neumann universe is infinite. The mathematical infinity, as glorious and fascinating as it is, as much as it might tangle with paradox, as much as it moves and inspires mathematicians, is still a *concept*. The infinity of God is not a concept; it is antecedent to concepts entirely. In the words of theologian David Bentley Hart,

> [God] is freedom as such, the fiery energy that liberates the flame from the wood. He is the very power of agency. He is the Good that makes the rational will exist. He is the infinite source of all knowledge and all truth, of all love and delight in the object of love, who enlivens and acts within every created act. [6]

At the start of this article, before getting into the potential for the spiritual benefit of mathematical activity, I mentioned the benefit of mathematics for forming moral character. Among those benefits is intellectual humility: since mathematics relies on a standard of logical proof, it tears down other psychological edifices of authority. Potentially, this builds an intellectual humility. If you can't find a proof, mathematics doesn't care what you think is likely the case or what you shout loudly should be true.

This potential for humility is true, but it is also true that mathematics can just as easily build an intellectual hubris. Compared with the objects of other studies, mathematical problems and constructions can feel universal and eternal. By relying on proof over inductive reasoning, mathematical statements can carry the kind of authority that other scientific conclusions cannot aspire to. Unchecked, this can lead to an ill-founded pride in the superiority of mathematics. I find it fascinating, here, that I'm talking about same aspects of mathematics: its formal structure, its proof-based reasoning, its abstract objects. In different settings, the very same aspects of mathematics can inspire humility or hubris.

I believe my project to inquire about the spiritual benefits of mathematical activity balances on a similar knife's-edge. In this article and in the future work, I want to walk that balance: to investigate what spiritual benefit exist in doing mathematics without edging into idolary and the apotheosis of mathematics, without claiming its superiority among the disciplines, without losing track of its many limitations.

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Setting up Students for Success: Analysis of Effectiveness of Mathematics Placement

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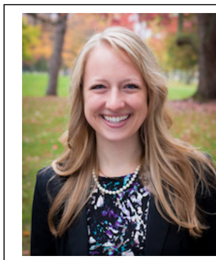

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Abstract

How do we effectively and equitably place students into math classes in a way that provides them the best chance of success? As many higher education institutions veer away from placement based on standardized testing, many departments are seeking placement alternatives that will properly support students. Additionally, math placement determines not only a student's mathematics courses but also influences their progress in related fields, including physics, chemistry, engineering, and more. This paper will describe three different existing placement systems at each of our liberal arts institutions as well as the affordances and constraints of each approach. Further, we analyze data from each institution to study the impact of the various placement strategies on student success. We take a data-driven approach to math placement with the hope of improving student outcomes through accurate and equitable math placement.

1 Introduction

With many institutions no longer requiring standardized test scores, new systems are needed to place students into courses in which they will both be challenged and successful. The importance of appropriate math placement is relevant not only for math courses but also for other related STEM disciplines. As authors, we represent not only the discipline of mathematics, but also math education, physics, and engineering. Despite this wide array of areas of expertise, there are also many commonalities. Our universities are all located in the Midwest and are liberal arts institutions grounded in a faith-based perspective. Although each school offers slightly different curricula, we will take a cross-institutional snapshot and focus primarily on Calculus I since that class is fairly uniform across our universities and primarily serves other departments. In addition, all three of our institutions have transitioned to an optional standardized test application system in recent years. Further, we each offer an in-house placement test through our respective Learning Management Systems (Canvas, Moodle, etc.) and are considering other placement options such as ALEKS.

In Section 2 we highlight some of the existing research on mathematics placement. Section 3 will then describe the setting and current placement system at Calvin University, followed by Dordt University in Section 4 and Marian University in Section 5. One of the most significant benefits to cross-institutional studies is the ability to combine data and look at overarching trends, so in Section 6 we consider student success as it relates to current placement as well as other features such as high school GPA. This analysis emphasizes some of the commonalities among our institutions as well as highlighting some of the nuances of each university context. Finally, Section 7 summarizes the results of this study as well as outlining plans for mathematics placement moving forward.

2 Literature Review & Methodology

Much work has already been done to study math placement in various institutional settings. We highlight but a small subset of the research here to emphasize both general themes as well as some of the variation that appears based on institutional context and other factors.

In her work at a Midwestern university, Coggin used machine learning models to analyze student success in the context of both intermediate and college algebra [7]. This study concluded that a combination of high school GPA and Math ACT score is best predictor of student success. Although this work was completed in 2018 before many schools shifted to test blind systems, it should be noted that according to their feature analysis, GPA was the most important factor in student success. In a related study, Geiser and Santelices found similar results at the University of California [10]. Their work found that high school GPA was the strongest predictor of success for all academic disciplines.

Work of Scott-Clayton in [18] showed that placement exams are more predictive of success not only in mathematics but also college-level courses more generally. This study was done in the context of an urban community college, and specifically found that, while placement exams were used in multiple subjects, they were more predictive of success in math than English. Following up on this work in [19], Scott-Clayton et al. then expanded their analysis to 56 community colleges. Based on this analysis, they found that adding test scores to high school data provided little additional benefit.

Ngo and Kwon also studied placement in the community college setting, this time considering those in California [17]. They found that multiple measures, including both GPA and prior math background, improved placement accuracy. In related work based on California community college data, Bahr et al. conducted a decision tree analysis and found that high school GPA is most consistently useful predictor of performance [2]. In particular, their analysis indicated that the most relevant factor was GPA, followed by information about a student's high school precalculus record. Burdman also investigated the placement practices at several colleges and universities in California. She concluded that most placement systems create a complicated maze for students to navigate and too often do not result in a pathway to eventual success in mathematics courses such as calculus [4].

For a perspective across institutions, Hsu and Bressoud surveyed over 100 colleges and universities about their math placement policies and procedures [12]. They discovered a wide variety of approaches with many institutions using some combination of a placement test, an ACT/SAT score, and high school or prior college mathematics course grade information. It is notable that only 14 of the institutions were primarily 4-year colleges and none were specifically identified as faith-based institutions. Importantly, their study found that placement in precalculus or remedial mathematics courses often do not adequately prepare students for later success in calculus. Similarly, Hsu, Murphy, and Triesman document the ineffectiveness of many precalculus courses offered at the college level, noting that frequently such placements impede rather than foster success in calculus courses [13].

In a related research study, Sonnert and Sadler found that students taking a precalculus course at a college or university prior to calculus performed only slightly better than comparably prepared students who ignored the recommendation to take precalculus prior to calculus [20]. Almora Rios and Burdman found that many public institutions continue to rely on both standardized placement tests and multi-course prerequisite sequences despite equity concerns, compared to a more asset-based approach to student placement [1]. We conjecture that a similar philosophy is practiced at many faith-based institutions.

While these above studies help to describe math placement trends across the nation more broadly, one cannot assume that similar results hold at every institution. In particular, many of the published studies focus on research universities or community colleges, and so there are unique factors (such as graduate student instructors and class size) that would not pertain to a small liberal art school. In general, it seems that high school GPA is the best predictor of student success, followed by ACT/SAT scores, although the addition of this information seems to have mixed success in providing more reliable placement. In the following sections we start by describing the current systems at Calvin, Dordt, and Marian before analyzing data from these universities in order to address whether the conclusions from these studies also apply to our institutions.

3 Calvin's Story

Calvin University is in Grand Rapids, Michigan and enrolls approximately 3200 undergraduates along with more in various graduate programs. Calvin is a Christian liberal arts institution that is affiliated with the Christian Reformed Church of North America (CRCNA). The mathematics and statistics department currently has 11 full-time members and offers majors in mathematics, mathematics education, and statistics. Many of the students enrolled in Math 171 (Calculus 1) are engineering or computer science majors. Most of the students majoring in mathematics, mathematics education, or statistics enter Calvin with Advanced Placement credit for Calculus 1 and begin with Math 172 (Calculus 2). Calvin also offers Math 132, an applied calculus course without trigonometric functions, for those majors requiring a single semester of calculus (e.g., biology, kinesiology, and a slimmed down version of computer science).

3.1 Current Math Placement at Calvin

Until the Covid-19 pandemic, Calvin relied on multiple sources of data to place students into a mathematics course. Incoming students were required to take the ACT or SAT. A calculus placement test was taken by most incoming students needing to take Math 171. High school transcripts were also provided. Based on these data, the department made of one of the following recommendations:

- Take Math 171 (Calculus 1).
- Take a two-course sequence (fall semester followed by a 3-week January term) that covered the material in Math 171 at a slower pace. One option was Math 160-161 (the interim was entirely calculus-based) while the other option was Math 169-170 (the interim also met a general education requirement and included some interdisciplinary readings).
- Take a precalculus course at a local community college prior to enrolling in Math 171.

It is important to note that Calvin did not offer a precalculus course at the time. In section 6.1, we present summary data for each of these data sources (HS GPA, SAT Math subscore, Calculus Placement Test score) plotted individually compared to the student's score on the Math 171 final exam.

During the Covid-19 pandemic, Calvin changed its policies regarding the ACT/SAT (becoming a test-optional institution) and placement tests (making them optional and online). As a result, very few incoming students provided these data, so placement decisions were made based solely upon a student's cumulative high school grade point average (HS GPA). In the past year, Calvin has adjusted its admission policies to more strongly encourage the submission of ACT/SAT data (using it to help determined merit-based financial aid), but the calculus placement test remains optional and online and hence is rarely taken by incoming students.

4 Dordt's Story

Dordt University is an undergraduate liberal arts institution based in the Reformed Christian tradition located in rural northwest Iowa. As of the 2023-2024 academic year, Dordt hosts a student population of 1528 undergraduates [8]. Dordt has broad geographical representation, with 64% of students coming from out-of-state, and 30 countries being represented on campus. Despite the broad geographical diversity, only 20% of students at Dordt identify as non-white. Many students at Dordt are also engaged in co-curricular activities with about 43% of undergraduate students being student athletes, and with more than 20% of students participating in some musical ensemble. Dordt's ABET-accredited Engineering Program has shown consistent growth over the last five years [9]; naturally, engineering students are highly represented in Dordt's introductory calculus and physics courses.

4.1 Current Math Placement at Dordt

Prior to the 2021-2022 Academic Year, Dordt required ACT or SAT scores for incoming students. Since then, Dordt has moved to a test-optional model [11]. Students applying without a ACT/SAT score could opt to take an in-house unproctorerd online assessment through Dordt's learning management system (LMS). Students wishing to take introductory calculus but coming in with test scores below threshold (ACT, SAT, or LMS) could opt to complete the ALEKS adaptive learning program through McGraw Hill [16] to satisfy prerequisites. Presently, Dordt has shifted away from using the unproctored online assessment and is instead placing students based on their high school math histories. Similarly to Marian University, Dordt University has implemented the *Active Calculus* textbook and curriculum [3] since Spring 2019 for its calculus I course [15].

5 Marian's Story

Marian University is a university located in the urban setting of Indianapolis, and it offers a liberal arts education based in the Catholic and Franciscan tradition. There are approximately 2900 undergraduate students, and of those who are full time students, about 38% of them are Pell grant eligible, 47% are first generation college students, and 34% of them identify as non-white. In addition, many students are balancing not only school work but also part- or full-time work or athletic commitments—about 44% of them are student athletes.

Enrollment at Marian has continued to grow over the past decade, and so it is more important than ever to place students in math classes in which they will be both challenged and successful. In particular, Marian started an engineering program in Fall 2022, and graduation timelines for these new incoming students are significantly impacted by the math course in which they are initially placed.

5.1 Calculus Sequence at Marian

Over the past six years or so, the Department of Mathematical and Computational Sciences at Marian has made dramatic changes in the calculus sequence in order to better support students with research-based teaching methods. This shift has been due primarily to departmental leadership as well as recent hiring as a part of the growth in enrollment. These changes include enabling active learning instead of traditional lecture through smaller class sizes of 20-24 students, which were previously capped at 36. In addition, all of the courses in the calculus sequence, starting at precalculus, use a *mastery-based grading* system of assessment centered around multiple attempts to show understanding and revisions to reflect on past mistakes. Such alternative assessment techniques may also be referred to as *standards-based grading*, *specifications-based grading*, or *grading for growth* [6].

Because many students at Marian are also employed in some capacity, it was important to the department to remove financial barriers from our courses whenever possible. For that reason, courses have transitioned to open-educational resources (OERs) in the *Active Calculus* series [3]. Not only is this textbook available to students at zero cost, it is also available online and in pdf form, as well as in print for those who wish to purchase it. The homework in these classes has also shifted from traditional pencil-and-paper homework sets to a mix of online homework, writing assignments, and other projects. Further, students can receive support through the free tutoring center on campus as well as through Student Instructors and Undergraduate Teaching Assistants.

5.2 Current Math Placement at Marian

The math placement process historically at Marian was primarily based on ACT or SAT scores, so when test scores were no longer required starting in Fall 2022, the math department decided to make placement more uniform. Based on the MAA placement recommendations, a Canvas placement test had been written that had previously been used for alternative placement. The topics on this math placement test include adding, subtracting, and dividing fractions; order of operations; decimals, percents, and ratios; computing distance, area, and volume; simplification and algebraic manipulation; inequalities and absolute value; slope and y-intercept; trigonometry, angle measure, solving systems of equations, factoring, substitution, function composition, and graphs.

Based on preliminary work with two undergraduate students at Marian in the spring of 2022, all incoming students were now expected to take the placement test. It should be noted that given the variety of student orientation dates, the placement test occurs in an online unproctored setting on an individual basis. In many cases, if students are unhappy with their placement score, alternative factors are taken into account, such as their previous math course, the grade they received in that class, and any transfer or AP credits. In addition, certain majors offer their own quantitative reasoning courses (e.g., psychology statistics, modeling in biology, etc.) that do not have any prerequisite courses, and so in those cases math placement scores have no impact on students' placement. In fact, despite the new requirement that all students take the math placement test, we have found that 41% of students are not currently taking the placement test, and so it is that much more important to find other factors that could reliably predict student success.

One difficulty in creating a more refined placement process has been access to other metrics related to student success. Currently high school transcript information is manually entered based on PDF scans, so recovering information about high school math courses or math GPA for all such students is not possible given the existing systems and time constraints. As a part of the opening of the engineering school, Marian started using the online platform ALEKS as an additional form of placement for students. The primary benefits of this system are that, like the LMS placement test, students are able to progress through the ALEKS modules at their own pace. In addition, students are able to have multiple attempts at problems and the reports can include a breakdown of mastery based on topic. The primary disadvantage is the cost, particularly with ever-increasing enrollment, and so the question remained whether placement was better based on ALEKS compared to the Canvas placement test.

From the plots in Figure 1, it appears that math placement, while loosely correlated, is equivalently bad at predicting student success in Calculus I based on either ALEKS or the Canvas placement test alone. More details about placement at Marian using ALEKS can be found in [5]. However since there are only two cohorts of students who have used this alternative placement, it cannot be assumed that these results generalize beyond this small sample. Thus, we sought to combine data with other similar institutions in order to see the similarities and differences in the effectiveness of our math placement.

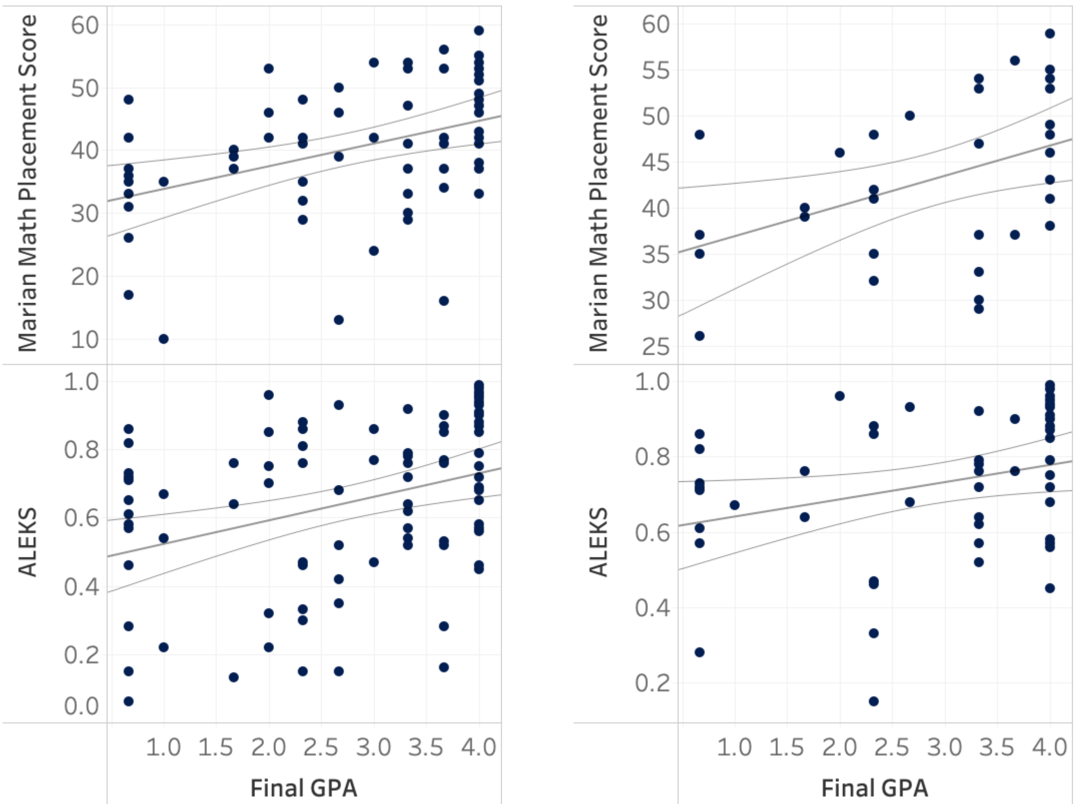

Figure 1: The left gives data from all students who have taken the ALEKS math placement test, with their placement test scores (on both the ALEKS and in-house Canvas placement tests) plotted against the grade students received in the course in which they were placed. On the right, only data from Calculus I was included, since that is the primary course into which students in this subset were placed. Note that no D- grades are assigned at Marian.

6 Data Analysis

The pandemic has introduced new challenges in education, and so while historical data can be useful, we chose to focus on the years 2020-2023 in order to more accurately represent current trends.[1] This choice is also relevant since all authors have been employed by the aforementioned institutions for that entire time frame, and this perspective can help provide context and insight.

6.1 Transitions from PreCalculus to Calculus at Calvin University

At Calvin, final exam grades were collected[2], and in the figures below we show those grades against the SAT Math scores (Figure 2), high school GPAs (Figure 3), and the placement test scores (Figure 4) respectively. Recall that Math 161 and Math 170 were both part of the two-term equivalent calculus course that Calvin offered over interim in January. In contrast, Math 171 is the classic calculus single-semester course offering.

[1]Data from spring 2024 was not available at the time of data collection.
[2]Whereas both Dordt and Marian reported final course grades. Thanks to Dr. Mike Bolt for these figures.

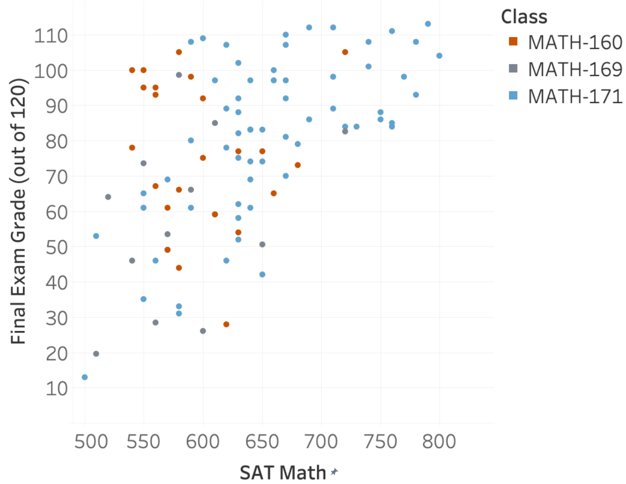

Figure 2: Calvin final exam grades against SAT Math scores.

Calvin documented relatively high rates of unsuccessful students in Math 171 (earning a D or an F or withdrawing from the course). One response was to create Math 071, a 1-credit support course for student to take concurrently with Math 171. A similar course, Math 072 was created to pair with Math 172 (Calculus 2). Students were advised to enroll in this support course, but relatively few choose to do so. After a two-year experiment, these support courses were discontinued and a 2-credit precalculus course (Math 110) was added beginning in Fall 2023. Math 110 meets during the second half of the fall semester. Some students were advised to take Math 110 prior taking Calculus 1 in a later semester, based primarily on HS GPA. Further, Math 171 professors advised students with failing grades to withdraw from Calculus 1 at midterm and to enroll in Math 110. Eventually, a total of 25 students enrolled in Math 110. Of these 25 students, nine registered at the beginning of the fall semester and did not enroll in Math 171, eight (out of a total of 15) withdrew from Math 171 at the recommendation of the Math 171 professor and enrolled in Math 110 (while four of the remaining 15 students withdrew from Math 171 without enrolling in Math 110 and the other three persisted in Math 171 against the professor's recommendation), and eight additional Math 171 students withdrew from Math 171 and enrolled in Math 110 without the professor's recommendation.

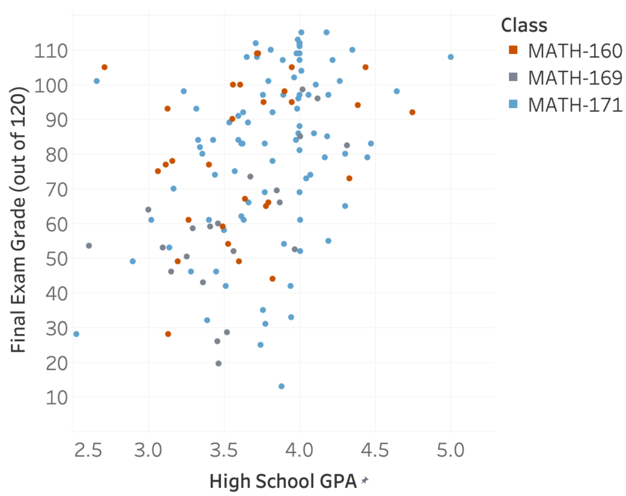

Figure 3: Calvin final exam grades compared to high school GPA.

At the end of the semester, a short survey was completed by Math 110 students on a voluntary basis. A total of 23 of the 25 students completed the survey. Here's a summary of the data:

- 70% has previously attempted Math 171 (Calculus 1)
- 74% felt more prepared to take Math 171 in a future semester after Math 110
- 78% planned to take Math 171 during the subsequent semester

Free response items asked what teaching strategies in Math 110 were most helpful to their learning and what suggestions would help to improve future offerings of Math 110. Most of the positive feedback focused on additional time provided to learn mathematical concepts and the professor's detailed explanations. There were fewer suggestions for improvement, but those offered focused on providing even more time to learn the mathematical concepts.

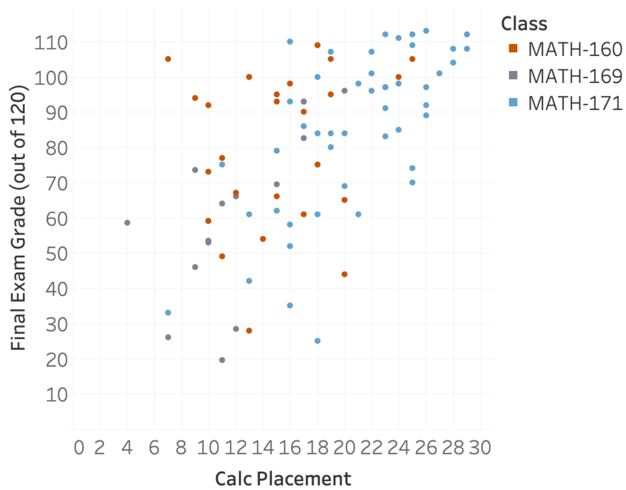

Figure 4: Calvin final exam grades and scores achieved on the university-conducted placement test.

We followed students who (re)enrolled in calculus during the subsequent semester (Spring 2024). Of the 25 students in Math 110, only 11 students took calculus. One student took Math 132 (Applied Calculus) and earned a B. The other ten students took Math 171 (some for the 2nd time). Of these ten, three students earned a C while the other 7 students earned grades of D or F or withdrew from the course. While these results are discouraging, there are plans to continue with Math 110 during the upcoming academic year. Notably, we will encourage students who can take Math 132 (rather than Math 171) to do so. This will eliminate trigonometry as a cause for struggle. If Math 110 were used primarily to review algebra and precalculus, then it would be ideal preparation for Math 132. A separate 2-week January intensive course on trigonometry might better serve the needs of students planning to (re)take Math 171 the following semester.

6.2 ALEKS Scores and Math Success

Unlike Calvin University, both Dordt and Marian offer semester-long precalculus courses. Further, both institutions have been using ALEKS for a smaller subset of students, typically those who are planning to study engineering. Since this assessment was used in comparable ways, data from both institutions was combined and analyzed. Because the ALEKS data for each school individually forms a small dataset, we strengthen the analysis by looking at the collective dataset for both Dordt and Marian.

Here, we focus primarily on Calculus I, since that is the class that most students taking ALEKS enroll in. First we consider the relationship between the grades students received in their calculus class and the score they received on the ALEKS placement test. Figure 5 shows these data from both Dordt (in yellow) and Marian (in blue). One nuance in conducting this analysis is the final grades are given as discrete letter grades converted to GPA, but a general loose correlation seems to hold for both institutions.

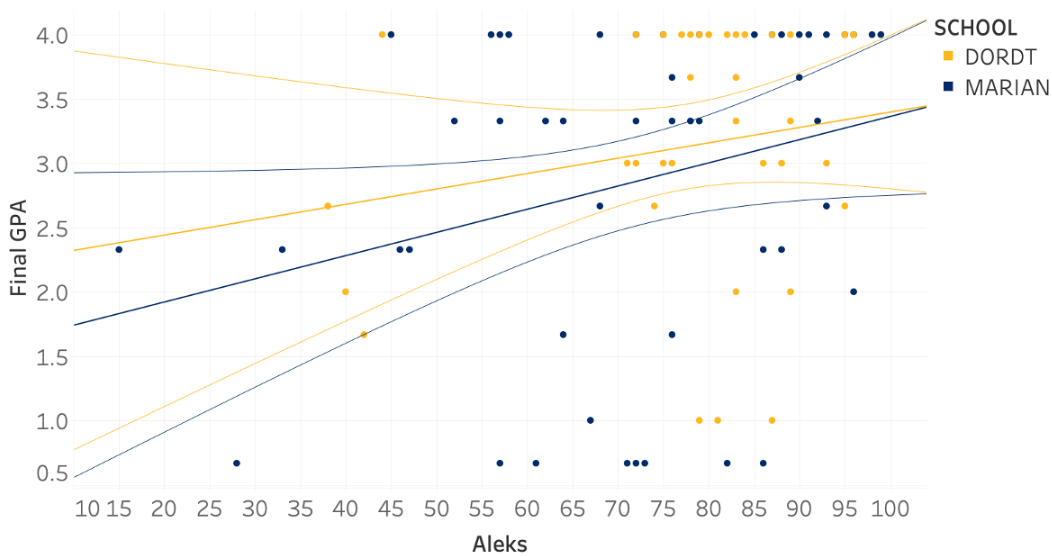

Figure 5: The plot above shows course grades in Calculus I and ALEKS placement test scores. Information in yellow pertains to Dordt University, while blue data are from Marian University.

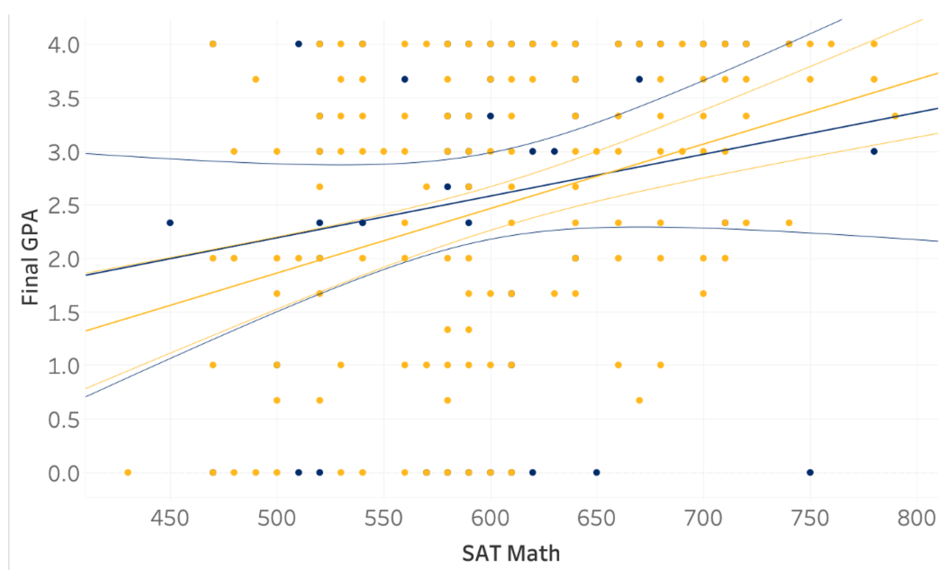

Figure 6: Calculus I grades and SAT Math scores at Dordt University (yellow) and Marian University (blue).

Previously, this placement was done based predominantly on standardized testing, so in Figure 6 we similarly compare these grades to the standardized test scores. Since students typically took one of the ACT, the SAT or a comparable standardized test, we converted this information to the equivalent SAT Math score for this plot.

Since many of the studies cited in Section 2 used GPA data in order to place students, we can compare each of the above plots to the that of Calculus I final grades versus high school GPA. Unsurprisingly, the confidence intervals for both institutions are considerably better than the previous plots, since there are considerably more students with high school GPA data than either an ALEKS placement or standardized test score. However, given disparities in school systems and variation in weighting of grade point averages, placement based solely on high school GPA also introduces potential nuances that do not capture the full picture of a student's math preparedness. Next, we compare this story of Calculus I success to that of physics, particularly as it relates to the ALEKS data. Since we've established that many of these placement measures demonstrate a loose correlation with student success, we then conduct a feature analysis similar to that of [7] to determine which factors are the most relevant based on these data.

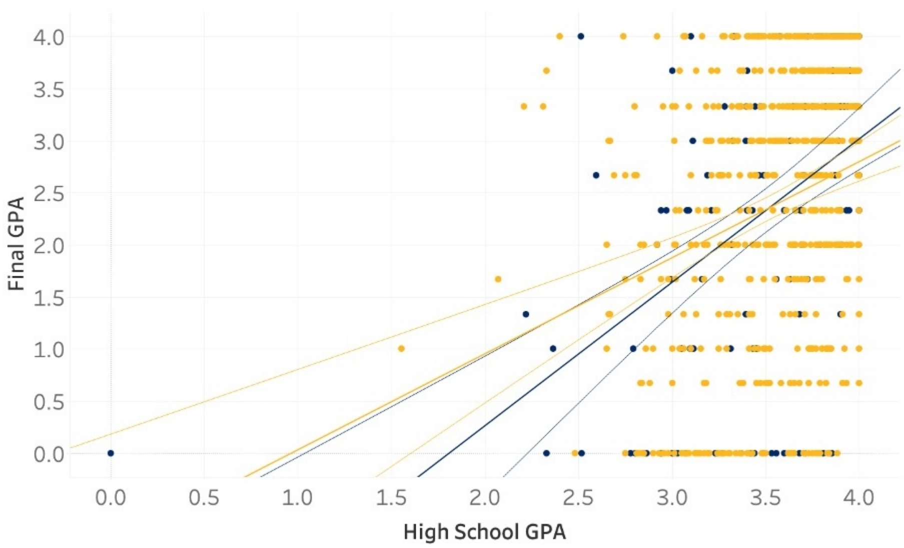

Figure 7: Plot of Calculus I grades and high school GPA. Dordt University data are shown in yellow, and Marian University data are in blue.

6.3 ALEKS Scores and Physics Success

Physics courses rely on some of the same mathematical background as many calculus offerings, but they also have unique challenges. For that reason, it is interesting to consider how preparedness for these courses relates to existing math placement systems. At both Dordt and Marian, there are two primary foundational physics courses: one that has precalculus as a prerequisite is predominantly taken by biology, chemistry, and math B.A. students; the other is offered as a co-requisite course with Calculus I that serves computer science, engineering, and math B.S. students.

We first consider the relationship between ALEKS scores and students' success in introductory physics, in this case roughly the same students who would be enrolling in calculus I. As before, discrete letter grades have been converted to a GPA scale. There are interesting institutional nuances here: while Marian's physics grades are generally correlated with ALEKS scores, Dordt's data tell a different story. The outliers here seem to drastically skew the data, but what this seems to suggest is that there is little to no correlation between success on the ALEKS to success in introductory physics, at least at Dordt. It should be noted that the ALEKS data probably are already biased towards students that have a lower level of mathematical comfort in this case however.

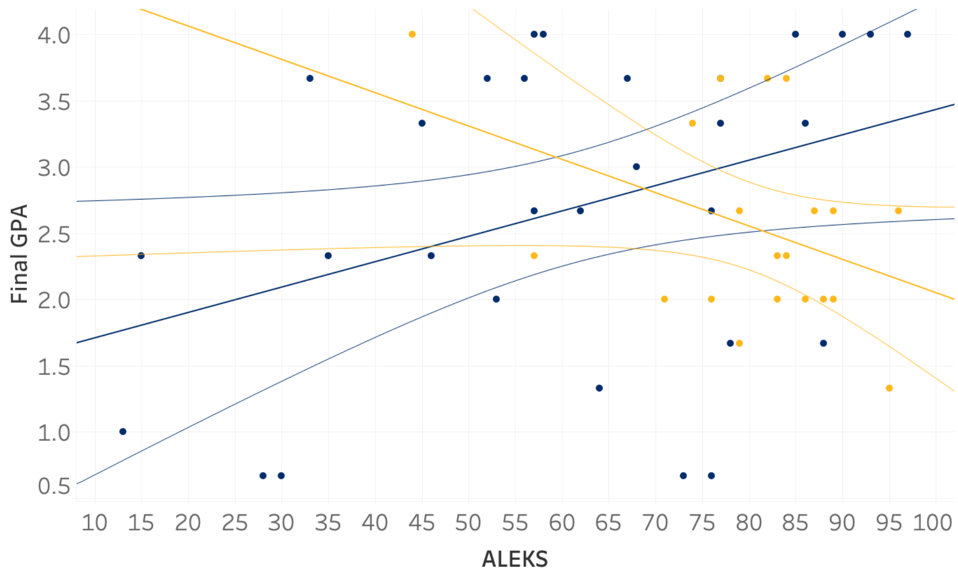

Figure 8: The plot above shows course grades in Physics I and ALEKS placement test scores. Information in yellow pertains to Dordt University, while blue data are from Marian University.

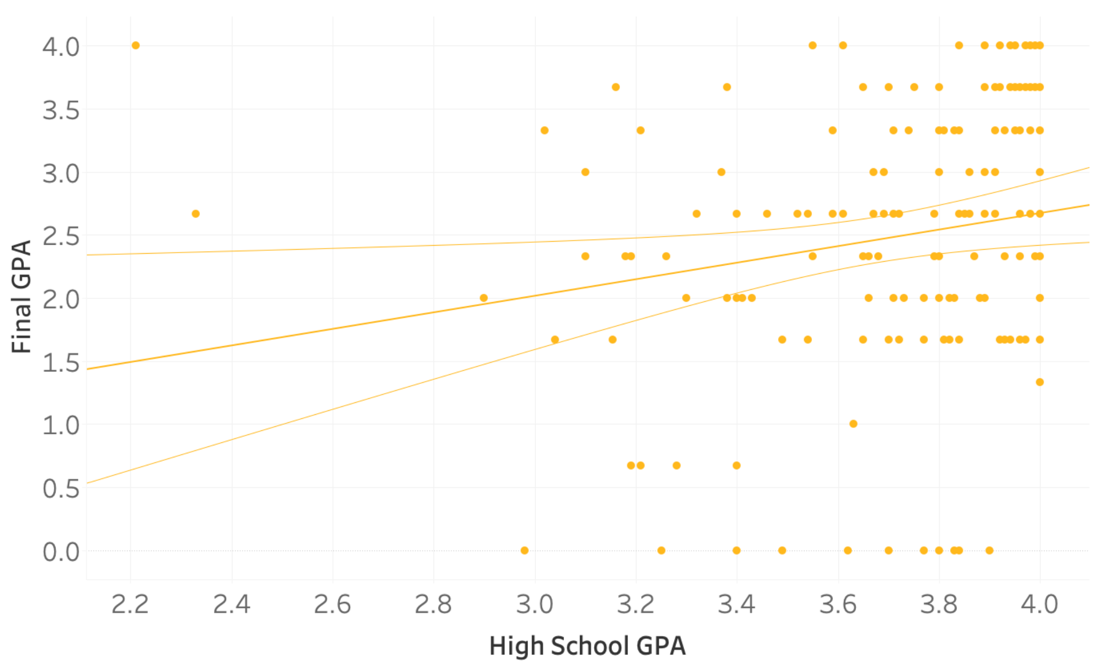

Figure 9: The plot above shows course grades in Physics I and high school GPA. Given the surprising correlation at Dordt University for ALEKS scores, we refine to that same subset of students here.

To provide further context for Dordt's data, we compare Figure 8 to both Figure 9 with high school

GPAs as well as Figure 10 with SAT Math scores. In these cases, correlation is better and clearer—success in introductory physics can be loosely correlated with high school GPA. We see similar and comparable trends in the SAT/ACT data as well, although that information is no longer provided as a part of Dordt application materials.

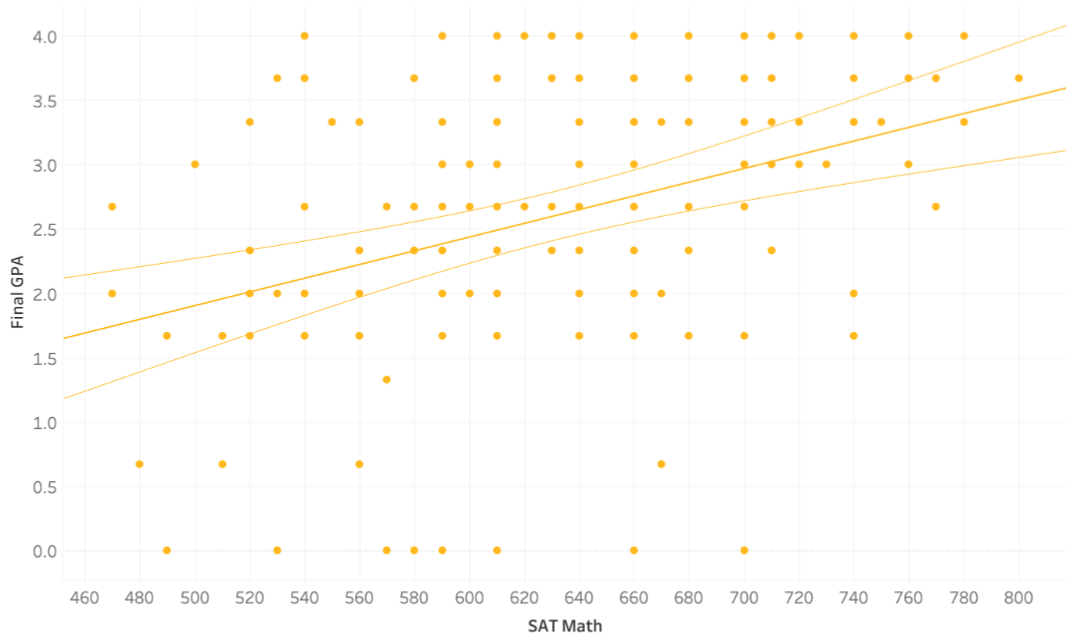

Figure 10: The plot above shows course grades in Physics I and SAT Math score. Given the surprising correlation at Dordt University for ALEKS scores, we refine to that same subset of students here.

6.4 Feature Importance

In Section 2, we referenced work of Coggin that studied student success in intermediate and college algebra using machine learning techniques [7]. Here we conduct a similar analysis using a machine learning model to investigate the relevance of high school GPA, standardized test scores, and math placement test information on determining a student's success in the math courses in which they were placed. Since many schools are most concerned with DFW (D, F, Withdraw) rates as a measure of student success, we used this binary to categorize and simplify the model, citing that many programs require at least a C- in order for a student to continue on in the major anyway. We then used 80% of our data as a training set, and then found around a 75% accuracy for all models on placing the remaining 20% of students compared to their actual success in the class. Although we used a variety of models in this process, the logistic regression was the best fit for the data in general. The overall feature importance is shown in Figure 11, for all data, the Dordt and Marian subsets, as well as the Calculus I courses at Dordt and Marian respectively. It should be noted that all standardized test scores were combined into an SAT Math and SAT Composite equivalent score, while the math placement score is based on the LMS math placement test.

As seen in Figure 11, the most important feature in predicting success in entry-level math courses is high school GPA. The level of importance varies on the sample of students, but in general about a third of student success can be attributed to high school GPA. As expected, some features ultimately reflect the student population—for instance, whether you attended Dordt or Marian and which math class you're enrolled in (e.g., college algebra, precalculus, calculus I, calculus II, etc.) are also strong indicators. Most fascinating of all, SAT Math score, SAT composite score, and math placement test score while relevant, are the lowest contributing factors not only for all of the data, but also for each of the subsets we considered.

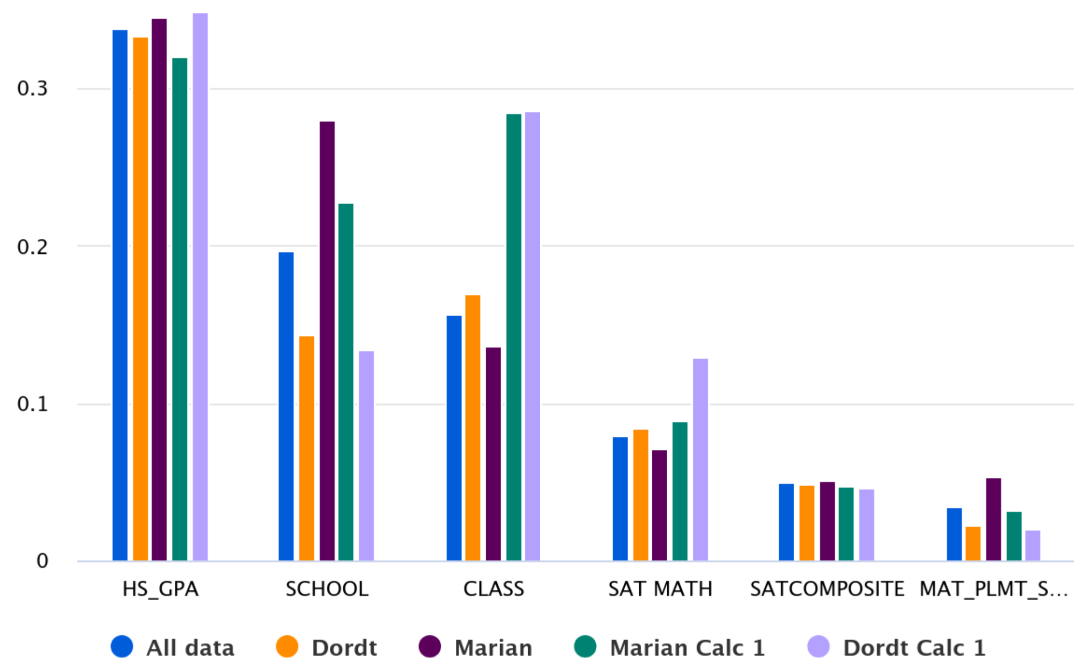

Figure 11: A machine learning model was built based on data from Dordt and Marian, and the plot above shows feature importance for all data as well as broken down by school and by Calculus I classes more specifically.

Given the results above, we can conclude that high school GPA is the most significant feature in determining a student's success at Dordt and Marian, although there are other factors that are also relevant. These findings are consistent with work of [2], [7], [10], and [19]. In this model, we found that there were 8 students predicted to DFW who did; 11 students who were predicted to DFW who passed; 71 students who were predicted to pass (i.e., earn a grade of C- or better) but received lower than a C-; and 452 students who were predicted to pass and did. Anecdotally it makes sense that any model will have limitations since no algorithm can capture emergency situations that come up, mental health issues that occur, or students who overcome a lack of college preparedness and are able to succeed despite perhaps being under-served in previous educational experiences.

6.5 High School GPA and Math Success

Since high school GPA is the most significant feature that contributes to our institutional DFW rates, we now dive deeper into these data sets to illuminate some of the underlying details.

First, we consider general trends in high school GPAs at both Marian in Figure 12 and Dordt in Figure 13. As before, we consider the years 2020 to 2023 since that captures a time frame while the authors were both employed at these schools as well as indicating some of the longer term impacts and challenges as a result of the pandemic. At both institutions, enrollment has increased over the past few years but the story looks a bit different at each school. As Figure 12 demonstrates, the median high school GPA of Marian students has decreased slightly over the past four years, perhaps a reflection of the larger spread of grade point averages. In contrast, Dordt's median high school GPAs in Figure 13 have remained fairly consistent, but the range of GPAs has narrowed over time.

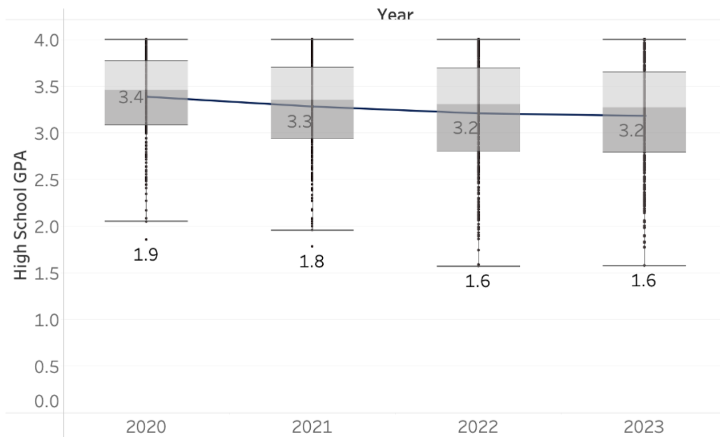

Figure 12: Incoming students' high school GPAs at Marian from the years 2020-2023.

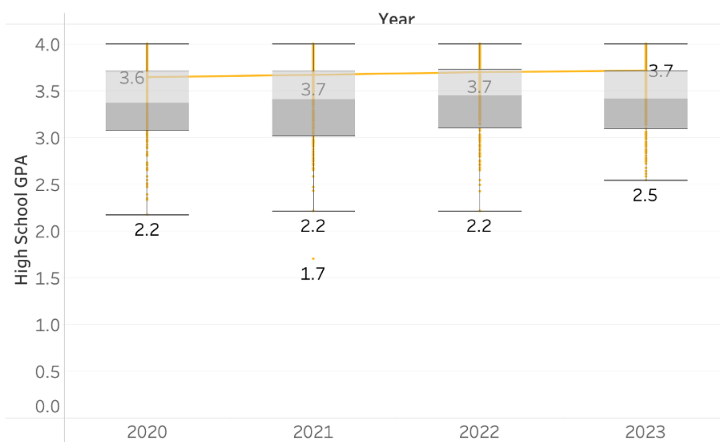

Figure 13: Incoming students' high school GPAs at Dordt from the years 2020-2023.

These differences in incoming high school GPAs between Marian and Dordt is also reflected in the Calculus I grades at both schools, shown in Figure 14. Students at Dordt are more likely to receive an A, with 86.6% of students passing and a DFW rate of 13.4%. In contrast, at Marian 66.3% of students pass and the DFW rate is 33.7%. One inherent bias in these data is the difference in withdraw dates. At Dordt, that date is about two months into the semester. At Marian, the last day to withdraw from a class each semester is the last day of classes. So in at least the past two semesters of Calculus I at Marian, between 20-30% of students withdrew; that is, of those students in the DFW category, approximately 2/3 of students chose to withdraw. From experience teaching that class, many of the students who chose to withdraw may have received a grade of C- or higher, and so a future direction of research would be to remove withdrawals from the DFW category in these analyses.

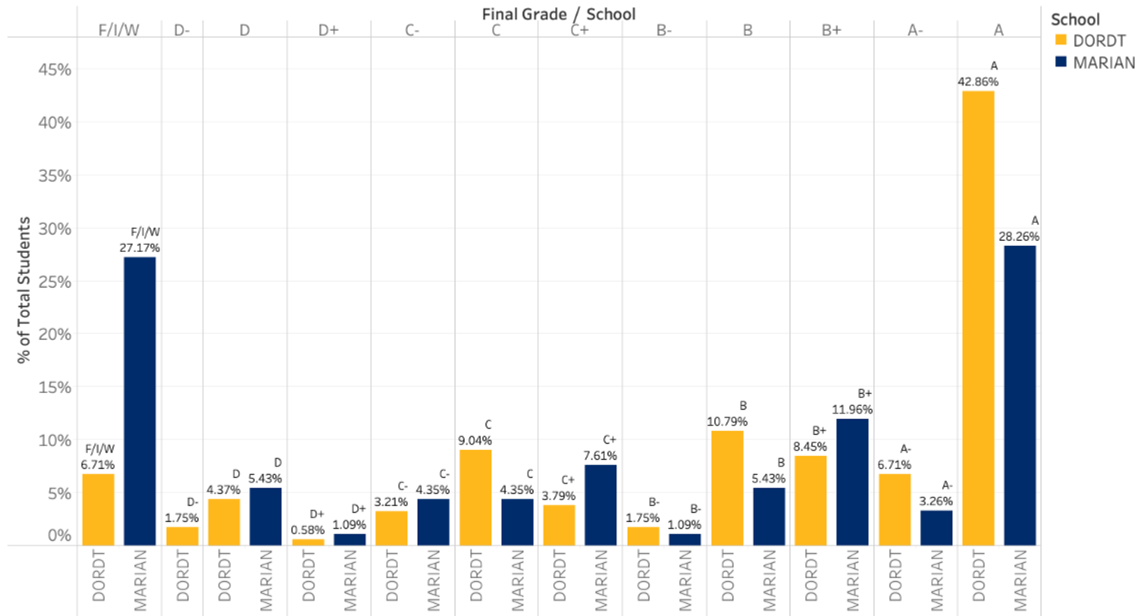

Figure 14: Calculus I grades at Dordt University are shown in yellow, and in blue for Marian University. It should be noted that Marian does not assign D- grades.

6.6 Limitations

There are many limitations of this study. Most notably, although we collected data from the past 5-10 years at each of our institutions, after cleaning the data even our combined data set was still small (less than 5000 students total). While some of the analysis was based on the grade that students received, a C- was the general benchmark for success in the machine learning model. This choice was made in part because of the salience of DFW rates, but many STEM programs also require this minimum grade in order to count toward and/or continue the major. Further, the overall dataset is skewed toward a passing grade in these courses, and while high school GPA remains the best predictor, this model has weak predictive power. There are many other variables that contribute towards students' success in these mathematics courses.

One hope of this study was to include information based on students' high school math experience, such as the classes that they took, the grades they received in those courses, or their math course GPA. However, one barrier to conducting this analysis is the lack of a database of such information. Currently such information exists in individual PDFs of each student's high school transcript, but efficiently acquiring these data in a way that respects student privacy was beyond the scope of this study.

7 Conclusion

Although our feature analysis indicates that high school GPA is the most significant factor in determining a student's success in math classes, no single variable is sufficient to reliably indicate whether a student will succeed. We conclude by summarizing next steps at each of our institutions.

For Calvin, we note that the high school GPA is as good of a predictor of success in Calculus I as is the placement test score and the SAT math composite score. Although we will use whatever data sources are available to us, it is likely we will rely on the high school GPA and prior math courses to advise future students regarding calculus placement. Given the existence of majors that require a single semester of calculus, we may also place more emphasis on the applied calculus (Math 132) for future students to avoid trigonometry as a potential barrier for success.

At Dordt, physics success and ALEKS scores have no convincing correlation but high school GPA is a much better indicator of success. Based on the above analysis, we can see that Dordt has relatively strong students who are generally doing fine in Calculus 1. While no single predictor can pinpoint one's chance of success in calculus, the combination of high school grades, ACT/SAT score, and ALEKS is probably the best available method.

At Marian, based on the analysis described in Section 6, we can conclude that high school GPA is indeed a more relevant feature than the ACT or SAT score or any existing placement test information. Further, the scores on the ALEKS placement modules and our Canvas math placement test seem to be equivalently inconclusive at successfully placing students based solely on that information. Therefore, we hope to find a more inclusive and holistic approach to placement moving forward. The math department would like to provide multiple attempts at a placement test and to improve communication around placement testing as a university (since students have anecdotally shared that they didn't realize the importance of the test in determining their math courses and potential delays to graduation). Currently the placement test is not taken in a proctored or distraction-free environment, and so it would be preferred to have a uniform testing setting for students, perhaps during their new student orientations over the summer.

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Running the Summer@ICERM Program

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Dr. Cara Jill Sulyok earned her Ph.D. at the University of Tennessee, Knoxville in 2021 before joining the faculty at Lewis University. Her research focuses on the development of dynamical models in immunology and epidemiology. She is especially passionate about guiding her students in interdisciplinary research projects.

Abstract

In this paper, we share our experiences running an undergraduate research program at the Institute for Computational and Experimental Research in Mathematics (ICERM), a national mathematics research institute at Brown University. The Summer@ICERM program is an eight-week summer research program during which four faculty research mentors work with 18 undergraduates and four graduate teaching assistants on research projects designed by the faculty. This paper focuses on the logistics of running such a program in addition to highlighting some of the successes, challenges, and lessons learned while mentoring a large group of student researchers. These reflections come from four faculty leaders as well as from a student who participated both as an undergraduate researcher and subsequently as a teaching assistant the following year.

1 Introduction

The benefits of undergraduate research experiences have been well-documented and supported by the Mathematical Association of America (MAA), encouraged by many undergraduate institutions of higher learning, and thoroughly studied [3–5, 8, 10, 12, 18, 20–23]. Providing opportunities for students to engage in research is especially beneficial for historically underrepresented STEM populations as these experiences can help students develop their identity and self-efficacy as scientists [4, 10, 12]. One common opportunity for students to conduct research outside of the classroom is through summer research experiences for undergraduates (REUs), funded by the National Science Foundation (NSF). Generally, the NSF grants approximately 1,350 awards annually, with around 175 of these awards for new REU sites for faculty members to host REUs at their home institutions [24]. While earning an NSF-funded REU is regarded as being highly competitive, there are other opportunities that fund summer undergraduate research. Many universities host an internal summer undergraduate research experience or run research courses during the summer and throughout the academic year. Additionally, there are opportunities provided by the Center for Undergraduate Research in Mathematics (CURM), which awards faculty to set up academic-year undergraduate research groups for students at their institution in collaboration with faculty and students at a partner institution (see `http://urmath.org/curm/minigrants/`). Additionally, the MAA and Society for Industrial and Applied Mathematics (SIAM) program PIC Math (Preparations for Industrial Careers in Mathematical Sciences) prepares mathematical sciences students for industrial careers by engaging them in research problems from applied mathematics.

The Institute for Computational and Experimental Research in Mathematics (ICERM) is one of six United States-based institutes that receive funding from the NSF with the goal of advancing research in the mathematical sciences, increasing the impact of the mathematical sciences in other disciplines, and expanding the talent base engaged in mathematical research in the United States. ICERM was founded in 2010 by professors Jill Pipher, Jeffrey Brock, Jan Hesthaven, Jeffrey Hoffstein, and Bjorn Sandstede through a major grant to Brown University from the NSF, Division of Mathematical Sciences. Their mission statement is as follows:

> "... to support and broaden the relationship between mathematics and computation: specifically, to expand the use of computational and experimental methods in mathematics, support theoretical advances related to computation, and address problems posed by the existence and use of the computer through mathematical tools, research and innovation."

ICERM provides a vast number of opportunities for mathematicians and scientists to collaborate in mathematics by hosting workshops and programs that range in length from one week to a semester. Targeting a wide variety of individuals, the program offerings include graduate training, math camps for high school students (GirlsGetMath), and research collaborations through both Collaborate@ICERM and Research Experiences for Undergraduate Faculty (REUF). One of ICERM's largest programs, which fosters and broadens participation in mathematical research between faculty and undergraduate students from institutions around the nation is their Summer@ICERM program.

Summer@ICERM (S@I) is an eight-week summer research program for 16-20 undergraduates, run by two to four faculty members and four teaching assistants (TAs). Unlike the average REU, where faculty mentors oversee the entire summer experience, S@I has the added benefit of start-to-finish logistical support. Faculty mentors are able to place their full focus on collaborating with students, while the staff at ICERM take care of the students' room, board, and stipends, as well as all other operational procedures. Additionally, the TAs help with planning various events and trips throughout the summer to grow camaraderie among the undergraduate student researchers.

In this paper, we highlight some of the successes, challenges, and lessons learned while mentoring a group of students working on various research projects. Additionally, we provide suggestions for developing an inclusive research environment, which may be helpful for anyone looking to run a research program or Course-Based Undergraduate Research Experience (CURE). The authors of this paper have a variety of experiences beyond running multiple S@I Programs. Among the faculty authors, we have mentored over 115 students in undergraduate research. These mentorships have taken place through a variety of means, including CURM-funded year-long undergraduate research projects, semester-long research courses, internal summer research programs at various institutions, PUMA-STEM (a summer research program for underrepresented STEM students to work with faculty in the greater Chicago area), and internal grants. Additionally, Horng provides insights through her experiences as both an S@I undergraduate researcher and teaching assistant.

2 Before the Program

Before applying for S@I, one needs to form a team of faculty mentors. ICERM allows two to four faculty members to run the program. These faculty mentors work closely during the eight weeks of the program with additional prep work, so it is *vital* to have a faculty group that consists of individuals who collaborate and work well together.

ICERM looks for faculty mentors who can provide successful student engagement in research. Before applying, it is advantageous to build up one's experiences in mentoring undergraduate research. We provide suggestions for doing so in Section 2.1. Finding other faculty collaborators can also be challenging, especially for faculty at smaller, primarily teaching universities. One program that can aid in this process is the Research Experiences for Undergraduate Faculty (REUF) Program that specifically supports undergraduate faculty who are interested in mentoring undergraduate research. This program includes a week-long workshop during the summer followed up by additional activities to support the continuation of research engagement initially sparked by the workshop (see https://reuf.aimath.org/). Harsy's first S@I was organized with her 2017 REUF group: Leyda Almodóvar (Stonehill College), Cory Johnson (California State University San Bernardino), and Jessica Sorrells (Converse University).

2.1 Building Experience

As previously mentioned, ICERM values faculty with a track record of successfully conducting undergraduate research. In this section, we provide suggestions and ideas for getting students interested and involved in research.

A first and most direct option is for faculty to invite students to conduct research with them. To this end, faculty should work to build strong student relationships both in and out of the classroom. Faculty can send emails to students who are majoring or minoring in mathematics or related field (such as computer science, physics, or engineering) to share their research and invite students to participate. We also acknowledge the inherent bias that exists in this process and encourage the broad advertisement of research opportunities whenever possible. Additionally, if possible, creating a mathematics research course, special topics course, or capstone course is a great way to provide students with an opportunity to get started in research with the added incentive of course credit for their involvement. In particular, course-based research experiences are a great way to engage entire classes in research experiences in contrast to more selective research internships or summer programs [2, 3]. For more information and examples of course-based research experiences in mathematical fields, we recommend referring to the December 2022 special edition of the *Mathematics Enthusiast* which includes a variety of papers describing the implementation of a mathematics course-based research experience [6]. In particular, Harsy et al. [13] provide details related to conducting a research course in the research area of the 2023 S@I Program. Running this research course was incredibly valuable and helped the organizers create the first-week S@I modules to introduce students to their research.

While offering course credit is a great way to acknowledge students (and potentially faculty) for conducting research, we also want to highlight alternative opportunities for faculty to find funding for undergraduate research. First, we recommend applying for internal grants from your university as some schools have initiatives to support undergraduate research. For schools without such opportunities, it is worth inquiring with a department chair, dean, or provost to find available funds. Lewis University's internal Summer Undergraduate Research Experience (SURE) Program was created by a group of faculty who worked with the Provost's Office to provide funding and locate donors. For any university with like-minded faculty across disciplines who share similar goals of supporting student research, this may be a great initiative. Faculty could also work with their Office of Sponsored Programs to reach out to local businesses that may be interested in supporting undergraduate research or help in securing external grants. We also recommend looking into the CURM mini-grant, the MAA's National Research Experience for Undergraduates Program (NREUP) and Tensor Grants, PIC Math, and SIAM-Simons Undergraduate Summer Research Program. These organizations not only support undergraduate research but foster collaborations as well.

2.2 The Proposal Application

ICERM solicits applications for S@I each fall, typically with a deadline of September 1. The S@I application is not particularly onerous as it requires a four-to-six-page proposal that includes the following (from `https://icerm.brown.edu/proposals/#summericerm`):

- Description of the program area/theme, written for a general mathematical audience
- Description of the central scientific challenges to be addressed by the program
- Discussion of the experimental and computational aspects of the program
- Description of three to five research projects suitable for undergraduates
- Plans for a computational research component such as computer experiments, visualizations, or coding projects.

The proposal should also include a residency plan for the faculty organizers that provides details/confirmation of an expectation to be on site at least six of the eight weeks. Additionally, it requires a description on how the TAs will be used during the program along with suggestions for three to five potential TAs, which ICERM can also assist with recruiting. Suggestions for finding and selecting potential TAs are discussed further in Section 2.3. Finally, it is suggested that interested faculty should contact the ICERM Director to discuss program ideas prior to submitting a formal proposal.

Past S@I programs typically have an overarching theme for the research proposal. For example, the 2023 S@I Program centered around projects on the combinatorial and graph theoretical properties of DNA self-assembly. The 2024 S@I Program featured diverse research projects using mathematical modeling to predict outcomes, inform preparations, and identify preventions in epidemiology, precision nutrition, and sports analytics.

Once faculty are chosen, they are asked to create a logo and provide a more detailed description of the proposed projects for advertisement to student applicants. Examples of such logos and descriptions can be found on past S@I webpages, including https://icerm.brown.edu/summerug/2023/ and https://icerm.brown.edu/summerug/2024/.

2.3 Student and TA Selection

Student applications are solicited through MathPrograms and require two letters of reference along with several short essay responses to prompts. For example, students are asked to discuss their goals and why they are interested in the particular research projects described.

Arguably the most stressful and challenging aspect of organizing S@I, and perhaps any undergraduate research program, is deciding which students to invite to the program. S@I student applications are due mid-February, and faculty mentors have approximately ten days to produce a ranked shortlist of students from the pool of applications.

The faculty organizers have full autonomy when choosing which students to invite and are encouraged to consider applicants for which the program will have a high impact and who may not have an REU opportunity, including students who are historically underrepresented in mathematics. In both the 2023 and 2024 S@I Programs, the organizers looked for applications that reflected strong writing skills, appropriate background, clear interest in the program (i.e., students who clearly read the project descriptions), supportive letters of reference, and compelling personal essays. We also aimed to create a balance among the type of university, experience/background, and gender of the cohort. This was challenging as we could not anticipate which students would accept our invitation. In particular, if two students from University X ranked in our top 30, it was possible that they could both be invited to the program and accept their positions. It was additionally challenging to choose students for the 2024 S@I Program since we had more varied projects within the theme than in previous years. One unique feature of the S@I Program is that student research teams are developed during the first week of the program, not prior to its start. As such, we had to ensure there was a balance of interest in all the projects. To help with this, the application for the 2024 S@I Program included a specific prompt asking students to discuss their interest in the projects. As the organizers, we also provided ICERM with an approximate target number of students for each project and included a list of the projects each student was interested in with our ranked lists. Invitations were extended to students through the ranked list while optimizing the target number of students for each project. For example, if a student who primarily was interested in sports analytics declined the invitation, the next student invited would be the highest-ranked student who had listed sports analytics even if it meant bypassing a student who had a higher ranking but had listed the precision nutrition project. The end result was a cohort with (roughly) balanced interest levels among the three projects. Of course, we could not precisely anticipate which students would join each project, and we ultimately had students work on projects that were not mentioned in their applications. Conversely, even though there were variations in the projects for the 2023 S@I Program, the projects were all within the area of graph theoretical modeling of self-assembling DNA, so it was possible to go through a singular ranked list of students.

Poring over a large number of student applications is incredibly challenging, especially when the majority of applicants are qualified and deserving of the position. We worried about missing a diamond in the rough or succumbing to unconscious bias. In an attempt to prevent this, both the 2023 and 2024 S@I Program faculty members divided the applications between all four faculty so that each application had two different faculty reviewers leave comments and score the applications based on a universal rubric. After this initial review, we created a shorter list of applicants that were reviewed by all four faculty. The faculty also met multiple times to discuss the applications. A coded spreadsheet was used to organize this work, and various ranking methods, such as ranked choice and Borda count, were then used to create our final ranked list of students. It is worth noting that the different ranking methods led to differences in the ranked lists, so we had follow-up discussions to determine our finalized list.

S@I faculty mentors are also tasked with selecting graduate TAs, whose role is to support the organizers in mentoring student research and coordinate social gatherings and events for the students. ICERM provides each TA with a stipend and housing for the summer, and they can also assist faculty in finding potential TAs. Our S@I TAs provided feedback on student presentations, reviewed code and proofs, and checked in with the students each morning of the program. Additionally, they organized a variety of social events and outings during the eight weeks, including student trips (to Newport, Boston, New York City, and Narragansett), movie nights, trivia nights, Math Jeopardy, Mario Kart tournaments, a PowerPoint party, and more.

When selecting TAs for S@I, we looked for students who had background experience in our particular research areas and who had the ability to organize and facilitate social gatherings for the students. We recommend reaching out to potential TAs early in the process before they make other summer plans. We also recommend choosing TAs who have worked with the organizers or come highly recommended by other faculty. It may be challenging to find a TA who has conducted research related to the specific applications included in the summer program, but we found that a strong graduate student can jump in and help with the research even without that exact experience, especially if they are adept at organizing events and creating a supportive research environment. For example, during the 2024 S@I Program, neither of the TAs for the sports analytics projects had experience in the research area; however, they had expertise in data scraping and statistics that significantly aided in developing students for the projects.

3 During the Program

ICERM provides a welcoming environment for conducting undergraduate research. Each faculty mentor is provided with their own office including breathtaking views of downtown Providence. There are also several shared office spaces for the TAs and students, each with equally beautiful views, chalkboard space, and monitors. Additionally, all participants have access to a kitchen and common space with numerous board games and math textbooks. Slack, a cloud-based team communication platform, was used throughout the summer for communications among the ICERM staff, faculty organizers, TAs, and student researchers. This allowed for easy communication with the entire group while simultaneously utilizing private collaboration channels for various subgroups, as well as using channels for general information, pictures, and social activities. With this method, communication was extremely organized throughout the entire eight-week experience and still is convenient now, months after the program conclusion.

Most of the program days shared a common structure. Each day started with students checking in at 9 a.m. with their TAs, who helped them streamline their goals for the day, divide tasks, and begin their research. Later in the morning, faculty mentors held team meetings with each research group. Each day also allotted students two 30-minute breaks, usually at 10:30 a.m. and 3:00 p.m., along with a 90-minute break for lunch since students had to walk approximately 10 minutes to the dining commons at the Rhode Island School of Design (RISD). The breaks also served as an additional time for community building among the students as they often used these times to play games or kick around the Hacky Sack.

One day of each week was devoted to student presentations on their research. These were short, approximately seven-minute talks about the work completed up to that point. Afterward, faculty mentors and TAs met with the research teams to provide feedback on the presentations and determine next research steps. The goal of these talks was not only to share their work with fellow students and receive feedback, but also to help each group prepare for a larger, more formal research talk. Finally, the concluding week of the program included a two-day symposium during which each research group presented their work as a talk and then subsequently participated in poster presentations during the Brown University Summer Research Symposium.

3.1 The First Week

While most weeks of the program followed the structure previously described, the first two weeks were varied to incorporate mini-workshops and modules that introduced students to the research techniques and goals of *all* projects. Creating the modules required the organizers to more formally introduce their research than what might typically be done in more individualized research settings. The 2023 S@I Program followed the approaches outlined in [13], but in a more condensed manner than typically done in a research course. The scaffolded approach to some of the mini-workshops can be found in the AMS Classroom Resource Materials book, *Teaching Mathematics Through Cross-Curricular Projects* ([1] for 2023 S@I and [14] for the sports analytics projects of 2024 S@I). The second week of the program included additional workshops on using LaTeX and Python, utilizing Brown University's high-performance computing cluster Oscar, and ethics in research training.

Students were placed into their research groups at the end of the first week. To aid in this process, we created a Google Form for students to provide feedback. Based on recommendations from past S@I participants and organizers, the form included not only the projects students wanted to work on but also their preferences for collaborators, including those they preferred or preferred not to work with. We also asked them about their research expectations to help us create successful groups. Many of these questions came from Dorff, Henrich, and Pudwell's book, *A Mathematician's Practical Guide to Mentoring Undergraduate Research* [9], and from workshops related to mentoring undergraduate research, such as the CURM faculty workshop and Creating a Better Summer Experience: A DEI Workshop For REU Directors And Faculty Mentors organized by Pamela E. Harris and hosted by the Center for Minorities in the Mathematical Sciences. See also [16] for additional reflections from `https://mathvalues.org`.

In our form, we asked the following:

- Please rate each project according to your interest. (1 = I would *love* to work on this project; it is my first choice. 5 = I really prefer to not be assigned to this project.) **Note**: You can rank more than one project a "1".
- Why do you want to do research?
- What are your current career goals? How do you expect this experience to help you achieve those goals?
- What does success in this research experience look like to you?
- If you start to struggle while participating in this research, what is your plan for moving forward?

- What is the best way for you to hear/receive criticism?
- In past group experiences, what characteristics of collaborative work have worked best for you?
- In past group experiences, what characteristics of the collaborative work have **not** worked well for you?
- Are there any other Summer@ICERM participants with whom you would really enjoy working?
- Are there any other Summer@ICERM participants with whom you prefer to **not** be in a group? Your response is confidential and will not be shared with any other participants.
- Please state your pronouns (he/his/him, she/her/hers, they/them/theirs, etc.).
- Do you have any dietary restrictions (possibly needed for social events)?
- Anything else the organizers should know?

3.2 Additional Academic and Professional Development Activities

Beyond facilitating the undergraduate research, the faculty mentors and TAs provided additional professional development, academic, and recreational opportunities. In particular, we hosted at least one academic and/or professional development activity and one social activity each week of the program. Both S@I programs invited in-person and virtual speakers to present their research. In addition to the speakers describing their research, we asked them to start by sharing a brief overview of their mathematical journey, which allowed students to learn firsthand about the various ways in which people arrive at their mathematical careers. During the 2023 S@I Program, we invited visiting scholars who were at ICERM for the Social Justice and Data Science Summer Research Programs to give research talks to our students and also hosted a panel where they discussed their research. Further, some of the graduate students participating in these programs served on our graduate school panel, which we held during both the 2023 and 2024 Programs. The graduate student panel consisted of both graduate students and faculty discussing their experiences and giving advice about graduate school. Some feedback we received from students in the 2023 S@I Program was that they wished they had more information about the admissions process for graduate school in addition to information about life in graduate school. As a result, during the 2024 S@I Program, we organized an additional virtual graduate school panel focused on admissions and invited faculty and staff involved in the graduate school admissions process from four universities.

3.3 The Student Experience

This section provides insights into the student experience from the perspective of Horng, a 2023 S@I student participant. Her team worked on problems related to DNA self-assembly, an unfamiliar field for the student participants. A defining challenge came when their algorithm's initial code failed to run, forcing them to think of a more efficient combinatorial approach for the algorithm. This required strong communicative teamwork of comparing alternative methods and coding together. Ultimately, the final algorithm reflected not only individual contributions, but also the strength of leveraging everyone's diverse skills in coding, abstract algebra, and mathematical modeling.

These moments of collaborative problem-solving were among the most rewarding aspects of the program. Daily discussions helped set goals, delegate tasks, and track research progress, fostering both teamwork and independent problem-solving. This approach ensured that everyone was contributing and staying on track. In particular, we found it helpful to always start the day by sharing which action items each team member was going to focus on and ended the day by sharing what was accomplished and any unfinished tasks to work on the next day. Additionally, we also constantly exchanged ideas, supported each other when stuck, and celebrated our successes. This experience taught us the importance of being resourceful in the face of challenges, collaborating effectively to maintain focus, recognizing when to work independently for maximum benefit, and staying motivated throughout the entire process. This experience underscored the importance of effective communication, resourcefulness, and perseverance in research.

Beyond research, the program emphasized professional and social development. ICERM's weekly receptions and workshops connected students with researchers across career stages, and faculty mentors offered guidance on both research and graduate school. Social activities, both organized by the TAs and initiated by students, helped build a strong community. Game nights quickly became a highlight, with Avalon emerging as a favorite, sparking intense strategic debates. Even following the program, many of us reunited at the 2024 Joint Mathematics Meetings to present our research, reconnect, and naturally, play Avalon once again.

3.4 The TA Experience

Additionally, Horng provides her reflections as a TA for the 2024 S@I Program. The role of the TAs extended beyond assisting with research - it involved mentoring students, facilitating discussions, and fostering a supportive and collaborative environment. Each morning began with informal check-ins, where we helped students set goals for the day and troubleshoot any immediate challenges. Throughout the day, we helped to debug code or clarify concepts. Naturally, some students came to points where they were not sure how to proceed. To address this, the TAs and faculty organizers asked guiding questions and mapped a flowchart of the research project, highlighting key findings and the questions that led them there. This exercise helped students refine their approaches and realize potential next steps. To support students, the TAs provided continuous feedback on their papers and GitHub repositories, which led to many groups submitting their papers to journals following the program. We also provided mentorship on graduate school applications, guiding students through personal statements and fellowship opportunities.

As TAs, we wanted to foster an inclusive and engaging community where students felt at home both academically and socially. At the beginning of the program, we shared a list of potential outings to gauge interest. The students enthusiastically embraced everything - from watching *Inside Out 2* on opening weekend to cheering on the Red Sox in Boston. A Juneteenth Math Jeopardy game at a faculty mentor's house was a favorite moment, alongside daily blackboard polls in ICERM's common space, which sparked friendly debates over everything from the best ice cream flavor to the "correct" folder color for math. The final week's PowerPoint Night, where students presented humorous topics ranging from assigning Disney characters to poems for each program participant, was a perfect way to wrap up eight weeks of learning, mentorship, and laughter. Looking back, my TA experience wasn't just about supporting research; it was also about helping students grow as researchers while building a close-knit, supportive community.

4 Mentoring for an Inclusive Undergraduate Research Program

One of the most important roles of a research mentor is the facilitation of an inclusive research environment. Dorff et al. [9] share a plethora of advice, and the articles by Callender [7], Ellis-

Monaghan and Pangborn [11], Hayes et al. [15], and Leonard [19] provide excellent reading material for the creation of a supportive and successful undergraduate experience. While the Hutchinson Lab Handbook [17] focuses its advice on science labs, its main themes can also be extended to mathematics research. Additionally, we personally have benefited from attending workshops related to mentoring undergraduate research, including some previously mentioned such as the CURM faculty workshop and the Center for Minorities in the Mathematical Sciences' Creating a Better Summer Experience: A DEI Workshop for REU Directors and Faculty Mentors. Below we summarize some of the suggestions from these resources.

Mentoring undergraduate research is more than just a chance to work on a fun research project. It is important to get to know mentees as *people* in order to build trust and comfortability in the collaboration. We like to ask students about their hobbies, jobs, families, and hometowns while also inquiring about their goals and how they prefer to receive criticism. Many of these questions were included in our Project Preference Survey described in Section 3.1, but were built upon throughout the program. Setting expectations and norms (see Section 4.1) helps to convey high standards and expectations. In doing so, it is best for mentors to provide a supportive environment that helps dispel doubts and assure students that they can meet these high standards. Meetings with students one-on-one or in small groups can also help students manage feelings of stress, and it is important to have conversations that address the needs of students not just with respect to the research but also in regards to their needs outside of research. It is imperative to ask students how they *feel* and listen to their personal needs in addition to providing research support. As mentors, we should value our students' individualities, embrace the diversity of their experiences, and support their sense of belonging by providing opportunities for breaks and fun. This could involve something as simple as inviting them for an outing to coffee or ice cream outside of the research lab. S@I had three breaks throughout the workday to facilitate this, allowing students to play games and build community with each other. In short, students need mentors who are *people*.

Another emphasis by CURM and Dorff et al. [9] is remembering there are many things we know that our students may not know about research. We have found it to be useful to remind our students that it is okay to make mistakes and to view mistakes as learning opportunities. Failure is part of the research process while discovering what does *not* work is also progress toward a research goal. We like to remind students it is not expected that they understand an idea the first time, and they should not be afraid to ask "Why?". We all get stuck and frustrated, and most tasks take longer than expected. Hard work is the most important feature of a successful student, and it is consistently the hard-working student researchers that take projects the furthest.

In our experience, we found that it is easier to create this environment when the faculty mentors have similar goals pertaining to inclusivity. While it is certainly not necessary for the faculty mentors to be from the same university, our background of working closely with each other at our home institution aided in our collaboration while at ICERM. For us, our shared values from Lewis University's Catholic and Lasallian Mission guided our decisions as we worked towards creating a student-driven experience which focused on establishing an inclusive community for a diverse group of students.

4.1 Goals and Norms for the Program

During the first week, we also discussed program goals and community norms. We share these below.

Program Goals

Participants will do the following:

- Work in a small team setting to solve mathematical research problems developed by the faculty organizers.
- Create career-building connections between peers, near peers (graduate students), and academic professionals.
- Attend lessons/workshops developed by the faculty organizers and graduate TAs to learn the fundamentals of the research for the summer.
- Compose drafts of articles suitable for publication in a peer-reviewed mathematics journal.
- Practice presentation and communication skills by crafting and delivering periodic "progress reports" and final end-of-summer presentations.
- Acquire skills and knowledge to help them enter graduate research programs and succeed in their scientific careers.

We also put together the following community norms and expectations. One way to facilitate this is by having students first put together their own norms and then allowing faculty to fill in anything they believe is missing.

Community Norms

All participants, TAs, and faculty will do the following:

- Respect personal boundaries.
- Be mindful of different communities and different experiences in them.
- Help each other when others need it and ask for help when *you* need it.
- Recognize when a break is needed and take it in order to promote focus later.
- Do not talk down about someone else's research or interests.
- Communicate and check in with one another.
- Leave space for others and also make space for yourself.
- Be respectful of others' time, both while working and outside of normal program hours.
- Be respectful when providing criticism/feedback.
- Be inclusive and invite others to join things.
- Be mindful that everyone has a different mathematical background.
- Be open-minded towards all ideas.
- Actively create an environment where disagreement is welcome (productive conflict).
- Acknowledge that mistakes are normal and serve as learning opportunities.

- Treat everyone with respect and kindness – critique ideas, not people.
- Be active listeners – be curious about the opinions and concerns of others.
- Share responsibilities and workloads.
- Help others and ask for help when needed.
- Cooperate and compromise with one another.
- Be mindful and respectful of other participants, TAs, organizers, staff, ICERM, your home institutions, and Brown when posting to various social media platforms – be aware that others might not wish to be posted about on these platforms.
- Do not claim work that isn't your own – be transparent about where information was obtained.

5 After the Program

Following the program, participants and mentors continued to collaborate through Slack and email. Students were encouraged to present at the undergraduate poster session at the Joint Mathematics Meetings, and many talks were also given during invited sessions. Many of our 2023 S@I students attended, and it was great to see the students reunited. Furthermore, many of the students also presented at other local, regional, and national conferences throughout the following year, and students and faculty continued to work to submit papers on the research. Additionally, many of the students reached out for letters of support for graduate school applications and/or other research or fellowship opportunities. Thus far, the same has also been true for the 2024 cohort. The role of the S@I faculty mentor is to be another point in their professional network, and it is vital for faculty to be available and support the students after the program. We reminded students that they can reach out for support even if it has been multiple years since the S@I program.

6 Conclusion, Advice, and Lessons Learned

Mentoring students in research is an incredibly rewarding experience. The opportunity to work in the environment facilitated by ICERM brought this experience to new heights. The work space was inviting and constructed in a way that facilitated group work and collaboration. Working with other faculty members and TAs allowed us to grow and learn from each other. Each mentor has their own style of leading undergraduate research, and the experience was improved by our collaborative efforts and discussions. When unexpected group dynamics or issues arose, it was beneficial to be able to discuss them with a team of faculty members. As such, we would like to share the following observations and pieces of advice from being part of this program:

- Everything takes longer than expected!
- Meet regularly and set expectations for meetings.
- Have regularly scheduled times for students to present their work throughout the program.
- Require a "deliverable." This could be a poster, final presentation, and/or paper.
- Community building is important. Make sure to schedule in social time. We found something as simple as asking a "Question of the Day" helped us get to know each other and build community.

- We benefited from being friends with our co-organizers. It helped to work with people we trust and knew that we valued each other as educators and researchers. Furthermore, our similar core beliefs helped us shape the program in a supportive way.

- Think about how you will divide the workload for S@I. It takes effort to manage 18 students. Decide beforehand how you will split up the time and focus of the faculty mentors during the program. That is, will each faculty work on an individual project, in teams with multiple projects, or will all faculty contribute to all projects? Remember the mentoring does not end after the eight weeks, and students will be working on their papers. Balancing multiple student papers is time consuming for faculty mentors as well, so it is beneficial to set timelines and discuss who has time to do what and when during the writing and editing process after the program.

- Think about authorship expectations ahead of time. Will all organizers be on each paper? Will all TAs? Just students? Share these expectations early in the program and explain why that decision was made. You may need to repeat these expectations.

- Think about the venue(s) where you will publish. Dorff et al. [9] has many suggestions for great places to publish with undergraduates.

- We highly recommend the following materials related to mentoring undergraduate research: Dorff, Henrich, and Pudwell's *A Mathematician's Practical Guide to Mentoring Undergraduate Research* [9], Ellis-Monaghan and Pangborn's "An Example of Practical Organization for Undergraduate Research Experiences" [11], Leonard's "Adventures in Academic Year Undergraduate Research" [19], Callender's "Keys to Successful Mentoring of Undergraduate Research Teams with an Emphasis in Applied Mathematics Research" [7], and Hayes et al.'s "Crossing Paths: Tips for Undergraduate Research" [15].

- Attend a workshop or complete some training in mentoring undergraduate research. CURM has a great workshop tied to their mini-grant, but often national and regional mathematics conferences like the Joint Mathematics Meetings, MAA MathFest, ACMS, and other organizations (Project NExT, UR SIGMAA, REUF, and SIAM) have workshops related to mentoring undergraduate research.

Many of these pieces of advice require close coordination and communication between the organizers of the program. One very important piece of a successful program is to have a good working group between the organizers. In our case, we found that having similar beliefs, core values, and goals helped us have a common approach and response to issues or challenges which came up during the program. In the 2024 Program, all of the organizers were from the same Catholic university so we had a similar work, faith, and teaching background. As one may note, the community norms we had for our program have a strong correlation with Christian values and beatitudes. With our common beliefs, we did not find ourselves in conflict throughout the program. But as with all things, sometimes it was challenging to keep a Christ-like approach to unexpected interpersonal situations.

Another lesson we learned was how important it is to know your TAs or have them be recommended by a trusted colleague. We worked closely with our TAs so it was vital for us to be able to delegate responsibilities to them. In any future research program, we would continue to use a similar student selection and project selection process. In particular, optimizing student groups not only based on project preference but also including student peer preferences was important and not always used in past S@I programs. Finally, we would continue to support and plan the activities described in the Student and TA Experience sections since we believe it aided in creating inclusive environments that extended beyond the programs.

Moving to Providence for the summer may seem overwhelming at first, especially if it means being away from some family members or if the temporary move includes pets and children. However, the staff at ICERM provide support in finding housing to suit the needs of the faculty mentors and their families. Further, while the distance was tough for the faculty mentors who were away from loved ones, Providence offers plenty of opportunities for the family members who were able to join, with an abundance of pet-friendly apartments and excellent summer camps for children of all ages. While it may not be feasible for family members to join for all eight weeks, some mentors had their family join for part of the summer to reduce the time away from each other.

In conclusion, Summer@ICERM is a great program! We highly recommend taking advantage of being involved in this or a similar program.

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Incarnational Mathematics: Applying Relational Ministry to the Mathematics Classroom

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Abstract

In this paper, I will share how I use my fourteen plus years of experience in the outreach ministry of Young Life to inform my approach to teaching mathematics. Just as Young Life stresses the incarnational ministry of Jesus, where "The Word became flesh and blood, and moved into the neighborhood" (John 1:14 MSG), I emphasize the role of real and authentic relationships in the classroom. I will share some of my experience, and simple, practical things I do to cultivate relationships with students, where the goal is to share not just my knowledge, but my life with students.

1 Introduction

My view of the relationship between my faith and anything else in my life is rooted in the preeminence of Christ (Colossians 1:18). He must be at the center of all things in my life, for only in Him do I find my true purpose. "For in Him we live and move and have our being" (Acts 17:28 ESV).

My faith integration within the discipline of mathematics takes two major forms: philosophical and relational. Many volumes have been written on the philosophical aspects of mathematical faith integration, (see [4][8][9]) and it is definitely an important aspect of my own integration. However, because of my extensive experience working with the ministry of Young Life (a world wide organization that specializes in bringing Christ to high school students that are disinterested in church), I feel like I can bring a lot more to the conversation on the relational side.

In this paper, I delve into the concept of incarnational ministry and how it has fundamentally shaped my approach to teaching. My perspective is deeply influenced by my 14 years as a volunteer leader with Young Life, where I've learned firsthand the power of relationships in transforming lives. These experiences have solidified my belief that meaningful, authentic connections are at the heart of any effective ministry or educational practice.

The paper is structured to reflect this journey. I begin by unpacking what incarnational ministry is—a model of ministry rooted in the life and mission of Jesus Christ, emphasizing deep, relational engagement. From there, I explore how this model has influenced my teaching philosophy, particularly in the context of higher education. Finally, we move on to the practical side, describing how I bring these principles into the classroom through specific intentional practices that help build genuine relationships with students.

This exploration is more than just a theoretical exercise; it is a reflection of how I have seen relationships impact students' lives, both in ministry and in the classroom. My aim is to show how even small, seemingly simple actions can make a profound difference, just as they have in my years with Young Life.

2 Incarnational Ministry

We begin by looking at the idea of incarnational ministry in general. Incarnational ministry, at its heart, is a concept rooted in the life and mission of Jesus Christ. It is a ministry model that emphasizes the relational engagement with those served. The term "incarnational" is derived from the Latin word *incarnare*, meaning "to make flesh." It is inspired by John 1:14: "The Word became flesh and made his dwelling among us." The essence of incarnational ministry lies in this profound truth: Jesus entered into our humanity. He fully immersed Himself in human life, living among us, experiencing our struggles, and demonstrating God's love in a tangible, personal way. At its core, incarnational ministry involves embodying Christ's presence by living among and alongside those being served, reflecting God's love and truth in the context of real relationships. Rather than engaging from a distance, incarnational ministry emphasizes being fully present.

A key passage that points to what this looks like is 1 Thessalonians 2:8: "Because we loved you so much, we were delighted to share with you not only the gospel of God but our lives as well." This holistic engagement goes beyond simply preaching the gospel; it calls for sharing in the everyday life. New Testament scholar Dean Flemming echoes this idea, noting that "Incarnational ministry is not simply about proclaiming the gospel, but living it out in ways that are visible, tangible, and transformative within the culture" [2]. Another prominent author, Alan Hirsch, describes incarnational ministry as a move away from "attractional" models of church, where the expectation is that people will come to a church building to hear the gospel, toward a model where Christians embed themselves within communities to live out the gospel. Hirsch writes, "Incarnational mission is about going into the context and being the gospel there, in ways that make sense to the people who live in that context" [3]. This emphasis on "going where kids are", as we say in Young Life, is what makes it stand out as a youth ministry.

This practice is deeply relational, aligning with Jesus' method of ministry. Just as Jesus did not isolate Himself but chose to dwell in the midst of people's joys and sorrows, incarnational ministry emphasizes the importance of proximity. Henri Nouwen (from whom my second son Phineas gets his middle name) described this as "ministry by presence", where the act of simply being with others can be a profound expression of Christ's love. Nouwen argued that "real care means the willingness to enter into someone else's world and be present to them where they are" [7]. I have had first-hand experience living this out through Young Life, but it also has played a major role in how I engage in teaching.

2.1 Incarnation in the Classroom: Relational Pedagogy

The way faith informs my teaching is primarily rooted in my belief of God as a relational God. Our God is a communal, Triune God who created us for relationships. I connect most with God by being in relationships with others. The platform that teaching gives me to have mentoring relationships with students is what particularly drew me to this vocation. Through teaching, I can use my platform to point students toward Truth.

Jesus is our ultimate Teacher, and in my philosophy of teaching I strive to emulate Him. Jesus challenged people, loved people, and brought them to deeper levels of understanding. He epitomized what it looks like to disciple people through His words and actions. When I teach others, I hope to mimic Jesus' interactions and model my classroom after one He might have had. Due to my experience in Young Life, my teaching philosophy strongly emphasizes holistic relationships. This resonates with the Mennonite faith tradition of "focusing on hands and heart", where "they therefore help us to see that higher education in the Christian genre must be multifaceted and holistic, helping students to develop every aspect of their being, not simply their minds" [5]. The goal of education is not just knowledge, but true character formation. When done right, the teacher-student relationship can lay the groundwork for transformative character development. The way this plays out in my teaching philosophy is through the application of incarnational ministry.

It should be noted upfront that this application in the classroom is not discipline specific. However, with a notoriously difficult subject such as mathematics, it is imperative to practice these techniques to help students feel known and safe. Being a math professor allows me to have a unique opportunity to walk students through the process of solving difficult problems and realizing their potential for logical and rational thinking. And it also requires an extra bit of care, because we know the negative association many of our students bring to the math classroom. Being extra intentional in our relationships with students helps to break down walls and barriers.

You have probably heard the old adage "people only care what you know, when they know that you care". Though it may be clichè at this point, from my personal experience, I have seen it ring true in almost all my interactions with students. Pursuing authentic relationships with students has become second nature to me and is the main way that I show students that I care. I don't just share knowledge with students, but also my life as well (1 Thessalonians 2:8). As an extrovert, I live for this stuff, but I know that not everyone is like me. Being an unpaid vocational minister for my entire Christian life has equipped me uniquely to pursue students relationally. But I still recognize the daunting task that it can be to "share our whole lives" with students. I have cultivated four relatively simple but effective practices to implement this incarnational and relational style of teaching: knowing students by name, knowing their story, sharing my story, and breaking bread.

3 Four Practices for Implementing Relational Pedagogy

I will examine the four practices by first justifying their use via Scripture, and then giving very practical ways to apply them. In order to make these practices actually possible without causing burnout or overtaxing teachers, I have given some ways that I have implemented them that make them more manageable. My desire is not to make teachers feel like they should be doing way more than they already are. I want to give simple, tangible ways to make real connections with students, that doesn't take an exorbitant amount of extra time and effort.

3.1 Known by Name

I love George Fox's Be Known promise, which states "At George Fox, each student will be known – personally, academically, and spiritually." To even begin to know our students, we must first know their name. There is power in a name. When Jesus calls out "Mary" in the garden, she finally recognizes who He is (John 20:16). Knowing students by name is fundamental to implementing a relational teaching ethos. This seemingly simple act fosters a more personalized and inclusive learning environment, signaling to students that they are valued individuals rather than anonymous participants. Just as Jesus, in John 10:3, calls each of His sheep by name, a professor's use of students' names can build trust and rapport, enhancing engagement and motivation.

Although I'm sure most people reading this are already aware of the importance of this practice, I think it is always good to be reminded of its fundamental value. Because I believe it is so foundational to building authentic relationships with students, I strive to implement it as quickly as possible. I make it a goal to have them all memorized by the second class day. Through this one simple gesture, I let my students know that they are not just a student to soak up information. They are a person, with a story and a name. By making the effort to remember and use students' names, professors not only acknowledge each student's unique identity but also embody the incarnational approach, where teaching is not merely about transmitting knowledge but about deeply connecting with and nurturing each student's growth.

There are many practical tips I have received over the years for remembering names (many of which have come from mentors in Young Life), but in my personal experience the simplest and most effective is to have some document that has their name and photo, and to constantly look over it and quiz myself. At George Fox, we are lucky enough to have photos built into our class grade rosters, so I can start learning my students' names before they even step foot in the classroom (though I generally don't reveal that I know their names before day 1, for fear of seeming obsessive). On the first day of a class I will always have students introduce themselves to the entire class. They are always so amazed when I quickly pick up all their names and can refer to them by their first name right away. They think I have some supernatural short term memory, when the truth is, I put in the time to know their names beforehand (and even pray for them!) before I meet them in person.

If you are not at an institution that provides you with student photos, another option I have seen implemented is to have one of the first assignments be students sending some form of a photo, so you can practice putting their name to their face. This could be part of a larger assignment (like the Who Am I PowerPoint which I will explain in the next section), or just a simple standalone assignment. You can easily justify such an assignment by letting students know upfront that you value knowing them by name.

One last practical tip I will give regarding names is to keep going back to your names and faces document. This is especially helpful for students that I have had in class before, but haven't seen in awhile. I will regularly go back over my old class rosters to remind myself of students' names if I forget them. The message this sends to students is meaningful: I see you, I know you, and I remember you.

Even with all these little tricks and tips, it is important to remember we will still mess up and forget a student's name, and that is okay. Give yourself grace, and move forward. And look back at your cheat sheet!

3.2 Know Their Story

Going beyond just knowing my students by name, I make it a point to know their story. In the spirit of incarnational/relational pedagogy, knowing students' stories is a crucial facet. In Jeremiah 1:5 (NIV), God says, "Before I formed you in the womb I knew you, before you were born I set you apart." This verse underscores the importance of knowing someone's background and potential from a place of deep understanding and purpose. Additionally, Psalm 139:16 (NIV) highlights, "All the days ordained for me were written in your book before one of them came to be." This illustrates how our individual stories and experiences are seen and valued by God. We can emulate Him by taking an interest in our students' stories. By taking the time to learn about students' backgrounds and personal experiences, we can better connect with them, address their unique needs, and foster a more empathetic and supportive learning environment. This is definitely a lot more difficult and time consuming than simply knowing their name. But it is worth it. I have seen its fruit both in my Young Life ministry as well as my ministry in the classroom.

One way to make this goal more accessible is an assignment that I adapted from my good friend and colleague, Pete Rusaw, called the "Who Am I" PowerPoint. I assign it the first week in all my classes, and it has several different sections where the students can share some different parts of their story. In order to help students feel the safety to be vulnerable and actually share meaningful info, it is communicated upfront that these PowerPoint slides will **not** be presented in front of the class, and that they are for my eyes only. I let students know that this is a fun assignment where they can be as vulnerable and creative as they want. It is an opportunity for them to share a little bit about themselves, because I truly care to know.

For the current version of the assignment I use the following sections:

1. **Basic Info**: Here I have them put down just some basic demographic info, such as name, age, birthday, hometown, and year in school. It helps get the ball rolling to start with easy, superficial data.

2. **Family of Origin**: Here I invite the students to share some info about the family they grew up in. From my time in premarital and marital counseling, I came to realize how important families of origin are. They greatly shape the person we are. Interacting with and reflecting on our family of origin is a great way to understand some of the reasons we are who we are. Inviting students to begin that introspection work is good in and of itself, but it is also illuminating for me to see the background that students come from, and try to better cater to their specific needs.

3. **One Story Marker**: Instead of asking for a bird's-eye view summary of their lives, I like to focus on just one specific story marker. Something that they are proud of, or some situation that they feel was especially impactful for them. I encourage them to be extra creative and have fun with this particular slide. I focus on a single story marker for two reasons. First, it is generally easier for me to remember and associate a student with one particular story than a quick summary of their whole lives. Second, because it is generally easier for students to write compelling stories when it is just a single thing to focus on. This slide is my favorite to read.

4. **Spiritual Background**: Here I invite students to share a little bit about their spiritual background. I ask them to state where they feel like they currently stand with Jesus, and what they would like in terms of their relationship to Jesus. Since George Fox is a Christian institution, I expected that most of the students would talk about their deep and vibrant faith backgrounds. I was so surprised at how many students either didn't have any sort of faith background, or have fallen out of a faith tradition. For me personally, this was actually an encouragement. It meant students felt like they could answer honestly. It also meant that I had an opportunity to share Jesus with them, and hopefully plant a seed.

5. **Personality and Character**: In this section, I ask students to put down some of their strengths and weaknesses. I also ask them to give their Myers Briggs and/or their Enneagram results if they know them. I know the giant rabbit hole these personality tests can lead to, so I encourage them to not worry about answering this unless they have already thought about them before.

6. **Academic Experiences**: This is a more practical section, where I hope to better understand students' view toward math and previous experiences with math. This helps me to better

calibrate how I approach things, recognizing that not everyone is going to have the awesome positive feelings towards math that I do.

7. **Personal Connection**: This is the final section, and in it I ask three simple questions: 1) "What are your expectations for me as an instructor this semester?" 2) "What should my expectations of you as a student be this semester?" 3) "How can I pray for you?" Being able to know what their expectations are, and then actually pray specifically for them is very helpful and powerful.

I won't sugar coat it, these take quite a bit of time to read through. However, the quality of insight I get into my students' lives is so beneficial for the rest of the semester, and it makes a difference. I have had many students tell me that they appreciated me getting to know their story, and they felt it really lived up to the "Be Known" promise.

3.3 Sharing My Story

I make it a point in my class to share who I am with the students. Not so much in a drastic or overbearing way, but by simply sharing snippets of my life, my family, and especially my recent struggles. The Bible underscores the significance of personal testimony and mutual understanding. In 2 Corinthians 1:4 (NIV), Paul writes, "He comforts us in all our troubles, so that we can comfort those in any trouble, with the comfort we ourselves receive from God." This verse emphasizes how sharing our own experiences can provide support and encouragement to others.

A verse I have already mentioned two times, 1 Thessalonians 2:8, really underscores how the sharing of our lives shows students we genuinely care. Making myself vulnerable and personable to students helps to break down walls, and make the subject matter seem less intimidating. This emphasis on relatability helps make the classroom a much more welcoming environment, and it lets the students know that I am a person too, not just a math-reciting robot. In both the classroom and Young Life, I have seen walls been broken down by simply sharing who I am, and how I make mistakes too.

One of the main ways I share my story is by filling out the "Who Am I?" PowerPoint assignment with my own info. This serves the double purpose of setting the bar of authenticity and vulnerability I can expect from the students, as well as inviting students into knowing my story more. Although there is definitely a fine line between being authentic and oversharing, I think in general, leaning towards authenticity and vulnerability will produce more meaningful connections with students, and ultimately better outcomes in our classes.

3.4 Breaking Bread

In an attempt to build relational clout with our students, let us not forget the power of sharing a meal. The Gospel of Luke notably emphasizes the importance of eating together as a means of building connection and community. Throughout Luke, Jesus is frequently depicted sharing meals with others, highlighting its role in relationship-building and hospitality. For example, in Luke 24:30-31, we see Jesus breaking bread with His disciples: "When he was at the table with them, he took bread, gave thanks, broke it and began to give it to them. Then their eyes were opened and they recognized him." Similarly, Luke 5:29 (NIV) recounts how Jesus dined with tax collectors and sinners: "Then Levi held a great banquet for Jesus at his house, and a large crowd of tax collectors and others were eating with them." This emphasizes Jesus' approachability. By sharing a meal with students, teachers can foster openness, build trust, and create a more personal and supportive learning environment.

Ideally, I would have a meal with each student to follow up and hear more in person on what they shared in their “Who Am I”, but logistically this is a bit more difficult, and oftentimes infeasible. Early in my career (before kids and tenure-track demands), when I was adjuncting at Westmont College, I ate with students three days a week. I had a Google Sheet sign up form that students would fill out, and I would grab lunch in the dining commons with them every time I was on campus. I was especially motivated to do so: Westmont had a program where students could take their professors out to lunch at the dining hall, cost free for both of them. This program, funded by their association of students, was unlimited! As a poor graduate student, nothing was more enticing than a free meal. Getting to know my students better and sharing meaningful connection over a free meal made it all the more sweeter, so I took advantage.

Now that I am at George Fox, where sadly no such program exists (though I am advocating for one!), and where I have much more on my plate, this level of intentional bread breaking with students is impossible. It would be borderline irresponsible of me to devote the time required to get a meal with every one of my students, and it would be just as irresponsible to suggest that other instructors do so. Like all my practices, the key is finding a sustainable mode of implementation. For me, that has looked like getting lunch with either one or more students in the dining commons once a month, and also inviting students from my smaller upper level classes to share a meal with me at my house. Whatever it may look like will vary from individual to individual. Regardless, my hope is that the significance of breaking bread with students would not be understated, and could be adopted in some form. Some of my best conversations with students from Young Life and the classroom have occurred over a meal. The relational clout built from a meal should not be underestimated.

4 Conclusion

In reflecting on the integration of incarnational ministry into my teaching, I recognize that these practices—knowing students by name, understanding their stories, sharing my own, and breaking bread together—can seem somewhat simple. There is something about these simple everyday acts that have power.

In my favorite novel, The Brothers Karamazov, this idea is exemplified by the short parable of the “little onion”. The parable tells of a wicked woman who dies and ends up in hell. Her guardian angel, searching for something good she had done in her lifetime, remembers that she once gave a beggar a small onion. The angel presents this act to God, who allows the angel to try pulling her out of hell using that very onion. The angel lowers the onion to her, and as she clings to it, she begins to rise.

This parable acts as a microcosm of a major motif throughout the entire novel: that small, mundane acts of kindness can have huge redemptive value. And we see many such “onions” given away throughout the novel. This is what Morson in writing about the “onions” called the “mythic prosaic” [6]. There is something profound and powerful when the Holy Spirit abides in our small acts of kindness and obedience. Just as a small gesture of kindness in The Brothers Karamazov—the “little onion”—can lead to transformative change, these everyday acts of relationship-building can have a significant impact on students’ lives.

In a world increasingly driven by technology, where AI can potentially impart all the knowledge students might need, the role of the teacher is evolving. But what remains irreplaceable is the human connection. The relationships that foster trust, growth, and true understanding are still crucial. These practices are not just about teaching; they're about embodying Christ's love in the classroom, offering students more than knowledge—offering them a sense of belonging and purpose.

By focusing on relationships, we're not just preparing students academically; we're helping to shape their character, guiding them to become whole and well-rounded individuals. And while the methods may seem ordinary, their impact is anything but. Just as my years with Young Life have shown me the power of being fully present with others, I believe these practices can have a lasting, transformative effect on students, long after they leave the classroom.

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A Divine Conceptualist Interpretation of Modal Structuralism

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Abstract

Structuralism is a contemporary philosophy of mathematics which asserts that the content of mathematics is not located in the objects of its study but rather the relations which hold between those objects. Modal Structuralism, in turn, emphasizes that mathematical truth concerns modal statements about what would be necessarily true of a system if such relations held. Finally, Theistic Conceptual Realism argues that possible worlds are in fact instances of divine self-knowledge, and thereby provides a means of grounding modality in the mind of God. In this paper, we argue that Modal Structuralism, as interpreted via Theistic Conceptual Realism, provides a satisfying view of mathematics which we hope will appeal to Christian mathematicians in particular.

1 Divine Conceptualism

Divine Conceptualism (DC) can be defined as the belief that objects such as propositions, properties, possibilities, etc. are best understood as thoughts in the mind of God. The divine conceptualist perspective, broadly conceived, has deep roots within the Christian philosophical tradition. For example, Augustine famously writes that:

> ... the ideas are certain original and principal forms of things, i.e., reasons, fixed and unchangeable, which are not themselves formed and, being thus eternal and existing always in the same state, are contained in the Divine Intelligence. And though they themselves neither come into being nor pass away, nevertheless, everything which can come into being and pass away and everything which does come into being and pass away is said to be formed in accord with these ideas. [3]

In this vein, the divine conceptualist believes God's idea of a circle provides the principle and original form of all things circular in shape, or that God's idea of the number two provides the principal and original form of all collections consisting of two objects.

James Bradley has explored the history of what he calls an *Augustinian view*, that is, a divine conceptualist understanding of mathematical objects such as circles, sets, and numbers. On this view, mathematical objects are "ideas in the mind of God that have been there from eternity and that God has used in creating the physical universe" [5]. Bradley cites Edward Everett, for example, who asserted that, "In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven" (quoted in [5]). More recently, Alvin Plantinga has posited that "... perhaps the most natural way to think about abstract objects, including numbers, is as divine thoughts" [27]. Likewise, Vern Poythress echoes the likes of Kepler in asserting that, "In thinking about arithmetic, we are thinking God's thoughts after him" [28]. Thus, in many times and places, Christian thinkers have embraced a divine conceptualist understanding of mathematics as a realist position on mathematical objects which views those objects as dependent upon God in the same way that thoughts are dependent on a thinker.

1.1 The Nature of Divine Thoughts

By identifying mathematical objects with divine thoughts, DC invites questions regarding the nature of such thoughts. For example, mathematical objects are typically classified as abstract—as opposed to concrete. However, DC, as we use the term, is a claim that these objects which are typically considered to be abstracta are not, in fact, abstract.[1] Rather, they exist as concrete divine mental states/events, that is, as divine thoughts.

Why think that divine thoughts are concrete? Based on his survey of common perspectives on the abstract/concrete distinction, David Liggins suggests that something be considered abstract if and only if it is acausal or lacks spatial location [21, p. 4]. Thus, if we accept Liggins's suggestion, we should view God's thoughts as concrete because we view them as both causally active and spatially located. Regarding their causal activity: In scripture, God's thoughts often appear to cause God to act in the way he does—and his actions bring about effects. Consider, for example, the text from Exodus 2:23-25 (ESV translation): "During those many days the king of Egypt died, and the people of Israel groaned because of their slavery and cried out for help. Their cry for rescue from slavery came up to God. And God heard their groaning, and God remembered his covenant with Abraham, with Isaac, and with Jacob. God saw the people of Israel—and God knew." The writer explicitly points out that God "remembered" his covenant and "knew" the Israelites were suffering. These are descriptions of mental activity. Moreover, this mental activity is presented as a reason that God then acted to deliver the Israelites from slavery. In other words, one could say that God's thoughts caused God to act. Regarding their spatial location: God himself is traditionally considered to be spatially located in that he is located everywhere—he is omnipresent. We would then also think that all of God's thoughts are located where he is located.

Therefore, our definition of DC agrees with that of William Lane Craig: "Divine conceptualism is a non-Platonic realism which substitutes God's thoughts in the place of abstract objects" [8, p. 123]. In other words, we believe it is valid to describe DC as a realist perspective on objects such as propositions and properties, but the version of DC we consider here does not view such objects as abstract in the traditional Platonic sense. In making this clarification, we attempt to heed Craig's exhortation:

> If [theistic philosophers] are realists, they need to be clear whether they mean to affirm not

[1] Here, we acknowledge that Plantinga's usage of the term "abstract" in the quote above conflicts with this contention.

just realism about propositions, properties, mathematical objects, and so forth, but whether such things are abstract objects created by God or concrete thoughts of various sorts which God has. [9]

But how much hangs on our assertion that God's thoughts are concrete? Although much of what we will argue is independent of this question, as will become apparent later,[2] viewing God's thoughts as concrete allows us to formulate an interpretation of mathematics that is neutral regarding certain ontological commitments. In particular, the position we will articulate and defend is independent of one's position on the existence of abstract objects.[3]

1.2 Structuralist Challenges to a DC Perspective on Mathematics

While DC has a strong theological appeal and one could argue that it brings with it certain philosophical benefits, there are various questions one might ask in response to divine conceptualist claims. In particular, one might probe the central ontological claim that objects such as numbers are thoughts in the mind of God by asking elementary questions such as "Which divine thought is the number 2?"[4] One would hope that there is a correct answer to this question, at least in principle. If there is not, one might legitimately wonder if one is justified in making the claim that the number 2 is in fact a thought of God.

This concern is intimately connected with the concern expressed by Paul Benacerraf in response to the ontological claim that numbers are sets. In his famous paper, *What Numbers Could Not Be*, Benacerraf argues that numbers cannot, in fact, be sets [4].

Benacerraf's argument begins with a description of both the Zermelo and von Neumann models for the natural numbers. The Zermelo model identifies the number 0 with the empty set and provides a recursive definition of the successor function as $S(n) = \{n\}$. Thus, in the Zermelo model, the number 3 would be identified with $\{\{\{\varnothing\}\}\}$. On the other hand, the von Neumann model identifies the number 0 with the empty set and provides a recursive definition of the successor function as $S(n) = n \cup \{n\}$. Thus, in the von Neumann model, the number 3 would be identified with $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. So then, does 17 contain 3? According to Zermelo, the answer is "no." According to von Neumann, the answer is "yes." If the numbers 3 and 17 do in fact exist as particular sets, this indicates that at least one model is wrong. In light of this situation, Benacerraf focuses attention on the question of identity: "Is $3 = \{\{\{\varnothing\}\}\}$?" He argues that, in general, such a question is utterly superfluous to our understanding and use of numbers via relational axioms (e.g. those attributed to Peano), and that there would be no way—even in principal—to decide the answer to such a question [4, p. 62]. Indeed, he argues that such a question is ill-posed (i.e. "unsemantical") to the point that it has no answer [4, Section III.A].

Benacerraf's *identification problem* is relevant in that his argument seems equally applicable to the DC claim that numbers are divine thoughts. Indeed, suppose that the number 3 is a divine thought. What is meant by this claim? If by this claim we mean that the number 3 is a divine thought about a certain set, we land squarely within the scope of Benacerraf's argument. If we eschew set-theoretic constructions when trying to identify the divine thought that is the number 3, upon what will we rely? According to Benacerraf, his argument against numbers being sets can

[2]See Subsection 5.1

[3]To be clear, we believe that our position would remain coherent, *mutatis mutandis*, if God's thoughts were viewed as abstract, but it would simply no longer be compatible with nominalism regarding abstracta.

[4]We thank David Liggins for bringing such a question to our attention.

be extended to imply that "numbers could not be objects at all; for there is no more reason to identify any individual number with any one particular object than with any other (not already known to be a number)" [4, p. 69].[5] In particular, if Benacerraf is correct, then numbers are not divine thoughts and the central tenet of DC is false.

Operating under the assumption that numbers are not objects, we are faced with a choice: abandon DC completely, or modify DC in some way so as to avoid a commitment to numbers as divine thoughts. We choose the latter. Our goal in what follows is to abandon the claim that mathematical objects are divine thoughts while nevertheless grounding mathematics in the divine mind. Of course, one might immediately object that in refusing to affirm the existence of mathematical objects as such we do more than merely *modify* DC but rather we *abandon* it, and we admit that in so doing we substantively depart from classical DC. Yet, insofar as we ultimately locate all of mathematics in the divine mind, we find it legitimate to retain the moniker of DC in describing our approach. In particular, we emphasize here at the outset that we are *not* articulating a divine conceptualist interpretation of mathematical objects, but of Modal Structuralism. In other words, we attempt to retain the desirable theological attributes of DC while avoiding the identification problem. We begin this pursuit with a brief description of Mathematical Structuralism.

2 Structuralism

Benacerraf's important contribution inaugurated a new category within the philosophy of mathematics known as *Mathematical Structuralism*. Structuralism seeks to sidestep Benacerraf's identification problem by denying that the content of mathematics is directed at objects of any kind. Instead, what matters is the relations which hold between objects.

In their introduction to Structuralism, Hellman and Shapiro characterize a *system* as any "collection of objects together with relations on those objects" [16, p. 1]. Examples of systems abound in the natural world and are studied by the various sciences, but we may also wish to include the set-theoretic objects considered by mathematicians. For example, both representations of the natural numbers (Zermelo's and von Neumann's) considered by Benacerraf could be considered systems.

In turn, a *structure* is a form of a system "which ignores or abstracts away from any features of the objects that do not bear on the relations" [16, p. 2]. The well-known Peano Axioms identify a structure precisely because they select specific relations which must obtain among a set of objects in order for that system to exemplify the structure we typically associate with the natural numbers. When given their natural interpretations, both of Benacerraf's set-theoretic systems equally satisfy the Peano Axioms and therefore equally exemplify the natural number structure.

However, once we move the content of mathematics from the level of addressing systems to the level of addressing structures, we may still ask for a deeper account of what constitutes a structure, and the various contemporary viewpoints within Structuralism may be distinguished by the ontological status which they give to structures. In his *Philosophy of Mathematics: Structure and Ontology*, Shapiro divides these viewpoints into three categories [30].

First, structures may be said to exist ontologically prior to the systems which instantiate them. This view is called *ante rem* Structuralism, and various objections can be raised against it along

[5]The reader may object that this inference constitutes an unjustified leap from an epistemological premise to an ontological conclusion. We agree that such an inference need not in general be valid. However, Benacerraf argues that this inference is in fact valid in the specific context of identity questions about numbers [4, p. 58].

the lines of the usual objections raised against Platonic realism. Second, structures may be said to exist ontologically posterior to the systems which instantiate them. This view is called *in re* Structuralism, and it has more in common with the Aristotelian position on abstracta. The objections here would center around whether or not there are truly enough systems to instantiate all the structures we normally talk about in mathematics. Finally, there is the *post rem* or eliminativist viewpoint on structures which denies that they exist at all. Instead, talk about structures is simply a shorthand way of talking about systems which share similar properties. With this view, one may wonder whether a satisfactory account of objective truth concerning statements about structures can be given if structures themselves do not exist.

While realism regarding structures may at first appear appealing to those with divine conceptualist impulses, we caution against identifying structures with thoughts in the mind of God. This is because many objections which can be raised against non-structuralist philosophies can be revised to apply at a higher level if we are realists about structures as objects. Typically, when structures are purported to exist as abstract objects, they are thought to consist of "places" or "positions" which can be filled by elements of a system. However, what prevents us from now considering these "positions" themselves as forming a system simply where all relations other than the structural ones have been abstracted away?[6] If the divine thought which is the structure of the natural numbers is construed as a thought *about such a system*, then it is still open to Benacerraf-style questions. If God were to "rethink" the natural number structure so that the second-place position and the first-place position were to trade places, would this be the same structure or a different one?[7] Essentially, "Which divine thought specifically is the number 3?" now becomes "Which divine thought specifically is the natural number structure?", and therefore Benacerraf's identification problem persists simply at a higher level.

To reiterate, a philosophy of mathematics which takes seriously the concerns giving rise to Structuralism is not easily obtained if we are realists about structures as objects. For this reason, we favor eliminativist Structuralism. In particular, we favor a variety of eliminativist Structuralism known as Modal Structuralism. We will see that this view does provide a satisfactory account of the objectivity of mathematical truth while avoiding ontological commitment to structures as objects. After describing this view, we will afterward discuss potential connections with DC.

3 Modal Structuralism

The key insight of Modal Structuralism (MS) can be illuminated by considering a typical yet instructive example. Goldbach's conjecture is a well-known mathematical conjecture which states that every even integer greater than 2 is the sum of two primes. While its truth has been confirmed by computer for even integers as large as 10^{18}, the conjecture itself remains far from solved. Suppose, for the sake of argument, we eventually find the conjecture to be false but that the smallest counterexample to the conjecture is extremely large. Humanity remains uncertain of the finitude of the physical universe, but suppose further that the total number of elementary particles in the universe is finite and that the smallest counterexample to Goldbach's conjecture is much larger than this by comparison. Then, in the language of Structuralism, there would be no physical system large enough to serve as the needed counterexample in this imagined scenario.

[6] While some structuralist philosophers would try to block this move, others, like Shapiro, are quite explicit in the fact that structures themselves can be considered as systems.

[7] Such "permutation arguments" against realist structural accounts were originally suggested by Hellman, but a more detailed exposition of them can be found in Assadian's [2].

However, if this were the case, most within the mathematical community would not consider this to have any bearing on the truth of Goldbach's conjecture. Most would feel strongly that the absence of a physical counterexample of a particular kind does nothing to save Goldbach's conjecture from being mathematically false. Why do we feel this way so strongly? Different philosophies would answer differently. The Platonist would say it shows that mathematics addresses eternally existing forms rather than specific physical entities. The formalist would claim that the falsity of the conjecture reduces to the existence of the derivation by which we verified the counterexample. The modal structuralist proffers a further alternative, namely the fundamental issue, the reason we feel that Goldbach's conjecture would still be false even if we lacked a physically realized counterexample, is that mathematics addresses all systems which could possibly exist rather than what systems actually do exist—physically or otherwise. This is the key insight of MS.

If we are willing to accept this insight, we can simply say that on MS a structure is a description of the way a possible system can be. However, we would need to change (or clarify) how we typically conceive of mathematical truth. We would need to grant that mathematical truth is *modal* in nature, having to do with what is possible and what is necessary in systems that may or may not be actual, and we would need to investigate the consequences of such a view. How do we view ordinary mathematical statements in this new modal context? Furthermore, how do we view proofs from our current formal systems in this context? One would hope that there would be some way of translating our ordinary theorems and proofs into this context in a way that would be natural and do minimal to no damage to their mathematical meanings.

Geoffrey Hellman, in his book *Mathematics Without Numbers*, has undertaken just such a task, first for number theory and later for real analysis and even set theory [10]. We will look briefly at his modal structuralist account of number theory to highlight the main aspects of his approach. Hellman employs second-order modal logic (specifically the modal logic S5 without the Barcan formula) to articulate two essential components of his modal structuralist interpretation.[8]

3.1 Two Components of Hellman's MS

One component Hellman refers to as the *hypothetical component*. Here, we translate ordinary mathematical statements into statements in second-order logic of a particular form. For number theory, Hellman states,

> Intuitively, we should like to construe a (pure) number-theoretic statement as elliptical for a statement as to what would be the case in any structure of the appropriate type. In this case, the structures are, of course, "progressions" or "ω-sequences", so what we seek to make precise is a translation pattern that sends a sentence S to a conditional such as: If X were any ω-sequence, S would hold in X. [10, p. 16]

After some analysis, Hellman eventually settles on construing a sentence A in the language of number theory as

$$\Box\,\forall X \forall R[\mathrm{PA}^2 \to A]^X(S/R) \tag{1}$$

[8] Elsewhere, such as [14, p. 96], Hellman is clear that he is *not* making strong hermeneutical claims to the effect that we should interpret other mathematicians as really meaning these modal statements *deep down*. Instead, Hellman sees himself as providing a "rational reconstruction" of mathematics. This reconstruction then *could* be believed by mathematicians concerning what their theorems amount to after philosophical reflection. We too are in line with this approach. (See also Liggins [21, p. 28-9]).

where PA^2 is the conjunction of the axioms of second-order Peano Arithmetic, the superscript X means that the implication is interpreted with elements in the domain X, and S/R means that the relation R fulfills the role of the successor symbol S. Therefore, (1) is a formula of second-order modal logic saying: It is necessary that for all domains X and relations R, if PA^2 is true of (X, R), then A is true of (X, R). This meets Hellman's intention stated above.[9]

As Hellman points out, "we are exploiting the well-known *superior expressive power* of second-order theories" [10, p. 21] in that second-order PA is *categorical*: any two structures satisfying second-order PA are isomorphic to each other. If we are inclined, we might also say that any structure satisfying the second-order axioms of PA must be isomorphic to the "standard model" whatever we take that to be. While this latter formulation is no longer in line with a purely structuralist view, the important issue is that those statements A which satisfy (1) are precisely those true of the structure we normally associate with the natural numbers. Hellman also rightly reminds us that the categoricity of second-order theories has been a subject of some controversy,[10] but he encourages us to distinguish between the use of symbolic logic in mathematics where the emphasis will be on proofs and using symbolic logic in philosophy where the emphasis will be on explaining what we mean by our assertions [10, p. 21]. The implication is that second-order logic may be useful for the present purposes of MS even if it may not be the right foundation for mathematical theories themselves. Similarly, when Hellman considers modal structuralist accounts of real analysis and set theory, second-order categoricity results play important roles.[11]

The other component of Hellman's modal structuralist interpretation is referred to as the *categorical component*. Thus far, we have made great progress towards an eliminative theory of structuralism in that we have taken normal mathematical statements to be true when they hold of all systems of interest there could be according to the scheme (1). However, the concern remains that no such systems are even possible! Then the antecedent of (1) would be always false rendering (1) vacuously true for all sentences A. Therefore, in order for Hellman's scheme (1) to be an accurate reconstruction of standard mathematics, it is essential that he include

$$\Diamond \exists X \exists R [\mathrm{PA}^2]^X (S/R) \tag{2}$$

which asserts that it is possible for there to exist a system where the second-order axioms of PA hold. The categorical component is a bold assertion which commits us, not to the existence of structures, but to their possibility. Therefore, while Hellman's MS avoids commitment to the existence of structures as objects, it is not without commitments entirely. And, of course, there are analogous, even stronger commitments when one begins to address real analysis and set theory. In these cases, one would be committing oneself to the possibility of a system satisfying second-order axioms for the real numbers or set theory. But according to Hellman, the categorical component is "absolutely essential to affirm" [10, p. 27] to preserve the objective non-vacuous truth of mathematical statements. In fact, it is precisely this component which separates his theory from a simple kind of "modal if-thenism" which he considers unsatisfactory [10, p. 26].

[9] Hellman also tries to make his quantifications over arbitrary domains and relations nominalistically acceptable, but these details need not concern us here. See [10, pp. 47-52] and [13].

[10] For an excellent overview of categoricity arguments in mathematics and a critical assessment of their potential uses in philosophical contexts see Maddy and Väänänen's [23].

[11] It should be noted that the case of set theory is much more delicate, partly due to the fact that the second-order axioms of set theory are only quasi-categorical – the structures are isomorphic relative to the choice of some strongly inaccessible cardinal height. Moreover, Hellman has recently suggested changes to his original translation scheme for set theory in response to the critique of Sam Roberts (see [29] and [12, Section 5]). Finally, we remark that Hellman is far from the only philosopher interested in modal reconstructions of set theory. In fact, modal set theory is an active area of contemporary research. See Christopher Menzel's [24] for an introductory survey.

In passing, we note that the choice of second-order logic also has implications for the meaning of the categorical component of Hellman's MS since (2) cannot be reduced to the simple assertion that PA^2 is formally consistent. While in the first-order context, Gödel's completeness theorem tells us that the formal consistency of axioms implies the existence of a structure satisfying them, this is not so in the second-order context.[12] Furthermore, the structure which (2) asserts to be possible cannot be nonstandard, the axioms of PA^2 being categorical.

Once we have both the hypothetical and categorical components in place, we can now give a modal structuralist interpretation of everyday mathematical proofs. From (2), one can apply (modal) existential instantiation to obtain an isomorphically unique (N, S) satisfying PA^2. In this set up, rather than being a specific object, N is merely a variable name. However, if all our continued arithmetical reasoning can be carried out with respect to this fixed N, such reasoning will appear *as if* referring to an object as is the case is standard mathematical practice. Furthermore, all instances of the first-order axioms of PA including those of the induction schema can be obtained from PA^2 with quantifiers relativized to N. So then, after completing a derivation in the usual way, we may apply universal generalization and necessitation in order to obtain a statement of the form (1). Hellman says,

> Obviously, it is not being recommended that ordinary proofs be carried out in this way. . . . The point is merely that the translation scheme is proof-theoretically faithful in that, in principle, ordinary theorems can be recovered in the appropriate form and within a framework in which apparent reference to numbers or to "the standard model" as an object has been entirely eliminated. [10, p. 26]

Importantly, everyday mathematical proofs can be seen as composing the "meat" of such a proof, located between uses of modal existential instantiation and universal generalization which are particular to the modal structuralist interpretation. Therefore, if one is favorable to such an interpretation, it will be easy to consider these steps as implicit and unnecessary in actual mathematical practice.

3.2 Implications and the Potential Relevance of DC

Having seen the importance of modality to Hellman's eliminative version of structuralism, one will naturally be interested in the interpretation of the modal operators. In particular, what precisely are we committing ourselves to in the categorical component if (2) is not reducible to the formal consistency of PA^2? Hellman says that in choosing S5 as his background logic, he intends the operators to convey "an absolute, mathematico-logical sense of possibility" [10, p. 17]. However, the operators themselves are not given any interpretation beyond this. In their introduction to structuralism, Hellman and Shapiro relate the following:

> The central problem with [MS] concerns the nature of the invoked modality. Of course, it will not do much good to render modality in terms of possible worlds. If one does that and takes possible worlds, and *possibilia*, to exist, then modal eliminative structuralism would collapse into the above ontological version of eliminative structuralism. Not much would be gained by adding the modal operators. The modalist typically takes the modality to be primitive—not defined in terms of anything more fundamental. [16, p. 6]

[12] One way to see this is via Gödel's first incompleteness theorem. Note that Gödel's incompleteness proof goes through for PA^2 just as easily as for PA, so let G be the Gödel sentence for PA^2 and consider $\Gamma = PA^2 + \neg G$. Then, assuming PA^2 is consistent, Γ is consistent because PA^2 cannot prove G. However, no model satisfies Γ since by categoricity any model satisfying PA^2 must satisfy G rather than $\neg G$.

Consequently, the usual objections to MS concern the interpretation of the modal operators and the accompanying epistemological concerns about how we come to know what is possible and what is necessary. Hellman and Shapiro admit that "the main new problem for MS is reliance on primitive modality.... One would like a formal criterion for these notions, but that is not to be hoped for" [16, p. 70].

This is where we believe DC can be of use to the Christian mathematician who finds MS attractive. While far from providing a "formal criterion" for the primitive modality employed by MS, we do believe that DC provides a compelling way to interpret that modality in light of the divine nature giving a theistic point of reference from which to address concerns raised about MS.

4 Modality and Divine Conceptualism

Our goal in this section is to ground modality in the divine nature via a particular perspective on DC provided by Greg Welty's *Theistic Conceptual Realism* (TCR). However, before describing Welty's view, we first discuss relevant notions of possibility and introduce the language of possible worlds.

4.1 Logical and Metaphysical Possibility

While other notions of possibility have been formulated, the options of *logical* possibility and *metaphysical* possibility appear most relevant to our present considerations.

To say that something is logically possible (what Brian Leftow calls "Leibnizian narrow-logically possible" [19, p. 37]) is to say that it does not entail a contradiction. Using this rather minimal[13] version of logical possibility, one would say that a collection of number-theoretic axioms is logically possible just if it is consistent. Something is logically impossible if it entails a contradiction. Something is logically necessary if its negation entails a contradiction.

To say that something is metaphysically possible (what Leftow and Plantinga[14] call "broad-logically possible" [19, p. 37], [26, p. 2]) is to claim more than mere logical possibility. To illustrate the distinction between metaphysical and logical possibility, Leftow points out on one hand that, necessarily,

AB. all bachelors are unmarried.

The statement AB is necessary in the sense that its negation, "some bachelor is married" implies a contradiction: "that some man is married and is not married" [19, p. 33]. Thus AB is logically necessary.[15] On the other hand, Leftow also points out that, necessarily

R. nothing is red and green all over at once.

Leftow argues that R is necessary in a sense independent of natural law. Indeed, he goes on to

[13] Plantinga's conception of narrow-logical possibility (obtained via narrow-logical necessity) is a bit less explicit than Leftow's [26, p. 1].

[14] Leftow aligns himself with Plantinga in describing broad-logical possibility [19, p. 37].

[15] Note that this conception of logical possibility/necessity allows for an analysis of terms such as "bachelor," and is not, in general, strictly formalist in nature.

assert, "This necessity does not seem relative to any further condition; it seems to rest solely on the nature of color itself, and so to be absolute. But though the matter has been controversial, ¬R does not seem to entail a contradiction, nor anything else contra-logical" [19, p. 34]. Thus R is a necessity, but not a logical necessity. It is, however, a metaphysical necessity.

4.2 Mathematico-Logical Possibility

Hellman's *mathematico-logical* possibility lies somewhere between logical and metaphysical possibility. In fact, Craig argues that mathematico-logical possibility lies *strictly* between these two alternatives. On one hand, Craig points out that Hellman's categorical component (2) is asserting more than the absence of formal contradictions in that it implies the possibility of a *standard model* for the natural numbers. As previously mentioned, by Gödel's completeness theorem, any consistent first-order theory has a model. However, the same conclusion does not follow from the consistency of a second-order theory. Therefore, Craig understands Hellman's mathematico-logical possibility as strictly stronger than logical possibility. On the other hand, Craig points out that Hellman relies upon the set-theoretic semantics of Nino Cocchiarella[16] as he provides a justification for his translation scheme between Platonist and modal structuralist interpretations of mathematical statements. In so doing, Craig argues that Hellman is utilizing a version of possibility that is strictly weaker than metaphysical possibility [8, p. 288].

As an aside, we believe it may be worth noting that Hellman and Shapiro (writing subsequent to Craig) repeatedly describe the possibility employed by MS as logical possibility (without the prefix "mathematico-"). For example, the categorical component (2) is described as "affirming the logical possibility of a progression" [16, p.63].[17] In any case, for the purposes of what follows, we view Hellman's mathematico-logical possibility as being at least as strong as logical possibility and strictly weaker than metaphysical possibility.

4.3 Possible Worlds and Modal Operators

In order to further our discussion of modality, we introduce the language of *possible worlds*. Intuitively, a possible world is a way things could have been. But in what sense is a possible world possible? Plantinga describes possible worlds in terms of metaphysical possibility, writing, "A possible world... is a possible state of affairs — one that is possible in the broadly logical sense" [26, p. 44]. Plantinga further explains that for a state of affairs to qualify as a possible world, it must be *maximal* in the sense that it either includes or precludes any other state of affairs. Leftow offers a description of possible worlds via *maximal possibilities*. On Leftow, possibilities are maximal if they are "so large that were they realized, for any proposition P, either P would be true or ¬P would be (assuming bivalence) or — if a third truth-value, indeterminate, be allowed — both would be indeterminate" [19, p. 38]. He then points out that "Philosophers sometimes call largest possibilities possible worlds" [19, p. 38]. Moreover, the notion of possibility used here by Leftow agrees with that of Plantinga (namely, metaphysical or broad-logical).

How does the language of possible worlds help us describe modality? Regarding the possibility that a given proposition P is true, Leftow explains, "One can ask about whether it is possible that P — that is, whether in any circumstances, however outré, it could occur that P. This is to ask whether

[16] See [10, pp. 36-37] and the references to Cocchiarella given there.

[17] Also see [13] for Hellman's independent usage of "logical possibility" on pp. 24, 33, etc. to describe the categorical component in his writings subsequent to [10].

in some possible world it is the case that P" [19, p. 44]. In other words, P is possible if P is true in some possible world. Furthermore, P is necessary if P is true in all possible worlds, or impossible if true in no possible worlds.

4.4 Possible Worlds and Theistic Conceptual Realism

The ontological status of possible worlds has been extensively debated. On one hand, one could argue that possible worlds are merely useful fictions (as does Leftow, for example [19, p. 41]). On the other hand, one could argue that possible worlds are existent objects of some abstract or concrete sort (as do Plantinga [26, p. 48] or Lewis [20], respectively).

Welty, in formulating his position known as Theistic Conceptual Realism (TCR), argues that possible worlds are best understood as "necessarily existing objects that represent ways the universe could be, and that are relevant in making it the case that the universe could be as they represent it to be" [32, p. 63]. He goes on to identify the nature of possible worlds by arguing that "God is an omniscient being. One consequence of this is that God perfectly knows the capacities of his own power, and therefore all possibilities... Thus, possible worlds are simply God's knowledge of his own power, of what he is able to instantiate. The notion of 'God's knowledge' is not just a useful fiction, and so neither are possible worlds" [32, p. 219]. Thus Welty's view of possible worlds is *theistic* in that possible worlds are located within God,[18] it is *conceptual* in that possible worlds are understood as God's thoughts about the scope of his own power, and it is a *realist* view insofar as God's knowledge is real.

A notable strength of a theistic conceptual realist approach to possible worlds is its *relevance* to modal truth given belief in "God as an essentially omniscient and omnipotent being who exists *a se* and is an intelligent creator if a creator at all" [33, p. 92]. On TCR, if it is possible that P, this is true *precisely because* it is within the scope of divine power to instantiate a world W in which P is true. This possible world W in which P is true (whether actually instantiated or not) is characterized by TCR as a divine thought because,

> The rational act of divine creation is always according to knowledge, in particular, according to God's knowledge of the range of his own power, such that (a range of) the divine ideas constitute all possible blueprints for any act of creation. There is a certain class of divine ideas which represent to God the entire range of his ways He could have created. [33, p. 93]

In short, "The possible features of any World are constrained (quite literally) by the content of the divine ideas" [33, p. 93].

Thus, TCR grounds modality in God by understanding possibility and necessity in terms of possible worlds. However, it is important to note that Welty does *not* attempt to provide a formal criterion for possibility. On the contrary, Welty argues that primitive modality is unavoidable[19] [32, Section

[18] In [34], while defending TCR against possible objections, Welty considers two possible approaches to divine thoughts and leaves readers to choose. First, he considers God's thoughts (e.g. his self-knowledge) as constituting "concrete parts of God" [34, p. 259], a strategy seemingly congenial to his interlocutor, William Lane Craig, who denies divine simplicity. Second, Welty considers an 'adverbial' account of God's thoughts as a timeless divine activity consistent with divine simplicity: "So perhaps the language of 'thoughts' just tells us how it is with God, with respect to this conceptual activity, rather than specifying with any metaphysical seriousness distinctly existing events, event-tokens, or parts" [34, p. 260]. (We thank Welty for clarifying his neutrality on these options to us via personal communication.)

[19] In particular, Welty argues that Lewis fails to provide a non-modal characterization of possible worlds.

3C]. “So what matters is *where* we locate our primitives, and if worlds just are the divine ideas that represent to God all the ways He could have created, then why not locate modality for our universe exactly there?” (emphasis in original) [33, p. 109].

4.5 Theistic Conceptual Realism and Possibility

But, on TCR, in what sense is a possible world possible? In other words, what sort of possibility is under consideration in a theistic conceptual realist approach to modality? Is TCR designed to address logical possibility, or metaphysical possibility, or perhaps something in between?[20]

Regarding metaphysical possibility, Welty agrees with Craig:

> ... theists in particular have good grounds for adopting a restricted view of metaphysical possibility, since, given God’s metaphysical necessity, there are worlds which seem *prima facie* to be metaphysically possible — for example, a world in which the highest life form is rabbits existing in a state of unremitting pain — but which are, upon reflection, seen to be metaphysically impossible, since they are incompatible with God’s essential goodness and so, necessarily, would not be permitted by God. [8, pp. 309-310]

The idea here is that God’s attributes constrain the notion of metaphysical possibility. Indeed, Welty differentiates between possibilia and impossibilia by arguing that “the relevant principle of differentiation here is that possibilia are possible objects of God’s will, while impossibilia are not, and that God (in his perfect self-knowledge) knows this about himself” [32, p. 239]. While it may be the case that there is nothing logically problematic about a world populated by pointlessly suffering rabbits (call this world SR), both Craig and Welty would argue SR is nevertheless metaphysically impossible due to the fact that God would never will such a world to exist (in light of his essential goodness, as Craig puts it).

Thus, one might argue that theistic conceptual realist possibility is essentially metaphysical in nature. Indeed, “the range of ways [God] could have created” is bounded by *all* the attributes of God, for God less any one of his attributes is not God at all. God could not create a world that in some way violates his goodness, his holiness, his justice, etc. This is not an external limitation imposed upon God, but is simply an acknowledgment that the scope of divine action is intrinsically determined by the divine nature. Therefore, apart from some sort of additional qualification, the possibility of TCR just is metaphysical possibility.

But what sort of qualifications would allow a theistic conceptual realist to speak meaningfully about forms of possibility that are weaker or stronger than metaphysical possibility? The idea here is to relativize the theistic conceptual realist conception of a possible world. Instead of understanding possible worlds as “God’s knowledge of his own power, of what he is able to instantiate,” we take possible worlds to be “God’s knowledge of his own power, of what he is able to instantiate *relative to A*,” where A denotes some consideration that restricts or broadens our understanding of God’s “ability to instantiate.” If A merely denotes the divine nature, then our relativization is trivial and we recover the original metaphysical possibility of TCR. If A includes more than the divine nature, then we obtain a stronger form of possibility. If A includes less than the divine nature (but nothing external to it), then we obtain a weaker form of possibility.

[20] To whatever degree we clarify this issue, we owe a great debt to personal communication with Welty. His willingness to answer our questions and share his ideas made possible our articulation of the content of this subsection.

For example, if A merely denotes the divine attribute of *rationality*, then (one could argue that) one recovers logical possibility. These possible worlds are precisely those God knows he has power to create and whose creation he recognizes would not entail a logical contradiction, regardless of whether or not they would contradict some other divine attribute such as mercy or justice. In other words, God knows that it is within the scope of his power to create the world SR (for example) without violating his own rationality. One could formulate other forms of possibility (all weaker than metaphysical) by relativizing God's ability to some other subcollection of his attributes.

To obtain notions of possibility stronger than metaphysical, suppose A denotes the divine nature *and* the stipulation that proposition P holds true (where P references something external to the divine nature). For example, suppose P states that the natural laws (as they are in the actual world) are in force. Then we obtain what is usually referred to as *natural* or *nomological* possibility. God knows that it is within the scope of his power to create worlds in which our natural laws are in force. Indeed, one could take P to be any true but not metaphysically necessary description of the actual world (e.g. such a proposition from the "book" on the actual world, using Plantinga's terminology in [26]) and thus obtain a notion of possibility that is stronger than metaphysical.

But we must note that there is a certain asymmetry between weaker and stronger notions of possibility (relative to metaphysical). Indeed, stronger notions of possibility represent what one might call *real* possibilities in that they are all metaphysically possible. They correspond to states of affairs that could be actual. For example, one could argue that God knows he could have created a world in which our current natural laws hold but there are no rabbits. On the other hand, possibility in a weaker sense is not real possibility. God knows he could not *actually* create a world whose creation would violate his goodness (or justice, or mercy, etc.), even if he knows that he has the power to create such worlds were his goodness (or justice, or mercy, etc.) not relevant. As we generalize the TCR conception of a possible world, we must "take note that we are abstracting away from reality in order to make a set of illuminating distinctions" as Welty puts it.[21]

In summary, by generalizing the theistic conceptual realist conception of possible worlds, we are able to accommodate Hellman's notion of mathematico-logical possibility within the scope of theistic conceptual realist modality. This is accomplished by relativizing God's creative power to the identifying characteristics of mathematico-logical possibility. Due to the location of mathematico-logical possibility between logical and metaphysical possibility, this relativization would only involve conditions *internal* to God's nature.[22]

5 Assessing a DC Interpretation of MS via TCR

Our goal has been to articulate a version of DC which takes seriously the concerns giving rise to structuralism. Our contention is that MS, where the modality is interpreted in light of the divine nature via TCR, is such a view. In this final section, we conclude by highlighting various mathematical and theological advantages of this view which Christian mathematicians may find appealing. Along the way, we also hope to address a few potential concerns.

[21]Via personal communication with the present authors dated July 2024.

[22]At a minimum, God's perfect rationality.

5.1 Ontological Neutrality

As a variety of Structuralism, MS avoids Benacerraf's identification problem by not privileging any system over any other when both equally satisfy the axioms in question. However, as a variety of eliminative Structuralism, MS goes further and avoids commitment to the existence of abstract objects altogether. Instead, via the hypothetical component of MS, mathematical statements are asserted about whatever systems (concrete or abstract) there might be. Then, via the categorical component of MS, we commit ourselves only to the possibility of the existence of such systems. Finally, via TCR, we understand such possibilities as concrete thoughts of God. It is true that we thus commit ourselves to the existence of God and this may add significantly to the ontology of the nontheist. However, the Christian mathematician already believes in God, and we propose our view with the assumption of such prior commitments.

To reiterate, a DC interpretation of MS remains neutral regarding the existence of abstract objects. From a purely mathematical perspective, this is the best that can be hoped for. On the other hand, we recognize that Christians may have theological reasons for rejecting the existence of abstract objects. Some argue that, if abstract objects exist according to the usual Platonic account, this fact would cause issues for traditional theism.[23] First, God would no longer be the only uncreated being, compromising his unique aseity. Likewise, God would have no control over what is true about such objects, compromising his sovereignty. Finally, since God himself possesses properties, this makes him somehow dependent upon the forms of these properties, which is far worse than their being merely independent of him. If one feels the force of such theological objections, denying the existence of such entities is completely compatible with a DC interpretation of MS.

5.2 Objectivity of Mathematical Truth

We agree with Hellman that preserving the objective truth of mathematical theorems should be one of the foremost priorities for a philosophy of mathematics. In his introduction to *Mathematics Without Numbers*, he says that mathematics as the "queen of the sciences occupies her throne because of the wealth and beauty of her results" and that "the queen merits the full respect of her handmaid" [10, pp. 1-2]. While not committing us to the existence of Platonic objects, MS is still able to maintain the objectivity of mathematical statements. By this we mean that well-formed mathematical statements have determinate truth values which are true or false regardless of human viewpoint. Furthermore, since care has been taken to preserve mathematical meaning, the mathematical statements which are objectively true on MS are precisely those which have been traditionally held by mathematicians. In our view, a philosophy of mathematics which labels all mathematical claims false without further explication is not a tenable philosophy. Even if the *prima facie* interpretation of mathematical statements as being about Platonic objects is denied, an adequate philosophy of mathematics must do more. Ideally, it should tell us how mathematical statements can be true. MS accomplishes this by construing mathematical statements modally, as being about what is possible and what is necessary regarding structures as structures.

While the objectivity of mathematics follows solely from MS, a divine conceptualist interpretation of the modality employed will bring further advantages for the Christian mathematician. In his book *God and Necessity*, Leftow challenges theists to take seriously all such "claims for which 'God is the ultimate reality' is a convenient shorthand" [19, p. 3]. In particular, Christian mathematicians should desire to understand how mathematical truths relate to claims of God's ultimacy. On our

[23] For instance, see Craig's [7] or [8].

proposed view, God's knowledge of his own rational nature and power delimits what is possibly and necessarily true about structures. Therefore, all mathematical truths are modal truths, and all modal truths are truths about God's nature. Grounding mathematics in the divine nature is the characteristic of divine conceptualist mathematics which we find most admirable and most worthy of preservation. Thus, in this sense, we can agree with St. Augustine that mathematics is "contained in the Divine Intelligence" even if we do not agree that the content of mathematics addresses divine ideas understood as mathematical objects.

5.3 Naturalness Regarding Practice

Over the past several decades, there has been a call for a more human-centered philosophy of mathematics which keeps central mathematics as practiced by actual mathematicians.[24] Furthermore, naturalist philosophers of mathematics have also emphasized that mathematics should be judged and evaluated by standards which are internal to it rather than external to it.[25] While we applaud sociological and historical viewpoints and welcome needed discussion of mathematical practice as an aspect of human culture, we are also comfortable with the fact that philosophers may wish to address questions which do not arise in the daily practice of mathematics itself. This is even more true of theological questions which might be posed by Christians who seek to know how mathematics relates to God and his nature. The most important thing, and here we are in full agreement with both humanist and naturalist philosophers of mathematics, is that our philosophy of mathematics, if true, should not turn out to make mathematics as practiced impossible or strange. Since this could be a concern for our proposed view, we are obligated to address it.

First, it should be reemphasized that MS, unlike intuitionism and other constructivist philosophies, does not advocate any changes to standard mathematical practice. This is because, as was pointed out previously while discussing proofs in the MS framework, all our current deductive proofs can be viewed as occurring between modal existential instantiations and modal universal generalizations. Admittedly, much of mathematical practice is actually non-deductive and is instead guided by intuition and heuristic principles. In articulating a DC interpretation of MS, we do not seek to justify all such principles, but merely provide an interpretive framework compatible with them.[26] Furthermore, Hellman himself would agree with naturalist philosophers like Maddy who eschew a "philosophy first" stance toward mathematics. While addressing work of Hartry Field, Hellman says "a more appropriate [stance]—especially in relation to so successful a subject as mathematics—would be 'Philosophy last, if at all!'" [11, p. 261].

However, because MS provides a reconstruction of ordinary mathematical discourse in a new modal context, this by itself might be considered a strike against the naturalness of MS. But, as practicing mathematicians ourselves, we are convinced that the initially unfamiliar modal nature of MS becomes more familiar after inspection.

For instance, one could point out that mathematicians do not speak of possible existence but only existence. It is true that if you ask mathematicians, "Do there exist infinitely many primes?", they will immediately answer, "Yes, and Euclid proved it!" However, if you ask, "Do there exist natural numbers?", they are likely to hesitate. What can be the explanation for this? One explanation

[24] A classic work along these lines is Reuben Hersh's [17].

[25] Here, Penelope Maddy's [22] is an essential text.

[26] As an interesting case of this, consider the principle of "reflection" in set-theoretic practice. As Hellman points out in [12], MS can provide for the possibility of structures where specific formal reflection principles hold without having to show that "reflection" holds globally among some collection of merely possible structures.

might be that, given our modal structuralist interpretation and specifically the hypothetical component (1), the first question is nontrivial in a way that the second isn't. The trivial nature of the second question causes the mathematician to wonder what exactly is being asked and whether or not the topic of the conversation has suddenly changed from mathematics to philosophy, a subject on which the mathematician is not prepared to speak with confidence. On the other hand, when hesitating at the question "Do there exist natural numbers?", the mathematician may be recognizing that the questioner is no longer asking for a proof but is rather questioning an assumed axiom. According to (1), the axioms are assumed to hold for the structure in question. If they do not, it is no longer a mathematical question. In fact, all mathematical existence proofs are relative in that they follow from the assumed existence of other mathematical objects,[27] and it is not difficult to imagine that all talk of ordinary mathematical existence is therefore also implicitly relativized in this sense.

Of course, most mathematicians will feel that Euclid's proof of the infinitude of the primes has important content and will still wish to assert that "There are infinitely many primes" and not merely a (potentially vacuous) implication such as: "If the natural numbers exist, then there exist infinitely many primes." But as long as we are construing mathematical existence as possible existence, this is precisely what the categorical component (2) enables us to do! While arguing against Maddy's enhanced if-thenism, Hellman says that, without (2),

> The concern is that the characteristic content of mathematics, inherent in its substantial theorems, is lost when we restrict ourselves to asserting conditionals even with suitable, well-motivated axioms as antecedents and those theorems as consequents. Rather than asserting, for instance, that there are infinitely many primes, we merely assert that if the Dedekind-Peano axioms of arithmetic hold, then there are infinitely many primes. Now that difference would vanish if we were to go on and assert that those axioms do hold or are true (or at least possibly true [as in MS]), as we would then be able to detach the consequent by modus ponens. [15, p. 246]

Therefore, when both (1) and (2) are employed, MS seems to us to be more consistent with mathematical practice than it first appears, even as it regards assertions of existence.

As mentioned previously, construing mathematics modally will raise epistemological questions concerning how we know what is possible and what is necessary. Now that we have articulated our TCR understanding of modality, these questions will be inextricably tied to other theological issues. We postpone these important and challenging questions for our final subsection. Regarding the "naturalness" of theological considerations, while we admit that they will be considered extra-mathematical or even extra-philosophical by some, we would also like to point out that they are not exactly foreign to the mathematical experience either. Throughout history, mathematicians have frequently cited the feeling that they encounter something transcendent when they do mathematics. This was the case with the ancient Greeks, but remains true of recent practitioners, even secular ones. Paul Halmos famously said that thinking about mathematics was like being "in touch with God," and Paul Erdős's whimsical remarks about a divine book containing all the most beautiful proofs of mathematical theorems are well-known. In any case, our concern in this subsection is only with whether or not MS or DC somehow *conflict* with mathematical practice, and our contention is that they do not.

[27] If a background of set theory is assumed, this is especially clear since almost all axioms of set theory concern the existence of sets.

5.4 Freedom from Paradox

Avoiding contradiction should be a goal of any mathematical or philosophical theory. Some relevant ones to avoid for a DC interpretation of MS are the Russell, Cantor, and Burali-Forti paradoxes. All of these have in common that they occur when unrestricted comprehension is permitted and we arrive at totalities which are especially large. When such totalities are present, contradictions arise. For this reason, Hellman is careful to articulate an appropriate principle of comprehension for his MS:

> In the modal context, unrestricted comprehension leads to intensions, transworld classes and relations. For example, suppose the predicate "planet" is available; then we would generate a class not merely of existing planets, but of those together with "any that might have existed" .... In particular, we would generate a universal class of all possible objects, and corresponding universal relations among possibilia.... Ordinary, mathematical abstracta seem tame compared to such extravagances. [16, p. 64]

Eventually, Hellman settles on a version of comprehension which allows collection of objects within a fixed system or world but not across systems or worlds. He states, "It simply makes no sense to speak of a collection or plurality of all structures or items in structures *that there might be*" (emphasis in original) [16, p. 66]. Instead, Hellman embraces a version of the "extendibility principle" according to which any totality is contained in some larger totality. In the modal context of MS, this principle becomes the compelling "any totality there might be might be extended" [16, p. 64].

Given our DC interpretation of the modality employed by MS, we believe this picture is even clearer. Here, it is important to emphasize that TCR does not identify possible worlds with just any thoughts of God but only of those of a particular type, namely those thoughts of God corresponding to worlds which he recognizes that he, relative to some aspect of his nature, could bring about. Therefore, while it may be possible for God to think about all planets that he could create, there is no reason to suppose that the collection of all such planets could coexist in a single world God recognizes as one that he could create. So, on our proposed view, while it may make sense "to speak of a collection or plurality of all structures or items in structures *that there might be*" we are not obligated to recognize such collections or pluralities as possible. This is not because we are restricting a comprehension schema, but rather because we can relate possibility to God's rational nature. In the case of the large totalities considered by the Russell, Cantor, and Burali-Forti paradoxes, the inherently contradictory nature of such totalities is sufficient for God, in his rational nature, to know that he is not able to bring them about. Therefore, they are not even logically possible, and ought not be regarded as mathematical possibilia (that is, as existing in a modal structuralist sense).

To be sure, these comments are far from giving a formal criterion according to which different possible structures can or cannot be aggregated together to form other possible structures. But, in giving a DC interpretation of the modality employed by MS, we are interested in articulating *how* it is the case that some structures are possible or impossible, not in providing formal criteria for telling *when* such structures are possible or impossible. Again, we will return to our stance on such epistemological questions in the final subsection.

5.5 Compability With Application

In Chapter 3 of *Mathematics Without Numbers*, Hellman addresses the relationship between mathematics and physical reality. He begins by noting that the applicability of mathematics provides a strong source of motivation for taking mathematical truth seriously [10, p. 94]. And, indeed, MS takes mathematical truth seriously! But how might one apply mathematical structures (as understood by MS) to the physical world? To this end, Hellman proposes various ways to formulate relations between hypothetical mathematical structures and physical objects of study. He also identifies a few technical issues that one might face along the way. As we shall see, it is in the consideration of these technical issues that DC provides another distinct advantage.

After indicating how one might relate a hypothetically entertained mathematical structure to a physical system, Hellman points out that the mathematical structure under consideration must not "interfere" with the physical system under consideration. For example, when attempting to compare the numerical size of two collections of objects in the physical world, one must stipulate at the outset that whatever hypothetical instantiation of the natural number structure is employed does not interfere with the desired comparison. Hellman offers the example of the statement, "There are more spiders than apes (and a definite number of each)," and describes how a hypothetical instantiation of the natural number structure consisting of infinitely many apes would interfere with this statement. To prevent such complications, Hellman asserts that "we must *stipulate* from the outset that the only possibilities we entertain in employing the '$\Box$' are such as to leave the actual world entirely intact" [10, p. 99]. Hellman refers to this as the *non-interference proviso.*

But this general formulation leads to questions about what is meant by the "actual world," and how much of the actual world must be left "intact" by the possible structures employed in a given application. Hellman considers two responses to these questions. On one hand, one can view such questions as moot via a commitment to a "realist view of nature that might be put roughly as follows: in science, we investigate a unique material reality, existing objectively and independent of our minds, our language, conceptual scheme, and so forth" [10, p. 127]. This simplifies our understanding of the non-interference proviso in that,

> When it comes to mathematics ... we need not regard its abstract structures as literally part of the actual world. It is sufficient that they be conceptually possible. In applying mathematics to the actual world, we appeal to mathematical structures as a way of carrying information about that world. In our own language, we may well not have any alternative way of expressing this information; mathematical language may be indispensable for any precise, detailed description, especially one which we can use in a theory for purposes of prediction and explanation. Nevertheless, this indispensability reflects our own language, perhaps our own capacities. It doesn't mean, for example, that abstract mathematical objects literally participate with the non-mathematical in making up material reality. [10, p. 127]

In other words, a realist view of nature allows one to view mathematical structures as useful in application without countenancing literal interactions between these structures and non-mathematical reality. Therefore, we need not completely specify which facets of the actual world are left intact by (and "literally participate" with) the introduction of possible mathematical structures. The actual world *just is* and is not determined nor affected by our conceptions of possible mathematical structures. We can affirm the non-interference proviso and view this as sufficient in itself.

On the other hand, Hellman notes that many would challenge the coherence of the notion that there is a unique actual world. They would argue that "the world is as many ways as it can be truly described" and one can at best use language such as "world-portrayals" (in the place of "the world") [10, p. 128]. On this metaphysically non-committal view, the goal is to provide a language by which one can "fix the actual material situation" [10, p. 129] in order to unambiguously interpret modal conditionals in the context of application (without commitment to objective physical reality). The pursuit of this goal is no simple matter. Indeed, in comparison with this approach, Hellman notes that "If one's 'realism' is sufficiently strong, the problems seem to evaporate" [10, p. 96].

We point the reader to these issues in order to make the following point: a realist view of nature does not expand the ontological commitments of one already committed to a DC interpretation of MS. Indeed, TCR modality is articulated via an understanding of God as creator. The actual world is precisely the world God *has* (not merely *could have*) created. The created world includes an actual physical world existing independently of "our minds, our language, conceptual scheme, and so forth." Thus, a DC interpretation of MS takes the simpler route to understanding mathematical applications not merely out of convenience, but as a matter of principle.

5.6 Epistemological Accessibility

Any philosophy of mathematics that argues for the objectivity of mathematical truth is, at least to some extent, then obligated to justify any claim to the knowledge of mathematical truth. In the case of MS, this question of epistemological access centers on modal assertions such as the categorical component (2) above. In other words, the central epistemological question is: "How do we know what structures are possible?" Hellman admits this is a somewhat challenging issue, not just for MS but for any version of structuralism. He writes that, in order to justify our knowledge of which structures are possible,

> It seems that we must fall back on indirect evidence pertaining to our successful practice internally and in applications, and, perhaps, the intuitive pictures and ideas we have of various structures supporting the coherence of our concepts of them. Perhaps this is the best that any version of structuralism can hope for. [16, p. 67]

Indeed, how might one know that an infinite totality is possible? (let alone a model of the second-order Peano Axioms). We agree with Hellman when he asserts that human mental constructions cannot produce a satisfactory answer because "we have only finite resources to work with and can only work so fast" [16, p. 70].

Beyond the finitude of human minds, Hellman also notes [16, p. 70] that the reliance of MS on primitive modality adds to the difficulty of such epistemological questions. While a DC interpretation of MS is reliant upon the same, it goes further than MS by explicitly grounding modal truth in God's knowledge of the scope of his own power. We believe this additional commitment provides additional insight into the epistemology of MS.

One additional resource available from the divine conceptualist perspective is the concept of *divine illumination*, whereby God reveals truth to those created in his own image. As Welty explains, "we subjectively apprehend the truth as God enables us to do so (whether his assistance comes to us dynamically and directly, via ongoing divine illumination, or more statically and indirectly, via our divinely created cognitive faculties)" [32, pp. 232-233]. This concept provides a means of explaining human knowledge of necessary (and thus possible) truths. As Robert M. Adams explains:[28]

> I do seriously entertain the hypothesis that there is a mind to whose nature it simply pertains to be able to recognize necessary truths. Indeed I am inclined to believe that such a mind belongs to God. And that opens the way for another explanation of our knowledge of necessary truths, an explanation in terms of divine illumination. Suppose that necessary

[28] And as quoted by Welty [32, p. 232] in describing a theistic conceptual realist approach to logic.

> truths do determine and explain facts about the real world. If God of His very nature knows the necessary truths, and if He has created us, He could have constructed us in such a way that we would at least commonly recognize necessary truths as necessary. In this way there would be a causal connection between what is necessarily true about real objects and our believing it to be necessarily true about them. It would not be an incredible accident or an inexplicable mystery that our beliefs agreed with the objects in this. This theory is not new. It is Augustinian, and something like it was widely accepted in the medieval and early modern periods. I think it provides the best explanation available to us for our knowledge of necessary truths. [1, pp. 750-751]

With specific regard for the possible existence of a completed infinity, one could argue that, via divine revelation (and perhaps illumination), humanity is given access to the knowledge that God is a transcendently perfect being. He is uncreated, existing without beginning and without end. He is supremely powerful and knowledgeable. He is not primarily understood in terms of bounds or constraints but rather in terms of his boundlessness and freedom. In light of these observations, we wonder what aspect of God's nature would preclude the possibility of a completed infinity.

Here we emphasize that we are *not* asserting the *metaphysical* possibility of a completed infinity. In fact, in order to make our proposed view attractive to as broad an audience as possible, we would like to remain neutral regarding this question in light of disagreements it has generated. For example, Craig has argued against the metaphysical possibility of a completed infinity in support of the claim that the universe has a beginning [6]. However, it is important to note that Craig's arguments along these lines have been widely criticized by mathematicians and philosophers alike. In particular, Wes Morriston, while not taking a position on the infinitude of the past, argues that Craig's objections to an actual infinity are not well-founded [25].

But can we remain neutral on this issue? Indeed, the reader may wonder if a DC interpretation of MS implies that the possible existence of an infinite totality unavoidably yields the concrete existence of an *actual* infinite totality. One might argue as follows: Welty posits that, "in God's case <knowing p> entails <thinking or having thoughts that p>" [32, p. 222]. Given this entailment (which we denote by W), suppose a completed infinity is logically possible. That is, God knows that (with reference to his rational nature) it is within the scope of his power to create a completed infinity. Denote this hypothetical infinity by H. Given God's perfect knowledge, for each item x in H, God knows that x is an item in H. One might then invoke W to conclude that the infinitude of x's in H implies the infinitude of God's thoughts about x's in H. Since God is concrete and his mental states are concrete (i.e. not abstract), we arrive at the concrete existence of an actual infinity. In other words, the logical possibility of a completed infinity implies the metaphysical actuality of a completed infinity.

One might find this line of reasoning perfectly satisfactory and the conclusion unproblematic. But, if one objects to the idea of an actual infinity, how might one respond? On one hand, one could note that Welty supports his assertion of W by citing Leftow's claim that "if God is a perfect knower, He does not forget what He knows or become unaware of what He knows: all His knowledge is occurrent, not dispositional" [18, p. 214, endnote 4]. One could then follow Craig, who questions Leftow's view that all of God's knowledge is occurrent [8, p. 206]. If Craig's challenge is sound, the above argument is undermined. On the other hand, one might agree with Welty and Leftow that God's thoughts are occurent and yet prefer an "adverbial" account of God's cognitive activity. One might then take the conclusion of W to provide a description of how God is thinking rather than an assertion of God having distinct thoughts (see footnote 18).

Morriston, however, argues that even something like a dispositional or adverbial perspective of God's knowledge seems to imply the reality of a completed infinity. Even if the whole of God's knowledge consists in a single divine thought (or way of thinking, we might add), Morriston asserts that the unity of God's thought "must be thought of as a unity within a multiplicity—a one in a many" [25, p. 159].

We do not attempt to settle these matters. But, regardless of one's willingness to countenance an actual infinite totality, we believe a consideration of God's boundless nature provides support for the (at least logical) possibility of such a totality. And, more generally we believe that the knowledge of God provides epistemological access to modal claims of modal structuralist mathematics. None of this amounts to a claim that human knowledge of necessary or possible truths is infallible, but fallibility does not preclude knowledge of truth.

We also note that a DC interpretation of MS yields a potentially helpful perspective on the distinction between divine and human knowledge of mathematical truth. On this view, God's mathematical knowledge is, ultimately, knowledge of himself. It is knowledge contingent upon nothing outside of God, ultimately resting upon God's knowledge of the scope of his own power relative to certain aspects of his nature. On the other hand, human mathematical knowledge is indirect in that it is knowledge of God's self-knowledge. Furthermore, in this way, we can recover the divine conceptualist notion that, when we are thinking about mathematics, "we are thinking God's thoughts after him" [28].

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Practicing Faith in the Public Sphere

Emily Sprague Pardee (The Integral Writer)

Emily Sprague Pardee studied and taught music then mathematics in public schools for half a century. Pardee has lived in thirteen of these United States and taught mathematics in five of them, retiring from Edinboro University of Pennsylvania, then serving as Visiting Lecturer at the University of the Pacific. She lives near Yosemite National Park with her husband, two cats, three banjos, and a post-retirement fascination with financial mathematics.

Abstract

In this paper I highlight the spiritual threads that informed my career as a professor of undergraduate mathematics in the public sector. I hope to provide a focus and some companionship for my fellow Christian professors still laboring in the field. The unifying thread is Luke 4:18–19, which I take in the King James version. I explore what it means to "preach the gospel to the poor."

1 Introduction

My education and career unfolded in public schools and universities. It never occurred to me to work anywhere else. My forebears and I received excellent education through the public schools; my grandparents devoted their careers to the professionalization of public school teaching and administration; my father taught in the City University of New York for more than forty years. I grew up understanding that education is a precious asset so I determined to follow my ancestors in offering the same hand up through public education that we received.

For some anecdotal insight into preferences for teaching in public and private spheres I asked Google whether teachers preferred working in public or private schools. The search turned up a Quora site [4] where the respondents were primarily K-12 teachers. The majority preferred teaching in the private sector and cited factors such as stronger parental support, fewer students with behavioral or learning difficulties, and more curricular freedom. The few who preferred the public sector mentioned a desire to be of service to those most in need of education, better pay and benefits, and more opportunities for professional development. One respondent believed that public education provides students with a better preparation for functioning in contemporary society.

I have always believed in the wisdom of separating church and state. Furthermore my training as a musician taught me that one's values and competence are best displayed rather than announced. This fits nicely into a paraphrase of a familiar saying, "Preach the Gospel at all times, and it is worth your career to do it without words."

The book *A Grander Story: An Invitation to Christian Professors* [2] by Rick Hove and Heather Holleman begins with Prof. Holleman's story of awakening from a self-centered dream of making a name for herself in academic circles to a decision to invest her life in God's grander story to bring hope to the world. The body of the book contains reflections of six professors in various disciplines who write about finding and sharing Christ with their students.

But how is it that we understand "preaching the Gospel" in context of teaching mathematics in public universities? Here I share stories from my own ministry that stem from my conviction that alongside the studying and teaching of mathematics lie avenues for spiritual ministry. As I have received healing, so the spirit of the Lord may use me as a conduit for Him to heal. My starting place is Luke 4:18–19, which I take in the King James version.

> "The Spirit of the Lord is upon me, because he hath anointed me to preach the gospel to the poor; he hath sent me to heal the brokenhearted, to preach deliverance to the captives, and recovering of sight to the blind, to set at liberty them that are bruised, to preach the acceptable year of the Lord."

2 Who are the Poor?

In the spirit of the adage "Give a man a fish and he'll eat for a day; teach a man to fish and he'll feed his family for a lifetime," we note that education is widely recognized as the route to socio-economic improvement, and that public schools and universities are frequently the portal to obtain that education that is most accessible in terms of cost and location. In particular, students who are the first in their family to embark on a college degree program (first generation students) are the likeliest to be seeking career and financial security as their primary educational goal.

According to Forbes [1]

- 47% of attendees at public four-year colleges are first generation students,
- 64% of attendees at public two-year colleges are first generation students,
- 50% of college students overall are first generation students.

Of all first generation college students

- 28% are older than 30 years of age,
- 30% are parents, or caregivers to a family member,
- 10% are newly immigrated Americans,
- 28% are of the first American born generation in their family.

Furthermore, without giving precise numbers, Forbes observes that first generation college students are more likely than the rest of the college population to be from low income and/or minority backgrounds, to have children, to be veterans, or to be newly immigrated or first generation Americans.

3 Preach the Gospel to the Poor

In this section I relate experiences in which I was confronted with opportunity to minister to first generation college students. I often found guidance from the Holy Spirit in how to reach the heart of their difficulties, sometimes with math help, sometimes to address issues underlying their academic problems. I've chosen two stories for each of the categories cited in the passage from Luke 4. For most of them, I have obscured the institutions and the student's age, race and/or gender while underscoring the characteristics of first generation college students described in the Forbes article.

3.1 Heal the Brokenhearted

A Determined Student from a Broken Home

A physics major who had been raised by an abandoned mother, studied with me in advanced math classes. When the time came, I agreed to serve as a reference "outside the major" for a research assistantship. One day I fielded a phone call from a research professor in the graduate program for which my student was applying who abruptly asked whether the student could "really do math." With no time for forethought I found myself describing the scope of our Real Analysis syllabus then detailing specific contributions the student had made to the collaborations of the Inquiry Based Learning groups. A couple of weeks later, my office door framed a radiant senior clutching an offer of the prized research assistantship. "It's funny, though," the student remarked, "They didn't call any of my references." I smiled. The student went on to complete a PhD and has a spouse and four children while building a career that spans biomechanics and artificial intelligence.

A Rock Star: Empathy or Enabling

A particular geology major had survived poverty while shielding a younger brother from an abusive step-father. The wider family, some of whom had never finished high school, neither understood nor supported the student's love of learning and actively discouraged the student's determination to earn a college degree. The student wanted to pursue graduate work in paleontology but Second Calculus was a steep rock face to be climbed. Although I strongly encourage students to form mutually supportive study groups, in this case the geologist was grouped with a candidate for an M.B.A. who had chosen to stop working on mathematics. The geologist was spending far too much energy trying to prod the laggard along to the detriment of their own studies. I empathized with this tendency to protect little brothers and intervened privately with each of them. The M.B.A. candidate and I parted with mutual respect while the geologist, unfettered, soared to success, eventually earning a Master of Science. In the course of cataloguing specimens for their thesis project, the student acquired additional credentials and career experience in Data Science which has developed into a career path.

3.2 Preach Deliverance to the Captives

Shackled by Misdiagnoses

A wheelchair-bound student, after three failures, petitioned to be excused from the General Education mathematics requirement for a bachelor's degree. A colleague and I proposed an independent study alternative as an accommodation to provide the student "equal access" in light of the student's physical disabilities. During our attempt to break down a process of factoring trinomials the student burst into tears of frustration. When the tempest subsided I asked the student to describe their experience of math in high school. The student began, "Because of my physical restrictions, they assumed I was also developmentally disabled. So, although my transcript shows that I had math all through high school, my Individualized Educational Program never took me beyond sixth grade math."

WOW!

We adjusted the parameters of the independent study to fill in the gaps. The student proved to be quite capable and mathematically imaginative. The next semester, on their own, they devised a graphical representation of the commutative property of multiplication and brought this trophy in to show us. The student finished their Bachelors and went on to earn a Masters degree.

Teaching Adult Priorities

There was a young man who stayed after class for help and asked about juggling full-time work with a full-time course load. As we began to add up all the commute times, work hours, class meetings, homework and so forth he shared that he had fathered a child and that his girlfriend would be required to resume taking classes that summer to avoid having to begin making payments on her college loans when her maternity deferment expired.

"I hope you aren't planning to take summer classes!" I yelped without forethought.

He replied with desperate panic about the need to finish his degree as soon as possible. I found myself delivering a speech about not missing these critical months of his child's young life and encouraging him to support the child's mother while she was heartsick about having to leave her baby with her mother. The young man listened. Much later that afternoon he returned to my office glowing. He'd spent the whole afternoon on the phone with his girlfriend and in his academic advisor's office where they'd worked out a path to graduation that allowed him to avoid summer classes.

3.3 Recovering of Sight to the Blind

Seeing the Problem

An international student finally prevailed over Developmental Mathematics as one of just two students in my special Summer session section. In Fall, the student arrived in my well-populated Business Calculus class. Although the student spent every possible minute in the tutoring lab, soon the student began to complain that the tutor was too busy to help. It turned out that the tutor was working with the students in groups and not able to work with my student individually.

I began to wonder whether my student had difficulty reading because when we discussed problems orally, the student drew beautifully clear diagrams and could sort out a pathway to solve them. I asked my student to tell me how they had learned to read. They replied with regard to their first experiences and I found out that their parents were illiterate but an aunt had provided reading lessons. I cleared some office hours to work with my student one-on-one and after a couple months my student was able to produce enough correct work on exams to pass the class. When I approached the director of the International Student Office to talk about the situation, I was told that my student had transferred to an art school.

Seeing Through the Lens of Prerequisites

I could not understand what was wrong with a first generation American math major in my Second Calculus class until I pieced together a set of administrative misadventures. Years before I was hired, the department had convinced the registrar to relax the automated prerequisite restrictions on math courses so that math majors' advisors could override them. As a result, our math majors did not understand the importance of taking courses in sequence. In this case, the student failed First Calculus during Spring, and promised their advisor they would retake the course during

Summer session. Their advisor had overridden the prerequisite and placed the student into Second Calculus for the Fall. But the student did not take actually take First Calculus that Summer, and so was attempting Second Calculus without the prerequisite. I convinced the student to withdraw without penalty from Second Calculus in order to pass through the sequence in order. In due time I saw this student gain success in two subsequent courses and earn a B.A. in Mathematics. My chairman received a note from the student saying that I was a teacher who, "...cared that (my) students really learned."

3.4 Setting the Bruised at Liberty

Restoring Self-Confidence

There was a forty-year-old Precalculus student whose daily preparation never wavered and whose eyes never glazed over during class. The student told me that they had emerged from a months-long coma induced so that the student could heal from injuries sustained in a devastating motorcycle accident. The student's mother suggested that passing some college courses would help the student regain self-confidence. While I was learning all of this I also learned that the student was intrigued by the so-called "Miracle on the Hudson," Captain Sully Sullenberger's landing of the disabled US Airlines Flight 1549 in the Hudson River. [3] The next semester when the student turned up in First Calculus, I encouraged the student to develop a project to use the data from the miraculous landing to an applied project exercise in Stewart's *Calculus*. [5] I brought the student with their project to a regional student research conference. This got the student sufficiently hooked on mathematical thinking that the next year, the student approached my colleague with an idea for a project in Differential Equations that they finished and brought to our regional Mathematics Association of America meeting.

Respecting the Disrespected

In my evening Elements of Statistics section there was a student in their fifties, none of whose family had ever gone to college, who volunteered their story. They had been told that if they didn't finish a college degree, they would be let go from a job they had performed successfully for decades. It took this student two tries through my sections to learn elementary statistics and more particularly how to pace themself through college courses. The student prevailed. I am so struck by the bravery of these older students.

4 When I was a Student

Here I share two stories of studying with gifted professors. The first was a practicing Christian, a lay preacher, who never spoke of his faith during class, but would address direct questions about faith outside of class. The second was an iconoclast who had been educated in Roman Catholic schools from first grade through his PhD. An ACMS colleague and I discovered that we each had known him when we were students. We agreed that despite his rejection of faith, this professor had one of the strongest Christian ethics we'd ever encountered.

4.1 Recovery of Sight to the Blind: Seeing in Three Dimensions

I am terrible at translating written descriptions into sketches. When I was in high school, long before graphing calculators were available, the curriculum did not insist that we learn the graphs of the elementary functions. Twenty years later I was navigating Integral Calculus with a significant blind spot. When we began to rotate curves around the vertical axis then calculate their surface areas, I was blind as a bat. I burst into tears in my professor's office. He scoured his bookshelves for another text with different illustrations. "Here," he said, "See? The rotation looks like a coffee cake." But to me it didn't look like a cake at all. My indignation over this mischaracterization kickstarted my imagination. I plunged a pencil through the center of his coffee cup and yelped "OH!! This is the y-axis and the cup is the rotated curve!" He smiled and never mentioned the coffee splashed over his desk. For my semester project I baked a tube cake, sketched a projection of its cross section onto the plane, calculated an approximation to the equation of the curve of the surface, then determined the area I needed to frost. Graphing became accessible.

4.2 Setting the Bruised at Liberty: Recovering from Defeat

The day I began the first of three, three-hour-long sessions of Graduate Qualifying Exams my brother died. I called my advisor in tears. "Go," was all he said. "See me when you get back." At the end of the subsequent semester I passed the exams at Masters level, ninety points shy of admission to the PhD program. I retook the exams the following semester and only improved slightly. My advisor called me into his office. Leaving no room for my excuses he decreed that I would have to decide whether I wanted to pass the exams or leave with a Masters degree. Furthermore, I was not to sign up for them until I knew I could pass well enough to advance to the PhD program. And finally, I had a year, but no more, to finish them.

I went home for a week to pray and reflect. On the last day of my visit my parents and I took an ill-mapped hike and found ourselves stranded up a mountain as the sun set. So. For five hours I guided us down feeling my way from one trail marker to the next. I sang every hymn I knew so that they could follow after me. The next day I knew that I would sit for the exams a third time. I understood that my approach to solving the problems had been undisciplined and that this time I would be able to focus through my confusion. But most importantly, I realized that my advisor had opened a spiritual understanding of how to do mathematics at a higher level.

5 Reflect and Respond

Whether we are ever to know the effect of the relationship we have with our students, we are called to minister to the wounded we encounter. A spiritual advisor once charged me with the task of praying for each of my students, by name, every day. I can't say that I accomplished this task, but I formed the habit of visualizing the room's faces and asking for the grace to minister as I collected my materials for each class.

- I invite you to remember times from your own student days when you have received ministry, then take a moment to give thanks what you have received.

I hope that this paper has awakened memories and awareness of instances of providing ministry to your students, and perhaps to your colleagues.

- I invite you to give thanks to our God who chooses to use you.

- Share your stories among colleagues when you feel it is spiritually and professionally safe to do so.

On the occasions of these stories, I found that most of the time, my interactions with my students were under direction by the Holy Spirit moving beyond my planning, experience, or logic.

- Have you experienced similar moments of awareness of divine inspiration?
- How has your awareness manifested in your conversation or actions?

In this paper I have used the specific characteristics of poverty cited in Luke 4 (broken-heartedness, captivity, blindness, and wounding) as metaphors to classify specific situations that we find our students to be in.

- Do these images ring true for you as examples of "preaching the gospel to the poor"?
- How do you understand the passage in Luke 4 to apply to your teaching?

For an overview of the legal principles that bear on Christian faculty sharing about their faith in public universities I refer you to an appendix in Hove and Holleman's *A Grander Story* [2].

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Teaching for Social Justice in Online Undergraduate Mathematics Courses

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Abstract

Teaching mathematics for social justice has become a heavily researched and highly published topic in recent decades. However, nearly all of the research has been conducted in the traditional classroom setting, and predominantly in K-12 school systems. My research sought to broaden this knowledge base by examining the impact of discussion forums on students' social justice beliefs in fully online undergraduate mathematics courses at a Christian university.

Quantitatively, students completed pre- and post-course demographic and Likert-scale surveys to determine pre- and post-course social justice scores. Qualitatively, students completed eight weekly discussion forums relating mathematical concepts to social justice issues. Analysis of the merged data resulted in optimistic outcomes and laid groundwork for future research in teaching mathematics for social justice beyond the classroom.

The study and results will be discussed, with extra emphasis on the care taken in planning and in discussion forum prompt creation.

1 Introduction

From religion to medical to political and beyond, the early part of the 2020s has presented many challenges in our country and our world. We are given more information than ever before, but with the challenge of deciphering truth from untruth and biased from unbiased. This has caused strain in communities, workplaces, and educational settings, and led me to question how we as educators can assist our students in learning to listen to others and work together, even if their viewpoints and beliefs are different. Or as Conway questioned, "How was I empowering my students to be agents of change in their lives and others' lives?" [1, p. 5].

One possible approach is to bring issues of social justice into our daily lessons. Through such we can work to strengthen students' understanding of connections between academic content and real-world challenges, as well as their own personal beliefs. By exposing students to issues of social justice in a controlled educational environment, we can work toward the goal of developing citizens focused on social justice for all in every situation. Gutstein states, "Students need to be prepared through their mathematics education to investigate and critique injustice, and to challenge, in words and actions, oppressive structures and acts – that is, to 'read and write the world' with mathematics" [8, p. 4]. As Christian educators, we can further extend this goal to showing God's love as taught in the Bible.

2 Planning, Methodology, and Findings

To better understand the research conducted, a brief overview of the problem, methodology, and findings will be presented in this section. Additional foci of this paper will be the care taken to address challenging topics with multiple entities while maintaining a personal Christian focus and ideas for future Christian-based expansion of this research. More detailed information on the research process can be found in [13].

2.1 Statement of the Problem

While teaching mathematics for social justice and equity has become a heavily researched topic in recent decades with Jo Boaler ([2–4]), Rochelle Gutiérrez ([5, 6]), and Eric (Rico) Gutstein ([7–11]) emerging as leaders in the charge, the focus has consistently remained on traditional classroom teaching (e.g., [2, 5, 7]). During the same time, enrollment numbers for online and distance learning have exploded in growth. This study sought to transition teaching mathematics for social justice from the classroom to the online learning environment and address the following research question: "How do social justice-based discussion forums in online undergraduate mathematics courses impact students' social justice beliefs?"

2.2 Methodology

All data for this study was collected in online undergraduate general education mathematics courses at a private, Christian university. The data was then used in the development of my dissertation for a Ph.D. in Educational Sciences with an emphasis in STEM Education at a public, state-funded university. Extra care was required to balance the needs and the mission of each institution.

The online mathematics courses in the study used a "master course" format, allowing for little change to the overall course structure. The courses were each 8 weeks, consisting of 8 modules. Each module consisted of class notes, recorded lectures, a discussion forum, and online homework assignments. Four of the eight modules (modules 2, 4, 6, and 8) also included exams.

To work within the above parameters, it was determined that modification of the discussion forums would best serve our purpose. Each discussion forum prompt was rewritten to relate the mathematics content covered in the module to a social justice based topic. The first and last forums also asked students to respond to the following statement: "Discuss what the phrase 'learning math for social justice' means to you." The responses to this statement were used for preliminary qualitative analysis.

Relying only on discussion forum responses provided valuable qualitative data, but a more complete picture was sought through the addition of quantitative data. Therefore, a pre- and post-course survey was adapted, with permission, from Ludlow's Learning to Teach for Social Justice – Beliefs Survey [12]. The pre-course student survey collected demographic data and responses to 13 four-point scale social justice belief statements, as well as consent and electronic signatures. The post-course survey collected only responses to the same 13 four-point scale social justice belief statements. After necessary reverse-coding, a pre- and post-course summative rating – called the social justice score – was computed from the 13 statements. A higher score was used to indicate a more socially just mindset. For this study, a "socially just mindset" is defined as a mindset where each and every individual is treated with equivalently high levels of respect and dignity by other individuals and society as a whole. Furthermore, "teaching math for social justice" is defined in this study as a method of delivering mathematical content in an intentional manner so as to encourage students to think critically, apply quantitative literacy skills, and take action in order create a more socially just world.

2.3 Findings

Quantitatively, students' pre- and post-course social justice scores as well as individual question pre- and post-course scores were analyzed using a paired-samples t-test. One-way repeated measures ANOVA was also conducted to examine interaction between time (within-subjects) and demographic categories (between-subjects). With a relatively small sample size of $n = 56$, no statistically significant results ($p > 0.05$) were found.

Qualitatively, students' responds to the following statement at the beginning and end of the course were coded and analyzed: "Discuss what the phrase 'learning math for social justice' means to you." Throughout the coding process, code selection was informed by the study framework, Gutstein's 3 C's: (1) community knowledge, (2) critical knowledge, and (3) classical knowledge [9]. The constant comparative method was used along with open coding and axial coding to arrive at five themes as shown in Figure 1.

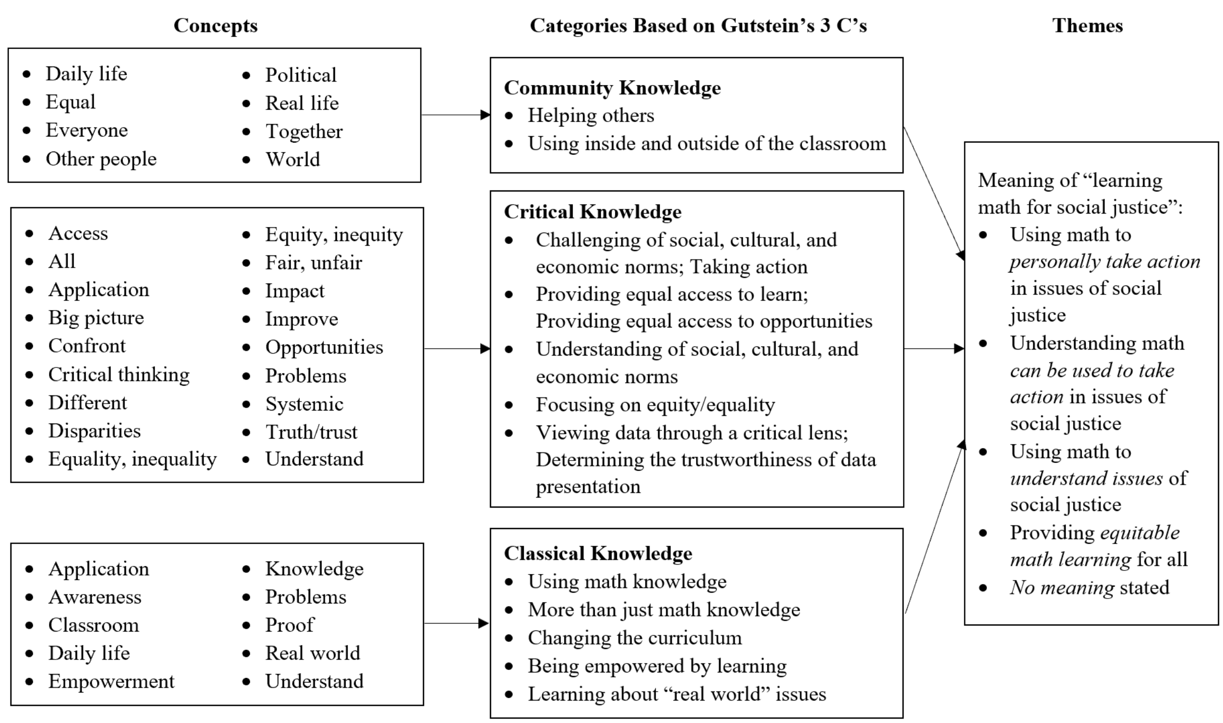

Figure 1: Constant Comparative Depiction.

The five meanings of "learning math for social justice" listed in Figure 1 were deemed levels as follows:

- Level 0: No meaning stated
- Level 1: Providing equitable mathematical learning for all

- Level 2: Using mathematics to understand issues of social justice
- Level 3: Understanding mathematics can be used to take action in issues of social justice
- Level 4: Using mathematics to personally take action in issues of social justice

The study saw a decrease from 31 students at levels 0, 1, or 2 on the first discussion forum response to 22 at levels 0, 1, or 2 on the last discussion forum response. The number of students at levels 3 or 4 on the discussion forum response increased from 15 to 24. This indicates that, while the quantitative data was not statistically significant for this small sample, there may have been an impact on students' understanding of learning mathematics for social justice.

3 Teaching Mathematics for Christianity

3.1 Personal Experience

When my doctoral advisor proposed a dissertation topic related to teaching mathematics for social justice in the spring of 2020, I must admit I wasn't sure how to address this topic. I wasn't familiar with this phrase and certainly hadn't experienced it in my many mathematics courses throughout K-12 (public), college (private, Christian), and graduate school (public). I planned to look for a different, more "comfortable" mathematics topic to propose, but then the chaos of the Covid-19 pandemic ensued. It seemed my topic was now set and I was left to explore far beyond my mathematics comfort zone.

At that time, I had over 10 years of experience teaching college-level mathematics content in the classroom and nearly 10 years of experience teaching the courses online, both at a Christian university. I had developed, taught, improved, and overseen many sections of online undergraduate mathematics courses. In the classroom, I had led rich discussions about faith and difficult societal issues. I had connected these with my personal Christian walk as well as the mathematical content we were covering. But I had fallen into the disconnectedness trap that so many find themselves in for online classes. I had never translated these valuable discussions to my online classes. With the pandemic sending everyone to remote learning, it seemed my path was being determined for me. I set out to learn all I could about teaching for social justice in online undergraduate mathematics courses.

Much to my surprise, there were many books, articles, and websites devoted to teaching mathematics for social justice. I questioned how I had been so oblivious to a topic that has been so heavily-researched in the recent past. Then I realized that every resource I found was focused on the classroom setting and nearly every resource was focused on the K-12 classroom. With my eyes opened to the importance of this topic and knowing the quickly growing online education enrollment, the gap my research needed to fill was seeming even more obvious. Not only would I research teaching mathematics for social justice, I would take it into the online setting at the undergraduate level.

With the research focus selected, I found my next challenge. My research was to be completed as part of the requirements for the completion of a Ph.D. at a state-funded, public institution. My data was to be collected at a private, Christian institution. With the words "social justice" commonly thought of as an agenda being presented, extra care was needed in discussion forum prompt development to avoid any bias. I needed to stay true to myself and my Christian institution, while keeping the topics general enough to be expanded in future research at Christian or non-Christian institutions. Each prompt was carefully selected to relate a social justice issue to the mathematics content being covered in that module. Only facts were given and students were asked to used their mathematics skills along with personal knowledge to analyze the situation (utilizing Gutstein's 3 C's [9]).

My personal preference would have directed the prompts to have a strong Christian focus. However, as mentioned above, it was decided by myself and the doctoral committee that more general prompts should be used to keep from limiting future expansions of the research by myself and others. There was concern that with strongly Christian focused topics, it could be inferred that the Christian focus provided the main impact on the results. While this would have been a very interesting result to me (and probably most who read this article!), these results would be very limited in scope and may not be as easily expanded to non-Christian focused research. Now, this may seem counter to my claim of maintaining a personal Christian focus in my research. Bear with me as I address this in sections to come.

3.2 Discussion Forum Prompt Examples

For the College Mathematics course, a common theme of health care was woven throughout the eight discussion forums. For the College Algebra course, a common theme of pay gaps was used. Three selected discussion forum prompts from the College Algebra course are included below. A complete list of the discussion forum prompts for both classes can be found in [13].

You will notice the topics of the discussion forum prompts are rather basic for this level of math. This was by design; prompts deemed "too difficult" or "too long" by students have been found, in my experience, to reduce student response rates when the relatively low point values of each prompt are considered. Choosing prompts that were more "student-friendly," but still related to the material and led to valuable discussion provided a good balance between participation rates and fair learning assessment measurements for this initial phase of research. While other factors can't be ruled out, these course sections did see greater student participation in the discussion forums than typical sections that had more difficult mathematical concepts in their prompts.

Example 1: College Algebra Module 2 Discussion Forum Prompt

In this module, we will discuss the difference between the "controlled gender pay gap" and the "uncontrolled gender pay gap." Namely, the "uncontrolled gender pay gap" measures median salary for all men and all women regardless of job type, seniority, location, industry, years of experience, etc. while the "controlled gender pay gap" measures pay for men and women with the same job and qualifications.

Study Figure 2. Discuss the slopes of the lines that represent each pay gap (e.g., Are they increasing? Decreasing? Staying the same? How do the slopes compare to one another?). Discuss what this means for the income of women compared to men. Discuss your thoughts on this gap.

Source: The Gender Pay Gap Over Time

Example 2: College Algebra Module 4 Discussion Forum Prompt

As we saw in a previous module, women earned 82 cents for every dollar earned by men in 2021. This is true when comparing the amount earned from white women to white men. However, the pay gap differs when considering other races and ethnic groups. In 2021, American Indian and Alaska Native women earned 69 cents to every dollar earned by white men; Hispanic and Pacific Islander women earned 76 cents to every dollar earned by white men; Black or African American women earned 77 cents to every dollar earned by white men; Asian women earned 95 cents to every dollar earned by white men.

Use the information above to set up functions to calculate the expected pay for American Indian/Alaska Native women, for Black/African American women, and for Asian women based on the pay of a white man. If the average white man was found to earn $20,000 in a rural area, $50,000 in a suburban area, and $80,000 in an urban area, use the functions you created to calculate the expected women's pay for each ethnic group in each location. Then plot the points all on one graph using the following guidelines:

- Label your x-axis as location and mark rural, suburban, and urban.
- Label your y-axis as income and mark in $10,000 increments ($0, $10000, $20000, $30000, ...)
- Plot one point above rural for white men, one point above suburban for white men, and one point above urban for white men. Then connect these three points. Repeat this process for each of the other three groups (American Indian/Alaska Native women, Black/African women, Asian women).

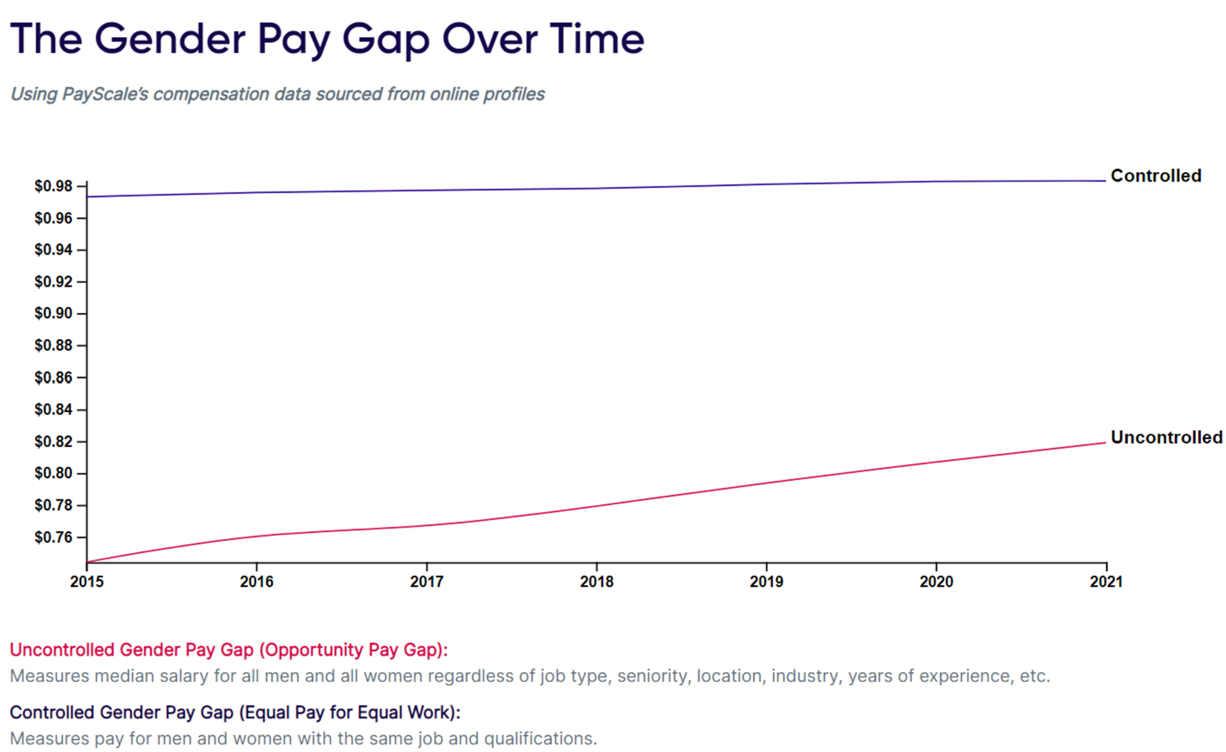

Figure 2: Gender Pay Gap Graph.

Discuss the resulting graph. Why do you think these differences might exist?

(Note: These values illustrate the "uncontrolled gender pay gap" which measures median salary for all men and all women regardless of job type, seniority, location, industry, years of experience, etc. The "controlled gender pay gap" which measures pay for men and women with the same job and qualifications is much smaller, but still exists. Visit the website provided below for more information.)

Source: A Deeper Dive into the Gender Pay Gap

Example 3: College Algebra Module 5 Discussion Forum Prompt

According to Census.gov, in 2019 there were approximately 2.7 million people working as janitors and building cleaners. Approximately 11% of these were classified as having at least one disability. The most common disability types for workers were ambulatory, hearing, cognitive, and vision.

Let j represents the total number of janitors and building cleaners. Let $d = g(j) = j \times 0.11$ where d represents the total number of janitors and building cleaners working with a disability. Let $a = f(d) = d \times 0.30$ where a represents the number of janitors and building workers who have an ambulatory disability. Find the function $f(g(j))$. Find $f(g(j))$ for 2019. What does this value represent?

The article cited below from Census.gov states the following:
Overall, workers with a disability earn less than workers who do not have a disability. Yet, depending on the types of work they do, much of the difference in median earnings disappear. Today, a record 9 million people with a disability work. While these workers, age 16 and older, are spread throughout the labor force, workers with a disability tend to concentrate in certain jobs depending on their age and particular disability.

Discuss the wage gap that results for workers with disabilities versus those without disabilities.

Source: https://www.census.gov/library/stories/2019/03/do-people-with-disabilities-earn-equal-pay.html

Ideas for Christian Focused Discussion Forum Prompts

Although I was not able to use Christian focused discussion prompts in this research, I did maintain my Christian focus while selecting prompts that provoked thought without any agenda or bias. I maintained my focus when corresponding with students through email and messaging. As a researcher, I intentionally limited my direct interactions with students in the study to reduce any possible impact on the research data. In future research, I would like to explore prompts with a Christian focus.

An obvious idea for adapting the above examples to have a Christian focus would be choosing an appropriate theme, such as church attendance. Church attendance data could be analyzed based on household composition, distance from church, geographic location, socioeconomic status, and many more factors. Likelihood of children attending church upon adulthood could be considered. A more specific idea might consider data related to Vacation Bible School. These ideas would work very well in many general education courses. A more in-depth application could pair with local faith organizations to collect data and then analyze the data, though this would likely be better-suited for a classroom section than an online section.

Another straightforward integration could examine religious groups over time. Students could analyze the population over time in specific regions by religion. Students could then illustrate this in an appropriate graph form and use mathematical terms to discuss the changes observed.

While not directly related to teaching math for social justice, other prompts could be used to relate the Christian faith to mathematics in the form of discussion forums. For example, when teaching a section on dimensional analysis, students might be asked to convert the dimensions provided for Noah's ark from the Biblical measurements to modern customary measurements. Discussion could then proceed regarding the size of the ark compared to the amount of animals onboard. The wrap-up could discuss how this might influence an individual's beliefs (e.g., students might debate the plausibility and therefore their thoughts on the validity of the text leading to rich discussion).

Care should be taken to have students practice their mathematical content and communication skills, as well as their critical thinking skills that expand beyond the mathematical content. Care should also be taken to ensure students are being respectful to one another. Differing opinions can lead to rich discussion, but can also lead to hurt feelings and poor student experiences.

3.3 Student Response

Being general education mathematics courses, the student population varied greatly with respect to age, ethnicity, income, and major. There were several students who responded very positively to the direct applications of the mathematics content to real world situations. But there were several students who did not respond positively. Thankfully, the literature I had read prepared me for this reality. In the United States, mathematics has been largely taught as a content-only subject, limiting realistic interactions with other areas of study. There are occasional examples, but strong interactions and applications have not been common in our mainstream textbooks. Further, most students are not accustomed to writing prompts in mathematics courses. Adding in the fact that many of the students had not taken a math course online before, the students in the study encountered an unfamiliar mathematics learning experience. Overall, the students were very respectful of one another and had very deep discussions. The interactions on the discussion forums were much lengthier and more numerous than in a typical section with mathematics-content only prompts.

I was fortunate to have very supportive administrators who evaluated my work, provided positive feedback, and responded to the student complaints. Discomfort seemed to be most common among parents of dual credit high school students enrolled in these online courses as it exposed their teenage children to ideas and issues they were not aware of prior to taking the course. I would recommend (as the books I read also did) that anyone who plans to incorporate these ideas in their classroom – in person or online – first seek guidance from their academic leaders and from those who have used these methods before.

3.4 Teaching Mathematics for Social Justice and Christianity

While, like my students, this topic first made me uncomfortable and nervous, I quickly came to realize that teaching mathematics for social justice does not (and should not) mean pushing an agenda or trying to influence thoughts. I know I was not alone in this confusion as I have had to explain my research many times to those unfamiliar with the idea and holding the same misconceptions. However, I now feel I was naive to even have thought the way I did. This research isn't something that should have created a personal challenge balancing the two institutions I was working with.

Teaching mathematics for social justice is giving students the tools, skills, and mindset to see a situation, gather information, use their mathematics skills in the evaluation of the situation, and then make an informed decision, possibly leading to action. Just as we as Christians are to go out into the world to show God's love and live the Word He has given us, we as educators have a duty to enable our students in doing the same. This research allowed me to see that incorporating social topics into learning environments is one unique way I can serve in this mission.

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Mathematical Theater and the Human Condition

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Abstract

Stephen Abbott's book *The Proof Stage: How Theater Reveals the Human Truth of Mathematics* explores how theatrical plays in the last century have incorporated mathematical concepts. Many of these mathematical ideas are included in liberal arts mathematics courses and have been explored theologically. We will examine what mathematical plays and their interpretations have to say about the human condition and what they may have to offer for the Christian scholar.

1 Introduction

The Proof Stage: How Theater Reveals the Human Truth of Mathematics [1] by Stephen Abbott is an excellent account of the interaction of theater and mathematics. Abbott, a Professor of Mathematics at Middlebury College, provides history, connections, and interplay between two seemingly disparate areas relying on his formal training in mathematics and his decades of experiences with theater. Throughout the book, Abbott devotes several pages of mathematical explanation for the uninitiated. He also provides backstory and snippets of dialog to help the reader get a feeling for a specific play or playwright. Since each era, each play, and each scene allow for multiple interpretations, it becomes difficult to fully explicate such themes of the book as the shift from transcendent certainty to equivocal relationships to truth. While there is much to be said, we will try to keep to a few of the book's main themes.

2 Why Math and Theater?

Math and theater seem to make strange bedfellows. As Abbott points out, mathematics appears to be authentic, forbidding, enduring, inevitable, and imbued with certainty. On the other hand, theater is fanciful, imaginative, temporal, and a momentary illusion for a live audience. Despite their differences, the two areas have more in common than is apparent at first glance.

In some sense, mathematics pursues truth or certainty. With this pursuit, questions of epistemology and ontology arise which philosophy and art can examine or incorporate. In particular, plays allow the exploration of philosophical questions in action. Rather than providing didactic resolutions, playwrights can ask deep human questions which they cannot answer but find important enough to explore in depth. While perhaps a stretch, Abbott proposes "[E]verytime a playwright who engages mathematical truths does so with the goal of accessing human truth."

Plays can provide multiple interpretations by its use of ambiguity to challenge perspectives and to present layers of meaning to alternative viewpoints. Whether a thought experiment or an acted-out experiment, theater is a type of experiment to see humans wrestle with live experiences and the implications of the subject matter. Actors through their tone, facial expressions, presence, and body language as well as features such as the set, lighting, and pace bring the play to life communicating through emotions, ideas, artistic expression, and plot. During live performances, actors adjust to the audience's demographics and their emotional response in order to connect the audience to the material.

What mathematics can provide are metaphors, an appearance of certainty, and mathematical results challenging that certainty. Throughout history philosophers, artists, and mathematicians have tried to make sense of mathematical ideas to see what implications they have on what it means to be human. These include irrational numbers, Newton's mathematical physics, non-Euclidean geometry, Heisenberg's Uncertainty Principle, chaos theory, and Gödel's Incompleteness Theorems.

I have found plays involving mathematics offer a change in perspective on how mathematics might be interpreted. They are ways to interact with deep philosophical ideas in action. There is an emotional component which is rarely associated with mathematics. Furthermore, plays are another outlet to share my love of mathematics with non-mathematicians to discuss the ideas and approaches mathematicians develop. I often hope theater can be a conduit to illustrate the relevance of pure mathematics and other mathematical sciences.

3 Shifts Away from Realism

The historical study of mathematics and theater often begins with Ancient Greece. Thales, Pythagoras, Euclid, and Archimedes were contemporaries of Ancient Greek dramatists Aeschylus, Aristophanes, Euripides, Sophocles, and Menander. Greek theater demonstrated timeless truths by providing compelling demonstrations of moral laws. Ancient Greek plays codified and communicated truths of the universe. As illustrated in Euclid's *Elements* mathematics was based upon timeless truths logically demonstrated from self-evident axioms. Up until 150 years ago, mathematics was often seen as immutable or at least as close to certain knowledge as one can expect.

During the nineteenth century and the early part of the twentieth century, the perception of the timeless truth of mathematics began to change. Similarly, theater was beginning to move away from its origins of certainty. The issue for both theater and mathematics was the relationship between the subject matter and the real world.

For mathematics, the question became amplified by the development of non-Euclidean geometry. Chapter 2 recounts the history of Euclid's parallel postulate in geometry as it relates to the evolution of mathematics. Non-Euclidean geometry and other mathematical developments opened the door for mathematics to go beyond merely reflecting and modeling the real world. Mathematicians could be free to explore new abstract worlds.

Meanwhile, theater was experimenting with its own relationships to realism. For most of the nineteenth century Romanticism reigned in theater which was characterized by melodrama and operettas featuring "intensely emotional acting and exaggerated scenic effects." [1] Heroes and villains were standard characters which came with standard arrays of voices and gestures. Henrik Ibsen's seminal work *A Doll's House* with its uncomfortable ending of a woman slamming the door as she leaves her family exemplifies a shift towards realism. Theater started to engage serious contemporary topics, the social and psychological realities, rendering life as it really was without cliche or euphemism. [2]

Perceiving a general cultural crisis, avant-garde theater and other movements pushed back against the status quo of realism. Artists began to free themselves from the constraints of the observable world to create new art forms with different emphases on language and values including tearing down the barrier between artist and audience.

Abbott compares and contrasts each area's shift away from the real world making thought-provoking parallels but also their effect on each other. Although challenging at times to understand, each pushed their areas into new productive worlds. By keeping their own internal integrity to produce new work, both did not collapse into nonsense. Pure mathematics and its unreasonable effectiveness to applied problems and avant-garde theater's foreboding of the future provided valuable insights. Each changed how we view and evaluate the products of their work.

Perhaps in the end, mimicking the world is not the only way to understand the world.

4 The Search for Meaning

With non-Euclidean geometry, Cantor's conception of infinities, and other increasingly paradoxical mathematical results, mathematicians attempted with greater vigor to place their subject on the solid ground of certainty through Foundationalism and Formalism. How can we be certain mathematical theorems are true?

Falling back on axiomatic systems, thinkers such as David Hilbert and Gottlob Frege began their quest for a perfected, formalized language so that every mathematical statement could be reduced to rules without any reference to the meanings of the formulas to determine its truth or falsehood. That is, the formalized language would provide a reliable foundation for truth. For Hilbert, the necessary and absolute confidence in mathematical results would follow from a complete and consistent axiomatization of mathematical ideas.

Abbott contrasts this mathematical search for meaning and certainty through formal languages with the literary work of Samuel Beckett and his exploration of language. Just as mathematical axioms no longer necessarily mirrored reality, Beckett viewed language as a veil unable to reach the essence behind it. In other words, language does not fully describe the undetected meaning behind the object.

Through his wide range of literary work, Beckett explored deep questions about human identity believing the deepest truths about the human experience lay beyond the reach of language. With plays like *Waiting for Godot* and *Endgame*, Beckett saw the search for meaning fruitless. He developed characters in hopeless situations who do not succumb to hopelessness through stoicism, self-delusions, and persistence. Our sense of self cannot be separated from Beckett's curious and strained relationship to language.

What is perhaps surprising is Beckett's study and use of mathematical ideas in his literary works. Part of this general lack of awareness is that mathematical ideas are more prominent in his lesser known-works and are hidden beneath the surface in those that are well-known ones.

Abbott shows how paradox and the language of axiomatic mathematics played a role in Beckett's view of language. Beckett was aware of mathematical developments such as those of Russell and Gödel, and his early avant-garde novels like *Murphy* incorporated mathematical ideas which would show up later in plays. In *Murphy*, rational numbers represented the universe accessible through linguistically dependent observation. We know rational numbers are not all there is, and irrational numbers stood for the unfathomable and indescribable which lay beyond. In Beckett's work, there are self-referential paradoxes, permutations and combinations, illustrations of the ineffectiveness of purely rational approaches to the enlightenment. There are logical quagmires of real experience and the fragility of logical structure. Abbott connects the death of Frege's logicism through self-reference to the prominence of self-reference in *Waiting for Godot* and *Endgame*.

Perhaps the densest chapter with respect to literature, chapter 3 has much more to explore in how twentieth century mathematics influenced Beckett. It presents Bertrand Russell's work as well as Gödel's Incompleteness Theorem. Abbott concludes we must acknowledge our languages' validity at the same time we recognize our languages' limitations. Our mathematical axioms and human languages are incomplete. Abbott ends the chapter with the contradictory logical expression $p \wedge \neg p$ and the paradoxical quote from *Waiting for Godot* "I can't go on, I'll go on."

5 Loss of Certainty

For the most part, Abbott summarizes the interconnections between Math and Theater, but in chapter 5 "Stoppard: The Logic of Self-Conscious Theater" he provides his own provocative claims and his personal story for writing the book.

One of Gödel's Theorems proves with sufficiently strong axioms there will always be statements which are true but cannot be proven within the system. This theorem is an odd mathematical statement since it is a theorem about mathematics itself—-a metamathematical theorem. Just as formal systems can be employed to probe their own integrity, plays frequently examine themselves. Suspending the imaginary wall separating the story from the real world, actors can break the fourth wall by making comments about the play that the audience is watching. There are also the plot element of a play within a play as exemplified by Shakespeare's *Hamlet* or *Midsummer's Night Dream*.

Abbott employs Tom Stoppard's 2025 play *Dogg's Hamlet* to demonstrate, evaluate and illustrate Gödel's Incompleteness Theorem using the play's self-referential and made-up language of schoolboys doing *Hamlet*. Abbott examines Stoppard's *The Real Thing* to demonstrate how theater and mathematics are able to "explore their own expressive potential" through the hierarchy of plays within plays and mathematics within mathematics.

Abbott then makes the leap to human's fundamental nature by examining Stoppard's 2015 work *The Hard Problem*. The Hard Problem of consciousness refers to the thorny question as to why humans have consciousness. In the play there are the materialists, who attribute consciousness to nothing more than neurons and Darwinian evolution. The soulists claim there must be more. Both sides enlist Gödel as an ally for their causes. Rather than coming to a definite conclusion, Stoppard seems to be more interested in exploring the complexities of the problem.

Much has been written on the various perspectives of the Hard Problem, but the author Abbott is partial to the materialists. He references Doug Hofstadter the author of *Gödel, Escher, Bach* [3] who has written extensively on Gödel's Incompleteness Theorems. An ego-possessing form of intelligence could be modeled in a purely mechanical way on a computer, and the brain is similarly a mechanical machine. Abbot posits

> . . . one implication of Gödel's work is that the hardware in the brain could very well be

> enough to explain the more mysterious aspects of the human experience such as creativity, emotions, and most especially, a sense of self. The mind may not be extra.

Specifically, Abbott ties the **proof** of Gödel's Theorem as to why computers could develop emotions. In the proof, a contradiction is reached when a statement in the language of the formal system is determined to be **undecidable** in the language of the system.

> The lower-level rules bring about the possibility of higher-level meta-interpretations which then, mysteriously, wield some influence back down on the system from which they were spawned [1, p. 269].

Thus, the proof of Gödel's Theorem allows the existence of art from a closed system. Humans are creative beings, and this creativity could be allowed by mechanical rules. We can formulate ideas within a system perhaps leading to a meta-theoretical statement which can then say something about the system itself and the conditions which set up the system. Like art.

6 Connection to Humanity

> Tolstoy says that free will is merely an expression denoting what we do not know about the laws of human life ... Perhaps it's an illusion. But without that illusion, life would be meaningless. Turing (the character) in *Breaking the Code* [4, p. 86].

Chapter 6 follows up with plays which illuminate the implicit humanity of Alan Turing and his mathematical ideas. Many philosophers and artists have explored Turing's life, the Turing Thesis, and the Halting Problem to present some of life's most challenging issues. From the realistic to the fanciful, here theater provides a deep connection, playing with art to examine the human condition.

Turing's life and work provides fertile ground for the playwright. There is the Turing Test of whether one could design a test to determine a machine's ability to exhibit intelligent behavior. Turing himself asked whether the mind could live without the body. The playwright can play with questions such as whether mental processes can take place in something other than a human. What is intelligence? What does it mean to be human? Turing trying to navigate between the abstract theoretical world and the practical lived-in world provides incongruity and tension. Turing's homosexuality, his struggle with human interaction, and his ultimate suicide provide poignancy.

One of our basic traits of humanity is consciousness and empathy. In Hugh Whitemore's play *Breaking the Code* Turing, one of the greatest minds, is portrayed as someone who has trouble feeling or empathizing with others. The play juxtaposes these struggles with whether a computer could ever "feel." For Turing, computer science was poised to answer some deeply fundamental questions about life. Is the human brain some sort of universal Turing machine? For the playwright Whitemore, how can humans build models of another human's psyche? If so, what about empathy?

Snoo Wilson's play *Lovesong of the Electric Bear* fancifully reenacts through a stuffed bear the events of Turing's life. The tone of the play meshes with Turing's eccentric personality but does more than that. Allowing a fantastical perspective rather than a rigid one, the play actually provides dialog that is historically accurate. However, Turing in Wilson's play is like an ancient among the Greek gods. Turing's fatal sin is his deeply held belief that there is no qualitative divide between minds and machines.

Although not about Alan Turing's life, *Half Life* by John Mighton illustrates Turing's Test as applied to our human nature. Two elderly people in a nursing home are experiencing what life is like when memory is fading. Not just a story of how society treats the elderly, there follows bigger life questions. How do we assess the nature or degree of human intelligence? Do we judge aging adults by their functionality? A romance, the play has a key moment whether an off-stage voice is a human companion or a computer—a Turing Test.

For Abbott, there is an implicit humanity in Turing machines. Referencing his conclusions from Gödel's Theorem, empathy is actually a more compelling argument for a mechanical model of the human mind. Just as formal systems and Turing machines acquire gaps and imperfections in their capabilities, they really might be part of intelligence: fallibility, self-awareness, empathy, charm, pleasure, grief. On the other hand, the plays are not trying to answer the viability of intelligent machines, but at a fundamental level what is intelligence?

From his examination of these plays, Abbott advances the ability of theater with live actors to provide for the audience empathy and human understanding. He cites the closing scene of *Half Life* as a powerful, distinctive capacity of theater to find human stories inside the mathematical ones. Saying the same thing in a different context, perhaps our imperfections are really part of our intelligence.

7 The Ability to Create

A subcontext to the book's claims is the ability to create art and the ability to create art from mathematical ideas. For example, the implications from Gödel's Theorem and Turing's work provides rationales as to why art could be created in a materialist universe.

There are many plays referenced in *The Proof Stage*, and we will look at three specific plays as to how they relate to creativity and epistemology.

7.1 Arcadia

Almost all of Thomas Stoppard's plays have a mathematical element, but *Arcadia* is perhaps the first broadly successful play which engaged so much mathematical content. First performed in 1993, *Arcadia*'s use of math metaphors were central to the passions and motivations of the characters. Like several other plays, mathematics was used for its propensity and illusion of certainty. Among a myriad of themes, there was a romantic incarnation of math in its service of paradox and unpredictability. It was not the usual debate between art and science but the Classical versus the Romantic or the Rational versus the Passionate. It employed chaos theory, fractals, non-Euclidean geometry, and Fermat's Last Theorem. It questioned whether nature is written in numbers but now with a new type of mathematics. It challenged a predetermined, Newtonian billiard ball view of the universe with the unpredictable but predetermined chaos theory created out of simple and profound patterns. The play was able to clear the hurdle in making mathematical ideas understandable. Humans may be headed to the end according to the laws of thermodynamics, but there is sex (the attraction Newton left out) and art to make us human.

Thomas Stoppard takes up the largest portion of *The Proof Stage* due to his inclusion of mathematical ideas even before he realized it. For Abbott, *Arcadia* was particularly inspiring. When he first saw *Arcadia*, the familiar was suddenly full of mystery to him. He wondered why it caused him such visceral excitement. This set him on his math and theater journey.

Arcadia is an example of a play which would not work as a movie due in in part to a literal chaotic scene between two time periods. If of good quality, I would highly recommend attending this play.

7.2 Copenhagen

Michael Frayn's play *Copenhagen* is able to accomplish one of the greatest challenges in theater, mathematics, and science. Frayn is able to make the science and the characters authentic. Abbott commented

> . . . with *Copenhagen*, Frayn has orchestrated a small miracle whereby the principles of modern science under discussion are interwoven with the insights into human nature under debate, what are in turn incorporated into the architecture of the play being formed.

Copenhagen is about an actual meeting in 1941 between nuclear physicists Niels Bohr and Werner Heisenberg. The question, "Why did Heisenberg go to Copenhagen?" frames questions regarding epistemology and the Heisenberg Uncertainty Principle. The more we examine the meeting the less we know and the more questions we have. On a human level, it is a play about the desire to know and to be known.

In 2002 the BBC adapted the play as a film by Howard Davies, and may be available for viewing. In my judgment, the film version is a fairly successful film adaptation among mathematical plays. I highly recommend attending the play or watching the BBC version.

7.3 A Disappearing Number

A Disappearing Number has a peculiar backstory in the process of its development. Inspired by reading *A Mathematician's Apology* by G.H. Hardy, Simon McBurney recognized the creative nature of mathematics. He and his theater company Complicité set out to "create a theater piece in which the concepts of continuity, partitions, prime numbers and the mathematical infinite were embedded into the fabric of the performance." [1] There was a problem to this plan. McBurney did not know any mathematics.

Instead, the theater would use improvisational techniques. In many ways, they would mimic the process of how mathematics is actually done: consider special cases and anomalous examples, experience false starts, and identify false conjectures. The theater company would experiment and discard numerous ideas and throw ideas together to allow the possibility of the unexpected. Here, Abbott connects mathematicians and practitioners of theater. According to Michael Frayn,

> Plays are not called 'plays' for nothing—they are a means of messing about, exploring the world without the restriction of actually moving about in it. [1]

For Abbott, he makes the analogy of theater to the creative process of pure mathematics.

> [T]he theater is an idealized environment where the artist can escape the limits of physical reality and explore the world according to his or her own chosen set of assumptions.

This play is an example of math and theater being creative processes. It is difficult to find a showing of this production.

8 The Problem

There is a wealth of connections of plays with mathematical themes. Whether it increases our ability to understand ourselves, to see mathematics in a new light, or to reflect upon deeper questions, there is unfortunately one big problem with math and theater. Accessibility.

Theater is a temporary, in-person, one-time event with an interaction between the actors and the audience. We cannot rewind, flip the pages back, or return to earlier scenes with theater. It is difficult to read scripts apart from the production. We can read the dialog from the scripts, but these fail to capture the setting, stage movements, the body language, and the facial expressions.

Furthermore, it is challenging to find a performance of a mathematical play let alone a particular one. These "intellectual plays" demand much from the audience. There are mathematical contexts, theatrical nuances, and artistic creativity which can easily be missed. The required careful attention makes intellectual plays difficult to draw audiences in comparison to lighter fare. For Christian colleges, liberal arts colleges, and courses exploring philosophical and ethical issues in mathematics, incorporating these mathematical plays into a curriculum is almost fruitless if a live performance cannot be found.

9 Conclusions

Stephen Abbott's *The Proof Stage* is a great resource for those in the mathematical sciences interested in how mathematics enters into humanity. Well-written, the book does an admirable job of juxtaposing mathematics with theater to demonstrate how each reflects the similarities of their respective developments. While I did not come to the same conclusions, I found my perspectives broadened as I learned something new about art and theater. At the same time, I believe one could read a section of the book and still gain strong insights.

For the Christian scholar, mathematical plays and Abbott himself take many of the same ideas as those who try to integrate faith and mathematics. Christians will likely come to different representations or conclusions on several issues. Christians would be wise to listen to alternative perspectives presented in different forms. Tentative rather than strong claims will helps us avoid the pitfalls of "God of the gaps," or even, "Gödel of the gaps." Mathematics and art are rarely meant for dogmatic statements out of context.

Part of this challenge involves the use of metaphors. As Abbott points out,

> Any time a precisely formulated conclusion of mathematics or science is employed in the service of an artistic metaphor there is the danger of deforming the original principle beyond useful recognition.

For the interested Christian, there is much to mine in the book particularly the chapter on Turing in light of the recent developments of generative AI. One can take in the philosophical arguments of artificial intelligence and the capabilities of machines to mirror humans, but we should also appreciate how plays portray humanity in Turing's life and in *Half-life*.

I maintain that plays provide a means to challenge our beliefs and reflect who we are. They remind us that being human involves both our emotions and our intellect. After all, are we not admonished to love God with all of our heart, mind, soul, and strength?

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Experiences with Standards-Based Grading

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Peter Jantsch (Ph.D., University of Tennessee) is an Assistant Professor of Mathematics at Wheaton College. He has interests in approximation theory, numerical analysis, data science and alternative grading methods, recently co-editing and co-authoring the open source text *Python Jupyter Notebooks for the College Math Classroom.* Outside the classroom, Peter loves his wife Lindsey, supporting the Arsenal, and winning at board games.

Abstract

One of the main ways students know what is important in a course is by what and how we choose to assess, and grading practices are a major part of the process by which students are formed in the classroom. Alternative grading—in its many flavors—aims to change the focus of our students from "getting the grade" to learning and growth, and is gaining a lot of traction in higher education. I will describe my experiences with standards-based grading in a range of math classes, from introductory to upper-level. I'll talk about why I made the change, what I've observed in my students, and give some advice on how you can get started with standards-based grading, too!

1 Should we rethink grading?

For many students, getting high grades is the end goal of education. Teachers—myself included—reinforce this notion in the ways that we set up our class to emphasize timed, high stakes tests and partial credit. We show that we value speed, memorization, and getting things completely correct the first time. It's little wonder that students spend so much time working out how to game the system, and this game tends to favor those students with particular family and educational backgrounds. Rather than discussing the subjects we are passionate about, we end up having long conversations about grades that have nothing to do with learning, negotiating with students for points and scores.

Most college math courses—and most courses in general, but I'll speak only about mathematics here—use a "traditional" grading system, where most of the grade is determined by timed exams or quizzes, with no reattempts or revisions allowed. The grades on these assessments are point values or percentages which are then alchemized into a final grade by some weighted averaging process—and typically converted again into a letter grade or 4 point GPA scale. These numerical and letter grades are an efficient way to assess large numbers of students, and their widespread adoption makes them easy to translate to other institutions. The systems of weighting to determine the final score seem to promote a kind of transparency that could be beneficial for the students.

On the other hand, it is not clear that these methods actually promote learning, nor do they show a student how they could grow in their understanding. A 75% score means that a student got 75% of available points, but points don't translate into information for a student to chart a way forward and improve their learning. In fact, the averaging process hides students' learning! A student who starts slowly but eventually learns everything may get a 50% and 100% on consecutive tests. But they will end up in the same spot as the student who demonstrates average work and little growth, scoring a 75% on each exam.

So we see that these traditional grading methods often fail to accurately reflect a student's knowledge and progress. Traditional grades may also be prone to bias, and tend to lead to demotivation in students, as they can be arbitrary and may not communicate what a student knows or needs to improve upon [2]. Additionally, the pressure of timed assignments and the concept of partial credit can allow students to pass without fully understanding the material. These flaws make the work of grading slow and frustrating. Personally, I was spending a lot of time making meaningless distinctions—determining if student work should get 7 or 8 points out of 12, when in the end there is no appreciable difference between their work, and no real meaning assigned to scores.

This illustrates part of the issue: grades are actually categorical data. Statisticians distinguish between numerical data, a number that represents a measurement on a scale, versus data that represents a name or a label—think body temperature (numerical) versus a zip code (categorical). Note that the latter example is a label represented by a number, just as many grades are! While grades can and should have some kind of ordering, they are qualitative data. Points are granted or subtracted for a variety of things, e.g., timeliness, effort, organization, and understanding of the topics. When we average these scores, we are averaging things that have different meanings. Even when done with care and the best intentions, the quantification of qualitative information will always lose something in translation.

An outsider to the math world might think that in mathematics there is always one correct answer, and grading should be objective. Using numbers as scores gives the appearance of objectivity, but even in mathematics grading still involves quite a bit of professional judgement on the part of the teacher. Did the student make a mistake because they are unfamiliar with the concepts, or is the mistake simply a calculation error? We do our best to infer student understanding from the evidence they present, but it's not a perfectly objective process.

The game of grades leads to negative student-professor interactions, where office hours become arguments about points and scores rather than learning and mathematics! Bargaining for extra points back is a normal post-test experience for many faculty. And even the students who do want to learn from their mistakes may not have another opportunity to demonstrate growth.

For me, these concerns began a journey to figure out how I could encourage students to be more resilient. I wanted them to be unafraid to fail, and show them that failure is a part of learning and can be productive. Mathematics is about exploring ideas, and no one can really explore without getting a little lost sometimes! I wanted my class to be more than just information transfer, and to set up my class to show that I think mathematics is worthwhile not just as a technical discipline, but also in how the practice of mathematics can form my students.

Alternative grading—with its myriad implementations—have been gaining a lot of attention in higher education, aiming to change the focus of our students from "getting the grade" to learning and growth. The goal is to mirror the way we actually learn: we try something, get feedback from a knowledgable source, process and apply the feedback, and then try again. To incorporate these feedback loops into the structure of our classes, these systems use the so-called "four pillars of alternative grading": (1) Clearly defined standards; (2) Helpful feedback; (3) Marks indicate progress; and (4) Reattempts without penalty [2]. All of these together help form the foundation of the feedback loops that are central to the learning process. They allow students to demonstrate growth in their skills and understanding. Alternative grading is a way to "think about grades built on growth over game-playing, learning over letters and numbers, and productive relationships over adversarial ones" [2].

With a little bit of digging, you might find that there are as many implementations of alternative grading systems as there are instructors using them! In the following sections I will describe the particular ways I implement these ideas in my classes, which are sometimes known as Standards-Based Grading (SBG). Though it might be known by other names, by standards-based grading I mean a grading structure focused on assessing students' understanding of specific learning objectives. Instead of averaging numerical grades from various assessments like quizzes, exams, and homework, grades in a SBG classroom are determined primarily by whether students have (or have not) satisfactorily met predetermined standards throughout the term, and students have multiple attempts to meet these standards. Because so many variations are possible, I'll also include some alternative ideas and links to other resources. Finally, I'll share about my experiences with the implementation, including some of the drawbacks and things I'd like to change.

1.1 Grading as faith integration?

In concluding the introduction, I want to make a few remarks about where I see these methods of alternative grading connecting with my calling as a Christian and an educator. I believe that alternative grading, along with any teaching practice that encourages student engagement and promotes ownership of learning, allows for more authentic engagement with mathematics in a way that helps the students build the virtues of mathematical practice. I do not mean to suggest that alternative grading is the only or best way to do this, but thinking through my assessment methods in terms of faith integration has been a helpful exercise that I believe will interest many of the readers of this journal.

In the excellent book Mathematics for Human Flourishing [8], Francis Su devotes chapters to the desires of exploration, meaning, play, beauty, permanence, truth, struggle, power, justice, freedom, community, and love. Each chapter shows how these desires may be fulfilled in mathematics, and how, rightly practiced, they might produce virtues in us that help to flourish. I want to talk briefly about three of these desires—justice, struggle, and truth—which I think align well with Christian values and motivate my use of alternative grading.

Justice is a Christian desire, as God is the God of justice. Proverbs 2:9 says that those who receive God's words of wisdom and walk uprightly will "...understand righteousness and justice and equity, every good path." Likewise Micah 6:8 asks "what does the LORD require of you but to do justice, and to love kindness, and to walk humbly with your God?" Many of our current systems of assessment were designed to sort students based on the belief that ability is innate and fixed. Likewise, the impact of how and what we choose to assess has a big effect on who continues to pursue math and science, and the ways we grade can mitigate (or exacerbate) inequities in the classroom [4]. For me, the work of moving toward grading strategies that are more equitable is one way to reflect the value of justice in the classroom.

Secondly, there is struggle. Just as one might engage in struggle through exercise with the goal to grow in strength or athletic ability, most types of growth require struggle. Here we are not talking about struggling due to suffering, trials or lack, but the kind of struggle to grow or move forward in a certain area. Struggle and growth are values that tie in closely with Christian faith: while Jesus paid the debt for our sins on the cross, he freed us to turn away from our sins and follow him. This is a lifelong journey that requires effort—not an effort to make ourselves right with God, but having already been made right to live our lives in light of that salvation. Yet the struggle to grow is not always promoted in traditional methods of assessment. What if getting a question wrong didn't have to mean failure, losing points, or "looking stupid", but could be another opportunity for growth? I think alternative grading is a fantastic way to provide opportunities for productive struggle and remove the gatekeeping mechanisms of timed, high-stakes testing.

Lastly, the pursuit of truth is an important part of the life of a Christian. While this does involve our reason and rationality, it likewise involves a deep faith and an ongoing relationship with God. The more we pursue truth, the more steadfastness we have when we experience doubts or uncertainty. The more we encounter truth, the more humility we have in the face of what we don't know. Truth, like love, should not be puffed up, but should draw us toward others in empathy, and build a desire to continue on this journey. Most discussions of truth in mathematics take place in the realm of philosophy or epistemology: what sorts of truths can we prove in an axiomatic system, what are the limits of knowledge in these systems? Here I am talking about something slightly different which is confidence in truth. Despite the complexity, ambiguity, or uncertainty in many aspects of our lives, it is important to not become disillusioned or cynical about truth. It's easy for students who fail to understand a concept to give up. Engaging in feedback loops—the kinds promoted by alternative grading—allows students to gain confidence in the truth of mathematics and their ability to use their God-given reason and rationality to understand it.

2 A brief overview of standards-based grading

In my courses, grades are primarily determined by the number of standards (I also call these "learning outcomes") in which a student has satisfactorily demonstrated understanding—hence the name, standards-based grading. For instance, in a Probability Theory course, one learning outcome might be: "I can apply the techniques of conditional probability, including Bayes' Rule, to find the probability of events," or "I can calculate probabilities involving order statistics." No partial credit is awarded for attempting a standard. However, students have multiple opportunities to reattempt each standard throughout the semester. At the end of the article you can find a link to my syllabi, which contain all of the learning objectives for my courses, as well as other details on my implementation of SBG.

With a little bit of digging, you might find that there are as many implementations of alternative grading systems as there are instructors using them! I will describe briefly the way that I have tried it in my classes, as well as some variations I've seen. Because so many implementations are possible, it's helpful to keep in mind the "4 pillars" from the introduction—each piece of this system has a purpose based in one or more of those ideas.

In the 2023-24 academic year, I used SBG in four different undergraduate mathematics courses. The first two made up a year long Probability and Statistics course sequence. The first semester there were about 18 students, with only 12 students in the spring semester. These were all upper division students working toward a math major or minor.

The third class was a fall semester section of Calculus II, where I had about 20 students—a low number for my institution. These were mostly math and science majors, and primarily freshman and sophomores.The final class was Introductory Statistics, where I had two sections of about 25 students, one in the fall and one in the spring. These students were primarily not mathematics students, though there were a few mathematics minors and mathematics education majors. Many of these students took the course to satisfy their general education quantitative reasoning requirement.

Though the overall grading scheme is quite different from traditional methods, I continue to assess students using quizzes and exams. Each question on every assessment is connected to one specific standard, and students can receive one of three marks: (M)eets Objective, (R)evise and Resubmit, or (N)ew Attempt Needed. If a student receives a mark of M (Meets objective) on a question, then they receive credit for achieving the learning objective, and do not need to attempt it again. A mark of R (Revise and Resubmit) means that the student's work demonstrates sufficient understanding of the concepts in the standard, but they need to correct minor errors to earn credit for the attempt. With an R, students can resubmit corrected solutions within one week to earn an M. Finally, a mark of N (New Attempt Needed) means they are missing some parts, need to clarify some details, or perhaps there are major errors/confusing communication. No credit is received for the attempt, but they can reattempt this standard, usually during office hours or preset times proctored by a TA. (Notice that the language is about progress!) Most of my classes will have 26-30 different standards, though many proponents of standards-based grading prefer fewer. Because I have so many standards, I only require students to meet each standard only once to receive credit. In the end, the final letter grade is determined by the proportion of Ms the student earned during the term.

Standards are typically tied to specific content in the course, but I have also used non-content standards. I've experimented with a few different ways to incentivize homework, and the most successful in my experience was to incorporate two homework standards. On homework, I will assign a mark of (I)ncomplete, (C)omplete, or (S)atisfactory—a grade of C meant that all assigned problems were attempted, while they earned S if all the problems were solved correctly. All assignments were marked with feedback, and revisions were allowed to earn a higher mark. To get the first homework standard, a student needed to earn a Complete or higher on 85% of homework assignments. To get the second, the student needed to receive a Satisfactory score on 85% of homework assignments.

Many instructors do not like to incorporate non-content based standards, and achieve a similar goal with more complex final grade rubrics. For example, a student might need to get 18/20 standards, *plus* satisfactory grades on 85% of homework assignments to get an A in the course. I am moving to that model in my current classes, but in getting started with SBG I chose to keep my classes as simple as possible.

2.1 Class Projects and Hybrid SBG

In the second semester of the Probability/Statistics sequence, I have students complete a project where they perform a statistical analysis on a data set of their own choosing. I decided to incorporate this by splitting the final grade into two parts: 85% of the final grade was determined directly from the proportion of standards the student achieved, and the final 15% from their grade on the project. Since the project already incorporates feedback loops with several rounds of revisions and reflection on feedback, I chose to grade it in a traditional manner. Though I did grade using points, I attempted as much as possible to give clear instructions and a detailed rubric on how I would score, especially on the portions related to group contributions and professionalism.

In a similar way, it's possible to use SBG in some parts of a course, with traditional grading in others. These hybrid methods are a great way to accommodate alternative grading in different contexts, such as courses that require a common final exam.

3 My Experiences

In my experience, SBG has been a great success in my courses! I made many mistakes along the way, and it certainly took a lot of work to get the system up and running, but overall the benefits are apparent.

3.1 What went well

Among the main benefits of SBG were clarity and consistency in my course design. To implement the standards, I was forced to look hard at my assessments and make sure that every question on assessments aligned with my objectives. This required a considerable amount of time (more on that in the next section!), but in the end my course became more coherent. I no longer asked questions about low-level concepts, and I eliminated a number of questions and topics that didn't align with my objectives. While my standards are not always as clear as I'd like, each semester they get a little bit better as I get more experience.

SBG seems to work well both for slow learners and quick learners. Students can learn at their own pace and to an extent can retake assessments at their convenience. The usual high achieving students still did well in the course, while other students—who might normally not be at the top of the class—saw how reassessments gave them an opportunity to persevere and improve their grade. I had more than a few students get no Ms on the first exam who were then able to catch up through retakes and revisions. On top of this, SBG is more equitable. With flexible deadlines, students with tough situations can make up work when they get a chance. It seems there are fewer unwritten rules that disadvantage first generation students who know less about how to play the game of college [2].

One criticism that is often leveled at SBG is that reassessments undermine the rigor of the course. Though there is a substantial discussion to be had about what 'rigor' really means [3], I found that, at least in my upper level classes, the quality of student work markedly improved. When a student realizes that they can no longer get partial credit for partial answers, they put a lot more effort into practicing until they start getting questions correct. For my part, I am able to ask more difficult questions, and am impressed at the ways in which I see my students raise the level of their work to meet the expectations.

In the end, the grade distributions in all of my courses were quite high, which is in line with other courses in the math department and the college as a whole. Because the number of students was so small, I can't make any big generalizations but I did notice the bimodal pattern that other SBG instructors have talked about: more students scored higher *and* more students scored lower [1, 5]. This seems to be because students in the middle understand exactly what is required of them to get a higher grade, and can put in the appropriate effort to get there. On the other hand, some students can't or won't put in the extra work to move past partial understanding.

3.2 Issues in Introductory Statistics

Though I observed an increase in the quality of student work in my upper division courses, there were a handful of students in my introductory statistics class who had issues raising the level of their work in this way. For whatever reason, they couldn't make the jump past partial understanding to earn credit in a lot of the standards. I don't have enough evidence to say anything concrete, but I would speculate that this is a combination of not being well prepared coming into the course plus inefficient study habits. In the future, I would try to mitigate this by providing more structure around retake attempts. One common practice in alternative grading is to require students to earn "retake tokens" to reattempt a standard. For example, a student might earn a token by submitting reflections about feedback they've gotten relating to the standard they want to reattempt, or by writing a short summary or study sheet. This promotes good study habits and encourages the students to engage with the feedback.

3.3 Other drawbacks and mistakes

Though I would call SBG an overwhelming success in my courses, it's not a perfect system. Even though I think the ideas behind alternative grading are sound, a poor implementation might be a bigger impediment to learning than traditional grading systems. From a student perspective, the biggest drawback is unfamiliarity. Initial research shows that SBG actually increases student stress at the beginning of the semester, though that decreases below the stress of their other classes as they become more familiar [7].

One of the main drawbacks for faculty is the amount of bookkeeping required. Scheduling reattempts and keeping track of which students have tried which problems can be cumbersome—not to mention the time required to create and maintain a large problem bank for each standard. On top of that, most learning management systems are not set up to keep track of grades for SBG. Many SBG practitioners, including myself, try to automate a lot of this process. For example, I have students use an online form to request a reattempt, where they choose which standards to attempt and suggest a few times that they want to come in for the attempt. These requests automatically populate a spreadsheet, as well as send an email to myself and the student. I then wrote a short script to read the student's retake information from my spreadsheet and typeset a unique quiz for them using problems that they have not yet seen.

It's especially important with SBG to set good boundaries for yourself and the students. Even when you try to show flexibility, it's possible to go too far to accommodate students. In my first attempts at SBG, I made the mistake of providing very few deadlines, creating a large amount of work for me and my students at the end of the term. While my deadlines are still flexible, I aim to provide a lot of intermediate deadlines throughout the semester for those students who need the external motivation—and to prevent overloading myself with grading. One practice I have found helpful is to allow students to request extensions on homework for any reason, but requiring them to submit a plan detailing how and when they will get the work done. This allows me to be flexible, but still requires the students to take responsibility for their work.

Another drawback of SBG for students is keeping track of their progress. Many learning management systems do not work well with alternative grading. When your LMS grade book can't be "hacked" for alternative grading, some instructors automate a spreadsheet to send students weekly email updates on their progress.

Finally, in my initial implementations of SBG I found that challenging the top students was more difficult. With my probability and calculus courses, I tried for the first time to incorporate a final exam. Beyond adding to the challenge of the course, I also wanted to promote some retention of the topics as these courses were in a sequence. Many students would get a standard on their first attempt, and didn't get to experience the reinforcing benefits of trying different problems and thinking about them in different ways.

The final exam was 10 questions from 10 different standards, and the final opportunity to earn Ms on learning objectives for which they previously had a score of N. On top of that, they had to earn Ms on at least 5/10 questions to maintain the grade they have in the course. A score of 4/10 would drop them one place on the scale, and 2 or fewer Ms will drop them 2 steps on the letter-grade scale—e.g., from a B+ to C-. No revisions or reattempts were available for the final exam.

I did allow each student to choose some of the standards for the final exam, as well as certain standards that they did not want to attempt. Of the ones that I chose, I always gave them learning objectives that they had not yet completed, unless they selected those as standards that they didn't want to attempt. Some students did experience a greater amount of stress with this requirement. Of course, my goal is not to remove stress entirely, but it seemed to bring back some of the extra test anxiety that SBG tries to remove. In the end, almost all students were able to improve or maintain their grade on the final exam.

4 Concluding Remarks

Alternative grading systems aim to rethink the traditional methods of grading that have been used in education for the last 100 years or so. These methods promote learning by using clearly defined standards and providing a structure of feedback loops that are characteristic of all learning. While there are some drawbacks and a large start up cost of time and effort—at least if you make the change wholesale—I have found the change to be overwhelmingly positive for my classroom and my students.

If you find SBG interesting, but think you might want a different perspective more aligned with your context, there are many resources for you. Many faculty publish their standards/syllabus—you don't have to reinvent the SBG wheel! My personal favorites are the website for The Grading Conference and blogs such as Grading for Growth. You can also find the syllabi for my courses mentioned in this article at the following link: https://drive.google.com/drive/folders/1rm-CkX_YP8VHAOL3Po0h_EmwZqGhpcJd?usp=drive_link.

It also helps to start small and simple. Simpler systems can be better for the students, as a complicated grading system adds to their cognitive load. If a complete overhaul of a course is out of the question, then perhaps working on aligning your assessments with clear standards, or allowing revisions or reattempts on certain assignments is a reasonable way to get started. Perhaps just allowing some flexibility on certain assignments could benefit some of your students.

In my view, standards-based grading provides a more accurate and fair assessment of student learning, emphasizing learning and growth over arbitrary scores. While it presents some challenges, the benefits of clear objectives, helpful feedback, and a focus on understanding make it a valuable approach in each of my different classes.

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Joyful Mathematics: Worship through Delight

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Abstract

In this paper, I characterize the response to the beauty and goodness of mathematics as joy. By providing several examples, I paint a picture of the wonder that mathematics can inspire and show how the response fits generally held notions of joy. I contend that the Christian ought to recognize that this joy comes from God and reflect it back as an act of worship. Finally, I discuss practical ways to train up joyful mathematicians in Christian higher education.

1 Introduction

Mathematicians have often claimed that mathematics is, or can be, beautiful. The rich literature on this topic has articulated the beauty of mathematics to a wide variety of audiences. Less has been written on how we ought to respond to those encounters with beauty and wonder in mathematics. The aim of this paper is to examine the emotional response to mathematical beauty, which I characterize as joy, and to propose that the Christian's response to that joy ought to be to direct it back towards its source, the Creator.

In the first section, I articulate my understanding of joy for the purposes of this paper. Next, I briefly discuss the nature of mathematics, as a deeper understanding of what it is is necessary to comprehend how it might inspire joy. Along with that I examine how mathematics can be understood as good and beautiful. After this I describe encounters with mathematical beauty and how they inspire wonder and awe. Then I examine the response to this wonder, looking at how joy comes from recognizing mathematics as good, and then contending that the next step for the Christian mathematician is to respond with worship. Finally, I outline ways in which we can bring our students into this mathematical joy, so that they might also direct it back to the Creator in praise.

2 Characterizing Joy

To begin, I wish to describe what I mean by joy in order to determine whether or not mathematics might bring joy. The intention is not to provide a comprehensive definition that will apply to all circumstances,[1] but to give a suggestion of what I mean by the term in this paper and also what I do not mean.

[1] For the reader interested in a more comprehensive analysis of joy, I recommend "Joy: a review of the literature and suggestions for future directions" [10] for a broad overview and *Joy and Human Flourishing: Essays on Theology, Culture, and the Good Life* [22] for a more theologically focused perspective.

Importantly, I do not mean a heightened feeling of pleasure or happiness. The average person's notion of joy is likely that it is more or less a synonym for happiness, though perhaps a more potent expression of it. Wikipedia describes joy as a state of being that gives rise to intense feelings of happiness, [12] and the Oxford English Dictionary includes a definition of joy that says it is "the feeling or state of being highly pleased or delighted." [11] However, the joy I am talking about is not a synonym for a potent experience of pleasure. It seems dubious that intense happiness would find itself amidst the fruit of the Spirit, as in Galatians 5:22-23. Certainly experiencing joy is pleasurable, but strongly felt pleasure alone is not sufficient to ascribe joy to the experience.

Theologian Miroslav Volf defines joy as "emotional attunement between the self and the world—usually a small portion of it—experienced as a blessing." He explains that joy relies on both subjective and objective criteria. When something positive happens,[2] the response is a joyful one. This can be tainted by refusing to acknowledge good things as good. If I perceive a gift as a bribe, for instance, I will not experience joy. On the other hand, joy is not entirely self-generated. Joy is a response to something external. [21]

This idea of joy as a response to something outside oneself is echoed by many other writers. Theologian Marianne Meye Thompson describes different types of joy, including "joy *because* of the good that one has or experiences." [20] C.S. Lewis writes, "Joy is never in our power and pleasure often is." [13] Thomas Aquinas claims that joy is pleasure that "follows reason", [1] as he suggests that animals may experience pleasure but not joy. We see that joy is not necessarily something we can control or force ourselves to feel. However, there is a sense that joy can be cultivated; Paul commands us to rejoice in Philippians 4:4, and this would be a strange command if joy was entirely outside our capacity to feel.

Philosopher Matthew Kuan Johnson gives a detailed overview of different facets of joy from several sources in his literature review. One aspect I want to highlight is the idea of transcendence. Johnson quotes Chris M. Meadows, who says, "In the midst of a joy experience one may feel she has transcended bounded space and or time" and "One... has the feeling that he is moving or has moved, soared, or passed beyond ordinary existence." Joy causes us to feel connected to something greater than ourselves. An important caveat is distinguishing this transcendence from ecstasy. According to Johnson, "Joy 'makes you more intensely you,' while ecstasy (or bliss) obliterates the self." In a joyous experience, we are responding to something good and beautiful external to ourselves in a way that feels like 'soaring' and yet is still grounded in reality. [10]

Joy is an experience of deep happiness that is more than just a passing feeling, [21] is intense but grounded in reality, [10] and is an ideally automatic response to some external good. [20] A command to see and respond to good things as good, although still challenging at times, is much more reasonable in light of Philippians and Galatians. This is the notion of joy I will be leaning on for this paper.

3 The Scope of Mathematics

Can solving the equation $x^2+3x+6=0$ really bring joy?[3] For those outside the field, the thought of mathematics and pleasure in the same sentence might make them squirm. I believe this is because

[2] Some examples that Volf cites are marriage celebrations or the birth of a child.

[3] Unlikely in this case, as this equation has no real solutions! Although anyone fond of the complex numbers may derive some joy in solving it.

many struggle to conceive of mathematics as more than arithmetic or solving an equation.[4] If asked to define mathematics, these would probably assert that it involves numbers, though perhaps some would begrudgingly acknowledge that geometry is a type of math for some reason. The presence of a single 'right' answer is often lurking in the background, even if not explicitly stated. To be fair, this is often the notion of mathematics that first attracts mathematicians to the field. Many of us are willing to acknowledge that we were first drawn to mathematics because we were 'good at' it. However, those who have had the privilege of making this subject our life's study have come to a greater appreciation for the scope and depth of mathematics.

A popular definition of mathematics is "the study of patterns." [4] These patterns manifest themselves in the laws of nature but are also present in human behavior and the structures and systems we have created. I characterize mathematics as the study of abstract structures and patterns which reflect, but are distinct from, the material world. A biologist may study the growth of a bacterial colony, while an economist looks at economic growth of a nation, but a mathematician is curious about the fundamental logic governing the abstract notion of 'growth' or 'increase,' which in turn may be applied to any situation. As a linguist studies language in its rawest form, so a mathematician studies the patterns of creation; though many mathematicians are also interested in applying their knowledge, much like a linguist with a passion for translation.

The prevalence of universal laws applying to both nature and man-made objects is likely due to the fact that we mimic the ultimate Creator in our own attempts at creation. As the famous quote popularly attributed to Galileo says, "Mathematics is the language with which God has written the universe." Those who study the philosophy of mathematics hold a myriad of views about the nature of what mathematics actually is. The book *Mathematics Through the Eyes of Faith* lays out two possibilities for a Christian understanding of the ontology of mathematics. One view, referred to as Christian Platonism, claims that at least some of mathematics, at minimum logic and the concept of number, are part of God's nature and therefore exist eternally in a Platonic-like fashion. The other view, which the book calls Christian mathematical empiricism, argues that the patterns of mathematics are part of creation—that God created and upholds these structures and laws. [9] These are not necessarily the only perspectives a Christian might hold, and I do not aim to take a position on this issue. I do not believe my claims in this paper rest on holding to one of these views in particular.

3.1 The Goodness of Mathematics

The question of what is "good" has been pondered by philosophers for millennia.[5] It is beyond the scope of this paper to truly tackle whether or not mathematics is good with the depth this question deserves. However, I claim that whether mathematics is created by God or a part of His nature, it comes from Him and therefore must possess some notion of intrinsic goodness. If one claims mathematics to be part of God's nature, saying that it must be good hardly requires further defense to a Christian audience. However, if one believes mathematics to be part of the created order, patterns and structures put in place by God but subject to Him, then it must possess the innate goodness declared over creation in Genesis 1.

Beyond this nearly universal goodness of creation, one can more specifically examine how mathe-

[4] Indeed, I have had many conversations about math research where I have been asked if I am just trying to come up with new equations or solve really complicated ones.

[5] See, for example, Plato's *Republic.* [15])

matics is good. It is widely acknowledged that mathematics is useful,[6] which is a type of goodness. From architectural and engineering wonders to lifesaving medical interventions to successfully sending a man to the moon, mathematics undergirds many of our greatest achievements, though I do not wish to claim that it deserves all the credit for every success of humanity. Indeed, despite the good things that have been accomplished with mathematics, its utility cuts both ways.

If mathematics is good as creation is good, it is vulnerable to the effects of the fall as creation is. As Christians, we hold in tandem that humans image the Creator and yet are in desperate need of salvation from our own fallenness. If we are capable of harm and destruction, then mathematics can also be wielded to do evil, despite its natural goodness. Even if we say that mathematics, at least to some extent, is part of God's nature, this does not preclude the possibility of using it for ill. People have used the Lord's name and words as weapons, so, by analogy, the application of mathematics is not automatically immune to similar misuses. If we give mathematics partial credit for humanity's greatest achievements, we must also acknowledge its part in our greatest failures.

Mathematics may be good in ways beyond just its utility. Francis Su points out several good virtues of mathematical study in his book *Mathematics For Human Flourishing*, among them truth, permanence, and even play. [19] Mathematics can be fun, which is also a type of goodness in its own way.[7] Beyond utility, though, one of the primary justifications of the goodness of mathematics lies in its aesthetic value, which I will consider in greater detail in the next section.

3.2 The Beauty of Mathematics

Above my desk hangs a banner; "Math is Beautiful" it proclaims to the world. I inherited this, along with my office, from the now professor emeritus of mathematics at Gordon College, Dick Stout, who was known on campus for his catchphrase "BMIB: Because Math is Beautiful." The idea of mathematics as beautiful is not new, though the explicit use of the term appears to have emerged in the 19th and 20th centuries. Henri Poincaré said in 1890, "A scientist worthy of the name, above all a mathematician, experiences in his work the same impression as an artist; his pleasure is as great and of the same nature." [8] Bertrand Russell wrote in 1919, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty." [17] In his famous 1940 work *A Mathematician's Apology*, G.H. Hardy claimed, "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way." [7]

Mathematicians and philosophers have spent a lot of time trying to categorize or characterize mathematical beauty by its attributes. An inexhaustive handful of characteristics of mathematical beauty are symmetry, elegance, power, and surprise. [5, 9] At its simplest level, mathematical beauty can be observed in mathematical art, such as the fractal in Figure 1. Su calls this "sensory beauty", [19] and one of its advantages is that it does not require much mathematical study in order to appreciate, unlike some other types of mathematical beauty. Some mathematicians have argued that one needs to understand mathematics in order to appreciate its beauty, [16] but I disagree. I think all humans have an innate appreciation for symmetry, order, structure, and patterns, which are frequently evident in sensory mathematical beauty, even to the untrained eye.

[6] Whether *all* of mathematics is useful is a potential point of contention. In fact, some mathematicians are almost proud of the uselessness of their work. [7] Here I am merely stating that mathematics, when taken as a whole, has utility, which seems difficult to debate.

[7] Connecting the theology of play to mathematical play is beyond the scope of this paper, but of interest to the author.

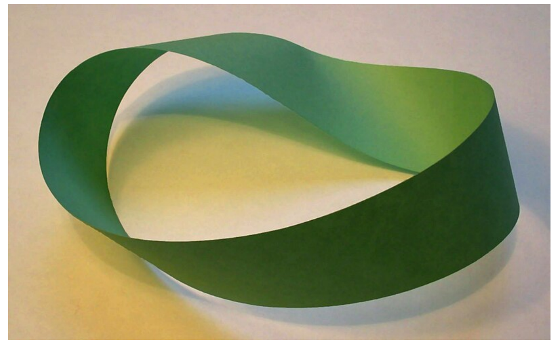

Figure 1: (Left) A fractal generated using free online software. [6] (Right) A Möbius strip. [2]

In addition to self-evidently beautiful mathematical art, mathematicians see beauty in elegant equations, such as $e^{i\pi} + 1 = 0$, in clever proofs, such as the classic proof that $\sqrt{2}$ is irrational, and in marvelous insights, such as the Möbius strip, shown in Figure 1. Experiencing mathematical beauty inspires a sense of awe and curiosity. A Möbius strip may be pleasing to the eye, but the true delight is felt with the knowledge that it is locally two-dimensional, and yet has only one side and hence no orientation. The equation $e^{i\pi} + 1 = 0$ is pleasant enough in its own right, but its beauty only grows when one observes that it contains arguably the five most important numbers in all of mathematics (0, 1, π, e, and i), three of the major operations (addition, multiplication, and exponentiation[8]), and the ubiquitous equals sign. As Carl Friedrich Gauss said in a letter to Sophie Germain in 1807, "The enchanting charms of this sublime science reveal themselves in all their beauty only to those who have the courage to go deeply into it." [3]

4 Encountering Mathematical Wonder

What happens when we encounter beautiful mathematics? What does it provoke in us? In this section, I will attempt to illustrate and explain the awe that comes with an experience of mathematical beauty and wonder.

In the fall of my junior year, I was fresh off a year of mathematical disillusionment. The previous year I had taken Calculus II, and as my students have heard from me many times, I despised it. It was the first time I had ever failed a math exam and it made me seriously question my commitment to majoring in math, despite choosing that path from a very young age. Regardless of these discouraging developments, I decided to persist and finally take Calculus III, multivariable calculus. In this course, one of the topics was quadric surfaces – equations that describe three dimensional shapes. Our professor used graphing software to help us visualize these shapes, and it featured an option to view them in stereoscopic 3D, provided one had the necessary 3D glasses. Given that this was the early 2010s and 3D movies were very popular, my professor was able to obtain enough for the class to experience this for ourselves. I will never forget my reaction to first viewing an ellipsoid (Figure 2) in 3D. I exclaimed, "You could find the equation of an egg?!" My mind was set alight with the way our world could be described mathematically. The image of the ellipsoid delighted me and captivated my imagination for what was possible with mathematics. That experience, along with the rest of that course, inspired me to continue pursuing a math major.

[8] An additional observation about these operations: multiplication can be described as repeated addition, and exponentiation can be described as repeated multiplication. It is amazing how many foundational ideas of mathematics one can find in this simple but true equation!

An experience of wonder in response to beauty can be extremely powerful when it follows a period of mathematical struggle. As any mathematician will tell you, much of researching mathematics involves long hours of frustrating confusion trying to make sense of a challenging problem. The moment of insight is beautiful as an oasis in a desert or turning on the lights in a dark room, as described by Andrew Wiles:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. You can see exactly where you were. At the beginning of September, I was sitting here at this desk, when suddenly, totally unexpectedly, I had this incredible revelation. It was the most—the most important moment of my working life. [18]

Much of what I have said so far can likely be applied to any field of study, not just mathematics. One of the things I think is distinctive, though perhaps not entirely unique, about mathematical wonder specifically is the connection to the transcendent. Su describes transcendent beauty in his book as beauty that "arises when one moves from the beauty of a specific kind of object...to a greater truth of some kind." [19] These experiences touch us deeply because they reveal something true about the world or, I believe, about God and cause us to feel connected to something greater than ourselves.

I remember the moment when my professor first told me that there were 'different sizes' of infinity. He said it as a casual comment, meant to provoke an undergraduate too comfortable in her understanding of the world. Later, in my analysis class, we would unpack the idea that the 'size' of the natural numbers is, in a sense, the same 'size' as the integers. Despite the appearance of doubling their quantity, the integers are still countably infinite! Most school children are comfortable with the idea that $2 \times \infty = \infty$, so this was easily swallowed, but the discomfort grew when we learned that the rational numbers are also countably infinite. However, the true transcendent beauty, in my opinion, arose with the discovery that the real numbers are uncountable, or *fundamentally more numerous* than the rational numbers. Together, these results paint a weird and wondrous picture

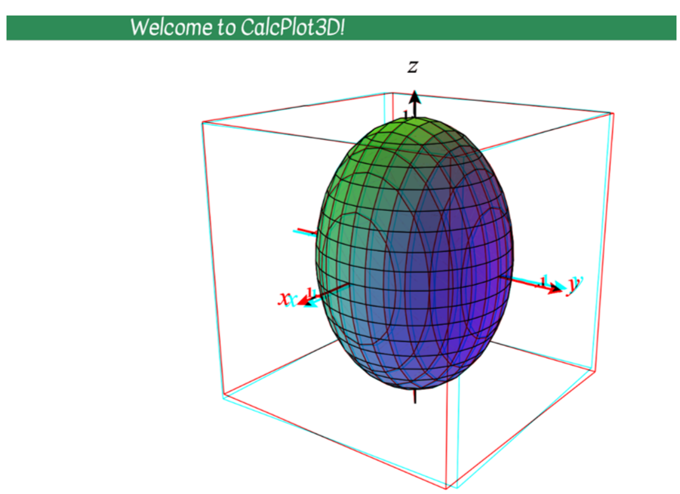

Figure 2: A screenshot of the graph of an ellipsoid generated using CalcPlot3D.

of what it means to be infinite. To this day, I delight in exposing my undergraduates to these seemingly paradoxical results and unpacking what I think it means for the believer. Mathematics points to a God who delights in structure and order, but with little counter-intuitive surprises sprinkled in. How wondrous indeed is an infinite God!

5 Responding to Mathematical Wonder

5.1 Mathematics Brings Joy

Having now witnessed the awe and wonder of an encounter with mathematical beauty, we set our sights on the primary question: can mathematics bring joy? I will remind the reader of our understanding of joy as a deep emotional response to a good thing independent of us. I believe that joy is the natural reaction to the goodness and beauty of mathematics. It seems apparent that these encounters with mathematical beauty cause us to experience pleasure, but I believe it is deeper than just powerful happiness. Mathematics can inspire wonder and a curiosity to know more. This leads to insight about the good ordering of creation at a fundamental level, which gives rise to rejoicing.

This progression, awe-inspiring beauty which leads to joy, is not found only in mathematics. When we see a work of art, read a poem, listen to a piece of music, or explore nature, we are often emotionally moved when we recognize something good and beautiful outside us. In the most profound experiences, perhaps we realize something true in a way we had not previously understood. Yet how many outside the field would include mathematical experiences in the above list? In the words of Bertrand Russell, "The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry." [17] The connection to some transcendent truth of the cosmos inevitably compels us to respond joyfully. Joy is a response to God's good ordering of the world, [20] both in the way He distinguishes good from evil and in the way He provides faithful and true structure as reflected in mathematics.

5.2 Worship as an Outpouring of Joy

I have one more story about an awe-inspiring experience in my undergraduate years and the joy that followed. Part of the core curriculum at my university was an introductory theology course. I was fortunate to be in an honors program that provided smaller, more discussion-focused sections of core courses that really elevated the experience of this course for me. We systematically studied the historic doctrines of the faith, from the trinity to the hypostatic union and much more. I cannot recall with the same clarity which of these topics inspired me, but I remember that same transcendent and wondrous feeling of joy that followed a new revelation. I immediately felt compelled to praise God for how marvelous He was and thank Him for unveiling this truth about Himself to me.

We were created for worship. When we experience joy over good things, it is part of our nature that we respond with worship. As religious scholar Charles Mathewes puts it, "The joyful act of praising God - a thankfulness flowing almost automatically from recognition of God's gifts - is the central action of the human." [14] We worship God for His attributes, such as righteousness, faithfulness, and justice, but I believe another way to praise God is to direct back towards Him our enjoyment of His good gifts. Insomuch as we are moved to thank the Creator when we experience the pleasures of creation, it is worship.

The Westminster Shorter Catechism states that "Man's chief end is to glorify God and to enjoy Him forever." [23] I think there is a misconception in some Christian circles of heaven as disembodied souls singing Sunday morning worship songs for eternity. Based on the goodness spoken over creation in Genesis 1 and the prophesied new heaven *and* new earth in Revelation 21, I believe we will glorify God and enjoy Him, at least in part, by enjoying the creation He has made.

The beauty and wonder of mathematics which unveils the good ordering of creation leads to joy. The acknowledgement and recognition of this goodness ought to compel us to direct our joy to God in praise. For in truth, it is not solely the mathematics that brings us joy, but the way in which God has structured His creation. Mathematics testifies to the glory of God, the faithfulness of God, and the goodness of God. The telos of the Christian mathematician is then to praise God for mathematics and the truths that are revealed by it.

The caution here is to be wary of idolatry. Our fallen nature bends us to elevate the created above the Creator. We need to be clear that we do not worship mathematics itself, and we ought to be vigilant against this temptation. With the rise of secularism and atheism in the modern era, some mathematicians have fallen prey to this impulse.[9] These still see the wonder and beauty of mathematics and are drawn to it, but they are unable to discern the source of its glory. The redemption of mathematics that the Christian scholar participates in is the explicit recognition and acknowledgment of who deserves glory for its wonder.

6 Forming Joyful Mathematicians

We are relational beings. Despite any stereotypes about mathematicians and our obsession with shoes,[10] as Christians we are called not only to worship the Creator but to do so corporately and to bring the gospel to those outside the Church. One could envision a way to use joyful mathematics to share Christ with others by pointing out the Creator whom mathematics foreshadows. However, in this section I want to focus on the task of those who find themselves in Christian higher education, namely to bring this joy to our students and remind them to Whom that glory is owed. To that end, I will outline some practical ways to catechize students into joyful mathematicians.

First of all, our students are unlikely to be joyful mathematicians if we are not ourselves joyful mathematicians. When something becomes routine, it can be easy to lose sight of the wonder that inspired that joy in the first place. If we find ourselves becoming desensitized to the joy that mathematics brings, we ought to seek out opportunities to cultivate it. I remember feeling particularly burnt out in graduate school and then attending a talk that reawakened my love of mathematics.[11] Whether it is learning about a new topic, rediscovering a prior passion, or working on a fascinating problem, we must find ways to maintain that sense of wonder for ourselves before we can bring students into this joy.

Some of the joy that we experience from mathematics may be naturally infectious to our students, though this may depend on the disposition of the educator. We cannot force our students to

[9] Still another temptation of postmodernism is removing the transcendence from mathematics altogether. Claiming mathematics to be a *fully* human activity, as some social constructivist philosophies do, will likely either lead to downplaying the transcendent or elevating humanity to divine-like status.

[10] There is a joke about how an extroverted mathematician is one who looks at someone else's shoes. At the risk of ruining the joke by explaining it, the implication is that introverted mathematicians spend their time staring at their own shoes.

[11] I cannot remember the precise topic, but I do remember the sensation of renewed passion for mathematics.

experience joy in anything, much less mathematics, but we can provide opportunities, like showing them beautiful mathematics and remarking on its beauty. Just as with art or literature, some universal examples are appropriate, such as the handful I mentioned earlier in this paper, but I also believe that personal examples contain a lot of power. Many students will be inspired by an instructor's individual encounters with joyful mathematics, so I believe that we should not skip over beautiful mathematics to make space for the purely practical. It is important to pause and give space to experience the wonder. Joy cannot be rushed.

I also recommend giving students opportunities to share their own experiences of mathematical beauty. In a freshman seminar course for mathematics majors that I taught recently, I had students do a short reading on beauty in mathematics, and then we discussed it during the following class. I asked them to give examples of mathematics that they thought were beautiful. I was surprised and delighted by the examples that they came up with, such as the equation $\sin^2 x + \cos^2 x = 1$. Because these students were earlier in their mathematical journey, they were able to identify examples of elegance that had become ordinary to me, which was a blessing. At the end of the semester, roughly half the class said that the day about beauty was their favorite of the course. The understanding of mathematics as beautiful is often the first step towards experiencing mathematical joy, so we should not be afraid to engage questions of beauty with students early on.

It is also vital to encourage our students that their inevitable struggles with mathematics can lead to even greater joy. So many students walk away from mathematical study because it is hard and they see themselves as incapable of overcoming. In many cases, I wonder if they do not see the end result as worth the difficulty. Particularly for those students who do not see themselves as 'math people', I think it is important to give them a new vision for what mathematics really is. I had a student who wrestled with severe math anxiety but absolutely loved fractals. The student did not really view fractals as mathematics, so I pushed back. My goal was to help the student recontextualize what mathematics was in order to see themselves as a math person. I have never met a mathematician that loves every single branch of mathematics, so I try to remind my students that it is okay to not like some parts of math. The key is to find the piece that gives them joy.

Finally, once students have experienced the joy of mathematics, the goal of the Christian educator is to call attention to the source of that joy. We ought to remind students where to direct their affections by giving God thanks for beautiful mathematics and remarking on aspects of his character that are revealed through the study of mathematics. We are created for worship and the greatest gift we can give our students is to point them back towards our Creator and Savior. Compared to the wonder of salvation, mathematics is just one small way in which He is worthy of praise, but especially for those to whom He has granted mathematical affection, we should never forget that He is the Giver of every good thing.

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Book Reviews

Reuben Hersh's *What is Mathematics, Really?*

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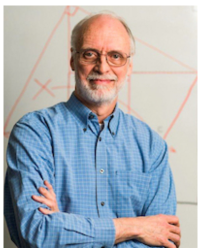

Cal's professional interests include the history and philosophy of mathematics and logic. Springer published his textbook *Introduction to Discrete Mathematics via Logic and Proof* in 2019. An essay recapitulating and updating his dissertation on Whately's revival of logic is the lead chapter in *Aristotle's Syllogism and the Creation of Modern Logic* (Bloomsbury Academic, 2023). He also maintains an annotated database on *Faith and the Mathematical Sciences*.

Abstract

Reuben Hersh's 1997 book *What is Mathematics, Really?* popularized a trend in the philosophy of mathematics that was gaining some traction at the time it was written. This essay examines Hersh's work within the broader historical context of 19^{th}- and 20^{th}-century developments in mathematics and philosophy of mathematics. It also focuses on Hersh's antagonism toward the influence of religion on (philosophy of) mathematics, concluding by briefly considering two positions taken by Christian mathematicians associated with ACMS in defense of such a connection.

1 What is Mathematics? A Twentieth-Century Perspective

Nineteenth- and early twentieth-century developments in mathematics exhibit a significant shift toward rigor, abstraction, and foundational interests: the arithmetization program in analysis begun by Cauchy and Bolzano around 1820 and carried forward by Weierstrass, Cauchy, and Dedekind half a century later to replace non-rigorous geometrical and infinitesimal bases for calculus; the rise of non-Euclidean geometry in the work of Gauss, Lobachevsky, and Bolyai before 1830, with its eventual acceptance after the contributions of Riemann, Beltrami, Klein, and Poincaré years later; the embrace of logic's importance and connection to mathematics in the diverse mid- to late-nineteenth-century approaches of Boole, De Morgan, Peirce, Schröder, Frege, and Russell; the increasingly abstract algebraic ideas from the late eighteenth century into the early twentieth century in the works of Lagrange, Gauss, Galois, Klein, Dedekind, Weber, Hilbert, Artin, and Emmy Noether; the independent abstract approaches to elementary algebra and number-system expansions in the mid- to late-nineteenth century by Peacock, De Morgan, Hamilton, and Cayley in Great Britain, Benjamin Peirce in the United States, and Dedekind and Peano on the Continent; the focus on rigor and the metalogical properties of geometry around the end of the nineteenth century by Pasch, Hilbert, and Pieri; and the creation of transfinite set theory in the last quarter of the nineteenth century by Cantor and Dedekind, with its subsequent axiomatization and use in the twentieth century as the theoretical foundation of mathematics by Zermelo, Fraenkel, von Neumann, and Bourbaki. All these developments, while far from exhausting mathematicians' concerns, imported novel ideas and methods into mathematics.

It is not surprising, therefore, that early twentieth-century philosophy of mathematics addressed such heady matters. The meticulous outworking of the logicist perspective by Frege, Russell, and

Whitehead; the metalogical pursuits of American postulate theorists such as Veblen and Huntington in axiomatizing mathematical theories; the metamathematical campaign of Hilbert to confirm the consistency of mathematics by means of proof theory; and the intuitionist counter-programming by Brouwer, Weyl, and Heyting—all raised mathematicians' keen awareness of and interest in foundational subjects. Gödel's *Completeness Theorem* for *First Order Logi*c (1929) and especially his unexpected *Incompleteness Theorems* (1931) for axiomatic theories that could support arithmetic signaled the promising potential of pursuing metalogical concerns related to mathematical theories, even while they seemed to deliver a damaging blow to Hilbert's formalist agenda. Gödel's 1940 results establishing the consistency of both the *Axiom of Choice* and the *Generalized Continuum Hypothesis* and Cohen's 1963 proof that these results are independent of *Zermelo-Fraenkel Set Theory* were only two further high profile results in the foundations of mathematics. While foundations of mathematics was being converted mid-century into a rather technical field of mathematical logic separable from philosophically grounded foundational programs, philosophy of mathematics understandably continued to be influenced by and attend to such matters.

During the same time, in addition to foundational advances, progress in specialized subfields of mathematics continued apace—in algebra, analysis, topology, logic, geometry, and other areas. A few authors popularized mathematics to give the general public some idea of all that was happening in the field and describe what mathematicians actually do. Hogben's 1936 *Mathematics for the Million* [14] gave a widely read elementary exposition of the main areas of school mathematics, placing mathematics within the broader context of civilization. Appearing five years later, Courant and Robbins' *What is Mathematics? An Elementary Approach to Ideas and Methods* [4] was more narrowly ambitious, pitched at a higher level and pursuing mathematical topics in more theoretical depth. It sought to capture the essence of modern mathematics as experienced by mathematicians, paying attention to intuition and constructive elements as well as axiomatics and deductive proof. And in 1981 Davis and Hersh published *The Mathematical Experience* [5], aimed originally at university-level mathematics students. Their book made no attempt to give a unified systematic treatment of the main fields of mathematics; rather, it presented impressionistic vignettes *about* mathematics based on the authors' experience by showcasing a wide variety of internal and external issues of interest to practicing mathematicians. It also included a few topics related to foundations of mathematics. Hersh's *What is Mathematics, Really?* [10], the book under review here, is a sequel of sorts to these last two works, picking up central themes in the philosophy of mathematics that had emerged in the interim. We'll look at these briefly before turning to the book.

2 Twentieth-Century Trends in Philosophy of Mathematics

The twentieth century saw a variety of challenges to the three main schools of philosophy of mathematics.[1] We just mentioned the impact that Gödel's stunning incompleteness results had on formalism. Intuitionism, which formalism had optimistically hoped to contravene by demonstrating that the non-finitary portions of mathematics were logically benign, had its own issues and detractors: many mathematicians found its doctrines too restrictive because it eschewed certain infinite collections and rejected key proof strategies involving negation and existence.

The remaining option, logicism, had also met with serious difficulties. Already at the turn of the century Bertrand Russell had discovered that Frege's derivation of arithmetic from logic was fatally flawed. While Russell, assisted by Whitehead, made prodigious efforts over the next two decades

[1] For an illuminating appraisal of these three schools, see [6]. As an aside, Hersh botches this bibliographic reference in [10], fusing its data with that of works by the nominalist Nelson Goodman and the historian Judith Grabiner.

to salvage the logicist program, in the end he admitted dissatisfaction with their own attempt since *Principia Mathematica* required adopting axioms like the *Axiom of Infinity*, which couldn't be justified as part of logic, yet were still needed to generate the theory.

By the 1930s, many mathematicians began to coalesce around a set-theoretic alternative to the three foundational schools.[2] Zermelo's axiomatization of Cantor's *Set Theory*, offered in the first place to substantiate his proof of the *Well Ordering Theorem* (1908), also explicitly fenced off known contradictions like *Russell's Paradox*. It, too, requires an *Axiom of Infinity* for developing *Peano Arithmetic*, but by abandoning the stringent expectation that set-theoretical axioms be self-evidently logical, this approach seemed to offer a way out of the foundational quagmire. *Set Theory* came to be largely accepted as an appropriate logical foundation for arithmetic, even as the gold standard for foundations of mathematics in general. *Set Theory* was adopted by Bourbaki in 1939 as the basic framework and language for developing theories of mathematics, and when *New Math* arrived on the scene around 1960, *Set Theory* was proposed as the conceptual basis for a streamlined unified approach to mathematics curricula across all levels. Most American research mathematicians and educators believed this was crucial for propelling the United States forward in the space race against the Soviet Union. But, generally speaking, practicing mathematicians (even Bourbaki) felt that once *Set Theory* had been granted privileged foundational status, they could safely ignore its niceties (and those of logic, foundations, and philosophy) as irrelevant to their interests—unless, of course, mathematical logic or *Set Theory* happened to be their speciality.

This didn't mean, however, that philosophy or foundations of mathematics had been settled to everyone's satisfaction. By mid-century, philosophical viewpoints began to unspool somewhat. The intellectual landscape is too complex to summarize briefly[3], but a few developments can provide the proximate philosophical background to Hersh's 1997 book. Some of these focus on traditional foundational matters, others introduce new themes for consideration.

Paul Benacerraf's seminal 1965 paper "What Numbers Could Not Be" [1] posed a significant challenge to set-theoretical foundations. It argued, based on the divergent accounts given by Ernst Zermelo and John von Neumann for treating natural numbers and their successors as sets, that neither approach should be taken as the approved way to conceptualize counting numbers since they lead to opposing conclusions on matters as elementary as whether 3 belongs to 17 or not.[4] Indeed, he found no convincing philosophical reason to identify numbers as sets at all, even if doing so would provide a way to set-theoretically model ordinary arithmetic. As an alternative, he suggested simply conceiving of the system of natural numbers as the abstract structure of any infinite recursive progression satisfying the *Peano Postulates*. This extended the core algebraic tendency to view formal axioms as implicit definitions (e.g., a group is whatever the group axioms require such a structure to be) to those structures with an intended (categorically specified) model. Benacerraf's perspective led later thinkers to develop the philosophical position known as *mathematical structuralism*, in which no consideration is given to the particular objects inhabiting a structure but only to their position within the structure and their relations to one another.[5]

[2]Set-theoretical foundations, however, carries forward tendencies of both formalism and logicism: sets can be considered the extensional content of logical concepts, and axiomatizing set theory fits a formalist approach.

[3]For instance, note the size and cost of Springer's 2024 *Handbook of the History and Philosophy of Mathematical Practice* [8], which is weighted (around 12 pounds worth) mostly toward one such trend, that in synch with Hersh's approach: 3300 pages, $1400.

[4]Benacerraf says that the answer is "no" for Johnny and "yes" for Ernie, though he confuses their positions on *successors* (maybe because von Neumann's definition was eventually adopted as standard in ZF *Set Theory*?), so their purported answers to whether $3 \in 17$ should be switched.

[5]Versions of this viewpoint are developed by Resnik [23] and Shapiro [24]. Shapiro [25] gives an overview of

Benacerraf's critique also applied to the extensional version of the logicist definition of natural numbers as equivalence classes of classes in one-to-one correspondence with one another.[6] But in 1983 Crispin Wright proposed a reformulation of Frege's logicism [28], introducing a second-order axiom known as *Hume's Principle* to replace Frege's problematic *Axiom V*, which had given rise to *Russell's Paradox*. Wright's work initiated a revival known as *neo-logicism*, giving new life to a logic-based foundation for arithmetic.[7]

Intuitionism also continued to exert a modest influence on foundational work. A central concern of Brouwer and his followers was the issue of mathematical existence: to assert the existence of a given object, one must construct it, not just show that its non-existence is contradictory. While this direct approach was too restrictive for many, Errett Bishop undertook to see just how much of and in what way classical analysis might be recovered using constructive methods. His 1967 work *Foundations of Constructive Analysis* [2] initiated investigations along these lines,[8] though mathematicians unconcerned with foundations could largely ignore these developments.

Philosophy of mathematics also veered off into some less trodden paths in the last quarter of the twentieth century. This trend, whose loosely affiliated contributors wanted to make the *practice* of mathematics an integral focus, found the concern for attaining secure foundations for rigorous deductive theories to be misguided.

Imre Lakatos's *Proofs and Refutations* [18], originally published in 1963-4 as a series of articles, used the case study of *Euler's Polyhedral Theorem*, $V + F - E = 2$, to argue that mathematical ideas and theorems dynamically evolve over time by modification and refinement rather than by axiomatization and accumulation, in ways unaccounted for by formalistic philosophy of mathematics. This work challenged traditional views on the nature of mathematics and made a deep impression on Hersh and others.

Philip Kitcher's 1983 *The Nature of Mathematical Knowledge* [17] was an ambitious attempt to formulate an alternative ("maverick") empiricist philosophy of mathematics in opposition to the reigning Platonistic perspective held by most mathematicians. Kitcher focused on how mathematical knowledge is acquired (the "logic of discovery") and historically developed over time. Mathematical concepts and theories, Kitcher believed, are not given *a priori* but originate in ordinary human quantitative operations (collecting, ordering, counting) and spatial experiences that are then idealized, generalized, proved, and tested through education and other social interactions. In this respect, he held, the practice and methods of mathematics are similar to those of other rational human endeavors, such as natural science. The mathematics of any given time exhibits a tacit commitment to a given *practice*, consisting of a technical language L, a set of metamathematical viewpoints M, a set of questions Q, a set of reasoning strategies R, and a set of statements S. The historical development of mathematics can then be seen as a rational transition from one practice $< L, M, Q, R, S >$ to another. In this way Kitcher, too, no longer focuses on a static collection of indubitable truths and sound proof procedures requiring a stable theoretical foundation but gives credence to more flexible aspects of everyday human mathematical activity.

With the focus shifting toward mathematics-in-the-making as created by human actors, philosophy

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mathematical structuralism.

[6] In fact, this approach succumbs to problems from the outset since the number 1, being the class of all singletons, would be at least as numerous as the notorious set of all sets: injectively match each set S with its singleton $\{S\}$.

[7] For later work in this tradition, see [7].

[8] See also [27], which covers Brouwer's intuitionism as well as Bishop's constructivism and a version focused on algorithms developed in Russia by Andrei Markov.

of mathematics could now acknowledge mistaken viewpoints, inadequate concepts and axioms, faulty proofs, etc. In other words, mathematics, like other human activities, had a subjective component and was not infallible. This philosophical approach also valued the insights contributed by history of mathematics and mathematics education: it encouraged the exploration of social and cultural factors that had been deemed either irrelevant or heretical by more traditional philosophies of mathematics.

The new interdisciplinary post-foundationalist perspectives set in motion by Lakatos, Kitcher, and others were carried forward in a number of later publications, still appearing today.[9] Tymoczko's 1986 *New Directions in the Philosophy of Mathematics* [20] was a beacon heralding the changing focus. An anthology of readings by well-known mathematicians, philosophers, and historians, it challenged foundational dogmas and argued that philosophy of mathematics should focus more comprehensively on mathematics as actually practiced. The book's lead essay [9] by Reuben Hersh set out a program for reviving philosophy of mathematics largely along the lines suggested by Lakatos. It presaged the acclaimed work he would write a decade later, his *What is Mathematics, Really?* [10], to which we now turn.

3 Mathematics: A Platonic or Social-Cultural-Historic Enterprise?

Hersh's book is a manifesto advocating a humanist philosophy of mathematics. It attacks the widely held foundational outlooks of Platonism and formalism and argues for conceptualizing mathematics instead as a social-cultural activity embedded in history. Hersh makes his case by first laying out conditions that a genuine philosophy of mathematics should satisfy. He then, at length, describes and evaluates the opposing perspectives of mainstream philosophers of mathematics and those he labels as mavericks and humanists, concluding that the latter's viewpoint is more faithful to mathematical practice. His treatment aims "to be entertaining and relevant, not definitive or exhaustive" (91), assuming "no mathematical or philosophical prerequisites" (xii) of his readers. Readers do encounter the scholarly opinions of experts on a range of topics, but Hersh often lets them speak for themselves, stitching together a diverse patchwork of lengthy quotes.

Let's begin by looking at the contrast he sets up. Hersh characterizes contemporary mathematicians as philosophical schizophrenics. While doing research, ordinary mathematicians tend to be naive Platonists, believing they are investigating an independently existing objective domain of abstract mathematical entities having real properties and relations. When pressed by students or outsiders to elaborate on or validate this belief, however, to explain where such objects are located and how one gains epistemic access to them, they tend to retreat into the bastion of formalism, where mathematics becomes a meaningless rule-based game played with formal symbols. They then claim they're merely working within an abstract theory based on an axiomatic system containing professionally accepted definitions and postulates, which theory may or may not have any application that goes beyond the symbolism. Such a shifting compartmentalized philosophical stance may be convenient, Hersh notes, but it obviously lacks conceptual coherence and integrity.

Hersh traces foundationalist philosophy of mathematics back to ancient Greece. A Platonic philosophy of mathematics joined to Euclidean axiomatics has dominated Western mathematical thinking since the time of Pythagoras and Plato. Mathematical objects such as numbers and shapes were taken to exist in some autonomous realm of forms; everyday marketplace measures and drawn figures are impure approximations of these mathematical realities. Pure mathematics is a science

[9] A sampling of these can be found in [8], [13], [16], [20], and [22].

deductively grounded in self-evident truths about ideal mathematical objects.

Medieval Christians were receptive to a neo-Platonic outlook, locating mathematical objects and truths in the mind of God. This continued the veneration of mathematics: it was an intrinsic and necessary part of God's eternal wisdom. God used mathematics, they believed, to structure the universe, providing creation with order, coherence, and stability. Humans, created in God's image, were able to think God's thoughts after him, and so gradually uncover the world's architectonic constitution. From Augustine and Aquinas to Kepler and Leibniz and Euler and on into the nineteenth century, European mathematicians managed to accommodate their philosophy of mathematics to deeply held theological beliefs. Mathematics was the model of absolutely certain knowledge, demonstrating humans' ability to gain access to infallible truths. This buttressed traditional Christian faith, which was based on spiritual truths revealed in Scripture.

The revolutionary nineteenth-century developments noted above were difficult to square with this outlook, however. The foundationalist philosophies that arose to deal with such problematic matters became unmoored in the twentieth century with the discovery of contradictory and undecidable results in Cantorian *Set Theory*. Philosophical programs—logicism, intuitionism, formalism, set-theoretic Platonism, neo-logicism—sought to address these crises, but many mathematicians found their solutions inadequate and judged them largely irrelevant to ordinary mathematical practice.

Regardless of their differing philosophical outlooks, all foundational programs shared a common inspiration and aim: to make mathematics absolutely secure by placing rigorous arguments upon an indubitable foundation unassailable by further discoveries. It is this goal that Hersh finds to be wrong-headed. In practice, mathematicians often encounter roadblocks, fruitless paths, misconceptions, and fallacious arguments as well as hazy intuitions and sketchy proofs as they work toward obtaining a significant correct result. Mathematics, while eventually self-correcting and virtually reliable, should be acknowledged to be a fallible human endeavor overall. Having developed professionally agreed-upon standards, mathematics doesn't need absolutely certain foundations any more than physics does.

Hersh therefore opts for a philosophy that explicitly takes daily mathematical practice as a pertinent given, underscoring the fact that mathematics is done by humans who learn it first through experience and education and who socially interact with one another professionally in developing their shared ideas. This perspective, which Hersh calls a humanist social-cultural-historic philosophy, incorporates subjective factors and takes into account educational theories, history of mathematics, philosophy of science, and sociology. Mathematical entities are conceptual objects that build upon previously accepted ideas, evolving over time through intersubjective social negotiation by experts in the field. In the end they're based on fallible human intuitions sharpened by scientific knowledge and everyday needs. Once created, these objects exist for all to consider and expand upon further. As communal knowledge, they have objective features that can be explored and discovered and connected with one another through hypotheses, counterexamples, informal arguments, deductive proofs, and axiomatics (19).

This humanist philosophy of mathematics, while not mainstream, also has a venerable history, originating in the philosophy of Aristotle. Aristotle analyzed mathematical notions like numbers and shapes as arising through humans abstracting quantitative aspects from their daily experience with physical reality. He believed science (i.e., knowledge) proceeded by intuiting self-evident first principles about a given subject matter and then systematically deducing consequences from them. Euclid's *Elements* of the next generation developed geometry and number theory in a way that was consistent with Aristotle's theory of knowledge, though it expressed no philosophical preferences.

After Augustine, however, Aristotle's approach was eclipsed in the Latin West by Platonic philosophy, which became strongly wedded with Christian notions of God. Consequently, an empirically based philosophy of mathematics wasn't able to take root until the nineteenth and twentieth centuries, when a secular worldview had become more widespread. Even then, it failed to gain much traction. John Stuart Mill's view of arithmetic is perhaps the classic example of an empiricist philosophy of mathematics, but it was dismantled by Frege's stinging critique in 1884. It was only in the twentieth century, when philosophers began more seriously to question how one might access the world of abstract mathematical ideas if not empirically grounded in experience that an alternative humanist philosophy of mathematics began to take shape. For Hersh, Imre Lakatos is the key thinker here, though he also draws upon the work of George Polya, Karl Popper, and Philip Kitcher, among others.[10]

Lakatos and Kitcher provide Hersh with a modern empirical perspective on how mathematics develops, aspects of which can be traced back to Aristotle. But mathematics does not unfold cumulatively by logical derivations from a set of unchallenged axioms, as Aristotle's epistemology held. It grows instead through a dialectical process of criticism and refinement (210), occasionally revolution, using deduction and counterexamples to unfold its implications. Popper's contribution to Hersh's viewpoint was his proposal that reality extends beyond the traditional dualistic categories of mind and matter to include an intelligible third world of ideas where mathematical objects reside (220ff.). Hersh transforms Popper's notion by arguing that mathematics is generated as a shared social-cultural realm of conceptual objects (ideas) that have science-like qualities and can be lawfully ordered (13). Such ideas can be accessed by anyone with the appropriate background and training; their impact on further thinking and on real-world developments is due to their meaning.

With his humanist social-cultural-historic philosophy of mathematics Hersh believes he has met the conditions he had posed for shaping a respectable philosophy of mathematics. In particular, it has breadth, it is consistent with contemporary philosophy of science and epistemology, and above all, it's faithful to how humans pursue and communicate mathematics. And it excises things considered extraneous to genuine mathematical interests, such as a transcendent Platonic realm for mathematical objects, infallible theories, indubitable foundations, and any affiliation with religion.

4 Hersh's Humanistic Philosophy of Mathematics: An Assessment

A group of a dozen ACMS members, more or less, met as we were able over Zoom in ten weekly sessions during the past summer to discuss Hersh's *What is Mathematics, Really?*. Although the book has been around for over a quarter-century, not many of us had read the book, so we expected we could learn from it and one another about a controversial postmodern perspective on mathematics.[11] The book is heavily focused on history and philosophy of mathematics, but it is written in a fairly popular style, so we all thought we could interact with it without being specialists in those areas. Several of us offered to lead discussions of the different chapters, but everyone chimed in with their questions and reactions. Afterwards, since the book was close to my background and interests, I offered to review the book, contextualizing it and giving an informed assessment of Hersh's viewpoint.

[10]Hersh describes his perspectival development in [11], especially in "Chronology," pp. 21-34.

[11]This topic was explored in the 2001 book coauthored by ten ACMS members, *Mathematics in a Postmodern Age: A Christian Perspective* [19], but Hersh's book itself received only glancing notice there (e.g., p. 63, fn. 21).

Participants found the book thought-provoking and engaging—all in all, a worthwhile read. In its time, it provided a healthy stimulus coaxing philosophy of mathematics out of its narrow foundationalist groove, though at times using over-the-top rhetoric to make room for its new approach. We found ourselves open to considering Hersh's emphases on mathematical practice as a social activity, on the salience of history of mathematics, on acknowledging that mathematical ideas are acquired through human experience and education, and on valuing heuristics and informal reasoning.

We were in less agreement with respect to his views on the nature and existence of mathematical objects, on downplaying the achievements and merits of logical rigor, and on the relative unimportance of foundations. We also wondered why one couldn't accept some of Hersh's social-cultural-historic emphases without throwing out the proverbial baby with the bath water. Can't one expand philosophy of mathematics to take into account social and cultural context, historical developments, philosophy of science, and learning theories without jettisoning foundational and logical concerns with truth and certainty? Couldn't the aim of mathematics still be to ultimately arrive at rigorous arguments for theorems, to provide a solid axiomatic basis for theories, and to seek a unifying conceptual framework for mathematics? Does the philosophy of mathematical practice need to be a competing alternative vision of philosophy of mathematics? Why do the insights of these different perspectives need to be taken as mutually exclusive?[12]

We took even stronger issue with those central features of Hersh's philosophy that touch on ontology and religion. While recognizing that mathematical objects (numbers, shapes, functions, groups, etc.) aren't concrete things, don't they have a more objective reality than that countenanced by seeing them as transpersonal intersubjective ideas? Don't these ideas refer in the end to some independently given underlying mathematical reality built into the universe? Can't one admit human participation in creating mathematics without seeing it solely as a collective human invention? This humanistic perspective seemed to us to concede too much to human agency.

Related to this is the matter of mathematics' relation to religion, a paramount interest of ACMS members. Hersh refuses to grant any valid link between them. This is in line with his antagonism to anything supernatural—to a divinity, to the existence of a spiritual realm. God and religion had no place in his thinking as he matured,[13] and he seems particularly aggressive in denying any connection between mathematics and religion. Mathematics' historic association with religion made possible by a Platonic philosophy of mathematics is something, therefore, that Hersh is eager to dissolve; in fact, it disposes him to oppose both religion and Platonism. While Hersh believes in a multifaceted reality that goes beyond material and mental realms to include a social-cultural realm inhabited by mathematical objects and truths, he rejects the existence of spiritual realities, sharing a humanistic outlook on religion with both modernism and postmodernism. This feature alone disqualified Hersh's humanistic philosophy of mathematics for ACMS discussion participants.

ACMS members have taken two main positions on this issue.[14] One approach, as mentioned above, is to accept a version of Platonic philosophy of mathematics as Christianized by Augustinian and Thomistic views of God and theology. This position takes mathematical objects and truths as abstract realities located in the mind of God. This gives them semi-divine status as essential

[12] In fact, Hersh softens his viewpoint in later publications, taking a more relaxed pluralistic stance toward foundationalist philosophies (cf. [11], p. 26 and [12], pp. 21ff.). This echoes Goodman's position in [6], mentioned above in footnote 1, that the various foundationalist schools overemphasize different legitimate aspects of mathematics.

[13] God and the supernatural were absent in Hersh's upbringing. While his father had been educated as a religious Jew, he relinquished his faith upon moving to Palestine. See [26], pp. 1-2, 6-7.

[14] See the chapter on "Ontology" authored by Bradley, Howell, and Vander Meulen in *Mathematics through the Eyes of Faith* [3] for a more extended discussion of these alternatives.

parts of God's rational nature. Mathematicians are able to gain epistemic access to these things, according to this view, by being created in the image of God and having an ability, though limited, to think God's thoughts after him. This perspective provides a religious foundation for mathematics by grounding it in God's nature. Those who hold this view tend to see their defense of a Platonistic philosophy of mathematics as integral to their defense of a connection between mathematics, God, and religion.

A second position on how God and mathematics are related is more in line with Aristotle's insistence that mathematical knowledge derives from an empirical process of intentional abstraction of certain aspects from everyday human experience of created reality. Abstracted ideas are then idealized, generalized, and systematized into a body of interconnected ideas and propositions. This need not be coupled with a humanistic approach to philosophy of mathematics, however. The basic reality underlying these abstracted ideas is a multi-dimensional universe and human experience that has been created and structured by God. This viewpoint is more aligned with Cappadocian and Reformational views of God, focusing on divine agency rather than on God's nature and attributes, refusing to grant mathematics and logic any quasi-divine status. It sees the relation between God, mathematics, and logic differently, locating mathematical realities in temporal created reality instead of God's eternal nature. This viewpoint is more sympathetic to Hersh's advocacy of the philosophy of mathematical practice and to his rejection of a Platonistic ontology and epistemology, though not to his humanistic stance on ontology or religion.[15]

Notwithstanding discussion participants' opposition to Hersh's humanistic and anti-religious worldview, reading his book together expanded our understanding of a contemporary trend in philosophy of mathematics that was fairly new to many of us. Hersh is more of a popularizer than an academic philosopher here, but his status as a mathematical insider gave some credence to his treatment of the philosophical issues he raised. Those desiring to follow developments initiated by Hersh's proposal for a new way forward have a number of recent works they can consult, though more traditional approaches to philosophy of mathematics have certainly not receded into the shadows.[16]

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[16] While [8] comprehensively covers numerous aspects of the philosophy of mathematical practice, this trend receives scant mention in the lengthy *Stanford Encyclopedia* entry on philosophy of mathematics [15]. Penelope Maddy's naturalism, for instance, looks at foundational issues in a way that takes mathematical practice into account.

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A Review of *The Mathematical Experience*

Brandon Bate (Houghton University)

Brandon Bate is an Associate Professor of Mathematics at Houghton University where he occasionally teaches courses on the relationship between mathematics, philosophy and Christianity. He completed his MA in Mathematics at Boston College, and upon discovering that advanced mathematics wasn't as scary as he imagined, completed a PhD in Mathematics at Rutgers University where he specialized in number theory.

Abstract

In this paper, I review *The Mathematical Experience* by Philip J. Davis and Reuben Hersh. In this book, the authors address topics in the philosophical foundations of mathematics, the history of mathematics and other assorted topics related to the experience of doing mathematics. I've used this text in my own teaching and discuss in this review its strengths and weakness in this regard, as well as highlight portions of the text that I've found helpful for providing natural opportunities for faith integration.

Faculty at many Christian universities and colleges are encouraged to engage students with a view of their discipline that is informed and enriched by a Christian faith perspective. I myself work at such an institution (Houghton University) and enjoy having the opportunity to engage with students in this way. Personally, I have found that our senior capstone course and courses in the history and philosophy of mathematics provide a natural settings in which this can happen since these courses tend to focus on questions that probe the meaning, purpose and motivations for pursuing mathematics.

There are multiple excellent texts that help students engage with this type of material. In my own teaching, I've used both *Mathematics Through the Eyes of Faith* by James Bradley and Russell Howell [2] and *Mathematics: The Loss of Certainty* by Morris Kline [3]. The former introduces students to topics in the foundations and philosophy of mathematics from a distinctly Christian perspective. The latter is a secular work that presents a historical narrative for the development of mathematics and science that touches on the attendant philosophical controversies and engages with these matters with a surprisingly strong emphasis on the role Christianity played in this. As much as I value these books, I can only re-read a book so many times before finding myself itching for a different take on the subject. Consequently, when I was asked to teach an honors seminar on the history and philosophy of mathematics and science, I decided to try out *The Mathematical Experience* by Philip J. Davis and Reuben Hersh [1]. What follows is a review of this book with special emphasis on its suitability for use in Christian higher education, both for students majoring in mathematics and also for students who have had little college-level mathematics, as was the case with many students taking the aforementioned seminar.

The Mathematical Experience was birthed from the authors' own experiences teaching courses in the foundations of mathematics. Their approach to the subject is interesting. Although the text touches on topics related to the history and philosophy of mathematics, its ambition is broader, and because of that, difficult to articulate. Its character is perhaps best exposed by its structure. The book itself largely consists of self-contained, yet interrelated vignettes that reflect on some particular aspect of the mathematical experience. These are loosely organized into chapters. The text is at times formal and reserved, and in other instances free-flowing and eclectic. Examples of the latter sort are perhaps best exemplified by the following three interrelated vignettes:

- "The Ideal Mathematician" gives a comical fiction of the "ideal" mathematician. In this story, an all too stereotypical mathematician at a university receives a grant to continue his study of "non-Riemannian hypersquares," an esoteric topic that appears to have virtually no scientific applicability or interest beyond a select group of like-minded mathematicians. This mathematician is confront by a university public relations officer, a student and a philosopher, all of whom are attempting to understand what exactly this mathematician "does." All of his interlocutors are genuine and reasonable, yet none of them are able to grasp the mathematician's work. Through their initial earnestness and subsequent disappointment, we are given a glimpse of how utterly foreign the modern mathematical enterprise appears to outsiders.

- "A Physicists Looks at Mathematics" is based upon an interview the authors had with a materials scientist. The scientist is asked questions about his use of mathematics in his research, his views of science, and his own understanding of ontology and epistemology. Like the "ideal mathematician," this passage presents a stereotypical scientist who views mathematics as a means to an end, that end being the understanding of physical reality. He largely denies an interest in mathematics beyond that and appears to harbor a philosophical commitment that borders on materialism.

- "Confessions of Prep-School Math Teacher" is based on an interview with, as its title suggests, a prep-school math teacher. The teacher understands mathematics as a sort of beautiful game that is fun in itself; its appeal is largely independent of its real world applicability. He appears to wear as a badge of honor his disinterest in the intersection of mathematics with science and philosophy.

I've found that vignettes such as these provide ample opportunity for classroom discussion. Rather than giving authoritative, abstract descriptions of Platonism, materialism or formalism, they instead give flesh and blood viewpoints that are portrayed as "works in progress" and rarely fully align with well-defined philosophical commitments. They are human, and because of that, easier for students to engage with.

In addition to these playful vignettes, *The Mathematical Experience* provides more formal content. The chapter "From Certainty to Fallibility" gives introductions to Platonism, formalism and constructivism typically found in a foundations or philosophy of mathematics course, but it is a decidedly terse and at times dry account. However, other readings (in separate chapters) help to flesh out these concepts in more relatable ways. The aforementioned interviews can serve this purpose, but there are other vignettes beyond these. The "π and $\hat{\pi}$" article gives a classic example by Brouwer that argues for the constructivist paradigm. Likewise, the "Formalization," "Non-Euclidean Geometry," and "Non-Cantorian Set Theory" vignettes can serve to highlight the impetus behind formalism. Lastly, the "Riemann Hypothesis" vignette not only offers a window into the current state of mathematics research, but according to the authors, gives a justification for a Platonist outlook. But even with these additional resources, I've found that the overtly philosophical topics broached by *The Mathematical Experience* are often difficult for students to grasp. In my own experience, most students have found Kline's presentation easier to comprehend.

The chapter "Selected Topics in Mathematics" introduces various topics in mathematics that are, in theory, accessible to undergraduates but are traditionally taught at the graduate level. This is in part to showcase the current state of mathematics research (as of 1980) and in part to provide material that interfaces well with the philosophical content elsewhere in the book. These topics include the Classification of Finite Simple Groups, the Prime Number Theorem, Nonstandard Analysis, Fourier Analysis and the previously mentioned vignettes on Non-Euclidean Geometry and Non-Cantorian Set Theory. I found these readings to be very informative since they often went into greater depth than other introductory content I had read. This was especially the case for the material on Non-Cantorian Set Theory and Nonstandard Analysis. Nevertheless, as personally satisfying I've found this content to be, some of it is much too difficult for the typical undergraduate mathematics student. Instructor discretion is advised.

More accessible content is found in the chapter "Inner Issues." This material is suitable for all readers, including students who are not on the verge of completing a mathematics major. The vignettes contained within communicate the heart of the mathematical endeavor through easy to understand examples. Readings cover topics on abstraction, generalization, mathematical objects, proofs, formalism, infinity, aesthetics/beauty, randomness, chaos, probability and more. For instance, the "Abstraction" vignette walks the reader through the process of understanding a maze (see Figure 1) as a graph (see Figure 2) and then as an incidence matrix (see Figure 3). Vignette's such as these are especially helpful to the uninitiated as it gives some concrete material that helps them to make sense of the more abstract content in the book.

Figure 1: A maze with labeled locations

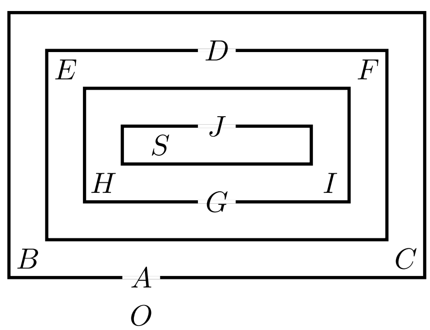
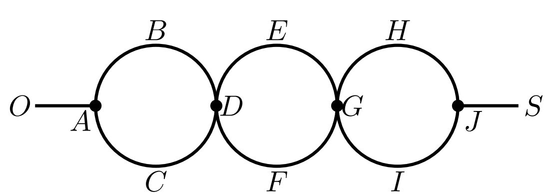


Figure 2: A graph representation of the maze.



The chapter "Teaching and Learning" gives a multifaceted view of pedagogy. The vignette "The Classic Classroom Crisis of Understanding and Pedagogy" gives an account of the struggles one of the authors had in communicating a slippery mathematical proof, his revisions and reattempts, and the social and emotional factors that come into play when learning and teaching mathematics. The vignette "The Creation of New Mathematics" shows in detail the process a group of students took in attempting to resolve an open-ended problem in number theory. In general, readings from this chapter highlight the mutual dependence that exists between learning, communication and advancement in mathematics.

Some of the most intriguing content of *The Mathematical Experience* comes in the chapter "Outer

Figure 3: An incident matrix for the graph.

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	1	0	2	0	0	0
C	0	2	0	2	0	0
D	0	0	2	0	2	0
E	0	0	0	2	0	1
F	0	0	0	0	1	0

Issues." The first half of the chapter explores questions related to the utility of mathematics. Here the authors remark that for many at the forefront of the scientific revolution, such as Newton and Leibniz, this was because "God is a Mathematician." This belief is folded into the Platonist view and contrasted with a semi-nominalist "mathematical model" view. The latter half of the chapter has the heading "Underneath the Fig Leaf." Here the authors explore aspects of mathematics that, as the heading suggests, is the cause of some embarrassment in the broader mathematics community. This includes the relationship of mathematics with mysticism, the occult and religion. In the vignette "Religion," we read of Hermann Weyl's belief that the desire to understand the infinite is tied with religious intuition. The authors expound upon this:

> Like mathematics, religions express relationships between man and the universe. Each religion seeks an ideal framework for man's life and lays down practices aimed at achieving this ideal. It elaborates a theology which declares the nature of God and the relationship between God and man. Insofar as mathematics pursues ideal knowledge and studies in the relationship between this ideal and the world as we find it, it has something in common with religion.

A brief review of concurring historical thinkers, including Nicolas of Cusa and Johann Kepler, follows. Similar sentiments can be found in the vignette "I.R. Shafarevitch and the New Neoplatonism" (located in a separate chapter). In speaking of the problems surrounding the existence of mathematical objects, the authors admit:

> Belief in a nonmaterial reality removes the paradox from the problem of mathematical existence, whether in the mind of God or in some more abstract and less personalized mode. If there is a realm of nonmaterial reality, then there is no difficulty in accepting the reality of mathematical objects which are simply one particular kind of nonmaterial object.

Also within the "Outer Issues" chapter, the vignette "Abstraction and Scholastic Theology" discusses how Saadia Gaon, a Jewish philosopher and theologian, sought to understand God through abstraction, and in so doing gives a proof for the uniqueness of God.

In summary, the "Outer Issues" chapter gives a natural opportunity to investigate interrelationships between mathematics and religious belief, albeit from a seemingly non-committal, agnostic viewpoint. In my own teaching, this material has provided opportunity to discuss motivations for Christians to pursue mathematics and discuss whether belief in God affords a means to rectify the seeming nonmateriality of mathematics with the functioning of our material universe.

The Mathematical Experience greatest strength is that it does not seek to establish a central thesis or put forth a historical narrative, but rather strives to describe many aspects of the mathematical experience that are too often neglected. Its weakness, when it comes to use in the classroom, is that portions of it are too difficult and technical for many students majoring in mathematics. I have yet to use *The Mathematical Experience* as the primary text for a course because of this, but I do regularly use it as secondary text. It remains a worthwhile resource in my own teaching, not simply because it dares to look at the mathematical experience beyond traditional historical and philosophical perspectives, but because through these explorations questions about the meaning and purpose of mathematics within the human experience naturally arise, along with an acknowledgment that religious belief and faith commitments are relevant when exploring such questions.

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—— Reports and Announcements ——

ACMS conferences conclude with a celebratory banquet. Depending on events that have unfolded since the previous conference, some of those in attendance may give homilies, tributes, and important announcements. The highlights from the banquet closing the conference at Dordt University included some updates by ACMS President Josh Wilkerson, and a project progress report on the *Bibliography of Christianity and Mathematics* by Cal Jongsma. Following the conference Dusty Wilson convened a summer reading group (meeting remotely), which he also led in previous years. His comments on those discussions are also included in this section.

Announcements by the ACMS President

Josh Wilkerson (Regents School of Austin)

1. The Brabenec Lecturer for 2024–2025 is Francis Su of Harvey Mudd College. See https://acmsonline.org/brabenec-lectureship/ for details.

2. We thank Dave Klanderman (Calvin University), Bryant Mathews (Azusa Pacific University), Victor Norman (Calvin University), and Valerie Zonnefeld (Dordt University), who are rotating off their membership on the ACMS Board. Victor will continue serving as the ACMS webmaster.

3. We welcome Amanda Harsy (Lewis University), Matt Hawks (Georgetown University), Mandi Maxwell (Taylor University), and Dusty Wilson (Highline College) to a four-year term on ACMS Board, and thank them for their willingness to serve.

4. The next ACMS conference (the 25th biennial) is scheduled to take place at Calvin University from Wednesday, May 27 until Saturday, May 30, 2026. Details can be found at the ACMS website, https://acmsonline.org/conferences-2/.

Bibliography of Christianity and Mathematics Update

Calvin Jongsma (Dordt University)

At the last ACMS Conference I gave a "Tribute to Gene Chase" [8] in which I talked briefly about our first edition of *Bibliography of Christianity and Mathematics* [6] and described what had happened to it since that time. In 2005 Gene had presented a draft introduction to a second edition of the *Bibliography* as an ACMS conference talk [4]. He had dropped out some ephemeral items from the first edition (student senior-seminar papers; unpublished faculty faith-learning tenure papers), reducing the total number of items in our *Bibliography* from 325 to around 280. Over time Gene had collected and annotated some 70 new items, increasing the number to 350. His 2005 ACMS presentation promised both a printout and an electronic second edition, but those never materialized. After Gene died late in 2020, his son John Chase located the data for these new items and passed them on to me to incorporate into a second edition, something Gene and I had once again started planning for.

Over the past two years, I've worked to move this project forward toward a second edition. I've now developed a Zotero database that contains about 1550 items, adding abstracts and keywords for most of them. To my surprise, I'm still finding new things—some that are short and not that deep—but also things that are interesting and significant.

What I'd like to do here is describe the database in its present state of completion.[1] Gene wanted to expand the *Bibliography* backward as well as forward in time and sideways in content and approach, and that's been my guiding understanding, too. The first edition of the *Bibliography* contained items from 1910 to early 1983; at the moment, the current database goes from 1699 to 2024, though this will no doubt change as I continue to develop and maintain it. Our initial intent was optimistically to include everything available on the topic of the booklet's title, found by manually searching bibliographic indices (imagine: books in libraries!) and asking others to share works they knew about. With today's access to Google's search engine and even AI, this goal seems both more manageable and also less attainable. Relevant items are continually being published, and I've even found a few earlier items that went undiscovered back then. One notable item is David Eugene Smith's retiring 1921 MAA presidential address, titled "*Religio Mathematici*," which can be found in four different forms [14], along with the related article "Mathematics and Religion" [15], which he published a decade later. I'll use these items to illustrate some overall features of the database.

Each entry in the database contains fairly complete standard bibliographic information as well as an *Abstract*, which annotates the item. I give a basic description of each item, crediting where the abstract comes from with bracketed initials at the end. [A, CJ], in the case of Smith's address, indicates that it has been excerpted and paraphrased (without fussing about precise scholarly attribution or indicating quotations) from what the author Smith wrote, supplemented by some of my own description. Occasionally I use a publisher's blurb, indicated by [P], but then I may have condensed and edited it to keep commercial hype to a minimum. Items that Gene abstracted are credited using [GBC], and a few others contributed abstracts as well to the first edition.

Keywords are located in the *Tags* field for an item; for Smith's talk, I added 15 of them manually, based on my reading. The list of all the tags I've introduced into the database is found in the bottom left-hand corner of the main Zotero page. Clicking on one of these will bring up all the items that have that keyword. A few of these tags are colored to highlight things I want to keep track of. For instance, I use an orange tag, which also appears as an orange square in the title column, to show which items still need to be annotated and tagged. Work on the *Bibliography* has expanded over time, as one item often led to several others. I thought I was nearly finished when I reached around 700 items, but that was obviously a premature conclusion, given the current count.

The *Related* field lists other bibliographic entries in the database related to the given item. For Smith's talk I've listed alternate publication printings, as well as the booklet *In the Shadow of the Palms* [16], which contains excerpts from a range of Smith's works. In the *Info* field for that item, I note that this booklet of selected writings includes a *Foreword* by Francis Su. I also wrote in the *Info/Extra* field that Francis recommended this booklet to ACMS members in a December 2022 Facebook post. Su's mention of Smith's address in his post was what alerted me to its existence in the first place, and it's also what made me offer to review the booklet for *Perspectives on Science and Christian Faith* [9]. My review is therefore listed in the *Related* field for the booklet. I've included significant book reviews in the *Bibliography*, when known. *Related* works is a symmetric relation, so you can get back to Smith's original address by clicking backwards through the relevant related titles.

[1]Bibliographic details have been updated in 5/2025 from my *Conference Banquet Talk* for these *Proceedings*. Zotero has also been updated to version 7, though that is not reflected in my description.

The central pane of the main database screen contains a number of columns that I've activated, exhibiting categories of bibliographic data listed in the *Info* field for the entries. The final column on the right, headed by a paper clip, shows a PDF icon for Smith's address. Double-clicking the icon loads the stored article. This feature realizes an exciting goal that I had for the original *Bibliography* project: to make items easily available for others to read, whenever possible. Even when no PDF is available, a URL link to a website where an item can be accessed, when available.

I'd like to talk a bit about the *Bibliography*'s criteria for inclusion. The title of the first edition, *Bibliography of Christianity and Mathematics*, gives a clear indication of its intended focus: it includes perspectival items relating Christian faith to mathematics in some way. But as you might imagine, this criterion is a bit slippery to apply.

Take Smith's presidential address, for example. His talk was in support of a rather ecumenical "religious attitude," an apology for "the reasonableness of a broad religious faith" that believes in something "infinite" and "eternal." It only considers some generic fundamentals of religion; it does not promote any particular creed or doctrine. Parallels, Smith says, can be made between mathematics and religion that help us reflect more deeply on our spiritual beliefs, whatever they are. Smith does not explicitly tell us what his personal religious affiliation is. Perhaps he was an orthodox Christian, as Francis Su seems to suggest, but he writes in rather cosmopolitan terms, taking care, it seems, not to offend believers in those non-Western religions he encountered on his many trans-oceanic travels.

All of this makes me wonder: do I include Smith's work in a *Bibliography* devoted to the connections between *Christianity and Mathematics*? At this point Smith's address is in the database; but then I probably need to broaden the explicit focus of the *Bibliography* to include anything that relates *Religious Faith and Mathematical Science*. The *Bibliography* could always indicate to the reader via the entries' abstracts and tags which items of the database are focused on or advocate for a Christian faith perspective on or connected to mathematics.

As you might expect, this is an issue for a number of the articles I've found. For example, I've come across works that deal with God and abstract objects. One of these is an article by Alvin Plantinga [13] that appeared in the February 2011 issue of the journal *Theology and Science* devoted to mathematics, guest-edited by James Bradley. Should I include this in the *Bibliography*? A number of these sorts of works seem to treat the topic from an adjacent theistic perspective without mentioning either the Bible or the Christian faith. For Plantinga's article, I haven't included a *tag* identifying it as addressing the topic from a Christian perspective. So, do I drop it out of the *Bibliography*? Or can I include it because I know Plantinga's religious background as a Reformed Christian philosopher? Regardless of the religious outlook for any such item, they may nevertheless reference (or be referenced by) something that *is* explicitly focused on Christianity – would that justify including them without a changed focus for the *Bibliography*?

Similarly, what about things written from a definite Unitarian or New-Age or Vedic perspective? That's obviously the case for Sarah Voss's book *What Number is God?* [18].[2] So, do I include this item in the *Bibliography*? Well, Zotero allows us to put this item into multiple collections, which I've done: it's both in the *Miscellaneous not in 1st Edn* and in the *Religion/Spirituality and Mathematics* collections. This feature allows me to err on the side of generosity in what gets

[2] Voss is a retired mid-western Unitarian pastor who was still active in 2023. While I disagree with Voss's religious outlook, I do like the very nice term *mathaphor* that she coined for *math*ematical met*aphor*s, something Christians often use to help them try to understand spiritual realities.

included in the database.

Of course, that just pushes the inclusion issue a bit further out: what qualifies as religious faith? You may be familiar with the 2011 Calvin and Hobbes comic [19] about mathematics being a religion—as a math atheist, Calvin tells Hobbes, he shouldn't be forced to learn mathematics.[3] This is humor that appeals to a neo-Calvinist mathematician named Calvin.

Note that this comic *Calvin-ist* perspective on mathematics and religion exaggerates a faith math-aphor that some of us ACMS folks like to use, though usually in a more reserved version: both mathematics and religion must start with statements accepted on faith, without proof.[4] Of course, adopting Calvin's extreme view would make the whole project of finding items relating religion and mathematics a redundant enterprise, which can't be right. While we may hold that mathematical developments do eventually get contextualized by a religious worldview, the *Bibliography* can still focus more narrowly on writing that seeks to spell out connections between a divinity standpoint and mathematical theorizing.[5]

Finally, let me say a few words about how the *Bibliography* might be used. Zotero has a rather robust search capability. Suppose you'd like to find all items in the *Bibliography* dealing with a Christian perspective on mathematics. Entering the keyword *Christian perspective* in the search box (I'd choose *All Fields & Tags*), 359 items come up. Here are some related search results: *Christian* (869), *Christianity* (109), *theology* (547). Choosing the *Everything* option for your search box brings up even more items: Christian (1439), Christianity (351), theology (840). Such searches are likely to give you more than you want: for instance, it may even bring up an item where the word is only found once in the stored attached PDF (which itself has a search feature). Adding more qualifiers will help you search for what most interests you.

There are many situations where you might wish to use the *Bibliography* to look something up. Maybe you're leading a section of a workshop for new math faculty and you need some readings on Christian perspective (e.g., [3], [7]). Or perhaps you'd like something to discuss in a senior capstone course or a faculty discussion group (e.g., [17]). Maybe, like I did just before the first edition of the *Bibliography* came out, you want to survey and analyze different approaches to relating Christianity and mathematics in order to further develop your own perspective (see [10]). Or perhaps you want a reading that offers a Christian assessment of *chance* for a Probability/Statistics course you'll be teaching (e.g., [1]) What about locating elementary or secondary mathematics curricula written from a Christian perspective (e.g., [2], [11], [12])? Or maybe you'd just like to browse the numerous items written by ACMS members (more than 275 by 2023) and soak in the rich variety of articles produced, like I did when I abstracted them for the database. It's my hope that this database will be a fruitful resource for ACMS members and others for a wide variety of uses in the coming years.

References

[1] Bradley, W. James. "Chance." *Mathematics Through the Eyes of Faith*, edited by W. James Bradley & Russell W. Howell. Harper One, 2011, pp. 93-114.

[3] See https://www.gocomics.com/calvinandhobbes/2011/03/09.

[4] In the case of the Pythagoreans, however, mathematics *was* their religion—number was what everything depended upon. And for many Christian Platonists, mathematics participates in the divine nature; even God is subject to the necessity of mathematical truths.

[5] Here I am drawing upon Roy Clouser's definition of religious belief. See [5], p. 24.

[2] Bradley, W. James et al. *Mathematics: A Christian Perspective*. The Kuyers Institute, 2007-2008. Nine lessons from a Christian perspective to supplement high school mathematics curricula. Available at https://calvin.edu/centers-institutes/kuyers-institute/education-resources/kuyers-math-curriculum/?dotcmsredir=1.

[3] James Bradley & Russell Howell, editors. *Mathematics Through the Eyes of Faith*. Harper One, 2011.

[4] Chase, Gene. "Bibliography of Christianity and Mathematics." *Proceedings of the Fifteenth Conference of the Association of Christians in the Mathematical Sciences*, 2005, pp. 81-87. Available at https://pillars.taylor.edu/acms-2005/18.

[5] Clouser, Roy A. *The Myth of Religious Neutrality. An Essay on the Hidden Role of Religious Belief in Theories*. Revised edition. University of Notre Dame Press, 2005.

[6] Chase, Gene & Jongsma, Calvin. *Bibliography of Christianity and Mathematics: 1910 - 1983*. Dordt College Press, 1983.
Reprinted online at https://www.asa3.org/ASA/topics/Mathematics/1983Bibliography.html.

[7] Howell, Russell W. & Bradley, W. James, editors. *Mathematics in a Postmodern Age: A Christian Perspective*. Wm. B. Eerdmans, 2001.

[8] Jongsma, Calvin. "Tribute to Gene Chase." *ACMS Journal and Proceedings* 23: May, 2023, pp. 324-325. Available at https://digitalcollections.dordt.edu/faculty_work/1487/.

[9] Jongsma, Calvin. "In the Shadow of the Palms: The Selected Works of David Eugene Smith, [Book Review]." *Perspectives on Science and Christian Faith* 75(2): 135-137.
Available at https://digitalcollections.dordt.edu/faculty_work/1469.

[10] Jongsma, Calvin. "Christianity and Mathematics: An Analysis of Different Approaches to Their Interrelationship." AACS Academic Paper, 1987, 13 pages.
Originally given as a sectional talk at the March, 1980 Jubilee Conference in Pittsburgh, PA.
Available at https://digitalcollections.dordt.edu/faculty_work/1491.

[11] Loop Hannon, Katharine A. *Revealing Arithmetic: Math Concepts from a Biblical Worldview*. Master Books, 2021.
This is a resource to be used alongside any mathematics curriculum.

[12] Nickel, James Duane. *The Dance of Number*. Sound Mind Press, 2018.
This four-volume series for grades 6 - 10 covers arithmetic, algebra,and geometry.

[13] Plantinga, Alvin. "Theism and Mathematics," *Theology and Science* 9(1), 2011, pp. 27-33.
Available from https://doi.org/10.1080/14746700.2011.547001.

[14] Smith, David Eugene. "*Religio Mathematici*: Presidential Address Delivered Before the Mathematical Association of America, September 7, 1921." *The American Mathematical Monthly*, vol. 28 (10): 339-349. Available at https://www.jstor.org/stable/2972153.
Reprinted in *The Teachers College Record* (1921) and *The Mathematics Teacher* (1921); slightly revised and abridged as a chapter in Smith's book *The Poetry of Mathematics and Other Essays* (1931).

[15] Smith, David Eugene. "Mathematics and Religion." *Mathematics in Modern Life*, pp. 53 - 60. *National Council of Teachers of Mathematics Yearbook*. New York, 1931.

[16] Smith, David Eugene. *In the Shadow of the Palms: The Selected Works of David Eugene Smith*, edited by Tristan Abbey. Science Venerable Press, 2022.

[17] Su, Francis Edward. *Mathematics for Human Flourishing.* Yale University Press, 2020.

[18] Voss, Sarah. *What Number Is God? Metaphors, Metaphysics, Metamathematics, and the Nature of Things.* State University of New York Press, 1995.

[19] Watterson, Bill. Comic for March 9, 2011. ["Mathematics is a Religion"]. Available at https://www.gocomics.com/calvinandhobbes/2011/03/09.

ACMS Summer Reading Group Activities

Dusty Wilson (Highline College)

At the ACMS conference at Indiana Wesleyan University (IWU) in 2019, a group of (mostly) junior faculty sensed that there was an interest in learning more together about the History and Philosophy of Mathematics. While a core interest of ACMS since its inception, this was a newer topic to many younger members. The initial conversation in a dark meeting hall was brief, but it got the ball rolling for serious subsequent activity including five summer-long book discussion groups and a pre-conference on the philosophy of mathematics at the 2022 conference at Azusa Pacific University (APU).

The initial group gathered on Zoom and discussed Morris Kline's *Mathematics: The Loss of Certainty* (Oxford, 1980). About 10 participants joined at least one of those weekly Zoom conversations. While we were growing in our understanding of the philosophy and history of mathematics, it should also be noted that the group was using Zoom regularly before the pandemic forced all of us online. So, in some small way, this discussion group providentially helped prepare us for that difficult season of teaching.

After a year hiatus in 2020, the group resumed and spent two summers discussing *Mathematics in a Post-Modern Age: A Christian Perspective* edited by James Bradley and Russell Howell (Eerdmans, 2001) and a third on *Equations from God: Pure Mathematics and Victorian Faith* by Daniel Cohen (Johns Hopkins, 2007). Participation continued to grow with 16 attending in 2021 and 24 participating in at least one Zoom meeting in 2022. The Cohen book was more specific in being a history of a particular part of math history and also less clearly tied to math-faith integration. As such, that group was a trifle smaller, but still the 13 who attended banded together over the course of a half-dozen summer Zoom gatherings including members from both coasts, as far south as Texas and stretching north into Canada.

Taking a step back, following the initial conversation at IWU, a group of senior ACMS members, Bob Brabenec, Russell Howell, and Dick Stout recognized the felt need for learning and community around the history and philosophy of mathematics as well as its connection to faith integration. Through a series of calls that later came to include Dusty Wilson as well as a face to face gathering at the 2020 JMM in Denver, the group developed and later hosted a pre-conference workshop on the Philosophy of Mathematics at the 2022 ACMS conference at APU. This group of 36 was noteworthy for unearthing a clear felt need among Christian mathematicians to understand the ideas underpinning our discipline.

In the summer of 2024, the largest ACMS Zoom discussion groups took place with 26 participants gathering over the course of an excellent series of discussions on *What is Mathematics, Really?* By

Reuben Hersh (Oxford, 1997). Sometimes asking questions, often voicing confusion or concerns over Hersh's writing, we gathered to grow as a community of Christians thoughtfully working to engage with our discipline. In addition to the topic at hand and prayer, we would check in across the nation which became particularly powerful when one of our members was impacted by a storm/flooding when a levee was breached.

In addition to the group discussions, numerous participants have gone on to give talks and/or write papers inspired by these topics. The Zoom community has extended to the physical as multiple collaborative projects have taken place. Looking ahead to summer 2025, the group expects to be meeting come mid-June, but has yet to choose a text. We have learned that the best and most enduring discussions seem to take place around books (not articles) that are generally new to the group, focused on broad topics garnering wide interest, and also written in a style allowing participants to come and go as summer and family schedules allow. If you are interested in joining and/or have a book suggestion, please reach out to Dusty Wilson at dwilson@highline.edu.

Appendix 1

Conference Schedule

24th Association of Christians in the Mathematical Sciences Biennial Conference
May 28 - June 1, 2024

Tuesday, May 28

Time	Event	Location
2:00 - 5:00 pm	Teaching Calculus with Infinitesimals Early Career Professional Development Teaching Introductory Statistics	SB 2602 SB 1641 SB 1604
5:00 - 6:00 pm	Dinner	The Commons
7:00 - 9:00 pm	Swimming and Board Games Social at Val Zonnefeld's House	807 4th Ave NE, Sioux Center

Wednesday, May 29

Time	Event	Location
7:30 - 8:30 am	Continental Breakfast	SB 1603
9:00 - 12:00 pm	Preconference Workshop 2	See Tuesday 2-5pm slot
12:00 - 1:00pm	Lunch	The Commons
1:30 - 4:30 pm	Preconference Workshop 3	See Tuesday 2-5pm slot
8:00 - 4:30 pm	Registration	Campus Center Information Desk
5:30 - 6:30 pm	Dinner	The Commons
7:00 - 7:45 pm	Opening Session	SB 1606
7:50 - 8:50 pm	Keynote: Joel Amidon *Teaching Mathematics as Agape*	SB 1606
8:50 - 9:30 pm	Refreshments	Science Building Atrium

Thursday, May 30

Time	Event	Location
7:00 - 7:30 am	Prayer Time	SB 2602
7:30 - 8:30 am	Breakfast	The Commons
8:40 - 9:10 am	Devotions and Announcements	The SB 1606
9:15 - 10:10 am	*Work, Life, Stress & the Goodness of God* *Moderator: Kristin Camenga* *Panelists: Danilo Diedrichs, Erica Oldaker, Michael Stob, Alana Unfried*	SB 1606
10:10 - 10:35 am	Refreshments	Science Building Atrium
10:40 - 11:55 pm	Parallel Sessions	SB 1603, CL 1321, SB 1641, SB 2602
12:00 - 12:45 pm	Boxed Lunch Pickup	Science Building Atrium
12:45 - 5:30 pm	EROS Center Excursion *Gov't Issued Photo ID Required*	Clock Tower
1:00 - 5:30 pm	Siouxnami Water Park	770 7th St NE, Sioux Center
6:00 - 7:15 pm	Dinner	The Commons
7:30 - 8:30 pm	Keynote: Judith Canner *Putting on Compassion: Ethics in a Data-Driven World*	SB 1606
8:30 -9:30	Refreshments	Science Building Atrium
8:45 - 9:15 pm	Choir Practice	SB 1606

Friday, May 31

Time	Event	Location
7:00 - 7:30 am	Prayer Time	SB 2602
7:30 - 8:30 am	Breakfast	The Commons
8:40 - 9:10 am	Devotions and Announcements	SB 1606
9:10 - 10:10 am	Keynote: Lydia Manikonda *The Pursuit of AI and the Future of Health*	SB 1606
10:15 - 10:25 am	Group Photo	Science Building Atrium
10:30 - 10:50 pm	Refreshments	Science Building Atrium
10:55 - 12:10 pm	Parallel Sessions	SB 1603, CL 1321, SB 1641, SB 2602
12:15 - 1:15 pm	Lunch	The Commons
1:20 - 2:45 pm	Parallel Sessions	SB 1603, CL 1321, SB 1641, SB 2602
2:50 - 3:10 pm	Refreshments	Science Building Atrium
3:15 - 4:50 pm	Parallel Sessions	SB 1603, CL 1321, SB 1641, SB 2602
6:00 - 7:30 pm	Banquet	The Commons
7:45 - 10:00 pm	Games and Social Time	The Bunker Kuyper Apartments Basement
7:45 - 8:30 pm	Choir Practice	SB 1606

Saturday, June 1

Time	Event	Location
7:30 - 8:30 am	Breakfast	The Commons
8:45 - 9:45 am	Keynote: Jason Douma *Does 3 Belong to 17? (and Other Absurd Questions)*	SB 1606
9:50 - 10:10 am	Refreshments	Science Building Atrium
10:15 - 10:45 am	ACMS Business Meeting	SB 1606
10:45 - 11:45 am	Worship Service	SB 1606
11:45 - 12:30 pm	Box Lunches	Science Building Atrium
12:00 - 1:00 pm	ACMS Board Meeting	SB 1604

Appendix 2

Schedule of Parallel Sessions

24th Association of Christians in the Mathematical Sciences Biennial Conference, May 28 - June 1, 2024

Thursday, May 30

Time	Math: Integration	Education	Pedagogy	Math: Research	Computer Science
Room	SB 1603	SB 2602	SB 1606	SB 1604	SB 1641
Moderator	Calvin Jongsma	Jeremy Case	Jerry De Groot	Mike Janssen	Vic Norman
10:40 - 10:55	**Karl-Dieter Crisman** The Crooked Made Straight: Alfonso's Geometric Work	**Curtis L. Wesley** Using Gateway Experiences to Reach Underrepresented Groups	**Joel Amidon** Redesigning Struggle to Foster Doers of Mathematics	**Samuel Alexander** Self-Graphing Equations	**Matthew Wright** Computational Mathematics: A Way of Thinking
11:00 - 11:15	**Anthony Bosman** Infinite Cardinality and the Possibility of an Infinite Past	**Peter Jantsch** Standards-Based Grading: The Why and the How	**Danilo Diedrichs** Mentored Research Projects in Applied Mathematics: Encouraging Creativity While Providing Structure	**Jocelyn Garcia (Zonnefeld)** A Computational Solution to the Game of Cycles	**Kari Sandouka** Engaging Students with ELAs
11:20 - 11:35	**Simon Tse** Equivalent Completeness Axioms in a Course of Real Analysis	**Patrick Eggleton, Richard Dusold** Building Thinking Classrooms - Research Based Instructional Strategies	**Rebekah Yates** Every Valley Shall Be Raised Up, and Every Mountain Made Low, and the Rough Places...a Crane?	**Kevin Grace** Dyadic Matroids with Spanning Cliques	**Matthew Hawks** Redeeming Statistics with DEI
11:40 - 11:55	**Melvin Royer** Examples and Counterexamples of "The Unreasonable Effectiveness of Mathematics"	**Valorie Zonnefeld, Ryan Zonnefeld, Luralyn Helming** Building Mathematics Teaching Efficacy in Preservice Elementary Teachers	**Sarah Klanderman, Dave Klanderman, Jason Ho** Setting Up Students for Success: Analysis of Effectiveness of Mathematics Placement	**Carlson Triebold** The Effect of Porous Vessel Linings on Red Blood Cell Interactions in the Microvasculature	**Fernando Santos** Exploring the Interfaces Between Worship and Work in Informational Technology

Friday, May 31 AM

Time Room Moderator	Math: Integration SB 1603 Russell Howell	Education SB 1606 Jason Ho	Statistics SB 1604 Alana Unfried	Computer Science SB 1641 Olaf Hall-Holt
10:55 - 11:10	**Jeremy Case** Mathematical Theater and the Human Condition	**Ricardo J. Cordero-Soto** The Catechism of the Mathematics Major	**Greg Crow** A Portfolio of Design Solutions in Excel	**Russ Tuck** Teaching Cultural Competency in a Christian Computers and Society Course
11:15 - 11:30	**Paul Lewis** Mathematics and Ethics	**Kevin Vander Meulen** Playing with Mathematics: Art as a Pedagogical Tool	**Will Best, Robin Lovgren** Divine Odds: Examining the Probability of Christ Fulfilling Prophecies	**Michael Kolta** Christian Ethics in Computers, Software, and Artificial Intelligence
11:35 - 11:50	**W. Scott McCullough** Was Copernicus a Heretic? How Protestants Decided "No" and Why It Matters.	**Steven Lippold, Amish Mishra** Lessons Learned from Artificial Intelligence Use in the Math Classroom	**Ryan Yates** Data Science that Welcomes Everyone	**Michael Leih** Using AWS Academy Learner Lab Cloud Technologies in Technology Courses
11:55 - 12:10	**Ashley Matney** Teaching Math for Social Justice in Online Undergraduate Math Courses	**Aaron Allen** My Experience Implementing UDL in College Algebra	**Jeremy Case** Ethics as Instruction	**Victor Norman** Do We Do Harm in Using Metaphors in Teaching Computer Science or Math/Stat?

Friday, May 31 PM

Time	**Math: Integration**	**Education**	**Math: Integration**	**Computer Science**
Room	SB 1603	SB 1606	SB 1604	SB 1641
Moderator	Bryant Matthews	Amanda Furness	Mike Stob	Russ Tuck
1:20 - 1:35	**Mitch Stokes** Plato, Symmetry, and the War for Contemporary Physics	**Dusty Wilson** Faith at a Public College; Ideas, Experiences, and Resources	**Brandon Bate** A Review of The Mathematical Experience by Philip J. Davis and Reuben Hersh	**Babafemi Sorinolu** Utilizing Technology as a Tool for Addressing Contemporary Challenges for Older Adults Living in Smart Homes
1:40 - 1:55		**Dave Klanderman, James Turner, Sarah Klanderman** Insight and Illumination in Mathematical Learning: Exemplars of Transitions to Understanding in the Classroom from Elementary to Graduate School	**Remkes Kooistra** Mathematical Practice and Christian Mysticism	**Lydia Manikonda** The Pursuit of AI: Is Technology Shaping Humans or Are Humans Shaping Technology?
2:00 - 2:15	**Elizabeth DeWitt** Mathematical Creativity		**David Freeman, Cory Krause** A Divine Conceptualist Interpretation of Modal Structuralism	**Shelley Zhang** Faith Lighting in the Shadow of AI
2:20 - 2:45	**Douglas Dailey** The Purpose of Mathematics according to Plato and Augustine	**Josh Wilkerson** Practicing Beauty as a Remedy for Math Trauma	**Jason Douma** The Wise Man Built His House Upon...Randomness?	**Devin Pohly** The Gift of Finitude in Computer Security

Friday, May 31 PM

Time Room Moderator	**Math: Integration** SB 1603 Brandon Bate	**Math: History** SB 1604 Kevin Vander Meulen	**Math: Research** SB 1606 Marissa Chesser	**REU + Statistics** SB 1641 Nicholas Zoller
3:15 - 3:30	**Emily Sprague Pardee** Finding Faith in the Public Sphere	**Brian Beasley** Alphametics: History, Inspiration, Application	**Mary Vanderschoot** Happy and Elated Numbers	**Russell Howell, Issac Jessop** Complex Analysis: REU Projects and an Open Source Text
3:35 - 3:50	**Erica Oldaker** Joyful Mathematics: Worshiping the Creator by Delighting in His Creation	**Calvin Jongsma** Logic's Modern British Upenders, Defenders, and Extenders: Whately's Revitalization of Logic	**Thomas Shifley** Is the Earth Actually Flat? An Introduction to Geometric Transitions	**Amanda Harsy** Running the Summer@ICERM Program
3:55 - 4:10	**Mandi Maxwell** Embracing The Mystery: How Mysterious Mathematics Bolsters My Faith	**Michael Stob** Gödel's Gibbs Lecture	**Lisa Hernandez** Analyzing Character Networks	**Judith Canner** A Pilot Program for Peer Intervention in Introductory Statistics
4:15 - 4:30		**Andrew Simoson** Trammel-Crafting for the Quadratrix	**Jesus Jimenez** An Identity for Fibonacci Numbers	**Alana Unfried** Statistics and Data Science Attitudes - A Platform for Collecting Data and Visualizing Results
4:35 - 4:50	**Daniel Kline** Mathematics and the Nature of Truth	**C. Bryan Dawson** Approximating the Master: Reconciling Euler's Power Rule Proof with Modern Infinitesimal Analysis	**Michael Porter** Integer Factorization Using the Method of Partial Factoring	**Tom Clark** Teaching Statistics with Quarto

Appendix 3

Presentation Abstracts

24th Association of Christians in the Mathematical Sciences Biennial Conference
May 28 - June 1, 2024

Preconference Workshops

Workshop 1: Teaching Calculus with Infinitesimals
Brian Dawson

Have you ever wondered what a modern approach to teaching calculus using infinitesimals might look like? This workshop is your chance to find out! Come and learn recently developed (within the last dozen years) notation and procedures that can make calculus look and feel more intuitive, how the algebraic burden with limits and other topics can be reduced, and how the course can be infused with fresh mathematical thinking. No prior experience with infinitesimals is needed; the workshop will include numerous opportunities to practice the techniques and learn how they are implemented in the classroom. Topics include infinitesimals, approximation, limits, the derivative and proofs of derivative rules, a simpler replacement for Riemann sums, a simpler test for series convergence, and more, all from the author of Calculus Set Free: Infinitesimals to the Rescue (Oxford University Press, 2022). A preview of Multivariable Calculus Set Free is also planned. Student reaction to these infinitesimal methods has been enthusiastic; come and see how your students could benefit!

Workshop 2: I Early Career Professional Development
Kristin Camenga, Derek Schuurman, Amanda Harsey Ramsay

Discuss and reflect on core areas of faculty work through workshops and stories from experienced faculty. Topics include teaching, scholarship, service, faith integration, the job search, and promotion & tenure.

Workshop 3: Teaching Introductory Statistics with GAISE: Statistical Investigation Process, Multi-Variable Thinking, Simulation-Based Inference
Nathan Tintle

In this workshop, participants will be immersed in hands-on activities for introducing students to statistical concepts, with a focus on: the statistical investigation process, multi-variable thinking and simulation-based inference. We will also provide experience with freely available software tools for exploring statistical concepts and analyzing real data, examine assessment items and techniques, and discuss common student misconceptions. To maximize post-workshop impact, all participants will be invited to join an NSF-sponsored (NSF-DUE-2235355) virtual community of statistics teachers which provides ongoing support, resources and additional professional development opportunities. Ultimately, we hope to move faculty participants closer to teaching introductory statistics effectively in accordance with the American Statistical Association's GAISE recommendations (www.amstat.org/gaise).

Keynote Speakers

Teaching Mathematics as Agape

Joel Amidon, University of Mississippi, Professor of Education

What does it mean to teach mathematics from a perspective of unconditional love? In this presentation, the speaker draws upon biblical principles, practitioner innovations, and inclusive teaching methods to conceptualize and illustrate an ideal relationship between students and mathematics. This relationship is characterized as functional, communal, critical, and inspirational, all stemming from a commitment (and passion) to teach mathematics as agape.

Putting on Compassion: Ethics in a Data-Driven World

Judith Canner, California State University, Monterey Bay Professor of Statistics

Data and compassion are not often paired together. We tend to think of data as some neutral record of the facts, and "facts" do not need "feelings" to support their meaning. But is that true? In response to our salvation through Jesus Christ, we are instructed to "put on compassion, kindness, humility, gentleness, and patience…" (Col 3:12), but the practical implications of that calling require us to examine our ethical framework as we approach our roles as both statistical educators and practitioners. If we integrate discussions of ethics and compassion into our classrooms through pedagogical frameworks that allow us to navigate difficult and sometimes divisive topics, we can train the next generation of statisticians and data scientists to bring about change to systems that seem unchangeable. With that goal in mind, we will explore compassion as the foundation for ethical practices in data collection and privacy; data analysis and inference; the communication of data; and pedagogical practices for teaching each of these topics.

Does 3 Belong to 17? (and Other Absurd Questions)

Jason Douma, Associate Vice President for Institutional Research and Professor of Mathematics, University of Sioux Falls

Paul Benacerraf famously told a story about two budding mathematicians (Ernie and Johnny) who could not agree on whether 3 "belongs" to 17. On its face, this seems a pointless argument to entertain. The question is marginally well-defined and most likely irrelevant to the actual practice of mathematics. However, the answer to the question—or more precisely, the difficulty in obtaining an answer to the question—does tell us something about the nature of the subject matter with which we work. This talk will examine what might be learned by pondering "absurd questions" in mathematics, including Benacerraf's little gem.

The Pursuit of AI and the Future of Health

Lydia Manikonda, Assistant Professor in the Lally School of Management, Rensselaer Polytechnic Institute

The growing ability to process enormous amounts of data at a highly efficient speed and with lower costs has paved the way for developing intelligent Artificial Intelligence (AI) systems that attempt to mimic humans. Despite this rapid growth of AI, it remains unclear how we can efficiently deploy this technology to solve real-world problems, especially in healthcare. Drawing on the insights from my inter-disciplinary research, this talk showcases the different aspects of modeling unstructured natural language data for decision-making and sheds light on different ethical challenges involved in this process.

Thursday Morning Panel

Work, Life, Stress & the Goodness of God

Moderator: Kristin Camenga, Juniata College
Panelists: Danilo Diedrichs, Wheaton University
Erica Oldaker, Gordon College
Michael Stob, Calvin University
Alana Unfried, California State University Monterey Bay

All of us face the stress of balancing the various parts of our lives – work, family, church, personal passions. As Christians, we also have wisdom and strength from God to direct us and carry us through the challenges we face. The panel will share their stories of how God has been good in the midst of the stresses of life and lessons they have learned. The session will conclude with all attendees having the opportunity to support each other as the Body of Christ by sharing and praying in small groups.

Parallel Sessions

Self-Graphing Equations

Samuel Alexander, US Securities and Exchange Commission

Can you find an xy-equation that, when graphed, writes itself on the plane? This idea became internet-famous when a Wikipedia article on Tupper's self-referential formula went viral in 2012. Under scrutiny, the question has two flaws: it is meaningless (it depends on typography) and it is trivial (for reasons we will explain). We fix these flaws by formalizing the problem, and we give a very general solution using techniques from computability theory.

My Experience Implementing UDL in College Algebra

Aaron Allen, North Carolina Wesleyan University

Universal Design for Learning (UDL) is an educational framework that aims to make learning accessible to all individuals; its primary goal is to eliminate barriers for learning. In this presentation, I will discuss my brief experience with implementing UDL principles in my College Algebra courses. In particular, I will discuss the changes I've implemented over the past couple of years and highlight how these changes have positively impacted student learning.

Redesigning Struggle to Foster Doers of Mathematics

Joel Amidon, The University of Mississippi

This session provides a brief overview of productive struggle incorporating research on the shame-pride axis and equitable mathematics teaching. Participants will be presented with how to reposition struggle, minimize shame, and foster students identifying themselves as doers of mathematics.

A Review of *The Mathematical Experience* by Philip J. Davis and Reuben Hersh

Brandon Bate, Houghton College

In this session I will discuss *The Mathematical Experience* by Philip J. Davis and Reuben Hersh. In this book, the authors address topics in the philosophical foundations of mathematics, the history of mathematics and other assorted topics related to the experience of doing mathematics. This book consists of a series of independent, and yet interrelated, vignettes. Rather than attempting to give an exhaustive historical account or build up a central thesis, the authors strive to expose readers to some of the subtler elements of the mathematical experience, including thoughts and observations on the interrelationship between mathematics and culture, theological/religious beliefs and education. I've used this book when teaching a course on the history and philosophy of mathematics and found the book's content helpful for spurring classroom discussion on these topics and for providing natural opportunities for faith integration. My goal for this session is to share about my experience using this book, and if time permits, have a broader discussion about other resources that are helpful for faith integration in courses like these.

Alphametics: History, Inspiration, Application

Brian Beasley, Presbyterian College (retired)

In this talk, we embark on a brief historical tour of alphametics, from the famous SEND + MORE = MONEY example published 100 years ago to their inclusion in two of the Wayside School books by Louis Sachar. Inspired by verses such as Psalm 119:160, we share a few Biblical alphametics for aficionados of verbal arithmetic to enjoy. We also make a case for including alphametics in Introduction to Proof courses, to provide potential math majors with some fun early exercises that help strengthen their reasoning skills. So come join us, and LETS + MEET + AT = DORDT.

Divine Odds: Examining the Probability of Christ Fulfilling Prophecies

Will Best, Belmont University
Robin Lovgren, Belmont University

In 1958, Peter W. Stoner published his seminal work Science Speaks, which discusses, among other things, Bible prophecies in relation to probability estimates and calculations. In this tribute to the original work, we consider the prophecies in Isaiah 53, often attributed as the gospel of the Old Testament. The Great Isaiah Scroll, discovered in Qumran Cave 1 in 1946, makes these prophecies especially important to the Christian faith as the scroll has been dated over one hundred years prior to Christ's birth. The probability of each of the prophecies in Isaiah 53 is calculated, and a discussion follows centered around the fulfillment of the prophecies in the life and ministry of Jesus Christ.

Infinite Cardinality and the Possibility of an Infinite Past

Anthony Bosman, Andrews University

Since Aristotle distinguished between the potential and actual infinite, there has been a rich interplay between philosophical reflection on the infinite and the study of infinity in mathematics. In this talk we will focus on a popular philosophical argument for temporal finitism, the view that the past is finite, based on the impossibility of an actual infinite, and its relationship to Cantorian set theory. While such arguments have a long history, dating back a thousand years to Muslim theologian al-Ghazali, we will focus on the contemporary scholarship of Christian philosopher William Lane Craig who has popularized this argument in defense of his Kalam cosmological argument. In particular, we will look at Craig's handling of infinite cardinality, including his appeal to Hilbert's hotel, ultimately finding his arguments interesting but unconvincing. We will conclude by suggesting the possibility of an infinite past consistent with a dynamic, social view of the Trinity.

A Pilot Program for Peer Intervention in Introductory Statistics

Judith Canner, California State University, Monterey Bay

Since the return to in-person teaching in Fall 2021, levels of engagement in the introductory statistics courses on our campus have been low, with poor attendance, no submission of basic assignments, and in general, poor performance with an increased DFW rate to 30% each semester. To break the cycle of student disengagement, we piloted a peer intervention program in two different introductory statistics courses, one that is general education and one that is for STEM-intending students. We will share the outcomes of the pilot program, including identifiable benefits and impacts on student success and reflections on its success.

Ethics as Instruction

Jeremy Case, Taylor University

I inherited and adapted an assignment involving ethics in a statistics course. As a mathematician transitioning to teaching statistics, I have tried to incorporate recommendations from the statistical education literature, but I did not expect the unintended benefits of the ethical assignment in terms of communicating statistical approaches and practices. This presentation will share the features of the assignment as well as its perceived benefits.

Mathematical Theater and the Human Condition

Jeremy Case, Taylor University

Stephen Abbott's book *The Proof State: How Theater Reveals the Human Truth of Mathematics* explores how theatrical plays in the last century have incorporated mathematical concepts. Many of these mathematical ideas are included in liberal arts mathematics courses and have been explored theologically. We will examine what mathematical plays and their interpretations have to say about the human condition and what they may have to offer for the Christian scholar.

Teaching Statistics with Quarto

Tom Clark, Dordt University

Quarto is an open-source publishing language that allows beautiful documents to be created that integrate statistical calculations and visualizations. In this talk we'll get a crash course in Quarto basics and see examples of what is possible. Additionally, we will hear some of the pedagogical lessons learned from teaching a Statistics course using Quarto for the first time.

The Catechism of the Mathematics Major

Ricardo J. Cordero-Soto, Jessup University

Most majors in mathematics can relate to a fork-in-the-road moment: the transition from computational mathematics to understanding and writing proofs. As an exercise in Christ-like compassion, we discuss teaching a set of renewed expectations for students to learn as they move through their upper-division mathematics courses. Students should internalize these renewed expectations much like a catechism. As students modify their expectations of the learning process in proof-based mathematics, students will potentially replace frustration with enthusiasm and improved performance. But if these values are to emulate a catechism, we will strive to mimic a catechism rooted in belonging, as proposed by Curtis W. Freeman in a Christianity Today article. This initial phase in developing such a catechism will include examples and challenges for program-wide implementation.

The Crooked Made Straight: Alfonso's Geometric Work

Karl-Dieter Crisman, Gordon College

At Christmas (especially in Handel's Messiah) we often hear Isaiah 40:4, that every valley shall be exalted - and the crooked be made straight. The New Living Translation, however, renders this, "Straighten the curves," and this is exactly how the 14th-century Jewish convert Alfonso of Valladolid thought of it in his Sefer Meyasher 'Aqov, a Hebrew treatise on the nature of geometry.

This talk will give some of Alfonso's background, his approach to the parallel postulate (which, like many contemporaries educated in the Islamic world, he provided an incorrect proof of), and especially his connection of his ideas on rectifying curves to Scripture and faith. I will lean heavily on the impressive critical work of Ruth Glasner, Avinoam Baraness, and others in translating and placing Alfonso's work in context.

A Portfolio of Design Solutions in Excel

Greg Crow, Point Loma Nazarene University

Excel is a versatile tool that can be used to design solutions to a wide range of problems for the classroom, the department, the institution, and the church. Classroom examples will include simulations and tools for statistics, genetics and bioinformatics. Administrative examples will include graphical tools for scheduling courses without conflicts as well as an analysis of four decades of future academic calendars. Examples will be shown that are used to convey to administrators the rotation of Final Exams, classroom usage, and discipline specific cost comparisons with other institutions. A denomination level model will be presented that church leaders have used to examine retention within their ministerial corps.

The Purpose of Mathematics according to Plato and Augustine

Douglas Dailey, Christendom College

In 1973, Russian mathematician I.R. Shafarevitch delivered a lecture to the Göttingen Academy of Sciences on the purpose of mathematics. The conclusion he reached in his address is that the ultimate purpose of mathematics must be religious. In this talk, we will explore a possible way in which this claim can be justified by understanding the purpose that mathematics served within a person's intellectual formation according to Plato. To place Plato's view into a Christian perspective, we will then investigate the thought of St. Augustine of Hippo, the great fifth century theologian and bishop. Augustine's insight on the role that number plays in the development of reason sheds light on how knowledge of mathematics conduces to knowledge of God.

Approximating the Master: Reconciling Euler's Power Rule Proof with Modern Infinitesimal Analysis

C. Bryan Dawson, Union University

A recent development in infinitesimal (nonstandard) analysis is the introduction of the approximation relation, which is utilized in the single-variable calculus textbook Calculus Set Free: Infinitesimals to the Rescue to simplify the calculation of limits (among other things). However, many of the resulting techniques are not new, but closely resemble those of the early masters. An example is Euler's proof of the power rule. In this presentation we use the modern infinitesimal methods to successfully analyze and justify Euler's proof, yielding insight that previously known methods are unable to provide.

Mathematical Creativity

Elizabeth DeWitt, Trinity Christian College

Creativity is a sought-after skill by employers. From a Christian perspective, creativity is connected to the cultural mandate. Thus, as Christian faculty, we should not only develop our students' creative skills in our mathematics classes but also help students to recognize this creative development. But that means we need words to describe mathematical creativity, so I will discuss literature.

Mentored Research Projects in Applied Mathematics: Encouraging Creativity While Providing Structure

Danilo Diedrichs, Wheaton College

Mentoring undergraduate students in research can be a daunting task for the faculty member. Furthermore, the students are often unsure how to concentrate their efforts productively when working on an open-ended research project. Here, we present a framework to approach a mentored research project in applied mathematics as a semester-long "course" structured around three milestones: project proposal, progress report, and final results. Students receive a syllabus, complete with instructions, deadlines, and rubrics for the assessments related to the progress of their project. This approach leaves space for creativity while providing a structure and opportunities for faculty guidance and feedback. Along the way, students learn how to frame their research question, collect data, learn from primary literature, generate a poster, and present their results orally and in writing.

The Wise Man Built His House Upon...Randomness?

Jason Douma, University of Sioux Falls

Albert Einstein famously expressed reservations about the fundamental implications of quantum theory in a letter to Max Born, stating, "In any case, I am convinced that he [God] does not throw dice." This talk will examine some essential and well-understood mathematical properties of randomness which suggest that chance (in some sense) could in fact play a foundational role in a well-ordered universe.

Building Thinking Classrooms - Research Based Instructional Strategies

Patrick Eggleton, Taylor University
Richard Dusold, Regents School of Austin

In 2021 Dr. Peter Liljedahl published the book "Building Thinking Classrooms in Mathematics," sharing research based instructional strategies that have radically shifted many pre-university mathematics classrooms. Strategies like using problem based tasks, having students stand and work at whiteboards, and using random collaborative groups are just 3 of the 14 recommended teaching practices developed from this research. This session is intended to introduce these researched strategies to faculty while sharing the responses of students both at the pre-university and at the university level.

A Divine Conceptualist Interpretation of Modal Structuralism

David Freeman, University of Cincinnati Blue Ash College
Cory Krause, LeTorneau University

Divine Conceptualism is the view that things such as mathematical objects are to be identified with thoughts in God's mind. While this view has a wide variety of supporters from Augustine to Plantinga, it suffers from various problems and ambiguities. For example, exactly which divine thought is the number 2? In this talk, we hope to show that Modal Structuralism, as interpreted via Divine Conceptualism, clarifies such ambiguities and provides a philosophically cogent and theologically satisfying view of mathematics in general.

A Computational Solution to the Game of Cycles

Jocelyn Garcia (Zonnefeld), Dordt University

In *Mathematics for Human Flourishing,* Francis Su introduced The Game of Cycles, a game played on a finite simple planar graph. In game play, opponents alternate adding direction to the edges of the graph with the goal of creating a cycle or making the last legal move. Recent work has sought to determine winning strategies on certain classes of graphs. This talk shares about a paper written in collaboration with Dr. Mike Janssen and Eliza Kautz. I will introduce a tabular representation of a game state and provide a computer program that determines which player has a winning strategy on any legal game board. The program builds a directed graph of all possible game states, utilizing concepts of impartial game theory in the labeling of game states and determination of winning strategies. Throughout gameplay, the player with a winning strategy can use program output to guarantee a victory.

Dyadic Matroids with Spanning Cliques

Kevin Grace, University of South Alabama

The work of Geelen, Gerards, and Whittle describes the structure of classes of matroids representable over a finite field. To use this to study specific classes, it is important to study the matroids in the class containing spanning cliques. A spanning clique of a matroid M is a complete-graphic restriction of M with the same rank as M.

We will describe the structure of dyadic matroids with spanning cliques. The dyadic matroids are matroids that can be represented by a real matrix A where every nonzero subdeterminant is $\pm 2^i$ for some integer i. Every signed-graphic matroid is dyadic. For these matroids, the entries of A are determined by a signed graph. Our result is that dyadic matroids with spanning cliques are signed-graphic matroids with a few exceptions. This is joint work with Ben Clark, James Oxley, and Stefan van Zwam. This talk will include a brief introduction to matroids.

Running the Summer@ICERM Program

Amanda Harsy, Lewis University

This talk shares experiences running a research experience for undergraduates (REU) at the Institute for Computational and Experiential Research in Mathematics (ICERM), a national mathematics research institute at Brown University. The Summer@ICERM program is an eight-week summer research program designed for undergraduate scholars. During this program, four faculty research mentors worked with 18 undergraduates and 4 graduate teaching assistants on research projects involving the combinatorial and graph theoretical properties of DNA self-assembly. This talk will focus on the logistics of running such a program in addition to highlighting some of the successes, challenges, and lessons learned involved in mentoring a large group of student researchers.

Redeeming Statistics with DEI

Matthew Hawks, US Naval Academy

Three major pioneers in the field of statistics promoted eugenics. How can Christian statistics instructors overcome this immoral foundation and attract students “from every nation, from all tribes and peoples and languages?” In this talk, gain three tangible classroom instruction tools you can use next semester! (1) Formulate a course policy that explicitly recognizes past exclusions. Such a statement addresses concerns that a student may not “belong” in the field. (2) Ask a daily question to nurture respect for each individual. Students will open up to their peers and have opportunities for daily validation. (3) Incorporate a simple biographical assignment to expose students to contributions from underrepresented individuals. Help students recognize that inherent, immutable characteristics are not predictors of future success in statistics. Redeem statistics and welcome the next generation to the field!

Analyzing Character Networks

Lisa Hernandez, California Baptist University

Basic graph theory tools are used to analyze and gain insights from the complex character network in Yaa Gyasi's "Homegoing."

Complex Analysis: REU Projects and an Open Source Text

Russell Howell, Westmont College
Isaac Jessop, Westmont College

"Flip a Coin, get an Annular Function?" This is the title of a paper we developed in the summer of 2023 as part of an REU effort. The paper is slated to appear in the American Mathematical Monthly sometime this year. In the first part of this presentation we will describe what the question in the title of the paper means, and what results we obtained. Painting with broad strokes, we develop a mechanism for picking a number in $[0, 1]$ and associating that number with a complex-valued function. We then prove that the probability that this function has a specialized property is either zero or one, and exhibit an uncountable collection of numbers that yield functions with and without this property. The second part of this presentation presents a new website (complexanalysis.org) that is planned to contain an open-source textbook as well as a list of potential REU projects.

Standards-Based Grading: The Why and the How

Peter Jantsch, Wheaton College

Alternative grading practices aim to change the focus of our students from “getting the grade” to learning and growth. I will describe my experiences with Standards-based grading in a range of math classes, from introductory to upper-level. I’ll talk about why I made the change, what I’ve observed in my students, and give some advice on how you can get started with SBG, too!

An Identity for Fibonacci Numbers

Jesus Jimenez, Point Loma Nazarene University

We give a generalization to the following identity. Let F_k denote the k-th Fibonacci number. The following identity $\frac{F_n \cdot F_{10n} - F_{2n} \cdot F_{5n}}{5 \cdot F_{n^2}}$ is congruent to zero modulo 10, appearing as advance Problem 891 in the Fibonacci Quarterly. We prove that $\frac{U_n \cdot U_{2pn} - U_{2n} \cdot U_{pn}}{p \cdot U_{n^2}}$ is congruent to zero modulo 2·p where U_n are the terms of a second order linear recurrence.

Logic's Modern British Upenders, Defenders, and Extenders: Whately's Revitalization of Logic

Calvin Jongsma, Dordt University

Logic developed dramatically during the last half of the nineteenth century. The baseline for these transformations in Great Britain was the revival of logic by Richard Whately around 1825. Whately successfully defended syllogistic logic as the science of valid reasoning against potent seventeenth- and eighteenth-century detractors – Bacon, Locke, Reid, Campbell, Stewart, and others. In so doing, he made logic an intellectually respectable field of investigation for the next generation of logicians to explore and extend. This included John Stuart Mill (inductive logic), Augustus De Morgan (logic of relations), and George Boole (algebraic logic; propositional logic).

Insight and Illumination in Mathematical Learning: Exemplars of Transitions to Understanding in the Classroom from Elementary to Graduate School

Dave Klanderman, Calvin University
James Turner, Calvin University
Sarah Klanderman, Marian University

Jacques Hadamard describes a four-stage epistemological process: preparation, incubation, illumination, and verification. Philosopher and theologian Bernard Lonergan builds on the writings of Aquinas to develop his theory of insight which elaborates the transition from incubation to illumination. Insight and illumination provide an analytical lens for transitional moments as students learn ever more abstract mathematical concepts. We offer exemplars of these transitions at the elementary, middle, high, undergraduate, and graduate school levels. We provide connections to theorists in mathematics education such as Jean Piaget (stages of development of understanding functions), Pierre van Hiele and Dina van Hiele-Geldof (levels of understanding geometry), and Dietmar Kuchemann (levels of understanding letters or variables). We also provide pedagogical strategies which align with Christian faith commitments and promote growth in understanding.

Setting Up Students for Success: Analysis of Effectiveness of Mathematics Placement and Support Systems at Liberal Arts Universities

Sarah Klanderman, Marian University
Dave Klanderman, Calvin University
Jason Ho, Dordt University
Jeff Carvell, Marian University

How do we effectively and equitably place students into math classes in a way that provides them the best chance of success? As many higher education institutions veer away from placement based on standardized testing, many departments are seeking placement alternatives that will properly support students. Additionally, math placement determines not only a student's mathematics courses but also influences their progress in related fields, including physics, chemistry, engineering, and more. This presentation will describe three different existing placement systems at each of our liberal arts institutions as well as the affordances and constraints of each approach. Further, we analyze data from each institution to study the impact of the various placement strategies on student success. We take a data-driven approach to math placement with the hope of improving student outcomes through accurate and equitable math placement.

Mathematics and the Nature of Truth

Daniel Kline, College of the Ozarks

In this talk, we consider the nature and role of objective truth in the Christian faith as we ponder belief, mystery, and beauty through the lens of mathematics.

Christian Ethics in Computers, Software, and Artificial Intelligence

Michael Kolta, Palm Beach Atlantic University

We are currently working with a publisher to write a book of the same title and seek feedback from like-minded peers. Permission has been obtained to share this copyrighted content at this conference. Topics include the ubiquity of computers and software, Christian ethics in software and AI usage and design, and a call to action. We search for what the Bible (both Old and New Testament) has to say about modern issues, such as content "sharing", seemingly endless notifications, and letting computers make decisions on our behalf. Ethical evaluations of these modern issues often change when Biblical principles are considered. We encourage users to use software ethically via example. We encourage designers to design software for ethical use and provide examples. In some cases, perhaps AI can "learn" directly from the Bible, especially Jesus' parables, and apply that "knowledge" to modern scenarios.

Mathematical Practice and Christian Mysticism

Remkes Kooistra, The King's University

In the moment-to-moment experience, what is it really like to do mathematics? Given the abstraction of its goals, the mental exertions of its methods, and the deep and frustrating intangibility of mathematical things, I might be tempted to say that doing mathematics is meditative. How far can I push that idea? Drawing from the deep well of Christian mysticism, this talk will investigate to what extent mathematics is like prayer and meditation, and to what extent it necessarily remains something less than these. I will focus on the practice of mathematics, as a human activity, and argue that foregrounding this practice with students can be a fruitful angle for the integration of faith and discipline.

Using AWS Academy Learner Lab Cloud Technologies in Technology Courses

Michael Leih, Point Loma Nazarene University

Creating a controlled computing environment in a technology course when students have different types of computers and operating systems can be a challenge. However, by leveraging cloud technologies, virtual environments can be created to have an identical and predefined configuration so labs and activities can have a consistent outcome for all students. This talk will review ways AWS Academy Learner Lab can be leveraged in technology courses to provide customizable virtual environments for each student, with no need to provide a credit card to establish an AWS account, nor additional cost to the student or university. In addition, each student's environment can be controlled and monitored by instructors to provide additional support.

Mathematics and Ethics

Paul Lewis, Houston Christian University

We will explore some connections between mathematics and ethics. In particular, we will examine recent work of philosopher Justin Clarke-Doane, whose arguments present a challenge to simultaneous positions of mathematical realism and moral anti-realism.

Lessons Learned from Artificial Intelligence Use in the Math Classroom

Steven Lippold, Taylor University
Amish Mishra, Taylor University

For decades, there has been research into Artificial Intelligence (AI), with the academic research starting with the advent of Machine Intelligence by Alan Turing in the 1950s. This has culminated into more recent developments, such as the large language model (LLM) ChatGPT developed by OpenAI, which greatly popularized the curiosity of AI by much of the general public. In this talk, we aim to outline several applications of AI in the classroom environment, focusing on two classes offered at Taylor University: Introduction to Statistics and Investigations in Mathematics (a liberal arts mathematics course). We cover how we implemented AI for lecture preparation, assignment preparation, and student evaluation by giving a particular example for each course and give our takeaways from our experience.

The Pursuit of AI: Is Technology Shaping Humans or Are Humans Shaping Technology?

Lydia Manikonda, Rensselaer Polytechnic Institute

The pace at which Artificial Intelligence (AI) systems are developing and being adopted widely in our everyday lives is significant. With this kind of adoption come severe ethical and privacy-related challenges that may potentially harm users more than help them in certain situations. Alongside this, there has been a major push to develop these technologies, so they are fair, transparent, and accountable. In this talk, I will first describe how individuals weigh privacy against convenience, as evidenced by some of the current research problems I have been working on. Then, I will compare these insights with biblical perspectives, which may in turn influence and propel the development of such fair and transparent AI systems.

Teaching Math for Social Justice in Online Undergraduate Math Courses

Ashlee Matney, Campbellsville University

Teaching math for social justice has become a heavily researched and highly published topic in recent decades. However, nearly all of the research has been conducted in the traditional classroom setting, and predominantly in K-12 school systems. My research sought to broaden this knowledge base by examining the impact of discussion forums on students' social justice beliefs in fully online undergraduate math courses at a Christian university.

Quantitatively, students completed pre- and post-course demographic and Likert-scale surveys to determine pre- and post-course Social Justice Scores. Qualitatively, students completed eight weekly discussion forums relating mathematical concepts to social justice issues. Analysis of the merged data resulted in optimistic outcomes and laid groundwork for future research in teaching math for social justice beyond the classroom.

The study and results will be discussed in this presentation, with extra emphasis on the care taken in planning and in discussion forum prompt creation.

Embracing The Mystery: How Mysterious Mathematics Bolsters My Faith

Mandi Maxwell, Taylor University

The connection between mathematics and Christian faith can often seem elusive to our students but I believe that mathematical modes of thinking can provide a helpful framework for engaging mysteries of faith. We routinely wrestle in mathematics with ideas that are challenging to wrap our minds around, results that reveal our finite, human limitations. Experiencing these challenges in the realm of mathematics can enable us to expect and embrace challenges in matters of faith and truths in God's Word that are difficult to comprehend. My purpose in this talk is to discuss some mathematical examples from my classes that foster my sense of awe and wonder for God which I hope students might find helpful as they explore the mysteries of mathematics and faith.

Was Copernicus a Heretic? How Protestants Decided "No" and Why It Matters.

W. Scott McCullough, Indiana Wesleyan University

Copernicus delayed for decades the publication of his heliocentric planetary system because he was concerned that he would be accused of contradicting Holy Scripture. Within three hundred years, Protestants who championed the inerrancy of Scripture had embraced the heliocentric system in spite of its apparent conflict with the letter of the text. The process that led to Protestant acceptance of the Copernican theory involved overwhelming evidence coupled with a willingness to reexamine theological positions. The history points to the need for Christian teachers of science to engage theology as well as scientific evidence to help students integrate faith and science.

Do We Do Harm in Using Metaphors in Teaching Computer Science or Math/Stat?

Victor Norman, Calvin University

In this presentation, we will explore the metaphors we use while teaching abstract concepts. Identifying the metaphors we use in teaching is the first step. Given that all metaphors break down, do we teachers (and do students) understand where the metaphors break down? What are the implications of misunderstanding where our metaphors break down? Do metaphors help understanding for all of our students, including those for whom English is a second language? Preliminary data from a survey given to computer science students will be presented to attempt to answer these and other questions.

Joyful Mathematics: Worshiping the Creator by Delighting in His Creation

Erica Oldaker, Gordon College

Mathematics is eminently useful. Using one’s mathematical talents in service to the kingdom is something I believe every Christian mathematician should seriously consider. But is this the only way to glorify God with math?

I want to contend for the enjoyment of mathematics as an act of worship. While being mindful of the potential for idolatry, I believe finding joy in the gifts of Creator with a rightly oriented heart can glorify Him and inspire reverence.

In this talk, I want to share a few examples of mathematics that have personally given me a small taste of the glory of God. I suspect most of these will be familiar, but I hope that meditating on their wonder will inspire joy and remind all of us to direct that adoration back towards its source.

The Gift of Finitude in Computer Security

Devin Pohly, Wheaton College

Human finitude—the set of limitations we have as created beings—is a surprisingly positive aspect of our existence. It is easy to interpret our limits in a negative light, but viewing them instead as a gift from God can reveal blessings that might otherwise be taken for granted. In this talk, we examine the field of information security, recognizing a number of its fundamental principles and techniques that are designed to account for our created finitude and exploit it toward productive ends.

Integer Factorization Using the Method of Partial Factoring

Michael Porter, California State University, Long Beach

Almost all modern methods for factoring a large integer *N* have two major phases. The first phase, sieving, generates relations, which have the form $a_1 a_2 \ldots a_m = b_1 b_2 \ldots b_n$ mod *N*. These relations are combined in the second phase, linear algebra, to find two integers, x and y, for which gcd(*x,N*) and gcd(*y,N*) might be the factors of *N* we seek. By balancing the effort for these two phases, the best asymptotic running time is achieved.

In this paper, a method is introduced which can be used for smaller *N*. A smaller factor base is used, so the linear algebra step can be run many times to monitor progress and feed back information to guide the sieving step. The sieving step will be longer because of the smaller factor base, but it might be possible to focus the sieving based on the information fed back from the linear algebra.

Examples and Counterexamples of "The Unreasonable Effectiveness of Mathematics"

Melvin Royer, Indiana Wesleyan University

In his influential 1960 article "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," physicist Eugene Wigner cited multiple situations in physics where existing but seemingly unrelated mathematics was found to "unreasonably" provide an effective model. Multiple authors have subsequently provided arguments and further examples from various disciplines that such effectiveness is unreasonable, while others have written rebuttals and interesting alternatives. (Some of this work was written from a theistic perspective). Rather than taking a side on the issue, I will summarize a selection of these later "unreasonable" examples and rebuttals from physics, biology, and artificial intelligence. There appears to be sufficiently varied evidence to make whatever case one wishes to presuppose.

Engaging Students with ELAs

Kari Sandouka, Dordt University

ELAs provide a way of teaching through 'learning by doing.' This presentation will showcase two experiential learning activities used to engage students with the content of the class. One activity is aimed at identifying and assessing risks and the second is aimed at closing the gap between the abstract concepts of object oriented programming and creating programs to represent the real world.

Exploring the Interfaces Between Worship and Work in Informational Technology

Fernando Santos, Calvin University

How should we approach the relationship between faith and work in the information age? In times when most work is done via interfaces (e.g., mostly screens) that manipulate digital information, new challenges arise regarding how to sustain Christian spirituality in the dynamics between body and symbol, or presence and "re-presence", in face of the various tasks we constantly perform in the informational world, such as clicking, reading/writing, or even programming. Therefore, we aim to highlight the importance of this theme and start tracing some directions for thinking about a virtuous interaction with information in a way that would acknowledge the worship dimension of human life.

Is the Earth Actually Flat? An Introduction to Geometric Transitions

Thomas Shifley, George Fox University

It is a scientifically confirmed fact that the Earth is a round (almost)sphere. However, intuitively when we look out, it seems flat. In this talk, I will address why it isn't such a crazy mathematical idea to think the Earth is flat. We will start by defining what exactly a "geometry" is, and then explore how we can take "limits" in a certain sense, leading to the idea of geometric transitions. Time permitting, I will show where my own work comes into play with transitions of three-dimensional Thurston geometries.

Trammel-Crafting for the Quadratrix

Andrew Simoson, King University

Hippias of Ellis (fifth century BC) is credited with inventing/discovering the quadratrix curve, a curve he used to solve one of the three famous impossibility constructions from Euclidean geometry, and which also solves a second one as well. How did he do this, and how can we fashion a physical trammel using easily available material so as to generate an ideal quadratrix curve?

Utilizing Technology as a Tool for Addressing Contemporary Challenges for Older Adults Living in Smart Homes

Babafemi Sorinolu, Houghton College

In this talk, I will discuss how the application of technology has served as a powerful tool for tackling contemporary challenges in the context of smart homes usage and the health of older adults living within their homes. Technology offers innovative solutions and opportunities for positive change in these areas. I will share past real-world examples of how I have applied cutting-edge technologies such as artificial intelligence, blockchain, and mobile application technology to address pressing issues and drive progress in this field. Additionally, I'll delve into the future roadmap of research, highlighting how we can further harness the full potential of technology for creating a better world. Join me as we explore the transformative power of technology in enhancing the quality of life for older adults and shaping the future of smart home healthcare.

Finding Faith in the Public Sphere

Emily Sprague Pardee, Edinboro University of Pennsylvania (retired)

Christ first got through to me while I was an itinerant cellist driving along an Interstate, then patiently taught me while I earned my mathematics degrees in measure theory in, then taught at, public universities. My talk will highlight some of the guides and roadsigns He planted in the people I worked with and in the higher mathematics I learned. The mathematics will be rigorous; the anecdotes will reflect my Christian growth in the public sphere.

Gödel's Gibbs Lecture

Michael Stob, Calvin University

Gödel gave the 25th Gibbs Lecture at the AMS meeting at Brown University in 1951. His title was "Some basic theorems on the foundations of mathematics and their implications." Of course, the basic theorems in question were his incompleteness theorems which he proved twenty years earlier. We don't know what he said (he never published anything based specifically on the talk) but he did leave a handwritten manuscript that probably formed the basis of the talk and that was resurrected and published more than forty years later. In this talk. I will look at a small part of this rich manuscript in which Gödel considers the "incompletability" of mathematics and presents a resulting "dichotomy" on the nature of the human mind and its relationship to mathematics.

Plato, Symmetry, and the War for Contemporary Physics

Mitch Stokes, New Saint Andrews College

The idea that mathematical beauty can be a legitimate guide to truth in physics skyrocketed in popularity during the 20th century with the remarkable successes of special and general relativity, quantum mechanics, and quantum field theory. There is, nevertheless, a heated debate in physics about whether beauty is in fact a reliable guide to truth. This debate is fueled by the persistent difficulty of finding a quantum theory of gravity, or even a unified theory of the other three fundamental forces. In this talk, I will argue that physicist's reliance on mathematical beauty is inescapable given the extent to which physics is based upon principles of symmetry. Moreover, I will argue that this aesthetic tradition in physics was actually inaugurated by Plato in his founding of the Western liberal arts tradition (in the Republic and Timaeus). In doing so I hope to show the necessity—and the naturalness—of integrating the sciences with the humanities.

The Effect of Porous Vessel Linings on Red Blood Cell Interactions in the Microvasculature

Carlson Triebold, Point Loma Nazarene University

The non-homogenous distribution of red blood cells (RBCs) in the microvasculature is key to oxygen transport throughout the body. This distribution depends on how RBCs are divided into diverging microvessel branches which is heavily influenced by RBC interaction with the porous endothelial surface layer (ESL) in those branches, and other RBCs. Here, we consider a two-dimensional model of RBCs comprised of interconnected viscoelastic elements suspended in fluid modeled by the Stokes equations for viscous flow. The resistance to flow in the porous ESL is modeled with a Brinkman approximation for porous media (hydraulic resistivity), and its resistance to compression is modeled with an osmotic pressure difference between the free-flowing fluid and the ESL. The presence of the porous ESL in simulations increased RBC distribution non-homogeneity, as well as RBC penetration into the layer and reduced RBC deformation, leading to clotting and decreased release of vasodilators (i.e. ATP and nitric oxide).

Equivalent Completeness Axioms in a Course of Real Analysis

Simon Tse, Trinity Western University

For the sake of defining $\mathbb{R}$, most undergraduate real analysis texts begin with either Dedekind completeness (existence of supremum) or Cantor completeness (non-emptiness of nested closed intervals). Then, a semester long course usually imbues students of a sense of how basic results of analysis, like the monotone convergence theorem, Cauchy criterion of convergence, Bolzano-Weierstrass theorem and the Heine-Borel theorem are proved, seemingly as landmark consequences of the axiom of completeness.

In this talk, I describe the use of insights from reverse mathematics in the classroom to show students that the mentioned theorems are actually logically equivalent starting points! Then, I pose the rhetorical question, "Can we believe one of them without believing the rest of these statements?" so as to provoke students to reflect on the role of faith in mathematical thinking – do we really have no need of such an assumption?

Teaching Cultural Competency in a Christian Computers and Society Course

Russ Tuck, Gordon College

This talk will describe experience adding a unit on cultural competency to a long-standing course on Computers and Society. The primary goal was to prepare students to welcome and work well with diverse students and future colleagues, by developing understanding and empathy to better obey Jesus' command to love our neighbors as ourselves. The new unit focused on the history, mechanisms, and current manifestations of discrimination and injustice, with particular attention to African Americans and women. It was well received by almost all students, and really appreciated by African American and female students.

The rest of the course was mostly unchanged, teaching some ethical theory (to prepare students to negotiate ethical issues with future non-Christian colleagues), exploring ethical issues in creating and using many types of computer systems, and indirect impacts of computer systems, such as changes in social interaction resulting from smart phone use.

Statistics and Data Science Attitudes - A Platform for Collecting Data and Visualizing Results

Alana Unfried, California State University, Monterey Bay

How do attitudes - both your students' and your own - matter in teaching students statistics and data science? This presentation is the culmination of a three-year NSF grant, MASDER (DUE-2013392), focused on developing instruments to assess such attitudes and describe the relationship between attitudes and undergraduate student achievement in introductory statistics and data science courses. My team has developed a family of instruments in each discipline, assessing student and instructor attitudes toward learning and teaching the discipline, respectively, as well as an inventory of the classroom learning environment. I will discuss the scope and validity of the survey instruments, our new account-based website for administering the instruments to your classes, and our automated, customized instructor reports allowing you to visualize your class results in comparison to our national sample of students and their instructors. The website will be available for public use beginning in 2024.

Playing with Mathematics: Art as a Pedagogical Tool

Kevin Vander Meulen, Redeemer University

Hardy has compared a mathematician's work to that of an artist: "a mathematician, like a painter or a poet, is a maker of patterns." This analogy has been used as a window to view the beauty of mathematics. I will describe the way I have used an art project as a method to have general education students think differently about mathematics. The intent was to change perceptions about mathematics as a mere technical tool, or as something static. And to open up space to encourage a playfulness toward mathematics by toying with more than the bare mathematics. This method also fits with a wider goal of describing mathematics as culture making.

Happy and Elated Numbers

Mary Vanderschoot, Wheaton College

The happy function maps a positive integer to the sum of the squares of its digits. A happy number is a number that maps to 1 under iteration of the happy function. Happy numbers have been studied for many decades. This talk will present some recent results about a variation of happy numbers called elated numbers.

Using Gateway Experiences to Reach Underrepresented Groups

Curtis L. Wesley, LeTourneau University

Growing up in my community, drug dealers would often sell gateway drugs such as marijuana as a stepping stone to trying more hard drugs like cocaine and heroin. This initial drug experience could be life-altering and lead the recipient down a path that they did not expect. In a more positive way, gateway experiences such as art, music, and games can be used to introduce underrepresented groups to the mathematical sciences. Framing the mathematical sciences through the guise of art, music and games allow for conversations that may not be feasible otherwise. In my presentation, I will introduce a card game that I created to be a gateway experience to learning trigonometry and more specifically the Pythagorean identities.

Practicing Beauty as a Remedy for Math Trauma

Josh Wilkerson, Regents School of Austin

We have all heard the phrase "I'm not a math person" from students/parents/colleagues at some point. When probed, the vast majority of people with this mindset can relate its development to very specific moments in their math education where they were made to feel 'other' than a mathematician. The recognition of these moments as math trauma have sparked an increase in proposed 'solutions,' mostly focused on building student confidence and making math 'fun.' However, I propose that math trauma isn't always overcome with more doing-of-math. Christian psychiatrist and neuroscientist Curt Thompson has a framework for healing from trauma through the experience of beauty that I believe has a potential analog in the math classroom. This talk will present some pedagogical practices of mathematical beauty as well as resources for further study.

Faith at a Public College; Ideas, Experiences, and Resources

Dusty Wilson, Highline College

ACMS has a tremendous legacy of helping Christian mathematicians think about the intersection of faith and mathematics. However those of us at public (or non-sectarian) universities can be left wondering how to apply these ideas in our setting?

After 20+ years of teaching at a public and ethically diverse campus (78% students of color) and more than a decade working with the ministry Faculty Commons including the last three as their lone Faculty Fellow from a two year college, I would like to share ideas, experiences, and resources to help others who are asking the questions, "What does it mean to be a Christian who happens to be a professor at a public institution?"

Computational Mathematics: A Way of Thinking

Matthew Wright, St. Olaf College

Computational skills are undeniably important for math majors, yet the development of such skills is often overlooked. In particular, many students lack the ability to take an open-ended math question, design and run computational experiments (in a computer algebra system or programming environment), and formulate conjectures based on their observations. In this talk, I will present some ideas for filling the gap in students' computational abilities, with examples from the Modern Computational Mathematics course at St. Olaf College. This course helps students develop computational fluency and skills that they can use to gain mathematical insight, discover patterns, ask good questions, test conjectures, and explore a result to see if it merits proof. Moreover, computational exploration can not only make advanced mathematical topics more accessible to students, but also provide transferable skills that students can use to tackle problems arising outside of mathematics.

Every Valley Shall Be Raised Up, and Every Mountain Made Low, and the Rough Places…a Crane?

Rebekah Yates, Houghton University

In Spring 2024, I taught a 300-level course on the mathematics of origami. In this talk, I'll share some of the mountains and valleys in the course, both the mathematics and the origami. Time permitting, we'll fold some things together, too!

Data Science that Welcomes Everyone

Ryan Yates

In this talk I will share my approach to introductory data science for majors and minors that made broad connections to many disciplines while giving a strong foundation for further study and useful skills for everyone.

Faith Lighting in the Shadow of AI

Shelley Zhang, Gordon College

As generative artificial intelligence (GAI) tools such as ChatGPT draw the world's attention to AI, the most intriguing subfield in computer science, concerns are also shared on the potential influence of AI on education, human life, and even society structure. This presentation aims to discuss the advantages and limitations of GAI, evaluate the potential impacts of AI from a Christian worldview perspective, explore the fundamental difference between AI and human life, and further consider the integration of faith in the computer science discipline.

Building Mathematics Teaching Efficacy in Preservice Elementary Teachers: The Power of a Mediated Field Experience Mathematics Methods Course

Valorie Zonnefeld, Dordt University
Ryan Zonnefeld, Dordt University
Luralyn Helming, Dordt University

In an effort to increase mathematics teaching efficacy among preservice elementary teachers, Dordt implemented a Mediated Field Experience (MFE) in their methods of elementary mathematics course. The MFE model moves content pedagogy courses onsite to K-12 schools with embedded teaching experiences. Attendees will learn about the design of the MFE and how it increased both preservice elementary teachers' self-efficacy for teaching mathematics and mathematical dispositions. Quantitative, pre- and post-surveys of mathematics teaching efficacy revealed statistically significant increases over the semester. A qualitative analysis of PSTs as sophomores and seniors further explored and supported this increase in the PSTs' self-efficacy for teaching mathematics as well as interesting changes in mathematical dispositions.

Appendix 4

Conference Participants

24th Association of Christians in the Mathematical Sciences Biennial Conference
May 28 - June 1, 2024

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