Association of Christians in the Mathematical Sciences

Eighteenth Biennial Conference Proceedings, June 1–4, 2011

Westmont College, Santa Barbara, CA

Edited by Russell W. Howell

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Introduction

Ninety participants, eight spouses, and three children came to Westmont College for the eighteenth biennial conference of the Association of Christians in the Mathematical Sciences. They enjoyed superb weather during the conference dates of June 1–4, 2011. Unfortunately, those who chose to spend some extra time in Santa Barbara and stayed through June 5 experienced a rare day of rain. Coincidentally, the last time ACMS convened at Westmont (1993) there was another rare rain event. It might be helpful to California if ACMS could schedule yet another meeting at Westmont when the state is in the midst of its next drought!

Among those attending, about ten were at their very first ACMS meeting. All participants enjoyed two presentations given by each of the the three invited speakers: Art Benjamin (Harvey Mudd College), Fred Brooks (University of North Carolina), and Glen Van Brummelen (Quest University).

As the conference schedule indicates (page 1), there were thirty-three contributed papers in parallel sessions dealing with a variety of topics in mathematics and computer science. Abstracts for the invited addresses and the contributed the papers can be found beginning on page 4. Twenty of the contributed papers, which begin on page 15, are part of these *Proceedings*.

There were also two panel discussions relating to the interaction between mathematics and computer science: within institutions, and within ACMS. A final panel session gave an an overview of a recently-completed ACMS book project: *Mathematics Through the Eyes of Faith*. This project involved eleven ACMS members and was published by HarperOne, San Francisco.

Several informal activities complemented the scheduled sessions. On Thursday evening about fifty delegates enjoyed viewing the skies from Westmont's observatory, which houses the largest telescope between Los Angeles and San Francisco. Early on Friday morning twenty-one hikers make a trek up the foothills to *Montecito Overlook*. A picture of their venture can be found in the conference photos section, which begins on page 210.

The traditional conference banquet took place on Friday evening. After Saturday morning parallel sessions dealing with faith integration issues the conference concluded with a worship service.

We at Westmont considered it a privilege to host this event. We wish you every blessing in your respective vocations and look forward to seeing many of you again in 2013 at the nineteenth biennial ACMS conference. Bethel University (St. Paul, Minnesota) has agreed to be the host site. In 2015 the conference is planned to convene in Canada for the first time, hosted by Redeemer University College (Ancaster, Ontario).

Russell Howell
Conference Coordinator

ACMS 18th Biennial Conference Schedule
Westmont College, June 1–4, 2011

Wednesday, June 1

5:30 – 6:30 p.m.	Dinner
7:00 – 8:30 p.m.	Opening Session (Winter Hall, Room 210) Introduction of first-time participants Invited Address: Art Benjamin, Harvey Mudd College *Mathemagics!*
8:30 – 9:30 p.m.	Refreshments

Thursday, June 2

7:45 – 8:45 a.m.	Breakfast
9:00 – 9:30 a.m.	Welcome from Gayle Beebe (President of Westmont); Devotional (Room 210)
9:30 – 10:30 a.m.	Invited Address: Art Benjamin, Harvey Mudd College (Room 210) *Combinatorial Trigonometry (and a method to DIE for)!*
10:30 – 11:00 a.m.	Break
11:00 – 12:00 p.m.	Invited Address: Fred Brooks, University of North Carolina (Room 210) *The Computer Scientist as Toolsmith*
12:00 – 1:00 p.m.	Lunch (ACMS board meets during this time)
1:30 – 2:30 p.m.	Contributed Papers: Assorted Mathematical Topics, I (Room 106) Nicholas Zoller: "The Mathematics of Cubic Sudoku" Brian Beasley: "Waring's Problem and Extensions" Bryant Mathews: "Tropical Mathematics" Contributed Papers: Assorted Computer Science Topics (Room 216) Nathan Gossett: "The Need for Graphics in Computer Science" Gene Rohrbaugh: "The Moral Status of Intelligent Machines" Wayne Iba: "Real Simulations and Simulated Reality"
2:30 – 3:30 p.m.	Break
3:30 – 4:30 p.m.	Contributed Papers: Faith and the Mathematical Sciences, I (Room 106) Jason Wilson: "Natural Law in the Secular vs. Biblical Mind" Jeremy Case: "Preacher-Kid Mathematicians" Loredana Ciurdariu: "Pascal's Thoughts and the Scriptures" Contributed Papers: Mathematics and Pedagogy, I (Room 216) Kristen Camenga: "Reflections on a Capstone Course" Rebekah Yates: "Linear Algebra as a Discussion Class" Steve Lovett: "Bringing REU into the Classroom"
4:30 – 5:30 p.m.	Discussion with Fred Brooks (Room 210)
5:30 – 6:30 p.m.	Dinner
7:00 – 8:00 p.m.	"Divide and Conquer or Merge and Maximize?" (Panel/Discussion, Room 106) Joint vs. separate Mathematics and Computer Science Programs Phillip Lestmann, Michael Stob, David Stucki, Maria Zack
8:00 – 9:00 p.m.	"Computer Science and ACMS" (Discussion, Room 106) Led by Matt DeLong and the ACMS board
9:00 – 10:00 p.m.	Observatory Viewing

Friday, June 3

7:45 – 8:45 a.m.	Breakfast
9:00 – 9:30 a.m.	Devotional and Announcements (Room 210)
9:30 – 10:30 a.m.	Invited Address: Glen Van Brummelen, Quest University (Room 210) *Mathematics Across Cultures: How Can Worldview Affect Mathematics?*
10:30 – 11:00 a.m.	Conference Photo (Winter Hall Lawn) and Break
11:00 – 12:00 p.m.	Contributed Papers: Mathematics and Statistics (Room 106) Mike Stob: "*R* on the Cloud" Sam Wilcock: "A Bayesian Secondary Analysis in an Asthma Study" Talithia Williams: "The Misapplication of Statistics in American Life" Contributed Papers: Computer Science in the Liberal Arts (Room 216) Gene Rohrbaugh: "The Place of Computing in the Liberal Arts" Ryan Botts: "Lessons Learned: A Journey in Computational Science" Lori Carter: "Lessons Learned: A Journey in Computational Science" Open Discussion: "Computer Science in the Liberal Arts"
12:00 – 1:00 p.m.	Lunch
1:30 – 2:30 p.m.	Invited Address: Glen Van Brummelen, Quest University (Room 210) *Trigonometry, Ancient Astronomy, and the Birth of Applied Mathematics*
2:30 – 3:30 p.m.	Break
3:30 – 4:30 p.m.	Contributed Papers: Assorted Mathematical Topics, II (Room 106) Michael Rempe: "The 25 Billion Dollar Linear Algebra Problem" Justin Marks: "Pattern Recognition in Large Data Sets" Jonathan Leech: "Skew Lattices in Algebra and Computer Science" Contributed Papers: Philosophical/Historical Topics, I (Room 216) Gordon Swain: "History of the Area Between a Line and a Parabola" Robert Brabenec: "Thinking Philosophically about Mathematics" Maria Zack: "Using Original Historical Texts in the Classroom"
4:30 – 5:30 p.m.	Contributed Papers: Faith and the Mathematical Sciences, II (Room 106) Mark Colgan: "Connecting Calculus with Biblical Ideas" David Stucki: "Teaching Infinity, and Beyond!" Josh Wilkerson: "Faith, Mathematics, and Process Theology" Contributed Papers: Mathematics and Pedagogy, II (Room 216) Mary Walkins: "History of Mathematics: An Exercise in Strengths" Judy Palagallo: "Calculus Communication Circles" Nicholas Willis: "Two Integrative Projects for Freshman Majors"
6:00 – 7:30 p.m.	Conference Banquet (Founders' Room in the Dining Commons)

Saturday, June 4

7:45 – 8:45 a.m.	Breakfast
9:00 – 10:00 a.m.	Contributed Papers: Book Projects Relating Faith and Mathematics (Room 106)
	Doug Phillippy: “A Text with a Christian Perspective”
	Panel: “ACMS Project: *Mathematics Through the Eyes of Faith*”
	Jim Bradley, Matt DeLong, Russell Howell, Kim Jongerius,
	Steve Lovett, Robert Myers, Ray Rosentrater, Kristen Schemmerhorn
	Contributed Papers: Philosophical/Historical Topics, II (Room 216)
	Donna Pierce: “Study Abroad Program: Worldview Perspectives”
	Nathan Moyer: “Connecting Students with Philosophy, History, Faith”
	Eric Gossett: “The Search for Hamilton”
10:00 – 10:30 a.m.	ACMS Business Meeting (Room 210)
10:30 – 11:45 a.m.	Sharing and Worship Service (Room 210)
12:00 – 1:00 p.m.	Lunch

Campus Map

Abstracts of Presentations

Brian Beasley (Presbyterian College)
Where, Oh Waring? The Classic Problem and its Extensions

In 1770, Edward Waring published his famous conjecture that every positive integer may be expressed as the sum of four squares, nine cubes, nineteen fourth powers, etc. Ever since, mathematicians have tackled not only Waring's original problem but also a wide variety of generalizations. This talk will sketch brief outlines of Waring's life and the history behind the eventual solution to his problem. In addition, it will present some of the related questions currently being studied, such as expressing sufficiently large integers as sums of powers, sums of powers of primes, and sums of unlike powers.

Arthur Benjamin (Harvey Mudd College)
Mathemagics!

In this fast-paced performance, I will demonstrate and explain how to mentally add and multiply numbers faster than a calculator, how to memorize 100 digits of pi, how to figure out the day of the week of any date in history, and other amazing feats of mind. I have presented his mixture of math and magic to audiences all over the world.

Combinatorial Trigonometry (and a method to DIE for)!

Many trigonometric identities, including the Pythagorean theorem, have combinatorial proofs. Furthermore, some combinatorial problems have trigonometric solutions. All of these problems can be reduced to alternating sums, and are attacked by a technique we call D.I.E. (Description, Involution, Exception). This technique offers new insights to identities involving binomial coefficients, Fibonacci numbers, derangements, and Chebyshev polynomials.

Ryan Botts and Lori Carter (Point Loma Nazarene University)
Lessons Learned: A Journey in Computational Science

Inspired by work on building a computational science program and student questions about modeling, we aim to discuss some of our experiences with computational science. We will first clarify what computational science is, why it is a legitimate science, why it is worth our students' time, and what makes it a challenging field. We will also discuss how computer scientists, mathematicians, and laboratory scientists each have something different to contribute to the field.

Robert Brabenec (Wheaton College)
Thinking Philosophically about Mathematics

I will present a broad overview of the important periods throughout history, from the time of the ancient Greeks until the present day, when mathematicians were encouraged to think philosophically about their discipline. While the three major philosophies of mathematics—logicism, intuitionism, and formalism—were proposed and developed during the period roughly from 1880 to 1930, there are several other distinct times when philosophical concerns influenced mathematics.

It is instructive to be aware of the mathematical developments during the nineteenth century that led to this emergence of formal philosophies after 1880. These include such items as the discovery of non-Euclidean geometry, the bringing of rigor into calculus, the development of abstract algebraic structures, the definition and construction of real numbers, and Cantor's theory of infinite sets. An understanding of the overall development of these major concepts in their historical context can help both faculty and students to better understand and enjoy the discipline of mathematics.

Kristin Camenga (Houghton College)
Asking and Answering Questions: Developing Independence in a Capstone Course

This talk will share ideas and reflections on a senior capstone course with a focus on helping students develop skills for lifelong mathematical learning by asking questions and seeking answers. Classroom activities and assignments will be shared that aim to help students learn to research, read a variety of sources, respond to speakers, and extend problems.

Jeremy Case (Taylor University)
Preacher-Kid Mathematicians

Euler, Abel, and Riemann were all children of ministers. Euler's and Riemann's fathers hoped their sons would go into the ministry, but mathematical potential won out. This talk will examine how mathematical historians portray these shifts away from clergy life towards mathematics. Furthermore, we will provide a survey of other interesting mathematicians who were pastor's kids, or PK's.

Loredana Ciurdariu (Polytechnic University of Timisora)
On the Beauty of Some of Pascal's Thoughts Seen in the Light of the Holy Scriptures

It is better to study Pascal's character from a different perspective as well, more precisely by his writings, by that which defines him. There are assumptions that his health might have deteriorated following the experiments he had done using mercury. He talks in his Pensées about faith, grace and purity of the heart, about the peoples and the way in which God leads them, about wisdom, dreams and hopes, and that which lies in the human heart. If statistics were gathered concerning the most used books and verses from the Bible in the Pensées, these would be: Ecclesiastes, Proverbs, Matthew, Mark, Jeremiah, Hebrews, Romans, Luke, Isaiah, Psalms. But those which occupy a central position are, of course, the ones used by Lord Jesus Christ, such as: John 3:16, when he speaks about faith, Luke 17:33, when speaking about the the wager, and Matthew 16:26, when talking about the thinking reed. For Pascal there is no antagonism between science and faith.

Mark Colgan (Taylor University)
Encouraging Students to Connect topics in Calculus with Biblical Ideas

My goals for the students in my Calculus I class go beyond learning the standard content of pre-calculus and calculus. Particularly since this is also a general education course, I also want them to learn to work in groups, to experience real-world applications, to learn to communicate effectively, and to understand mathematical life lessons, many of which are supported by Scripture. To help students explore these ideas, I ask them to write short reflection papers on topics such as working in groups, the fourth dimension, things we should maximize and minimize, financial stewardship as it relates to exponential growth, etc. I will share how I design these assignments, some sample reflections, and how students have responded to this approach.

Eric Gossett (Bethel University)
The Search for Hamilton

In the summer of 2009, my wife and I were in Dublin. I decided we should visit the bridge (location) where Hamilton scratched the quaternion equations. This turned out to be quite difficult. I will present a photo journal of that (eventually successful) search, as well as discuss the annual walk to the bridge sponsored by the mathematics department of the National University of Ireland, Maynooth.

Nathan Gossett (Bethel University)
The Need for Graphics in Computer Science

This talk will focus on the benefits of offering a course on programming Computer Graphics in an undergraduate Computer Science curriculum. A sample course outline will be provided, as well as a discussion of ways to conduct lectures, labs and a list of suggested assignments. A discussion of "dos and dont's" will also be presented, including a list of required prerequisite courses and skills that students would need in order for the course to be a success.

Wayne Iba (Westmont College)
Real Simulations and Simulated Reality

Movies such as *The Matrix* have stimulated popular interest in "brain in a vat" scenarios. Amidst the traditional questions of mind, we tend to overlook an integral enabling component—the world simulation—which merits consideration in its own right. When facing the simulations in these imagined scenarios, we struggle with conceptual muddles regarding what's real and not. In this paper, I argue that simulated worlds are every bit as real as the one we inhabit. This turns out to be important when considering the possibility, as suggested by Nick Bostrom, that the world we experience as "real" is actually a simulation. Can such a prospect be reconciled with an Orthodox Christian perspective? While the metaphysical status of simulations that I posit moves us towards an integration, significant obstacles remain to be addressed. I consider some of these remaining challenges and explore the associated stakes.

Jonathan Leech (Westmont College)
Skew Lattices in Algebra and Computer Science

Noncommutative variations of lattices have been studied since the physicist Pascual Jordan introduced the subject in 1949. The talk will survey recent developments, indicating connections to structural aspects of idempotents in rings, as well as to some research in computer science.

Stephen Lovett (Wheaton College)
Bringing REU into the Classroom

Mathematics graduate programs and companies that employ math majors often want to ascertain an applicant's potential for research. However, in many undergraduate courses, assessments consist only of regular exercise sets, quizzes, and in-class tests. Without doing a senior research thesis or landing an official REUs, students do not regularly gain experience in or an appreciation for research. Courses in the humanities regularly require students to write in the discipline, progressively preparing them methodologically for "writing in the field." This begs the question: could math departments do a little more to prepare students to use mathematics beyond college? In this talk, we explore options for incorporating undergraduate mathematical research and writing into regular course assignments. The speaker will present some investigative project questions he has used in a variety of courses and illustrate the results with actual student work.

Justin Marks (Colorado State University)
Pattern Recognition in Large Data Sets

Pattern recognition in large data sets, such as a collection of digital images or videos, is often achieved through considering the data as residing on a matrix manifold. We illustrate using digital images of human faces collected at Colorado State University. Building on recently established geometric properties of the Stiefel and Grassmann manifolds, we propose a new approach for computing retractions, i.e., maps from the matrix manifold's tangent space to the manifold. The result is a family of new algorithms that can be used for geometric optimization.

Bryant Mathews (Azusa Pacific University)
Tropical Mathematics

Tropical mathematics is an alternate version of mathematics, obtained by appending the number infinity to the real numbers, replacing addition with minimum, and replacing multiplication with addition. Many theorems in linear algebra and algebraic geometry have analogues in the tropical context. My students and I have been working on the tropical version of the minimal rank problem, which asks for the minimal rank of a matrix subject to certain constraints imposed by a graph. I will introduce the problem and report on preliminary results.

Nathan Moyer (Whitworth University)
Connecting Students with Philosophy, History, Faith

This talk describes a class project that helps students wrestle with the deep philosophical questions of mathematics in a relevant way. Through article readings and group discussions, various historical views of mathematical philosophy are examined and critiqued. An emphasis is placed upon students making meaningful connections between their own views of mathematics and their personal Christian faith or worldview.

Judith Palagallo (University of Akron)
Calculus Communication Circles

A Calculus Communication Circle is a communication network for Advanced Placement Calculus teachers. In northeast Ohio the Circle provides a forum where teachers meet to share ideas about mathematics and the teaching of calculus. In addition, faculty members from area colleges conduct workshops for the Circle two or three times per academic year. I will discuss the creation of the Circle and progress it has made over its three year existence. Perhaps you will want to start a Calculus Circle in your area!

Doug Phillippy (Messiah College)
What Mathematics has Taught me about God and Running

This talk will focus on the third draft of my text: The Study of Mathematics: Developing a mature understanding of mathematical thought with consideration of Christian faith and Vocation, a text targeting first-year mathematics majors attending Christian institutions of higher education. The talk will give an overview of the development of this text which has taken place over a seven-year period, but it will focus on the work done to complete the third draft of the text, work that was completed while on my sabbatical during the spring of 2011. Unlike the previous two drafts of this text, the third draft, which is more than double the size of the second draft, will mark the completion of this project and be a product ready for publication. This talk will describe the three-fold nature of the text, which is intended to tell some of my own journey (hence the title of the talk), to introduce mathematical topics that are both of interest and importance to first-year mathematics majors, and to help students understand what it means to consider their studies and future careers from a Christian perspective.

Donna Pierce (Whitworth University)
Math History Study Abroad Program: Worldview Perspectives

In January 2011 fifteen Whitworth University students participated in a program where they studied math history in a historical context on-site in Italy, Germany and England. Walking amongst the Roman engineering wonders, experiencing the mathematical beauty of Renaissance art and architecture, seeing the inventions, original papers and instruments of Galileo, DaVinci, Euler, Newton and others, and learning from experts about the effects of war and oppression on society, gave the students new insights into the worldview perspectives of men and women responsible for major developments in mathematics. In this presentation we will share some of the highlights of that trip, focusing on what the students learned about the faith and worldview perspectives of these mathematicians.

Michael Rempe (Whitworth University)
The 25 Billion Dollar Linear Algebra Problem

Soon after its debut in 1997, Google quickly became the most popular search engine on the web, largely because of its fast response rate and the quality of its search results. Compared to other tools, Google seemed to do a better job of placing the most relevant results near the top of the results list. In this talk I will present an overview of PageRank, one of the factors Google uses to arrange its search results. This surprisingly simple and intuitive algorithm is based on a familiar linear algebra topic, and therefore can be used as a motivating example or an instructional module in a Linear Algebra course.

Gene Rohrbaugh (Messiah College)
From Augustine and Aquinas to Asimov: The Moral Status Of Intelligent Machines

The recent victory of IBM's Watson over human Jeopardy champions is but one sign of the progress that is being made in the field of artificial intelligence. Intelligent machines are not only growing in their cognitive capabilities; they are also being given greater autonomy to act in the real world. Autonomous machine agents are being deployed in areas such as vehicle navigation, military reconnaissance, health care, space exploration, and even weapons systems. Secular and Christian computer scientists likely agree that actions in such domains are ethically consequential; however they may not agree on the prospects of building machine agents that can be expected to act ethically over the long run. Whether ethical behavior is possible, let alone likely, depends in part on one's view on the nature of ethical behavior as well as its source.

This paper juxtaposes the ideas of Augustine and Aquinas, representing traditional Christian thought, with those of Asimov, representing modern secularism. A self- identified atheist and open critic of evangelical Christianity, Asimov might be expected to have little in common with a fourth century Catholic Bishop and a thirteenth century Dominican priest, yet on many points they find agreement. Even more surprising, in some ways, the two Church fathers seem to grant such machine agents a higher moral status than did Asimov. Even though they obviously did not write about robots per se, Augustine and Aquinas articulate clear principles on the moral status of beings both human and non-human. After beginning with a brief introduction establishing the moral import of machine agents, the paper is structured around two central questions: (1) Are such agents morally considerable, either directly or indirectly? and (2) Are they truly moral agents? The paper closes by making a connection to student learning, describing how these and other engaging philosophical questions can be used to stimulate student interest in grand challenges in computing.

The Place of Computing in the Liberal Arts Curriculum

Modern liberal arts curricula, designed to promote the development of a broad knowledge base and critical thinking skills, commonly include requirements in literature, languages, philosophy, history, mathematics, and science, all of which have had a longstanding presence in western education. In contrast, computer and information science, a relative newcomer to the academic table, is often marginalized as having only professional or technical merit, of value primarily to specialists. But the chronological happenstance of a field's discovery should not be the sole measure of its value in educating liberally. Instead, each field should be evaluated in terms of how it can contribute to meeting the foundational objectives that drive an institution's academic programs.

In this paper, I explore the place of computing in the contemporary liberal arts curriculum. I begin by reporting the results of an exploratory poll of computing educators conducted earlier this year among participants of the CS-CHRISTIANITY mailing list. Participants were asked a series of questions designed to elicit their descriptions of how computing fits into the curricula at their respective institutions. The stories gathered in response were grouped into four distinct categories: (1) no computing requirement; (2) a requirement that can be fulfilled by computing courses as well as other courses; (3) a general distribution requirement taught by faculty in a variety of disciplines; and (4) an explicit computing requirement developed and taught by faculty in computing. The remainder of the paper outlines some challenges one might face in advocating for incorporation of computing into the curriculum, as well as strategies and approaches one might take to do so.

Michael Stob (Calvin College)
R on the Cloud

The program *R* is a system for statistical computations and graphics. It is quickly becoming the system of choice both for professional statisticians and classroom instruction. We use it at Calvin for statistical instruction at all levels because it is free, easily user-extensible, professional quality, and relatively easy to learn. During this past semester four classes at different levels used R on the cloud—that is *R* was accessed via the web from a remote server. The web interface to *R*, called RStudio, made *R* simple to learn and use for our students. In this talk I will demonstrate how, through RStudio, *R* can easily be used in an introductory statistics course.

David Stucki (Otterbein University)
Teaching Infinity, and Beyond!

Otterbein University, as a part of its transition to a semester calendar for the academic year 2011-12, has redesigned it general education Integrative Studies program from the ground up. A signature feature of the new program is a freshman year seminar (FYS) that is structured to meet common learning objectives for entering students that are independent of specific content. This allows faculty from any discipline to propose "passion" courses that will engage students in ways they are unlikely to have encountered in high school. In the autumn of this past year I piloted a course, offered to sophomores as a religion/philosophy credit, that will be packaged as an FYS this fall as one of our inaugural FYS offerings. Taking its title from the catch-phrase from Toy Story, the course *To Infinity and Beyond* is open to students from any major and carries no prerequisite.

I will present the course design, along with a bibliography of resources, and talk about adapting the course to various audiences. I would also like to report some of the anecdotal evidence of its impact on students (and on me).

Gordon Swain (Ashland University)
History of the Area Between a Line and a Parabola

The quadrature of the region bounded by a line and a parabola was first accomplished by Archimedes. But many mathematicians since then have also solved this area problem, using it as a "test case" for their methods. We will survey the math, the historical development, and the people themselves.

Glen Van Brummelen (Quest University)
Mathematics Across Cultures: How Can Worldview Affect Mathematics?

As Christians we often assert that worldview affects and informs every aspect of our lives. Within mathematics this is usually explored on a philosophical level. But broader cultural assumptions about the structure of knowledge have had surprisingly deep effects on the nature and practice of mathematics. Concentrating mostly on ancient Greece and pre-modern China, we shall compare practices to witness the effect of societal practices on beliefs concerning the nature of mathematics.

Trigonometry, Ancient Astronomy, and the Birth of Applied Mathematics

Trigonometry, one of the oldest of the mathematical sciences, was born in ancient Greece from the need to predict the positions of the heavenly bodies. The arrival of an efficient place value number system from Babylon allowed geometry to become quantitative, changing the astronomical game entirely. Astronomers and geographers in Greece, and later in India and medieval Islam, now had unprecedented powers to describe mathematically the world they observed around them. Not least among trigonometry's accomplishments was its role in one of the most successful predictive theories in the history of science: Ptolemy's epicyclic model of planetary motion. The fact that this model was completely wrong, yet its mathematics proved fundamental to the growth of science for at least two millennia, leads to some interesting speculations on Wigner's observation of the "unreasonable effectiveness of mathematics in the natural sciences."

Mary Walkins (Lee University)
History of Mathematics: An Exercise in Strengths

As a leader in strengths-based education, Lee University encourages all new students, since fall 2003, to take the Gallup StrengthsFinder to determine their top 5 signature themes (out of a possible 34). At Lee, the syllabus for the History of Mathematics course calls for students to write a paper on a mathematician. As an added dimension, students were asked to think critically about—and incorporate—the strengths they believe that mathematician may have. Each student was required to compare and contrast his or her strengths with those of a mathematician. This was done with the hope that, as aspiring mathematicians, students may be inspired to persevere to make their mark in the history of mathematics, since math is still evolving. In this presentation, through an exercise in strengths, I will share examples of how students were inspired by each mathematician selected.

Samuel Wilcock (Messiah College)
A Bayesian Secondary Analysis in an Asthma Study

A recent study published in the New England Journal of Medicine by the Asthma Clinical Research Network (ACRN) compared three different treatments for their effectiveness in treating adults with uncontrolled asthma. This talk will briefly describe the study design and its results, then detail the beginnings of a secondary analysis using Bayesian methods to estimate the parameters of interest. The methods will be explained, and the preliminary estimates given and contextualized. The talk will conclude with a discussion of the next steps and the goals for further analysis of the data in this study.

Josh Wilkerson (Texas A&M University)
What We Can Learn from Process Theology: Integrating Faith and Mathematics

There are numerous examples of great thinkers attempting to harmonize mathematical advances with the canons of the historical Christian faith in an attempt to make Christianity relevant to modern, intellectual society. Process theology arose because its adherents believed it to be the best of such attempts. Upon close inspection, however, process theology can only be labeled as a departure from Christian Orthodoxy. Yet the process perspective still has something to offer for the construction of a framework within which a distinctly Christian perspective of mathematics might be developed. As contemporary Christian mathematicians wrestle with integrating their faith and their discipline, it is the contention of this paper that they will benefit greatly from studying process theology and, in particular, from critically examining the ways in which it departs from orthodoxy.

In the first section of this paper I will summarize the tenets of process theology and examine the deep interplay between this school of thought and developments in the field of mathematics. Process theology will be presented as a clear historical example of how theological foundations have significant impact on the practice mathematics.

In the next section, I will critique the tenets of process theology in light of scripture and the historical teaching of the Christian church. This critique will focus specifically on the doctrines of divine revelation, God as Trinity, and the person and work of Jesus Christ. From this analysis the paper will conclude that process theology radically departs from Christian orthodoxy and therefore its proposal for integrating orthodox Christian faith and mathematics cannot be accepted.

Jason Wilson (Biola University)
Natural Law in the Secular vs. Biblical Mind

The scientific community agrees that there are “laws” in nature, for example, the “law of gravity.” Scientists also agree that mathematics is the language within which these laws are written. Differences arise in the understanding of the nature of natural laws. This presentation will briefly consider some secular viewpoints. The majority of the time will reflect on biblical texts leading to a proposed model of the nature of natural laws. The model centers on Jesus Christ as the pre-existent creator who is the source of life (Proverbs 8:22–35; John 1:1–4). What are the implications of this model for having, and living, the life of Christ as a professional mathematician / scientist (John 17:3, 21–23)? The presentation will close with remarks on this question.

Talithia Williams (Harvey Mudd College)
The Misapplication of Statistics in American life

H.G. Wells once said, “Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.” The widespread use of statistics plays an influential role in persuading public opinion. As such, statistical literacy is necessary for members of society to critically evaluate the bombardment of charts, polls, graphs, and data that are presented on a daily basis. However, what often passes for “statistical” calculations and discoveries are often meant to be manipulative. This talk will examine applications of statistics in American media and give examples of where statistics has been grossly misused.

Nicholas Willis (George Fox University)
Two Integration of Faith and Mathematics Projects for Freshman Mathematics Majors

Two projects will be presented that integrate faith and Mathematics in a freshman Introduction to Proofs class at George Fox University. The first project asks students to look at the life of a Christian Mathematician. The focus of this project is to show students that many great mathematicians also had immense faith. The second project asks students to take a close look at their own life. How do they plan to live a life of Christian faith in their chosen profession? Both projects are designed to encourage students to look at their careers in Mathematics as a vocation.

Rebekah Yates (Houghton College)
A Second Course in Linear Algebra, or Teaching Students to Read and Speak Mathematics

In this talk I will describe my experience in teaching a second course in linear algebra as a discussion class. Students read the text and presented both material from the text and assigned problems each day. I’ll share the successes and failures of this experiment, including the structure of the course, how the course was graded, and student responses to the class.

Maria Zack (Point Loma Nazarene University)
Using Original Historical Mathematics Texts in the Classroom

Like many mathematicians who teach courses in the history of mathematics, my degree is in mathematics and not the history or philosophy of mathematics. For the last fifteen years, as part of our institution’s commitment to the liberal arts, I have been teaching courses in the history of mathematics as well as incorporating historical topics into a wide variety of mathematics courses in our curriculum.

This incorporation of historical material into mathematics classes has been a way to help students make connections between their general education courses and their major course work. In the last three years I have experimented with using original historical texts (in translation if needed) in four classes: number theory, linear algebra, history of mathematics, and a course on mathematics, art and architecture. These experiments have included pilot testing some existing material that is part of an NSF sponsored program and well as developing materials of my own.

This talk discusses the successes, failures and lessons learned from a "non-expert" working with original texts in a number of settings. It also provides some practical information about resources, translation and the development of student activities.

Nicholas Zoller (Southern Nazarene University)
The Mathematics of Cubic Sudoku

In the last decade the Sudoku puzzle has fixed itself in America's puzzle consciousness. Sudoku puzzles share space with crossword puzzles and word finds in newspaper puzzle sections, and several books have been written for the Sudoku playing community. Mathematicians are among the most dedicated Sudoku players. Although some are content with simply solving puzzle after puzzle, others have used tools from combinatorics and algebra to study its important properties. The popularity of Sudoku has spawned several variants. We examine one such variant called Cubic Sudoku. It is played on a board set up to look like three adjacent faces of a cube. The regular Sudoku rules apply, and a new rule is added. Following the lead of other Sudoku researchers, we use Gröbner bases to attempt to count the number of distinct puzzles. We also show how to use the rules of Cubic Sudoku to deduce strategies for solving puzzles efficiently.

Where, Oh Waring?
The Classic Problem and its Extensions

Brian D. Beasley
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Brian Beasley (B.S., Emory University; M.S., University of North Carolina; Ph.D., University of South Carolina) has taught at Presbyterian College since 1988. He became a member of the Mathematical Association of America in 1989 and joined ACMS in 2007. Outside the classroom, Brian enjoys family time with his wife and two sons. He is an enthusiastic Scrabble player, a not-so-avid jogger, and a very shaky unicyclist.

In the 2009-2010 academic year, one of our mathematics majors, Olivia Hightower, became interested in the history of Edward Waring and his famous conjecture about expressing positive integers as the sum of kth powers. Olivia's investigation eventually led to her honors project on Waring's Problem, in which she focused on the history of the conjecture, the eventual proof that all positive integers may be written as the sum of at most nine cubes, and the work of Hardy and Wright in establishing lower bounds in the case of sufficiently large integers. Her research renewed her professor's own interest in Waring, leading to the following article. This paper will sketch brief outlines of Waring's life and the history behind the eventual solution to his problem. In addition, it will present some of the related questions currently being studied, such as expressing sufficiently large integers as sums of powers, sums of powers of primes, and sums of unlike powers.

We begin with a short summary of the biography of Edward Waring. Born in Old Heath, Shropshire in 1736, he excelled in mathematics from an early age. Waring eventually became the sixth Lucasian Chair in Mathematics at Cambridge University, following Isaac Barrow, Sir Isaac Newton, William Whiston, Nicolas Saunderson, and John Colson [9]. In 1770 he published his masterpiece, *Meditationes Algebraicae*, in which he studied symmetric polynomials and cyclotomic equations, stated Wilson's Theorem, and gave the famous generalization of the four squares conjecture [17]. As a member of the Royal Society, Waring was awarded the Copley Medal, but he resigned from the Society in 1795, citing poverty [14]. Waring died just three years later in Pontesbury, Shropshire.

At this point, the interested reader may wish to take the following short quiz on Waring. Answers will be provided at the end of the four questions.

Waring Quiz

1. What was Edward Waring's educational background?
    (a) no formal training—self-taught prodigy
    (b) tried both Oxford and Cambridge but dropped out
    (c) graduated from Oxford with high honors
    (d) graduated from Cambridge with high honors

2. Waring became the Lucasian Professor of Mathematics:
   (a) at age 26, before receiving his M.A.
   (b) at age 30, right after receiving his M.A.
   (c) at age 42, having been passed over once
   (d) at age 58, having been passed over repeatedly
3. Between 1767 and 1770, Waring:
   (a) resigned as Lucasian Professor
   (b) took up the study of medicine
   (c) took up the study of astronomy
   (d) proved the Four Squares Theorem
4. Waring was described as:
   (a) one of the greatest analysts in England
   (b) having an awkward and obscure writing style
   (c) a man of pride and modesty, with pride predominant
   (d) all of the above

Answers (see [9], [14], and [17])

1. (d) Waring attended Magdalene College of Cambridge University, and graduated with high honors.
2. (a) Waring was actually named the Lucasian Chair before officially receiving his master's degree.
3. (b) Waring earned his M. D. but never fully entered into practice as a physician.
4. (d) Each of these descriptions comes from a different contemporary of Waring.

To set the stage for Waring's Problem, we briefly examine the history behind expressing integers as sums of squares, using Burton's text [1] as our main source. One may trace this problem all the way back to Diophantus, although Claude Bachet in 1621 produced the first conjecture that four squares suffice for all positive integers. It should come as no surprise to students of mathematics history that Pierre de Fermat claimed to have a proof of this four-square conjecture yet never shared it. Later, Leonhard Euler would contribute two crucial lemmas, enabling Joseph-Louis Lagrange to complete the proof in 1770.

In that same year, Waring published *Meditationes Algebraicae* and generalized the four-square conjecture to higher powers. He claimed that every positive integer may be written as the sum of at most 9 cubes, at most 19 biquadrates, and so forth. The standard translation of Waring's Problem is: Given $k \geq 2$, there is a least positive integer $g(k)$ such that every positive integer may be written as the sum of $g(k)$ kth powers. In particular, Lagrange proved that $g(2) = 4$, while Waring conjectured that $g(3) = 9$, $g(4) = 19$, etc.; however, it would be many years before mathematicians found a proof of Waring's claims for $k \geq 3$.

Indeed, before 1909, the only known value of $g(k)$ remained $g(2) = 4$. In fact, the only values of k for which the *existence* of $g(k)$ had been established were $k = 2, 3, 4, 5, 6, 7, 8$, and 10. In 1909, David Hilbert produced an impressive breakthrough, proving the existence of $g(k)$ for every $k \geq 2$. Also in 1909, Arthur Wieferich published a proof that $g(3) = 9$; in 1912, Aubrey Kempner filled a gap in Wieferich's proof [13].

Although progress was slow in establishing exact values for $g(k)$, Waring and others quickly found lower bounds. For example, with cubes, 23 provides a "worst-case scenario":

$$23 = 2(2^3) + 7(1^3) \qquad \Longrightarrow \qquad g(3) \geq 9.$$

We note that 23 is not only smaller than 3^3 but also is one less than a multiple of 2^3; similarly, 79 serves as an example for fourth powers:

$$79 = 4(2^4) + 15(1^4) \qquad \Longrightarrow \qquad g(4) \geq 19.$$

In 1772, Johann Euler, the oldest child of Leonhard, generalized this pattern: If $q = \lfloor 3^k/2^k \rfloor$, then

$$g(k) \geq (q-1) + (2^k - 1) = 2^k + q - 2.$$

According to Delmer and Deshouillers [2], Euler may have believed that equality holds in this lower bound result. Consequently, they label the following claim as

Euler's Conjecture. Given $k \geq 2$, if $q = \lfloor 3^k/2^k \rfloor$, then $g(k) = 2^k + q - 2$.

As previously noted, this conjecture agrees with Lagrange's result $g(2) = 4$ and Waring's claims of $g(3) = 9$ and $g(4) = 19$.

In their survey article [16] on Waring's Theorem, Vaughan and Wooley provide a summary of the history of establishing Euler's Conjecture for various values of k. In 1936, Dickson and Pillai showed that the conjecture holds for $7 \leq k \leq 100$; four years later, Pillai proved it for $k = 6$ as well. In 1957, Mahler made a crucial contribution by showing that the conjecture fails for at most finitely many values of k. Stemmler extended the range to $6 \leq k \leq 200,000$ in 1964, while Chen proved the case $k = 5$ the next year. In 1986, Balasubramanian, Deshouillers, and Dress established the case $k = 4$. Three years later, Kubina and Wunderlich showed that the conjecture holds for $4 \leq k \leq 471,600,000$.

We now shift our attention from expressing every positive integer as a sum of kth powers to expressing every *sufficiently large* integer as such a sum. For $k = 2$, this still requires four squares, as every positive integer congruent to 7 modulo 8 requires four squares as a sum. However, for $k = 3$, the only positive integers to require nine cubes as a sum are 23 and 239. In fact, only finitely many positive integers require eight cubes as a sum. To answer the question of finding the smallest number of cubes needed to express all sufficiently large integers as a sum of cubes (as well as generalizing to kth powers), we define $G(k)$ as follows.

Definition. Given $k \geq 2$, let $G(k)$ be the least positive integer such that every *sufficiently large* integer may be written as the sum of $G(k)$ kth powers.

We note that $G(k) \leq g(k)$, with equality only for $k = 2$. In addition, as Vaughan and Wooley [16] observe, only two values of $G(k)$ are known, namely $G(2) = 4$ (Lagrange) and $G(4) = 16$

(Davenport). For cubes, Linnik showed in 1942 that $G(3) \leq 7$, while a modulo 9 argument establishes $G(3) \geq 4$. The actual value of $G(3)$ remains one of the most famous open questions in number theory. In 1851, Jacobi [1] conjectured that $G(3) \leq 5$, while the current consensus is that four cubes suffice; in fact, 7,373,170,279,850 may be the largest integer that requires more than four cubes [3].

Hardy and Wright [7] established the following lower bounds for $G(k)$:
(a) If $\theta \geq 2$, then $G(2^\theta) \geq 2^{\theta+2}$.
(b) If $\theta \geq 0$ and p is an odd prime, then

$$G(p^\theta(p-1)) \geq p^{\theta+1} \qquad \text{and} \qquad G(\frac{1}{2}p^\theta(p-1)) \geq \frac{1}{2}(p^{\theta+1}-1).$$

(c) For any $k \geq 2$, $G(k) \geq k+1$.

In 1922, Hardy and Littlewood proved that $G(k) \leq (k-2)2^{k-1}+5$, while the current best upper bound for $G(k)$ for sufficiently large k is due to Wooley (1995):

$$G(k) \leq k\left(\log k + \log\log k + 2 + C \cdot \frac{\log\log k}{\log k}\right),$$

where C is some positive constant [16].

The following lower bounds come from Hardy and Wright's results [7] and are conjectured to be the actual values of $G(k)$; the upper bounds are due to Vaughan and Wooley [16]:

$$6 \leq G(5) \leq 17;\;\; 9 \leq G(6) \leq 24;\;\; 8 \leq G(7) \leq 33;\;\; 32 \leq G(8) \leq 42;\;\; 13 \leq G(9) \leq 50.$$

Indeed, much work remains to be done in calculating $G(k)$, even for small values of k.

In order to discuss another variant of Waring's Problem, we first state the following definitions. Given a subset A of the set of positive integers, let $A(n)$ be the number of elements in A which are less than or equal to n. Then the *natural density* of A is

$$\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

In particular, if A is the set of positive integers having a property P, then *almost every* positive integer has property P if and only if $\delta(A) = 1$. This motivates the next step, as found in [16].

Definition. Given $k \geq 2$, let $G_1(k)$ be the least positive integer such that *almost every* positive integer may be written as the sum of $G_1(k)$ kth powers.

We note that $G_1(k) \leq G(k) \leq g(k)$. Again following Vaughan and Wooley [16], we observe that only six values of $G_1(k)$ are known:

$G_1(2) = 4$ (Lagrange); $G_1(3) = 4$ (Davenport); $G_1(4) = 15$ (Hardy and Littlewood);
$G_1(8) = 32$ (Vaughan); $G_1(16) = 64$ (Wooley); $G_1(32) = 128$ (Wooley).

For the next version of Waring's Problem, we pause to note two connections between Waring and Christian Goldbach. In 1742, Goldbach conjectured that every even integer greater than two may be written as the sum of two primes, and every odd integer greater than one is either prime or the sum of three primes. Waring became the first to publish this conjecture [17]. Furthermore, Waring's problem and Goldbach's conjecture may be combined into the "Waring-Goldbach problem" [16] as follows: Given $k \geq 1$, let $H(k)$ be the least integer s such that

$$p_1^k + p_2^k + \cdots + p_s^k = n$$

has a solution in primes p_i for all sufficiently large integers n satisfying certain congruence conditions.

In 1937, Vinogradov established his famous result that all sufficiently large odd integers may be written as the sum of at most three primes; equivalently, $H(1) \leq 3$. The next year, Hua [8] proved that $H(k)$ exists for every k, with

$$H(k) \leq k^2(4 \log k + 2 \log \log k + 5)$$

for $k > 10$. Hua also showed that $H(k) \leq 2^k + 1$ for every k; in particular, $H(2) \leq 5$ and $H(3) \leq 9$ remain the best known results in those two cases [11]. In 1987, Thanigasalam established $H(6) \leq 33$ and $H(8) \leq 63$, while in 2001, Kawada and Wooley proved that $H(4) \leq 14$ and $H(5) \leq 21$ (see [16] and [11]). A recent result due to Kumchev [11] gave the current best known upper bound for $H(7)$ as 46, an improvement over the previous "record" of 47.

Many more variations on Waring's Problem exist, including the use of mixed powers. In 1949, Freiman conjectured the following generalization:
Let $\{n_i\}$ be a sequence of integers with $2 \leq n_1 \leq n_2 \leq \ldots$. For any n_j, there exists an integer r such that all sufficiently large integers N are representable in the form

$$N = x_1^{n_j} + x_2^{n_{j+1}} + \cdots + x_r^{n_{j+r-1}}$$

for positive integers x_i if and only if $\sum_{i=1}^{\infty} n_i^{-1} = \infty$.

In 1960, Scourfield proved this result, and in 1966, Thanigasalam established a similar result for prime powers [15]. As noted in Ford's paper [4], in 1951, Roth considered the special case of mixed powers using successive powers. He showed that there is a number s such that all sufficiently large integers N may be written as a sum of s successive powers, starting with a square; that is,

$$N = \sum_{i=1}^{s} x_i^{i+1} = x_1^2 + x_2^3 + \cdots + x_s^{s+1}.$$

Roth not only showed that $s = 50$ works but also established that for *almost all* integers, s can be taken to be 3. Roth's upper bound of 50 for s has subsequently been improved [5], as follows:

Thanigasalam, 1968: $s \leq 35$; Vaughan, 1971: $s \leq 26$;
Thanigasalam, 1984: $s \leq 20$; Brüdern, 1988: $s \leq 17$; Ford, 1996: $s \leq 14$.

It is conjectured that $s = 3$ should hold in the sufficiently large case, as it does in the almost every case [4].

We mention just three more of the variations of Waring's Problem currently being researched.

1. J. Liu, M. Liu, and Zhan [12] have studied expressing integers as sums of squares of primes and powers of 2: $N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}$.

2. Ford [6] has researched the question of polynomial summands: Given $f(x) \in \mathbb{Z}[x]$ with no fixed integer divisor $d \geq 2$, write all sufficiently large integers N in the form $N = f(x_1) + f(x_2) + \cdots + f(x_s)$, where each x_i is a positive integer.

3. Finally, Kononen [10] has obtained results for solutions over finite fields: Given the finite field $\mathbb{F}_q$, find the minimum s such that we may write every element in $\mathbb{F}_q$ in the form $x_1^k + x_2^k + \cdots + x_s^k$ with $x_i \in \mathbb{F}_q$.

In conclusion, given the remarkable number of generalizations arising from Waring's Problem, we venture the conjecture that Waring himself would have been quite pleased to see so many results extending from where it all began, over 240 years ago:

"Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c. usque ad novemdecim compositus, & sic deinceps."

– Waring (1770)

"Every integer is a cube or the sum of two, three, ..., nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth."

– Waring (1991 translation by Weeks [17])

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Lesson's Learned: A Journey in Computational Science

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Ryan Botts and Lori Carter (Point Loma Nazarene University, San Diego, CA) have worked together with the other members of their department on an interdisciplinary major in computational science. They have conducted many interviews with industrial computational scientists and analyzed computational science programs at other institutions.

Abstract

Inspired by work on building a computational science program and student questions about modeling, we aim to discuss some of our experiences with computational science. We will first clarify what computational science is, why it is a legitimate science, why it is worth our students' time and what makes it a challenging field. We will also discuss how computer scientists, mathematicians and laboratory scientists each have something different to contribute to the field.

1. Introduction

Recent work in our department towards implementing a computational science minor and student interest in research projects in computational science has led to many interesting conversations and research about computational science. Our aim here is to discuss our observations of this experience. In particular, we aim to address what computational science is, why it is worthwhile as a discipline and as a skill, and what makes implementing such a program difficult. We hope that such an exploration can provide some guidance when implementing computational science components into your curriculum.

2. Defining Computational Science

Computational science, as a discipline, is highly interdisciplinary in nature, but some clarity as to what it is will help aid in identifying good problems and developing such a curriculum. We consider two definitions here to highlight several significant aspects of the field. A clear understanding of what the field is becomes critical for identifying problems that fall within the scope of computational science as opposed to, only say computer science.

According to, [5] computational science "combines computer simulation, scientific visualization, mathematical modeling, computer programming and data structures, networking, database design, symbolic computation, and high-performance computing with various scientific discipline." Nearly any subject in mathematics or computer science can be combined with a scientific discipline to obtain a computational science problem. Computational science includes, but is not limited to bioinformatics: it can involve many of the other sciences. It involves problems that go beyond just modeling, it can include problems dealing with data handling and manipulation. Although it is possible that a computational science problem involves only an applied mathematician or a computer scientist coupled with a bench scientist, it is much more likely to involve collaboration between experts in many of these fields.

As we have solicited interdisciplinary problems from the bench scientists at our university we have found the need to clearly distinguish between problems that are simple applications of math or computer science and those which are truly interactive. There are many problems that have roots in the bench sciences, but once formulated as computational problems, no longer require interaction with the bench science. That is once the problem has been abstracted a mathematician or computer scientist can perform all of their analysis without any knowledge or interaction of the underlying science. For example, an initial idea for a computational science problem proposed by a scientist in our building was essentially a problem of learning how to use GIS to display biological information. Although a good computer science problem, the fundamental solution to the problem did not require knowledge of the underlying biology. Although this problem was an applied problem in computer science it required little interaction between the disciplines.

Thus we can see a need to more clearly emphasize the nature of the interaction between the computational disciplines and the bench sciences. A second definition for computational science is, "The integration of knowledge and methodologies from mathematics, statistics, and computer science to analyze and solve problems from the STEM disciplines [1]. This definition highlights the symbiosis between the STEM disciplines and the computational methodologies. Problems in which the computational methods directly influence the nature of the scientific pursuit, and vice versa, make good computational science.

What is meant by integration of these disciplines can be clarified through an example. Suppose a chemist wishes to build a new material with some optimal properties. Constructing and testing new materials is both time consuming and costly, thus trying to directly build enough combinations of new materials to find an optimal one would be nearly impossible. Another approach would be to construct an algorithm that could predict the properties of unconstructed materials using the properties of known materials. A scientist could then find which materials are optimal inputs and then physically construct the best materials. In this problem, the models are directly influenced by the laboratory science, and the results of the computational methods directly affect the nature in which the laboratory science is performed. Clearly the computation and the bench science are in constant interaction with interaction with each other.

3. A Worthwhile Pursuit

While not easily defined, computational science is definitely being talked about and the conversation is coming from both the mathematical scientists and the bench scientists. In 2006 the Society for Industrial and Applied Mathematics (SIAM) published a report entitled Computational Science and Engineering for Undergraduate Students in which they report on current undergraduate programs in the computational sciences, and encourage the formation of additional programs [1]. Another report, BIO2010, published by the National Academy of Sciences, also encourages the inclusion of curriculum in computational science noting that while much of the postgraduate research in biology is enhanced using computational methods, biology educators have not adapted curriculum to introduce students to these computational tools [2].

3.1. More Complete and Expedient Results

There are definitely problems that cannot be solved by one discipline alone. A classic example is the mapping of human genome, completed in 2003. While scientists contributed the genomic understanding, the pattern matching algorithms and computing power required was largely a result of contributions made by mathematicians and computer scientists.

In some cases, computational methods can facilitate an experiment, allowing the scientist to reach a conclusion in a shorter amount of time. While a computer model is not a replacement for laboratory experimentation, it can be used to guide the scientist in the most promising directions. A simulation can be performed multiple times with different parameters. The results can be analyzed to identify which design parameters are of greatest interest, helping the bench scientist perform fewer experiments at a lower cost. Often there are tasks, such as image processing, that can be automated, freeing the scientist up for more bench work.

3.2. Improved Tools

The computer scientist and mathematician can work with the bench scientist to provide essential data storage and manipulation tools. There are numerous large repositories of information collected by bench scientists. These repositories contain information related to such topics as genetics, different species, and disease characteristics. Even if functional databases have already created to house this data, it is often the case that more information could be gleaned from these sources if they could be combined. Furthermore, it is often helpful if the result of a query can be automatically translated into a different form so it can be fed into a second program automating the workflow for the final result.

Mathematicians and Computer Scientists tend to approach research in ways that differ from bench scientists. Those in the quantitative sciences like to explore broadly. They ask questions like 'What variables affect student retention in the freshman year of college?" The research of a bench scientist often, of necessity, has a very narrow focus. They might ask a question such as "What is the effect of detergent toxicity on soil bacteria?"

Often, finding the right narrow research question could be facilitated by computational tools such as data mining which produces association rules based on a breadth of data. For example, a scientist could collect a variety of different soil samples, analyze the components of that soil, including the bacterial content and then run those results through a data mining tool that produces association rules. The output would be a set of rules such as: "when components X and Y are found in the soil, 75% of the time the level of bacteria K is below level J." The rules with the highest confidence levels might be worth pursuing as research questions. Statistical analysis is another area where computational personnel might be of assistance. Not only could they help with the analysis itself, but they could make recommendations regarding the setup of the experiment. For example, they might have more knowledge as to how many samples might be useful to produce a reliable result.

As mentioned earlier, simulations and models can prove to be useful tools for scientists. They can provide direction for a procedure, but they can also help with teaching. Models can allow scientists and students to visualize a complex structure speeding comprehension and encouraging exploration.

3.3. Benefits to Your College or University

Recruiting students as majors in mathematics and computer science has been an area of concern for many years [3]. The concern is especially great for females. A 2006 report revealed that only 0.6% of females in the incoming freshman class were interested in a mathematics major, compared 0.9% of males. For computer science, 3% of all males in the freshman class intended to major in computer science, but for women, the percentage was an alarming 0.4%. There is evidence that computational science is a way to interest more women in computational majors.

A 2006 study involving over 600 high school students in pre-calculus and calculus courses included a survey asking questions about a potential major in computer science [4]. This study revealed that the number one reason female students would consider a major in computer science was if it was in the context of using it in another field. Computational science provides that context.

One of the first computational science programs for undergraduate students was developed as an emphasis of a computer science major at Wofford College [5]. Of the 18 students who graduated from the program between 2002 and 2007 in this small college in South Carolina, 44% of them were women. Many of those interested in interdisciplinary research teach at smaller liberal arts institutions. One of the benefits of a liberal arts education is that students get a broad education, and can see how their intended major fits into a larger picture. Since computational science is an interdisciplinary pursuit, it fits well with the liberal arts education paradigm.

Finally, at the time of this writing, many research funding opportunities are drying up for economic reasons. Funding for research and curriculum development in the computational sciences is a notable exception. The National Science Foundation (NSF) currently has grant opportunities available for both undergraduate research and curriculum development [6]. Funding is available from private organizations as well.

4. Challenges

4.1. Challenges as a Discipline

Although there are many ways in which collaborations are clearly difficult, our conversations with scientists in other disciplines have pointed out some unexpected challenges. These challenges stem from differences in training and perspective: exploratory versus a well-guided approach to solving a problem, language barriers, awareness of other disciplines and biases against "simulation" are greater challenges than one might expect. Acknowledging these challenges is necessary for developing effective collaborative projects.

Laboratory scientists and in silico scientists approach problems in a quite different manner.

Bench scientists design well-guided experiments with very precise goals, making full use of experimental control. Although mathematicians and computer scientists may have goals for a project they often take a more exploratory approach to experimentation using post hoc analysis to find our answers. This leads to drastically different approaches to project design and expectations for a project.

This difference in approach can also lead to biases. A scientist expressed this bias well, "they are just simulated experiments on simulated results yielding simulated answers, they aren't real." A bench scientist may not see simulation as a valuable exploratory tool, which can lead to a devaluation of the computational methods as legitimate laboratory tools.

In [2] it is noted that training in the computational disciplines is insufficient within current biology curricula. Similarly mathematicians and computer scientists are usually not trained in a bench science. Thus there will be language barriers between the disciplines. For example, a biologist would use "function" to indicate the role of an object within some physical system. A mathematician would use the term to indicate a very precise type of object having inputs and outputs. This has been a large barrier to effective interdisciplinary conversations we have attempted.

Finally, by definition computational science covers a broad range of disciplines, disciplines which are quite broad themselves. Laboratory scientists may not know the types of computational tools available and mathematicians or computer scientists may not be aware of all of the tools within their own discipline. The range of problems included in computational science leads to a reduced awareness of what can be done.

4.2. Curriculum Challenges

Since computational science is a fairly new field, textbooks and other teaching resources are limited. Repositories of pre-designed teaching modules involving computation and science can be found, however, if one is tenacious. Links to examples of such modules can be found at [7].

Combining bench scientists and mathematical scientists in the same classroom can present a challenge. If the professor is attempting to present both science and computational topics, there are times when he will need to be teaching at a more remedial level than one of the groups needs. In addition, the two groups are used to different types of learning. The typical science class often requires a great deal of memorization. Conversely, the majority of computational classes involve derivation rather than memorization.

One of the jobs of a computational scientist is to be able to choose the correct tool to facilitate the work of the scientist. Tools might involve different programming languages, databases, statistical analysis tools, and visualization and modeling tools. Consequently the professor of computational science course will be to required to learn, and to present to the students a large variety of tools. Presenting these tools in the context of a meaningful project could present quite a challenge.

5. First Steps

During a panel discussion held during the 2011 conference for the Special Interest Group on Computer Science Education (SIGCSE), four participants talked about the success or failure of the computational science programs at their universities. Top determinants for success or failure included buy-in of the science faculty and buy-in of the administration. Of course, student buy-in is essential too. Interest from all three of the players would be influenced by the potential of future jobs and internships in the local area.

5.1. Science Faculty and Administrative Buy-in

A computational science curriculum cannot be taught exclusively by the computer science and mathematics faculty. At the very least, the science faculty must help provide project ideas, and encourage science students to explore the possibility of entering the program. They must be aware of the math and computer science students who will now be taking some of their classes. Ideally, they will tailor their classes so that they are more accessible to the computationally oriented students, and include computational modules.

At Point Loma Nazarene University (PLNU) the mathematics and computer science faculty have very close working relationships with our biology, chemistry and physics faculty. All of our departments are housed in the same building. The faculty members that we talked to were willing to work with us, but had no idea how computation could be included to enhance their research. Initial brainstorming sessions were tentative at best. But, as we encouraged the scientists (usually on an individual basis) to talk about their work, ideas began to take shape. We now have a summer research student and math/CS faculty mentor working on image processing to support 2 separate science research teams. We are talking about providing some software and analytical support for another project. But, we know that much more can be done.

We believe that it is up to us to find ways to help the scientists. Applied mathematicians and computer scientists are used to working in interdisciplinary teams, but bench scientists are not. The BIO2010 report [2] presents ideas for including mathematics and computer science into biology curriculum. There are several Computational "X" (computational physics, computational chemistry, etc.) journals in publication. We should be reading and listening, and then talking.

While there is currently funding available for starting computational science programs, sustainability must be considered. Consequently, administrative and marketing support of the endeavor is essential.

5.2. Understanding the Needs of Industry in Your Area

The SIAM report [1] emphasizes the value of internships and real-world projects in a computational science program. Furthermore, if we are to prepare students for future jobs, we need to understand what the skill set is that computational science companies in our local region are looking for. In order to find projects and internships, and to know what the skill-set is that we should teach towards, talking to computational science companies is essential.

One way to find these companies and to find the skill sets that they are looking for is to look at job requisitions using employment search engines such as Monster.com and CareerBuilder.com. A little creativity is required to find the positions that are related to computational science as many terms are used for these jobs. Some of the terms include computational scientist, computational biologist, computational chemist, computational physicist, bioinformatics specialist, biostatistician, scientific programmer, and R&D scientist.

Preliminary research in our area has shown that employers are looking for both hard skills and soft skills. Frequently listed hard skills include MATLAB, Python or some other scripting language, database knowledge, statistics, modeling, and the UNIX operating system along with basic knowledge in biology or another science. Soft skills include excellent written and verbal communication skills, ability to work with people in other disciplines, and good problem solving abilities.

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Thinking Philosophically about Mathematics

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Bob Brabenec will begin his 48th year of teaching at Wheaton College in the fall of 2011. His main academic interests are in analysis and the historical development of mathematics, including the relationship of mathematics and the Christian faith. His most recent book, published by the Mathematical Association of America, is *Resources for the Study of Real Analysis*. He enjoys being active in the lives of eight grandchildren.

1. Introduction

In my early years as a teacher of mathematics, the history of mathematics was seldom mentioned in the classroom. It was viewed as an unworthy topic that would detract from the presentation of mathematics itself. This opinion has dramatically changed over the years, and the history of mathematics is now embraced and used by many mathematicians in their teaching and even research. We might choose to ask a related question. How much philosophy is necessary or helpful for a mathematics teacher to know, and to use in his or her teaching? We see a growing interest in the philosophy of mathematics in our postmodern age, as evidenced by the paper sessions on the philosophy of mathematics at recent joint meetings, and by the number of books with this theme written during the past few decades.

While I have found it easy and profitable to introduce the history of mathematics into my classes, it is much more difficult to do the same thing with the philosophy of mathematics. So as an alternative to using the philosophy of mathematics in our teaching, I suggest we substitute thinking philosophically about mathematics. This has been easier for me to do. I began recently by making a list of philosophers whose views had some impact upon mathematics. I noticed that to a large extent, most of these philosophers lived in one of three distinct time periods—the Greek period (600–200 BC), the early modern period (AD 1600–1800), and the 50 years between 1880 and 1930 when three main philosophies of mathematics were developed by mathematicians. One might add a fourth period (i.e., a postmodern period) since 1965 or so, after mathematicians had time to recover from and process the implications of Gödel's theorem (see Figure 1).

For me, thinking philosophically about mathematics simply means to look at the development of mathematical ideas and to ask questions about why this development happened as it did, and what implications this may have for mathematics as a whole. This paper is a collection of brief comments about the seven time periods indicated in Figure 1. The key period from my point of view is the time from 1800 to 1880 when mathematics assumed much of its present form as an axiomatic development of ideas. This happened when mathematicians looked at the mathematics developed prior to the nineteenth century and thought philosophically about the nature and foundations of this discipline.

2. Period 1: 600–200 BC

Notice the following list of mathematical and philosophical concerns during the Greek period from 600 to 200 BC that were still of interest two thousand years later during the nineteenth century period described in the paragraph above.

1. Calculation of areas and volumes
2. Nature of the infinite (potential or actual)
3. Euclid's *Elements* and the parallel postulate
4. Number and its properties
5. Philosophical and religious ideas about mathematical concerns
6. Nature of the physical world, especially astronomy
7. Logic and reasoning

3. Period 2: 200 BC–AD 1600

Consider the historical richness suggested by the following list of some transition features from the time of ancient Greece to the beginning of the modern period. These features are not part of mathematics itself, but they had a significant influence upon mathematics by shaping the culture in which mathematics developed.

1. Roman empire and Graeco-Roman culture
2. Founders of major religions
3. Hindu and Arabic contribution to number and algebra
4. Rise of the church and its authority
5. Preservation and translations of Greek manuscripts in monasteries
6. Crusades and other conquests
7. Rise of the universities
8. Invention and use of the printing press
9. Renaissance and Reformation and Enlightenment periods
10. Interest in the motion of the planets

4. Period 3: 1600–1800

The early universities served as repositories and encouragers of Aristotelian thinking about science. This view was supported by the Catholic church and its rigidity discouraged investigation of new ideas by individuals such as Copernicus and Galileo. The emergence of radically new ideas in the seventeenth century led to the establishment of scientific societies, such as the Royal Society in England, to investigate the viability of these ideas. Universities took on this role and later became the major setting in which mathematicians would do their work. The intense interest in the nature of the universe and the motion of heavenly bodies by such men as Copernicus, Kepler and Galileo ultimately led to the discovery of what we now call calculus for a fuller explanation and understanding. From the results of Newton and Leibniz in the 1660s and 1670s and throughout the entire eighteenth century, the development and application of calculus dominated all other mathematical concerns in this period.

5. Period 4: 1800–1880

In the senior capstone course at Wheaton, we concentrate on five developments during the nineteenth century which significantly raised the level of inquiry and investigation into the nature of mathematics, as the rate of discovery of new results in calculus slowed towards the end of the eighteenth century. Questions raised during this process led mathematicians to propose several possible philosophical approaches to mathematics towards the end of this century. These five items are listed in roughly chronological order in Figure 2. I include below a few words about each of them. In Figure 2, notice also the identification of items prior to 1800 that influenced the development of these mathematical results, as well as the completely new nature of the mathematical concerns at the beginning of the twentieth century.

1. The preservation and translations of Greek manuscripts such as the *Elements* throughout the Middle Ages, the printing of the first copy of Euclid's work in the late 1400s, an obsession for centuries with the search for a proof of the parallel postulate, and a new indirect approach for a proof of this result suggested by Saccheri in the mid 1700s—all these features served as catalysts for the eventual discovery of non-Euclidean geometry. This in turn led to a new level of questions about the nature of mathematics such as: Which geometry is the correct one? How does math relate to issues of truth? Does Euclidean geometry describe the physical world? Looking back, we see this development began the move towards full acceptance of the axiomatic method as the way to do mathematics.

2. Although efforts to place a rigorous foundation underneath the calculus occurred at the same time as the discovery of non-Euclidean geometry, the primary catalysts for change were very different. We attribute the start of this process to Cauchy's textbook of 1821 in which he introduced the idea of limits as the means to define the basis concepts of calculus. Then he and others developed the theory of convergence and a definition for the definite integral over the next half century.

3. First geometry and analysis, and then algebra came under the influence of axiomatic thinking and the study of structures. Creative individuals such as Abel, Galois, Hamilton and Liouville began to work with ideas that gradually led to formal definitions for the structures of groups, rings, and fields. Some examples of these ideas include the attempt

to solve the general quintic equation and the attempt to trisect an angle according to the rules of the Greek geometers.

4. The arithmetization of analysis program as developed by Weierstrass and Dedekind used the algebraic structure of a field as the means for developing for the first time a careful and precise definition of real numbers, either in terms of equivalence classes of Cauchy sequences of rational numbers or by using cuts of rational numbers.

5. The concept of the infinite is involved with many of these developments—lines extending to infinity from geometry, the limit process from analysis, and the definition of irrational numbers from algebra—but Georg Cantor made mathematicians come to grips with the concept of the actual infinite. It is interesting that it took a man of faith to undertake this challenge and we see even more clearly today how the concept of infinity permeates much of our knowledge. Indeed, mathematics and the Christian faith are inextricably linked by a fascination with the mystery and meaning of the infinite.

6. Period 5: 1880–1930

All these developments during the nineteenth century gave mathematicians a new level of concern with the nature of their discipline, leading to one desire to provide an adequate foundation for mathematics and to a second desire to expand and generalize the discipline. In retrospect, we speak about the presentation of three possible philosophical explanations over a period of about 50 years that were each proposed, developed, evaluated, and found to be wanting.

Logicism, intuitionism, and formalism are the names usually given to these philosophies of mathematics. Logicism developed first following Boole's presentation of the algebra of logic, which was advanced by the work of Peano, Frege, and Bertrand Russell. The complexity of this program, along with an unavoidable need for non-logical axioms, led to a search for a simpler explanation, which was found in the view of intuitionism. This view was developed by Poincare and Brouwer, but its paucity of results prompted David Hilbert to begin his program of formalism around 1915. After some successes, his ambitious program ended with the impact of Gödel's results in 1931.

7. Periods 6 and 7: 1930–2011

It took time for the results and implications of Gödel's incompleteness theorem to be understood and accepted by mathematicians. Other possible philosophies of mathematics were suggested, but the former view of mathematics as the source of truth and complete knowledge of our physical world was gone forever.

We continue to search for an acceptable answer today. Perhaps *Mathematics Through the Eyes of Faith* will be a catalyst for present-day faculty and students to wrestle more successfully with these questions. I believe an understanding of the historical and philosophical overview of mathematical development is needed in order to be prepared to treat these questions in context. I suggest the topics described in this paper as a possible starting point for your investigation, and welcome dialogue with anyone interested in discussing these ideas further.

Seven Periods of Philosophical Involvement with Mathematics

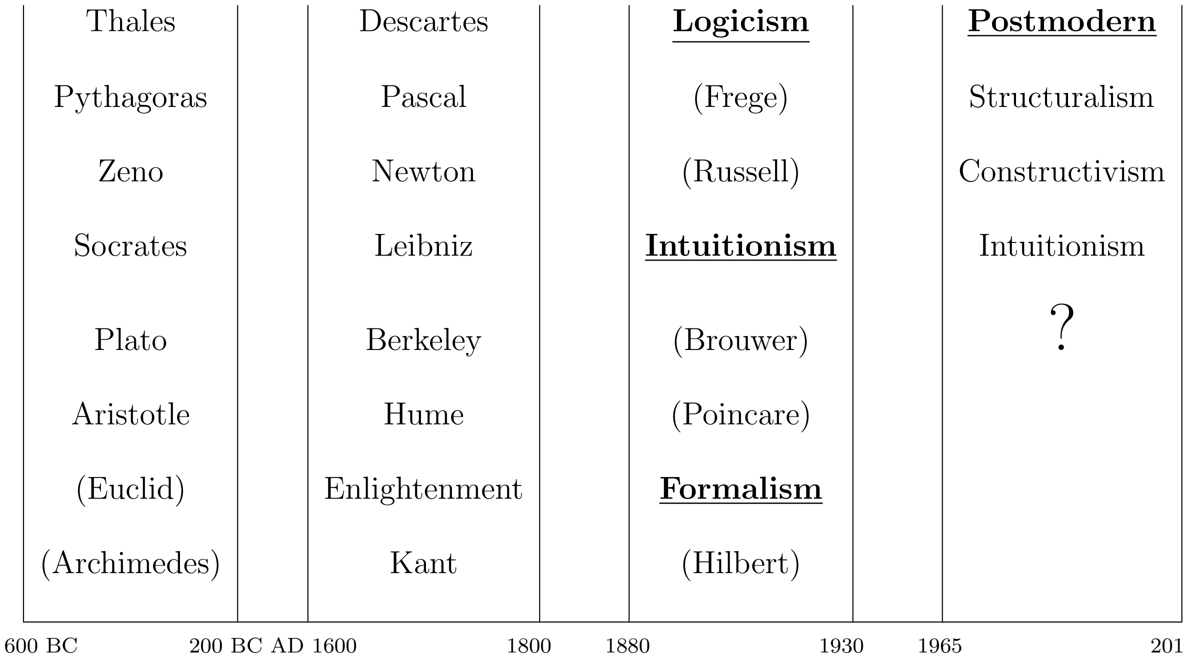

Figure 1

Overview of Mathematics and Its Foundations (Period 4)

	Discovery of non-Euclidean Geometry	
New math content in the modern period $\Longrightarrow$		$\Longrightarrow$ Philosophies of mathematics
	Bringing Rigor into Calculus	
Input from philosophers $\Longrightarrow$		$\Longrightarrow$ Dominance of the axiomatic method
	Beginning of Algebraic Structures	
Contribution from Greek mathematicians $\Longrightarrow$		$\Longrightarrow$ Generalizations and extensions
	Definition of Number	
Topics from science $\Longrightarrow$		$\Longrightarrow$ Gödel's theorems
	Theory of Infinite Sets	

1800 1880

Figure 2

PK Mathematicians

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Jeremy Case (Ph.D., University of Minnesota) lists linear algebra and mathematics capstone as his favorite courses to teach. He serves on his local library board as well as the ACMS board and the editorial board for the Dolciani Mathematical Expositions Series published by the MAA. Jeremy lives in Marion, Indiana with his wife, Felicia, and their three daughters. His father, a United Methodist pastor, still uses Jeremy in some of his sermon illustrations.

Pastors' Children live in strife;
Nobody know the PK life.
—Temporal Sunburn

1. Introduction

The term PK stands for "Pastor's Kid" or "Preacher's Kid." In many instances, the pastor is a prominent and visible member of the community so the children are then scrutinized by association. I Timothy 3:4–5 says that a minister "must manage his own family well and see that his children obey him, and he must do so in a manner worthy of full respect. If anyone does not know how to manage his own family, how can he take care of God's church?" PKs have a reputation of not living up to this biblical standard and have developed several negative stereotypes. Being holier-than-thou or rebellious are the two most common stereotypes [80].

Rebellious PKs include NBA coach Phil Jackson, film director Ingmar Bergman, and atheist philosopher Friedrich Nietzsche. Kim Il-Sung, dictator of North Korea from 1948–1994, claimed his grandfather was a Presbyterian minister and his father was a Presbyterian elder. War criminal Slobodan Milošević was the son of an Orthodox deacon [79].

However, pastors' children have gone on to prominence. With the notable exception of Martin Luther King, Jr., PKs who do become famous go into areas other than the church. Presidents Woodrow Wilson and Grover Cleveland, scientists Ivan Pavlov and Carl Jung, inventors Nikola Tesla and the Wright brothers, and world leaders German Chancellor Angela Frankel, Secretary of State Condoleezza Rice, and former United Kingdom Prime Minister Gordon Brown were all children of ministers. Other PKs include Aaron Burr, John Ashcroft, and Vincent van Gogh. Writers Jane Austen and the Bronte sisters were PKs in the liturgical Anglican Church. With its expressive worship style, the African-American church has produced PK singers Marvin Gaye, Aretha Franklin, Sam Cooke, and Nat King Cole as well as actor Denzel Washington [79].

I am a PK myself, the son of a United Methodist minister. When I came to Taylor University as a faculty member in 1995, four of the six members of the math department were PKs as well. I wondered if there were a connection between mathematics and being a PK. After all, prominent mathematicians Leonhard Euler, Niels Henrik Abel, and Georg Riemann were all sons of ministers. Other math PKs include

- Sophus Lie (Norwegian Lutheran), Lie algebras
- Émile Borel (French Protestant), whose name is attached to Heine-Borel Theorem, Borel measure, Borel sets, and Borel algebras
- Charlotte Angas Scott (Congregational), in 1885 became the first British woman to be awarded a doctoral degree in mathematics
- E.H. Moore (Methodist), first chair of the University of Chicago Math Department
- Harold Fine (Presbyterian), namesake of Princeton's Fine Hall
- Saunders Mac Lane (Congregational), Category theory
- Albert Tucker (Methodist), known among mathematicians for work in game theory and known among the general public as John Nash's thesis advisor

Is there a relationship between being a PK and a mathematician? Why did some select mathematics rather than a career as a clergyman? How is their religious background portrayed? It turns out that the answers to these questions have something to say about the historical development of mathematics and how the focus of mathematical history has changed.

2. Calculus

Isaac Newton (1643–1727) was not actually a true pastor's kid. His father died before Newton was born, but his mother married a minister in a nearby village when Newton was two. His mother moved leaving him in the care of his grandmother, and there is some evidence of bitterness towards his mother and stepfather. While Sir Isaac Newton is generally credited as being one of the two discoverers of calculus, some other PKs were instrumental in the development of calculus in Great Britain.

John Wallis (1616–1703) was the preeminent mathematician in England before Newton. Like his father, Wallis was ordained in the Church of England and came to mathematics somewhat late in his life. His mathematical and historical texts on conics and algebra prepared the soil for later mathematical advancements, and he made substantial contributions to the origins of calculus. His text *Arithmetica Infinitorum* helped inspire Newton to form his underlying ideas of calculus.

James Gregory (1638–1675) was the son of an Episcopalian Church of Scotland minister, but it was his mother who is credited with his mathematical ability and early mathematical education. Gregory worked on the ideas of calculus at the same time of Newton and was very close to coming up with the major calculus ideas. If Newton had not accomplished it first, there is some speculation that Gregory might have been the famous discover in England.

The son of a rector and tutored by his minister uncle, Roger Cotes (1682–1716) helped Newton edit his second edition of *Principia* which led to several numerical techniques such as the

Newton-Cotes quadrature formula. He invented the radian measure of an angle and made advancements in numerical methods, the theory of logarithms, and the integral calculus before dying at an early age. Newton said of Cotes, "... *if he had lived we might have known something.*"

The father of Colin Maclaurin (1698–1746) was a minister in the Church of Scotland and died when Maclaurin was very young. He was taken in by his uncle, another minister. Maclaurin thought about going into the Presbyterian Church but the dissensions in the church caused him to select another path which directed him to the University of Edinburgh. While Maclaurin did not independently discover Maclaurin's series, he did give the first systematic exposition of Newton's work in response to Bishop Berkeley's attack on fluxions. Maclaurin also made contributions in geometry, actuarial science, and numerical methods, but a major theme in his writings was how order demonstrated the existence and nature of God.

Two other PKs show up in a standard sequence of calculus.

- Robert Hooke (1635–1703), whose father was a curate and ran a small school on the Isle of Wight, is associated with Hooke's Law.
- Gabriel Stokes (1819–1903) is known for Stokes' Theorem. His father was a Protestant minister in Ireland. As a counterexample to PK stereotypes, Stokes reportedly strongly held onto his faith [45].

3. Bayes

One 18th century defender of fluxions made his name in the field of probability. Bayes' Theorem, named after the Reverend Thomas Bayes (1702–1761), continues to be a source of keen controversy in probability to this day. Bayes was the son of one of the first six nonconformists ministers ordained in England. He trained at the University of Edinburgh since nonconformists could not matriculate at Oxford or Cambridge and became a Presbyterian minister himself. Since there is little known about Bayes' personal history, there is also debate about what his philosophical writings meant, whether Bayes really understood the theorem's implications, or even whether he first discovered it [73]. It is also unclear what motivated Bayes on the famous theorem in probability. His friend Richard Price, the son of a Congregational minister, inserted his own introduction when he published the result after Bayes' death. Price saw the result relating to theology and providing evidence for the existence of God. (Laplace, the principle developer of Bayesian probability, presumably had no need of that hypothesis.)

Why were so many British pastors' children active in the role of mathematical research in the 17th and 18th centuries but so few afterwards? For most of the second millennium, the clergy were seen as educational leaders in a community and were in one of the few positions requiring education. A minister's son would likely go to school where the opportunities included either the church or education, which often required a theology degree.

Secondly, the time was ripe for calculus. Advancements and communications in algebra, geometry, and mathematical notation had laid the groundwork for its discovery. During that period, the university principally trained clerics and educators. With education primarily in the care of the church, it was natural for sons of ministers to enter similar educational paths where they would be exposed to these mathematical ideas.

Furthermore, British political shifts had their effects on the educational process. After the restoration of the monarchy in 1660, Oxford and Cambridge began discriminating against religious Nonconformists. In response, Dissenting or Nonconformist academies began to train their own clerics. With new vigor to defend its theological and political stances against the established Church of England, the academies developed a climate paving the way for an outpouring of intellectual and scientific advancements.

While mathematics had been interwoven with theology and philosophy prior to the 17th century, the discipline drew nearer to a Newtonian philosophy and a scientific approach to knowledge. As the disciplines became more specialized, mathematics became a detached and separate discipline from theology. The emphasis of mathematics in Great Britain became more practical and less theoretical. Dissenting vocational academies began providing new opportunities in law, medicine, engineering, commerce, and other fields. As Great Britain developed as an industrial and political power, a larger class of society went to school, and the ministry was no longer a primary source of educated individuals.

One exception to the separation of mathematics and theology involved Bishop Berkeley's critique of fluxions. Since the issue connected theological arguments with mathematical ones, the backgrounds of several individuals allowed them to contemplate this issue and develop further mathematical ideas. As a side note, Abraham Robinson addressed the philosophical problem of Newton's fluxions in 1964 with his development of infinitesimals and nonstandard analysis. Robinson could be considered the Jewish equivalent of a PK. His father was a Hebrew scholar but died before Robinson's birth. His stepfather was a known Zionist preacher but was not formally a rabbi. When his stepfather died, Robinson was taken in by a rabbi [18]. There do not appear to be many children of rabbis who have made names for themselves in mathematics which could be due to historical and social factors involving Judaism.

4. East of Eden: The Playfairs John and William

The story of the brothers John and William Playfair illustrate how mathematics interacted with the shift towards empiricism. In his time, John Playfair (1748-1819) was a well-known minister, geologist, mathematician, and professor of natural philosophy at Edinburgh University. Educated at home by his father until he was 14, John Playfair went to school to become a minister in the Scottish church like his father. He succeeded his father as Parish Minister, and then went to Edinburgh during the Scottish Enlightenment. Mathematicians today associate his name with Playfair's axiom, a version of Euclid's fifth postulate.

> **Playfair's Axiom:** Given a point and a given line, there exists a unique line to the given line through the given point.

Even though Proclus referred to the statement in the 400s, the axiom appeared in Playfair's widely used geometry book thereby attaching the name to Playfair.

While John Playfair was well-respected as one of Britain's foremost mathematicians and scientists, his brother William Playfair was a roguish self-promoter whose work was less well-received. William appears to have rejected Christian and gentlemanly behavior in his own life, pursuing fame and fortune through sometimes questionable means. Yet William's work has made a pronounced influence on the mathematical sciences. Ian Spence and Howard Wainer [64, 65, 66, 69, 77] point to William Playfair as the progenitor of the pie chart, the bar chart, and the time-series line graph—three of the four most fundamental statistical graphs. (The scatterplot was developed at the end of the nineteenth century.) Spence and Wainer claim that William Playfair developed these graphs without significant precursors.

In 1786 Playfair published *Commercial and Political Atlas*, which examined English trade during the eighteenth century. The Atlas contained no maps, but did contain several time-series line graphs and a bar chart. According to Spence [68]

> The graphs in the Atlas differ little from those in use today: hachure, shading, colour coding, and grids with major and minor divisions were employed. The charts were used to depict actual, missing, and hypothetical data, the various forms being differentiated by the kind of line used, solid or broken. In addition to functional variation, many charts emphasized area, indicating accumulated or total amounts. All included a descriptive title in the body of the chart and labelling of the axes.

In a later tract William Playfair introduced the pie chart.

William Playfair thought his graphs were an effective means of communicating data and information, which was a new train of thought in the mid-1700s. With a personality disposed to a disdain of convention, he argued repeatedly for his graphs but his graphical approach would not be not accepted in England for several decades. Rhetorical arguments were the primary means of persuasion and investigation. Illustrations had been viewed with suspicion in philosophy and science since Descartes and other philosophers mistrusted the senses. Playfair's graphs did find a better reception in commerce in France and in Germany in part because the diagrams transcended language barriers. Even as the mistrust of pictorial representations lingered, Playfair's work contributed to a shift towards an empirical method of investigation.

William died penniless, whereas John had over 500 mourners at his funeral and was hailed as a great teacher and expounder of theories, admired by all men. It may be too easy to paint John Playfair into the "holier-than-thou" PK stereotype and William as the rebellious PK, but there do seem to be some correlations. While John was the oldest, William was fifth of the eight children. Their father died when William was 12, and John took responsibility of the family's education. William greatly benefited from John's tutelage. William said of his older brother [69], "He taught me to know, that, whatever can be expressed in numbers, may be expressed in lines." John's expertise in geography and mathematics helped as well as his connections to the Scottish empiricist school of philosophers.

While John's academic pursuit led to the Church and later the university, William took an apprenticeship leading to a variety of experiences in engineering, publishing, commerce, map-making, and printmaking. For example, he worked for James Watt preparing and copying drawings for what would become the driving force of the Industrial Revolution: the steam engine. He went on to a series of business ventures with "an often-to-be-repeated pattern of grand purpose, conflict with others, suspicion of wrongdoing, and ultimate business failure [68]." There is evidence he attempted to blackmail another individual. According to Wainer, there was a "money-grubbing, opportunistic, and reckless aspect to [William] Playfair's character," but that these attributes contributed to the invention of statistical charts [77]. His brashness and disregard for the opinions of others allowed him to explore and promote new ideas. The "rebellious" PK ended up making a universal language useful to science and commerce alike.

5. Mathematics within Society

Until recently, there was not a high priority of understanding the social forces around which mathematics developed. When mathematics is discussed in a larger framework, it has been part of the narrative of scientific developments. Since Newton, mathematics has been seen as a pristine discipline, detached from the social, cultural, and political world. Meanwhile, mathematical disciplinary writing has emphasized the ideas rather than the lives of the mathematicians. There often will be a short sentence concerning the mathematician's life and then a much greater exposition of the mathematics.

There are several reasons why the discipline has downplayed the personal and social influences surrounding mathematical developments. Mathematical exposition has been primarily used for understanding how to move the mathematics forward. It is more difficult to determine the thought process and motivation of a creative mathematician than it is to follow a mathematical argument. For a wider audience, the technical nature of mathematics makes it generally inaccessible to the general public, and it is a challenge to connect the mathematics to other areas. Yet recent mathematical historians and expositors are attempting to do just that with greater frequency.

Three of the greatest mathematicians, Euler, Abel, and Riemann, were all PKs. How their lives are portrayed, particularly in their choice to select mathematics over theology, provides a means of understanding changes in mathematical biography and how authors treat the intersection of religion and mathematics.

6. Euler

Most mathematics books keep the biographical sketch of Leonhard Euler (1707–1783) short to move onto his prodigious mathematical output. Furthermore, as George F. Simmons notes [64], Euler's personal life was as "placid and uneventful as is possible for a man with 13 children."

One inconsistently reported piece of information is to which denomination his minster father belonged. The denomination has been reported to be Swiss Reformed, Calvinist, or Lutheran. In 1897, the *Mathematics Monthly* founder, B.F. Finkel, wrote a biography of Euler for the *Monthly* in which he attempted to clear up the manner in the following remarkable footnote [28].

> The Encyclopedia Britannica says Euler's father was a Calvinistic minister, while W. W. R. Ball, in his History of Mathematics, says he was a Lutheran minister. Euler himself was a Calvinist in doctrine, as the following, which is his apology for prayer, indicates: "I remark, first, that when God established the course of the universe, and arranged all the events which must come to pass in it, he paid attention to all the circumstances which should accompany each event; and particularly to the dispositions, to the desires, and prayers of every intelligent being; and that the arrangement of all events was disposed in perfect harmony with all these circumstances. When, therefore, a man addresses God a prayer worthy of being heard it must not be imagined that such a prayer came not the knowledge of God till the moment it was formed. That prayer was already heard from all eternity; and if the

> Father of Mercies deemed it worthy of being answered, he arranged the world expressly in favor of that prayer, so that the accomplishment should be a consequence of the natural course of events. It is thus that God answers the prayers of men without working a miracle."

It is difficult to imagine a theological statement in the *Monthly* or any mathematical journal today. The footnote assumes that the readers have enough of a religious understanding to determine the difference between Lutheran and Calvinist beliefs to conclude that Euler was indeed a Calvinist.

How Euler selected mathematics over theology seems pretty straightforward. Before entering the ministry, Euler's father had been a student of Jacob Bernoulli. When Euler went to the University of Basel to study theology, arrangements were made for him to study with Jacob Bernoulli's son, Johann. As C.H. Edwards in *The Historical Development of Calculus* [26] notes, "Although his father, a clergyman who had studied mathematics under James Bernoulli, preferred a theological career for his son, young Euler learned mathematics from John Bernoulli, and thereby found his true vocation." Euler's calling to mathematics and his exemplary life continues to have a Christian impact today even if Edwards did not mean true vocation in a Christian sense.

Essentially the same narrative shows up in the retelling of Euler's decision to pursue mathematics. His father wanted him to study theology, but Euler was attracted to mathematics and his father relented. This lack of variation is explained by Euler's transcription of his autobiography to his son late in his life [27].

> Thereafter, at the discretion of my family, I had to register at the Theological Faculty; since I was then expected to apply myself not only to theology but especially also to the Greek and Hebrew language, which however did not get on very well since I devoted most of my time to mathematical studies and, fortunately, the Saturday visits with Johann Bernoulli still continued.

Euler then proceeds to describe how he ended up in St. Petersburg.

Joseph Ehrenfried Hofmann remarks, "EULER is one of the most astonishing personalities of the 18th century ... Widely admired by some as the great teacher of Europe, who left his mark on the "mathematical century;" widely despised by others who want to see in him only a living computing machine and make fun of his peculiar philosophical views [27]." For example, E.T. Bell in *Men of Mathematics* claims Euler's theological treatises as the most mathematically unpractical side of his genius. In the end, many mathematical biographies, even those without a religious bent, are quite praiseworthy of his personal life.

7. Abel

Unlike Euler and Riemann, there is no memoir from Abel to explain how he chose to pursue mathematics. Whereas Euler's life is presented in a straightforward fashion, many writers take advantage of Niels Henrik Abel's life as a Romantic tragedy to liven up their writing. As Ioan James [35] notes,

> All the elements of a melodrama seem to be present: the penniless genius dying of consumption in the arms of his childhood sweetheart, while the selfish academicians sit on his masterpiece and the news, that he has been offered the position he so desperately needed, arrives just too late. The true story is tragic enough.

In trying to make sense of Abel's misfortunes, writers explored wider issues than just the ideas of mathematics, providing another emphasis of mathematical biography.

Both Abel's father and grandfather were ministers. The grandfather, Hans Mathias Abel, was a pious man with deep religious convictions. A well-respected man of the community, he held to traditional views regarding the church. Abel's father, Søren Abel, went to seminary in Copenhagen where he was introduced to more rationalistic ways. According to [75], Søren Abel had faith in human being's ability to solve life's problems through reason. Returning to Norway, he became a strong nationalist and eventually a member of parliament. He strongly supported education both in his local parish and at the national level. He favored economic development to make Norway self-sufficient to the point he pledged an annual donation to Norway's first university, the University of Christiana, which would later cause the family great financial difficulty.

Abel's brother, the oldest in the family, went to the university but suffered from depression and had to be sent home. It fell to Abel to be the standard bearer for the family. The University of Christiana focused on preparing educated workers and civil servants for Norway's desired sovereign state rather than preparing ministers. Abel was not sent to study theology but rather to fulfill his father's wishes for education. The development of his mathematical talent began when he was 16 and his tutor had to be replaced for beating another student to death. Abel's new tutor, Bernt Holmboe, gave students independent tasks to challenge them, and Abel excelled in mathematics under this plan. While he was mediocre in other disciplines, his engagement and ability to solve problems impressed his teachers and opened up his path to mathematics.

Family troubles contributed to his focus on mathematics. The marriage of his parents, Søren and Anne Marie, had started in more optimistic economic times but had declined with Norway's fortunes. At the time of their courtship, Norway had been in a favorable position providing materials to warring Europe. The daughter of a successful merchant in Norway, Abel's mother had an affinity for parties and social gatherings. However, the political and economic landscape changed when Britain attacked Denmark-Norway and instituted a blockade. Norway's economy waned, affecting the financial situation of the local parish and the Abel family.

By the time Abel was thriving under Holmboe's tutelage, Abel's father had suffered a series of declines in fortune. His father's catechism was attacked as too rationalistic and as an example of wrong thinking and mistaken theology. As a politician he disgraced himself by accusing another politician based upon flimsy evidence. Drinking heavily, his father became a beleaguered figure in Christiana and in Norway. Not unlike other pastors' children with a visible father, Abel retreated. His peers noted a sense of "exaggerated cheeriness." He poured his energies into mathematics as another defense mechanism.

In 1820, his father died, leaving Abel's mother and siblings in poor financial shape and with an annual financial pledge to the university. His mother, also an alcoholic, was deeply in debt, living a society life when the economy could not support it. Her late father's wealth had evaporated even before his death so there was no one to support her and her children except for Niels Abel.

Niels faithfully made every effort to assist his family. Finding a job suddenly became even more urgent. A new school, the University of Christiana was not yet a strong school, and Abel soon became the most knowledgeable person about mathematics in Norway. He tried to find a university position in Norway and in the major learning centers. He traveled to Europe with hopes of having his work recognized by the major mathematicians. Before his genius was recognized, he died at the age of 26 without an opportunity to develop fully his mathematical ideas.

While the focus here has been on the relationship of growing up in a minister's family and mathematics, Abel's biography can be used to illustrate several types of books regarding mathematics and its relationship to religion.

The book *Why Beauty is Truth* (2007) by Ian Stewart [70] tells the story of symmetry and how it became an important concept in science. Stewart's book is an example of a book which tries to make mathematics accessible for an audience with a minimal background in mathematics. By tracing the history of an idea, such a book tries to craft a compelling narrative. Downplaying the technical details, it humanizes the people behind the mathematics and provides a context in which the mathematics developed. Reflecting a recent postmodern trend, Stewart's book recognizes the historical and religious influences in Abel's life. While Christians may see this as a positive step, they may not be happy with the new alternative. In the last chapter "Seekers After Truth and Beauty" Stewart raises the question of whether the universe is genuinely mathematical. He wonders about its "unreasonable effectiveness" as to why mathematics is so useful for purposes its inventors never intended. In the end, he never even considers a theistic explanation for beauty as a viable philosophical option. It is as if faith is fine for an individual's life but Christianity is irrelevant for important ideas.

The second type of book is the standard biography. Øystein Ore's 1957 biography *Niels Henrik Abel: Mathematician Extraordinary* [54] goes into depth of the lives of Abel's father and grandfather. Born in Norway himself, Ore communicates an understanding of Norway and its social and religious fabric as he commemorates its greatest mathematician. Ore, a research mathematician at Yale, does not detail Abel's mathematics but instead writes for a general audience in order to understand Abel and his life.

Stubhaug's definitive biography *Niels Henrik Abel and his Times* [75] has a slightly different purpose as a cultural biography. Stubhaug writes more as a historian than a mathematician. Judith V. Grabiner writes "…the historian's view of both past and present is quite different from that of a mathematician. The historian is interested in the past in its full richness" while the mathematician "instead is oriented to toward the present, and toward past mathematics chiefly insofar as it led to important present mathematics [29]." The historian Stubhaug uses biography not only to give a portrait of the man but to use him as a springboard to describe an era and a concomitant mentality. "Behind this lies a basic conception of what constitutes the human" Stubhaug said in a 2005 interview [55]. In his writing, Stubhaug recognizes that religion is a part of understanding the mood, what was happening, and how that happened during the time period. Having studied the history of religion, Stubhaug provides a good description of the theological issues surrounding Norway during Abel's time but does little to discuss how it affects him as a mature human being.

It will be interesting if mathematicians embrace this style. If mathematical history has traditionally been used to motivate further mathematical study, a cultural study with a focus beyond mathematics may not serve this purpose. According to Stubhaug in [55], "what makes it different to write about mathematicians rather than artists, politicians, and other classical subjects of the genre is that the subject matter of mathematics is incomprehensible to most readers." Stubhaug goes on, "one easily confuses biography with a celebration of genius. Many have a romantic tendency to make its subject into an object of wonder rather than simply trying to understand the individual concerned as a human being." The wonder may be the part which helps motivate the reader to study mathematics.

As mathematics is placed within its social context, the religious lives of individuals will be part of the story. Unfortunately, as in much of contemporary historical scholarship, religious claims will likely be viewed as something to note but not to be taken seriously.

8. Riemann

Like Euler, Riemann had a father who wanted him to study theology. Like Euler, Riemann instead selected mathematics with his father's blessing. Unlike Euler, the interpretation of Riemann's selection varies.

E.T. Bell's popular book *Men of Mathematics* (1937) [6] had a great influence on mathematical exposition by providing biographical profiles of great mathematicians with lively tales. Written during a time when money was tight and mathematics was under attack as an educational "frill," the book was Bell's defense of mathematics. "In the coming tempest only those things will be left standing that have something of demonstrable social importance to stand on. Mathematics, as we mathematicians believe, has so much of enduring worth to offer humanity on all sides from the severely practical to the ethereally cultural or spiritual that we feel secure—until we stop to think [57]." By humanizing mathematicians, the book's intentions would help others see to the importance of mathematics and how mathematicians operate.

A talented writer, Bell had a impressive understanding of religion, but his writing portrays religion as something that ought to be jettisoned in favor of a scientific or progressive perspective. He blasts Pascal as a religious neurotic for "his self-torturing and profitless speculation on the sectarian controversies of the day." In his table of contents, Euler is "snatched from theology." Constance Reid says in her biography of Bell [57] that she could "give no reason for Bell's often extravagantly expressed hostility toward various religions, and religion in general." She goes on, "Throughout his life, however, he enjoyed coming up in unexpected situations, including the mathematical, with an always appropriate quotation from scripture."

Bell uses inverted religious language in his portrayal of Riemann's switch from theology to mathematics. The outline in the table of contents has "Poor but happy. Riemann's chronic shyness. Destined for the Church. Saved." In Riemann's chapter, he writes "By the end of his Gymnasium course it was plain even to Riemann that Great Headquarters could have but little use for him as a router of the devil, but might be able to employ him profitably in the conquest of nature. Thus once again, as in the cases of Boole and Kummer, a brand was plucked from the burning, *ad majoram Dei gloriam*." Here Bell is inverting the phrase "a brand plucked from the burning" that a religiously informed, although increasingly skeptical, audience would likely have recognized during that time. John Wesley often likened his Christian conversion to his experience of a childhood fire. All of the family were safely out of the burning house except for five-year old John, who was seen in an upstairs window. The neighbors climbed on each other's backs to save John, and the house shortly exploded into flames after his rescue. John Wesley referred to himself "as a brand plucked from the burning," quoting Zechariah 3:2, which says, "Is not this a brand plucked out of the fire?"

Bell references Boole and Kummer who considered going into the ministry but selected a mathematical career instead. As a symbol of being rescued from the alternative, Bell continues to play on this image ending with the Latin phrase *ad majoram Dei gloriam* translated as "All things work together for good," or to the greater glory of God if it is dedicated to God."

Morris Kline in his *Mathematics in Western Culture* [37] juxtaposes Riemann's "conversion" to mathematics with mathematics loss of certainty through the parallel postulate. "The sickly, precocious Bernhard Riemann (1826-66), who had to beg his father, a German Lutheran pastor, for permission to abandon his training for the ministry so that he might study mathematics, undertook to pursue possible alternatives to this axiom."

Kline compares religion in ancient societies to the position mathematics occupied as "revered and unchallenged" in Western thought. He goes on, "... in the temple of mathematics reposed all truth, and Euclid was its high priest. But the cult, its high priest, and all its attendants were stripped of divine sanction by the work of the unholy three: Bolyai, Lobatchevsky, and Riemann." Later in the chapter, Kline writes, "At this stage in its history mathematics scrubbed the clay of earth from its feet and separated itself from science, just as science had broken from philosophy, philosophy from religion, and religion from animism and magic." By placing these together we see Kline hinting that just as Riemann begged his father, non-euclidean geometry begged mathematicians to come to a more mature view of the world.

Most short descriptions of Riemann include Riemann begging his father to allow him to study mathematics. In following the typical myth of the hostility of science and religion, one could speculate that Riemann was asking to leave Christianity in favor of an unholy, antithetical lifestyle. Many include Riemann receiving his father's blessing. What most of these short descriptions leave out is a full explanation of *why* Riemann had to beg his father.

Here are multiple examples.

> His early studies were directed towards following in his father's footsteps, but an innate intellectual affinity and talent for the sciences (as well as a shyness dramatically at odds with the duties of a minister) soon overcame filial devotion. With his father's approval, Riemann took up a formal study of mathematics and physics [61].

> ... at 19 went to the University of Göttingen with the aim of pleasing his father by studying theology and becoming a minister himself. Fortunately, this worthy purpose soon stuck in his throat, and with his father's willing permission he switched to mathematics [64].

> In 1846, in accordance with his father's wishes, Riemann matriculated at Göttingen University in the faculty of theology. His interests in mathematics was so strong,

> however, that he asked his father to allow him to transfer to the faculty of philosophy [50].
>
> In poor health throughout his life, Riemann had to beg his father, a German Lutheran minister, for permission to study mathematics rather than prepare for a career in the church [76].
>
> Riemann's father had in mind for young Riemann to study theology, but Bernhard begged him to instead study mathematics. Fortunately for all of us, the elder Friederich granted his permission [40].
>
> Following his father's wishes, he moved to Göttingen in 1846 to study theology and philosophy. After attending various mathematics courses, he realized he had an aptitude for the subject, so after receiving permission from his father, Riemann switched fields [51]. His father, a Lutheran pastor, encouraged him to study theology at Göttingen. But even as a child Riemann had shown a tremendous aptitude for mathematics, and in 1847, he persuaded his father to let him go to Berlin to learn mathematics from the likes of Karl Jacobi, Peter Dirichlet, and Jakob Steiner [17].
>
> Ever dutiful to his parents, Riemann enrolled at the University of Göttingen planning to take a degree in theology and philosophy so that he could follow his father into the ministry. Much as he tried to satisfy his family's wishes, Riemann could not resist the attraction of Göttingen's one and only star mathematician: Carl Friedrich Gauss. Riemann was enthralled by Gauss's lectures on the method of least squares and decided to make mathematics his career [34].
>
> His father had encouraged him to study theology and so he entered the theology faculty. However he attended some mathematics lectures and asked his father if he could transfer to the faculty of philosophy so that he could study mathematics. Riemann was always very close to his family and he would never have changed courses without his father's permission [45].
>
> Encouraged by his father, he entered the theology faculty, but attended some mathematical lectures. After getting permission from his father, he transferred to the faculty of philosophy and took courses in mathematics from Moritz stern and Gauss [14].

The book *Scientists of Faith* [32] includes the following quote. "Initially Riemann's father, a Lutheran minister, pressed his son to enter the ministry. Georg was a devout youth who would not have been averse to becoming a clergyman. But he loved mathematics and pleaded with this father for permission to pursue a mathematical career instead. His desire was granted." This book, written in 1966, ironically suggests reading E.T. Bell's *Men of Mathematics* for further reading which has since been shown to be lacking in historical scholarship [62].

The question may be raised as to the motivation of writers including Riemann's begging his father and gaining his father's permission. Could it be a story for aspiring mathematicians that there may be family resistance to mathematics? Could it be a story of religious faith being initially incompatible with mathematics, but in the end it is okay? There must have been some tension between faith and mathematics in Riemann's decision, but to focus exclusively on this tension provides an incomplete picture. There were economic considerations as well.

John Derbyshire's book, *Prime Obsession* [21], like Stewart's *Why Beauty is Truth*, examines the history of a mathematical idea for a lay audience–in this case, the Riemann Hypothesis. Derbyshire alternates chapters on the mathematical ideas with the historical developments at the time. While others have noted the economic hardships of Riemann's time and its affect on his health, Derbyshire recognizes how the economic conditions influenced Riemann's reluctance to pursue mathematics. Without the job security in a more prosperous time, the Riemann family had to consider the difficulty of finding a job. Being a minster, while not being well paid, was a more secure means of supporting oneself.

The primary source of Riemann's life is a short memoir by Dedekind [20] who enlisted the help of Riemann's sister. There we find that Riemann's father was his teacher until high school, and that his father's natural desire was for Riemann to study theology. His father hoped Riemann could help support his family, which would be a relief to his father. Apparently, in those days it was easier to get a contract for a job with theology. Nevertheless Riemann's talent for mathematics was so obvious and overwhelming that his father relieved him of this duty.

Despite forgoing theology, even E.T. Bell recognizes that Riemann "persevered in his faith and remained a sincere Christian all of his life [6]." From Dedekind's private letters and Dedekind's memoir, he is found to be a strong Christian in a German Lutheran style. To Dedekind, the essence of religion for Riemann was a daily self-examination before God. Riemann always thought deeply about philosophy and often put his mathematical work within a larger philosophical context.

Some find Riemann to be a tragic figure. As a human being, Riemann was extremely shy, a hypochondriac, and in constantly poor health battling depression. For Laugwitz [42], the daily self-test before God must have been an ordeal for him. Yet Derbyshire resists the temptation to find Riemann "a rather sad and slightly pathetic character." This would only consider the outward appearance and manner of the man. "Within that diffident, withdrawn exterior was a mind of great brilliance and staggering boldness." Later he notes, "Outwardly he was pitiable; inwardly, he burned brighter than the sun."

Derbyshire [21] also finds Euler to be one of his favorite people in mathematics. Euler is a pleasure to read, but he is also a very admirable person. In reading about his life, "you get a strong impression of serenity and inner strength." Euler faced ridicule throughout his life about his Christian ideas but always maintained a rock-solid religious faith. While Derbyshire speaks admiringly about Euler and Riemann and sees himself as broadly sympathetic to religious expression, Derbyshire proclaims himself to be an unwilling unbeliever to the Christian faith [22].

In terms of the three greats, Euler, Abel, and Riemann, there is not a sense of rejection of faith in selecting mathematics over theology. Unlike the typical science narrative, their pastor fathers were not anti-intellectuals but were strong supporters of education. Instead, each recognized in their son the ability and natural inclination towards mathematics. In some ways, the sons carried over aspects of their father's vocations. All three were dedicated to their families. Euler and Riemann never left their faith, behaving in many ways like pastors.

Thus the study of the lives of these PK mathematicians has provided a glimpse of how mathematical writing of history has changed. Finkel in the 19th century wrote to an essentially

American Christian audience. In the early part of the twentieth century, the technical ideas of mathematics are primary and religion is to be ignored. When Bell humanizes mathematics, he urges the rejection of religious ignorance and superstitions in favor of a more progressive and scientific mindset. Around the turn of the 21st century, writers like Stewart, Derbyshire, and Stubhaug recognize the importance of religion in the social and historical contexts but in the end find it inconsequential and trivial.

At the same time, the Christian lives of Euler and Riemann continue to make an impact. While many are drawn to their mathematical insights and output, a closer examination of how the two approached life yields admiration. While not perfect, their lives in many ways are a model for Christian mathematicians.

9. Personal Observations

I never had any intentions of studying to become a minister. Growing up, I saw how parishioners behaved, how they failed to listen, and how fallen the church was in general. Mathematics was more suitable to my temperament. Yet I was struck by the following passage in Saunders Mac Lane's autobiography where he writes the following passage about his minster father.

> I started college with the aim of finding a career that would be scientific rather than ministerial. ... However, the effects of the ministerial background were actively present. I had seen my father struggle with the draft of his sermons, perhaps not so different from the draft of scientific papers. And teaching resembled preaching, since the student's mind is like his soul—both need repeated refreshment.

In some ways, I do find teaching to be very much like being a minister, and I would like to think that I am refreshing both my student's mind and her soul.

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Pascal's Thoughts Seen in the Light of Scripture

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Abstract

In this paper we study Pascal's character via his writings. There are indications that his health might have deteriorated following the experiments he had done using mercury. He talks in his Pensées about faith, grace and purity of the heart, about the peoples and the way in which God leads them, about wisdom, dreams and hopes and that which lies in the human heart. If a statistics was done concerning the most used books and verses from the Bible in Pensées, these would be: Ecclesiastes, Proverbs, Matthew, Mark, Jeremiah, Hebrews, Romans, Luke, Isaiah, Psalms. But those which occupy a central position are, of course, the ones used by the Lord Jesus Christ, such as: John 3:16, when he speaks about faith, Luke 17:33 when speaking about the bet, and Matthew 16:26 when talking about the thinking reed. For Pascal there is no antagonism between science and faith.

1. A Short Biography

Blaise Pascal was born on the 19th of June 1623 in Clermont and died on the 19th of August 1662 in Paris, being one of the biggest and most famous mathematicians of all time, bearing this unbelievable treasure, his life, in a clay pot. He was a child prodigy and had two sisters: a younger one, Jacqueline; and an older one, Gilberte.

Although Pascal's greatest contributions to the physical sciences were in the construction of mechanical calculators, the study of fluids and the clarification of the concepts of pressure and vacuum, his first love, and his first focus area since as early as childhood, was mathematics—especially geometry. He took part in the creation of two new major research branches, among which was the theory of probability. Among the best known geometry papers written during adolescence are "Essai pour les coniques" and "Traité du triangle arithmétique." Many of his ideas, such as the construction of a calculation machine capable of making very fast addition and subtraction operations, was implemented several centuries later, revolutionizing the science to date. Following a religious experience on the 23rd November 1654, which marked the beginning of his spiritual transformation (that was the day Pascal started fearing God), Pascal turned his thought to a religious life. His first great love was replaced with an even greater one, the love for God and Christ. He subordinated to this end all of his previous concerns, thus becoming a philosopher, a forerunner of existentialism, and very interested in theology and especially the Christian feeling. In 1647, Pascal published "Expériences nouvelles touchant le vide," where he presents in detail the basic rules that describe "to what degree various liquids could be supported by air pressure." It also provided reasons why it was indeed a vacuum above the column of liquid in a barometer tube, contrary to what was said by "Aristotelian trained scientists of Pascal's time" (see [6]). Another prominent French scholar of the time, Descartes, was in conflict with Pascal because the latter insisted upon the existence of the vacuum and was a strong opponent of Descartes' rationalism.

> The weather was chancy last Saturday... [but] around five o'clock that morning ... the Puy-de-Dome was visible ... so I decided to give it a try. Several important people of the city of Clermont had asked me to let them know when I would make the ascent ... I was delighted to have them with me in this great work ... at eight o'clock we met in the garden of the Minim Fathers, which has the lowest elevation in town... First I poured sixteen pounds of quicksilver... into a vessel ... then took several glass tubes ... each four feet long and hermetically sealed at one end and opened at the other ... then placed them in the vessel [of quicksilver] ... I found the quick silver stood at 26" and $3\frac{1}{2}$ lines above the quicksilver int the vessel ... I repeated the experiment two more times while standing in the same spot... [they] produced the same result each time... I attached one of the tubes to the vessel and marked the height of the quicksilver and... asked Father Chastin, one of the Minum Brothers ... to watch if any changes should occur through the day ... Taking the other tube and a portion of the quick silver ... I walked to the top of Puy-de-Dome, about 500 fathoms higher than the monastery, where upon experiment ... found that the quicksilver reached a height of only 23 and 2 lines ... I repeated the experiment five times with care... each at different points on the summit ... found the same height of quicksilver ... in each case ... [4].

This new focus area appeared in 1646 when a family friend presented him Torricelli's experiment (according to Julia Chew, see [2]). In the winter of that same year, 1646, his father, aged 58, broke his hip. Two of the most famous doctors in France, Mr. Deslandes, and Mr. De la Bouteillerie treated him very well for three months, succeeding in making him walk again. The two doctors were avid Jansenists. His sister Jacqueline became a nun at Port Royal monastery after their father's death in 1651, and for a short period (1651–1654) Pascal lived a bachelor's life, even considering marriage. In that period he wrote the "Conversation about the Passions of Love," which he later described as the lowest of the conditions of life permitted to a Christian. Nevertheless, after his father's death he felt responsible for the future of his sister Jacqueline, especially because she had given up her side of the fortune when she had joined the monastery. Actually, Pascal felt responsible during his entire life for everything he did: towards his father and towards his fellows he felt he had to be a role model, at least in the things which depended on him. And he was one. He was also an honest man, because when he saw the Jansenist with whom he had an argument, whom he thought to be a danger for society and whom he had denounced (he did not fear God back then) and went to jail, he felt sorry and tried to straighted out the consequences.

Let us have a look now at the following passages from [6], page 4, "Religious conversation" and then at [2], page 2, "In the summer of 1647, Pascal fell ill due to being overworked."

> In 1647, a paralytic attack so disabled him that he could not move without crutches. His head ached, his bowels burned, his legs and feet were continually cold, and re-

> quired wearisome aids to circulate the blood; he wore stockings steeped in brandy to warm his feet. Partly to get better medical treatment, he moved to Paris with his sister Jacqueline. His health improved, but his nervous system had been permanently damaged. Henceforth, he was subject to deepening hypochondria, which affected his character and his philosophy. He became irritable, subject to fits of proud and imperious anger, and seldom smiled. (In conformity with [6], see [5])

According to Pascal, since the age of 18 he had pains every day and poor health (see in Pensées, when he speaks about illness, he says it puts humans in close relation with God because it is then that man feels helpless and sickened by this world's pleasures and man would then be in a spiritually appropriate state, especially because God pities man. Then, men meditated about their condition and saw things differently).

In those times, it was not known that mercury is very harmful and seriously damages health. Some doctors even used it to treat certain diseases, and so it was thought it did not represent any kind of a risk. There is a chance that in that period his health may have been seriously affected for this reason too. The symptoms of an intoxication with mercury can be established very easily today and we can study the writings of those times about Pascal in order to determine if the symptoms were or were not present. There was no treatment back then. The effects are supposedly long lasting. Later on, even workers in the barometers and thermometers factories had been through these situations and nobody knew what the diagnosis was. The situation improved after the working conditions became better and gas leaks were eliminated (and mercury vapors ceased to appear; see the internet). It is said it can provoke cancer and it is very dangerous for the nervous system.

According to [2], page 2, Pascal died on the 19th August 1662 of an undiagnosed illness. According to [6], page 6, an autopsy has been made which reveals a great problem related to the stomach and damage to his brain, but the reason of his permanent state of non-health was never precisely determined, page 7.

> An autopsy performed after his death revealed grave problems with his stomach and other organs of his abdomen, along with damage to his brain. Despite the autopsy, the cause of his continual poor health was never precisely determined, though speculation focuses on tuberculosis, stomach cancer, or a combination of the two. The headaches which affected Pascal are generally attributed to his brain lesion.

Some say that Pascal was a hypochondriac, but even so, the experiments made in which he used mercury must have damaged very badly his health. Torricelli also died at a young age and he too worked with mercury approximately in that time, see his date of birth and death. Even Abraham Lincoln, two centuries apart (see historical dates), was intoxicated with mercury when he took a medicine, but interrupted the treatment because he did not feel well. He was very irritable in that period (his temper was also affected), and so was Pascal in the later period of his life and had some other similar symptoms (see Abraham Lincoln's symptoms, [7]).

> The active ingredient of blue mass is elemental mercury—a substance now known to be a neurotoxin in its vaporic state.[41] Whether mercury poisoning may have affected Lincoln's demeanor before or after he ceased its use in 1861 is unknown, but still remains the subject of conjecture by some historians. [42]

Some of the most important features of Pascal—very determined, sure of himself, very intuitive, visionary, very correct, introverted, a very fine spirit, multilateral, did not make any mistakes concerning mathematics, responsible, and loyal—can be observed not only in his life, but also in his writings.

2. On Some of Blaise Pascal's Thoughts

The collection of thoughts entitled *Pensées* represent Pascal's notes which he intended to use in a book defending Christianity, showing through logical arguments that Christian faith is not unreasonable.

According to what the author said in *Introductive Study*, page 6, 7, the text presented to the Romanian reader in the book entitled *Pascal's Thoughts*, translated by Maria and Cezar Ivanescu, see [3] which we have used, is a translation of that established by Pascal's manuscript in France's National Library used by Leon Brunschvicg for his edition in 1897. In the *Translator's Word*, page 147, it is mentioned that this book is the complete edition of Pascal's Pensées translated for the first time in Romanian. The translators underline they used Brunschvicg's edition because it is the first edition which recovers in its entirety the text of the Pascal's thoughts and represents an "excellent group topic of Pascal's fragments." Nevertheless, they claim that what this edition gains in clarity, it loses in fidelity, because it does not keep only the classified fragments (the only ones destined to the Apology), but the unclassified ones as well, which have an uncertain statute. The first edition which contains Pascal's Thoughts appears in 1670, by the care of a committee of sympathizers from the school of Port-Royal.

2.1. The Misery of Man without God

In the section entitled "The Misery of Man without God," Pascal begins by meditating in *Thought *67* (the vanity of sciences) at the fact that any toil of man in searching to know everything, in understanding the principle of all things is vain. The more he discovers, the more he comes to understand how little he did and how many more things are hidden behind what he has just discovered and thus, he becomes humble if he takes a look at the path which he should cross and if he is honest to himself. Or at least this should be the natural position because then he, as part of God's creation, discovers some things created by God, who is infinite and has left his mark upon these. In some of his thoughts (see the Latin note in *Thought 165*) he refers to the book Ecclesiastes. Philosophy is also approached here, and he continues to speak about imagination as the source of error and important factors determining action in *Thought 82*. He speaks, using here a style that could be found later in the writings of another prominent French moralist, La Bruyere, about prejudice in *Thought 98*, about self-love in *Thought 100*, self-deceit in *Thought 108*, levity in *110* and *111*, boredom and disorder in *129, 130, 131*, and finally about wickedness and vanity in Thought *161, 163, 171*, and *180* as a culmination of the previously mentioned. Especially in *Thought 180* there appears the idea from Ecclesiastes 9:2, 7, i.e., eat and live your life happily on earth because a big evil under the sun is like the sage and the fool, they vanish alike. *Thought 83* makes us think about another teaching from the Bible from Jeremiah 17:9, "The heart is deceitful above all things, and desperately wicked. Who can understand it?" The way in which all of those things are described in these thoughts, the arguments used, the pleading can but remind us that Pascal was the son of

Etienne, president of The Court of Appeal back when they used to live in Clermont. In *Thought 139* entitled *Divertissement*, he describes the cause of all our unhappiness as earthlings, also stating its well founded reason which relies on "the natural unhappiness of our condition of weak, mortal people" who cannot be caressed by anything. As an argument he then uses the example of a king who has everything at hand, a lot of fame, wealth, power and diversions, and still becomes miserable when thinking about his internal state. This king makes us think about King Solomon who had all of these things and then said that all are "a van rush." Nevertheless, Pascal argued that for the pagans and the atheists, who haven't heard about or don't believe in the existence of God, Pascal cannot to or not willing to use the Bible yet in order to prove the validity of his claims, hence this image of hopelessness, while Solomon, as a Jewish, had at least the ten commandments. In *Thought 109, 109bis* he talks about the disappearance of pleasure in a state of illness, but especially, see *Thought 174*, about the connection between wickedness and illness and wickedness and futility and what they have in common (see the Book of Job and Ecclesiastes) urging us to think more about the Creator and look less at our interior state. Many Thoughts of Pascal that were meant to best describe the lack of goals, the frustration and despair of an unbeliever that really sees inside himself (see Nietzsche) were used by his opponents to catalogue Pascal as being such a person, but Blaise Pascal's goal was to show that the only way to salvation was to believe in Jesus Christ. He planned to show the falsehood of the philosophical teachings of certain people in fashion at the time, like Montaigne (see *Thought 63*) and Epictetus, using the writings of Saint Augustine.

2.2. On the Necessity of the Wager

In section three, entitled "On the Necessity of the Wager," which is about the necessity of faith, the author begins by using a specific process which he uses brilliantly, fact that is proved by the success of the "The Provincial letters," i.e., he introduces in his work texts written in the form of letters. The respective paragraphs follow, in my opinion, the method entitled by the author *The Art of Liking*, addressing here human interest and intelligence, wishing to convince that religion is a pleasure. This is obvious, especially when he uses words like "The end of this talk . . . " or "What evil would you endure if . . . " which resemble the ones of the apostle Paul from one of his epistles. The pleading built in *Thought 194* has for subject the rhetorical question from *Thought 193*: "What will happen to men who despise the smallest things and do not believe the greater?" It shows first that the Bible speaks the truth about itself and that the people who truly wish to find out whether it is true come to be convinced by it. Then, he argues about the importance of clarifying the subject of the immortality of the soul, subject of which depends all of our behavior in its essence. Afterwards, he states what the Christian faith has got to prove: the fall of human nature and the redemption through Christ. He also gives as an example to support his first claim: the character of the opponents of Christianity. He then speaks about carelessness, about the fact that certain people are too sensitive as for small things and too insensitive towards the big ones, claiming that they are not actually like this, but they feign it and that if they were at least honest, if not good Christians, they would realize there are only two kinds of people who can be called reasonable: the ones who serve God with all their heart because they know him, and those who search Him from all their heart because they don't know Him. Moreover, *Thought 211* describes the condition of man in the middle of his peers: alone he dies, suffering (see also the Book of Job).

Thought 216 includes an idea surprisingly proved which can lead to *Thought 220*: "Sudden death alone is feared, hence confessors stay with lords."

"The falsehood of philosophers who do not discuss immortality of the soul. The falsehood of their dilemma in Montaigne."

The central part of this chapter, see *Thought 223* (the so-called Pascal's wager) one of the most fashionable in his time and which always comes into actuality under different forms is probably stemmed from the letters to his friend Chevalier de Mere, back when Pascal was very concerned with gambling. He used a probabilistic argument in order to prove the necessity of faith and of a godly life. In the opinion of W.W. Rouse Ball, from *A Short Account of the History of Mathematics* (4th edition 1908),

> Pascal made an illegitimate use of the new theory in the seventh chapter of his Pensées. In effect, he puts his argument that, as the value of eternal happiness must be infinite, then, even if the probability of a religious life ensuring eternal happiness be very small, still the expectation (which is measured by the product of the two) must be sufficient magnitude to make it worth while to be religious. The argument, if worth anything, would appear equally to any religion which promised eternal happiness to those who accepted its doctrines. If any conclusion may be drawn from the statement, it is the undesirability of applying mathematics questions of morality of which some of the data are necessarily outside the range of an exact science. It is only fair to add that no one had more contempt than Pascal for those who changes their opinions according to the prospect of material benefit, and this isolated passage is at variance with the spirit of this writings.

It is not allowed to apply mathematics to morality for which certain data are beyond the study area of this exact science. Nevertheless, lately, applied mathematics is used for the study of human behavior: in psychology and biology, totally unexpected branches of knowledge. Thus, Pascal may have only anticipated in his characteristic way what would happen in the future, just like in the case of the computing machine. Nevertheless, it would be interesting to study more in detail what W.W. Rouse Ball says, that this kind of argument would be anyhow used in all the religions that promise eternal life (the more it means that it bears generalization, being even more useful in this point, regarding the importance of the search of eternal life) because not all of them have the same set of doctrines, they do not use the same means. In addition, maybe we should see the wager as an allegory, as in the Bible, in Psalms for example, not in its literal meaning. Maybe the same kind of mockery and humor (attributes of a moralist) appear in Pascal's wager as in the Provincial Letters being used for the same goals, especially because it was said that at that time it was a very popular story having the same subject as Pascal's wager.

For the beginning, in *Thought 233*, the author restates what the Bible says, that it is only through faith that we can know the existence of God and through grace His nature, because He has no size nor limits. What we can know is the existence and the nature of the finite, being ourselves finite, and of the same size as him. He also claims that we can know the existence of the infinite, but we don't know its nature, due to the fact that He has a size, just like us, but is not bordered like us. He then shows that we must bet because we live, we are in the middle of all events, to make a choice, and reason would tell us not to bet if it is possible. We could lose truth and the right and our nature has got two things to avoid: error and wickedness. Due to the fact that we are forced to chose, reason will not be touched in any way and happiness will belong to those who chose to believe, states Pascal.

> Because we speak about an infinite of lives, infinitely happy to gain, about a chance of winning faced with a finite number of risks of losing and that what anyway you put at stake is finite. This removes all doubt: if we are to gain that what is infinite and for this, the number of risks of losing is finite compared to the chance of winning, do not stand in doubt, offer everything. And thus, being forced to play, we must give up the reason which tells us to keep our life; we should rather count on it in order to gain the infinite approaching us with steps as quick as nothingness ...

We are allowed to believe that in the passage above appears the idea expressed in the Holy Scriptures that: "If only for this life we have hope in Christ, we are to be pitied more than all men," the Bible also underlining the necessity of choice.

Finally, as a final and most powerful argument, he confesses that he chose to believe in Him and to devote himself to Him.

2.3. The Means of Belief

Chapter IV is called "The Means of Belief." For the beginning, in *Thought *242* the author shows that for the irreligious people it is not a good thing to try to prove that God exists using nature's works, although many fall into this trap, except for the canonical authors (see *Thought 243*). This is because, says he, it means making them believe that the testimonies of Christian belief are very weak and indirect, I would add, which would determine the irreligious to despise us. (Additionally, pagans also used nature in order to demonstrate their goals, they have come to adore it and divinise animals, even humans as themselves. Therefore, this would be a trap, especially because man was created by God to be a master over the animals and all human beings are equal.) Rightly, because it is just like hiding and forcing someone weaker than you to be your protection.

Furthermore, because he speaks about the means of Christian belief, the best place to be looking for an answer is Scripture. It tells us in its essence that since man fell into sin, he stepped into a spiritual blindness, and God hid His face from man, the only means to get to know God being through Jesus Christ. (see Notes 1, page 275). "No one knows the Son except the Father, and no one knows the Father except the Son and those to whom the Son chooses to reveal Him." (Matthew 11:27).

It is also said in many places that the one that truly is in search of God will find Him. The key word of the section is belief. We are then presented in *Thought *245* three means of acquiring belief, i.e., reason, tradition and grace, underlining the necessity of grace. The three means are to be discussed in depth in the thoughts that follow. Then, more aspects are presented and analyzed which help in creating belief, others which impede it and others that lead to the formation of a wrong or formal belief. Then we are presented some correct and some wrong attitudes of men concerning belief, as well as signs that indicate its presence or absence, other religions being referred to tangentially. Superstition, a kind of formal religion is mentioned as well, which unfortunately has deep roots into our sinful nature. Its fears are also presented in *Thought *249*, being mentioned in *Thought 254* that it is extremely dangerous. In *Thought 255*, piety is approached and this could have as a reverse superstition. Thus, "the bad fear" spoken about in *Thought *262* could lead to superstition, while "the good fear" might lead man towards piety. The common root is nevertheless fear. In *Thought 250* and **251* the author speaks about the balance that should exist between our faith and its exterior manifestation, given through its external signs ensuring peace and reconciliation with ourselves, and without which there is no perfection. And he gives as examples pagan religions that he says are made only of external signs, thus being easier to practice, with fewer hard demands and more popularity. An example of hard requirement would be to love your enemy, and as far as I know this is not present in the Muslim or Mosaic religions.

On the necessity of resting in Christ, see the quote from the Bible and then *Thought *252*: "... we must get an easier belief, which is that of custom, which, without violence, without art, without argument, makes us believe things and inclines all our powers to this belief, so that our soul falls naturally into it ... "

Afterwards, see *Thought *253*, we are presented two excesses which only lead to heresies that certain philosophers have embraced throughout history: the first is to exclude reason, and the second not to take into consideration anything but reason. The explanation lies in *Thought *273*. Another *Thought *267*, referring to reason, establishes the limits it may reach, i.e., to admit the existence of an infinity of things that are beyond it.

*Thought *257* is in relation with *Thought *245*, the means of acquiring belief leading thus to three kinds of people: some that serve God because they have met Him, others that struggle to look for Him because they haven't yet, and the others who are not in search of Him because they did not find Him.

In the following thoughts we are explained how people generally relate to faith. A common human feature, described in *Thought 259* is that they select the things they listen to, which is related to will, that is they can impose to themselves not to think about what they do not want, tending to keep what they already have, whether it is true or false, but others, on the contrary, the more limited access they have to this, the more they think about it and challenge it. These latter, says Pascal, do not love the truth and hide behind the many who deny Him, lying to themselves that everybody is just like them, that they form the majority. This conservatism of the world often leads to the vice that supposes too much submission (see *Thought 254*). People also like to see as many miracles as they can and their expectations to be achieved, see Lord Jesus when he fed the crowds. Jesus also says the world is deceitful and changing, the crowds often being compared in the Bible with water and sand.

In *Thought *265*, the author expresses the belief that faith is beyond senses, not against them and in *Thought 266*, he gives an example of his time in this matter. The definition of belief, from the Bible (see Hebrews 11, 12) brings the final argument in favor of the statements mentioned above. Starting with *Thought *274*, the author begins to speak about belief gained through grace, the only one sufficient for salvation, see *Thought *245*. First he speaks about the confusion usually made in the field of faith between fantasy and feeling and then between imagination and heart in *Thought *275*. Nevertheless, Pascal spoke before more in detail about the mistakes and utility of the imagination and the opposition between it and the reason in "The human wickedness without God" beginning with *Thought 82*, calling it "the mistress of error and falsity" which "creates beauty, justice, happiness that are everything in this world," having the great gift of persuading people's will, like a lawyer who pleads. In the Bible, we can find references regarding this matter in Proverbs, Ecclesiastes and Jeremiah, where it is said "The heart is deceitful above all things and beyond cure. Who can understand it?" (Jeremiah, 17:9), and also Proverbs 4:23, "Above all else, guard your heart, for it is the wellspring of life."

In the New Testament, Luke 12:34, Lord Jesus Himself says: "For where your treasure is, there your heart will be also."

Thus, in *Thought *278*, faith through grace is also presented as being "God manifested in the heart and not in reason," focusing on revelation, i.e., a gift from God, unlike other religions and it is said that truth can be known not only through reason , but also through the heart. People make and consider real principles they infer and they prove with the help of reason the propositions that derive from them. We are then presented some characteristics of the people God likes. The most important would be that God gave them love for Him and the power to hate themselves and probably an interior mood towards a greater holiness. It may be that some of them could not know the testimonials and prophecies, but may judge as well as those who know them. Anyway, it seems Pascal speaks knowledgeably about all of this, for he lived the experience from the night of 23rd November 1654 that marked all his life.

2.4. Justice and the Reason of Effects

Section V: "Justice and the Reason of Effects" begins by showing the relativity and inconsistency of any law regarding justice established by people. As a first argument is the fact that every country has its own laws and customs, the law makers not being able to agree among them. Others would think that justice should be searched in the natural laws available in all countries, but none is universal, the reason perverting them. He further states that the law themselves create errors (see, also in the Bible, Old Testament and Apostle's Paul writings) and those who are blindly submitted "listen to the justice they imagine, but not to the essence of the law," underlining what is the art of revenge that doubts and contests in it essence the authority and justice of the established traditions. The relation between justice and force is studied in *Thoughts 298–308*. Thus are presented the reasons which relate to utility, for which in general society prefers (admires) force instead of justice, see *Thought 298*. Another reason is that justice requires much time and force produces immediate results.

> It is only fair that what is just be followed and it is necessary that the most powerful be followed. Justice without force is powerless, force without justice is tyranny. Justice without force will always be contested because bad people are permanent; force without justice is condemnable. Force and justice should therefore stay together and for this, let us act in such a way that what is just be strong, or what is strong be just. Justice is always submitted to contestations, force is recognized without resistance. Thus, justice could not be given force because force contested justice and said it was unfair and that only it, the force, is just. Not being able to give strength to justice, justice was given to the strong.

In *Thought 300* it is explained that man conforms to the majority because the majority has the power and he feels more secure and the opinion of the majority uses power, see *Thought 303*. Of course, majority has even more reason, but this is not the motive.

Thought 304 speaks about the role of the imagination when maintaining the force in a certain group for a period, usually after the power is in other hands. Further, in *Thought 308* the peoples of that time are given the true explanation of the fact that the face of the King inspired fear and respect to his subjects, even when he is alone, and this is because they were always used to seeing him exercising his power. Thus, we speak about a custom and not a "natural force." There is then a well-known gnome in *Thought 309*, fashion can make justice as well as it creates pleasure. Pascal used to consider civil wars as the worst evil of all, see *Thought 313*. *Thoughts 315, 316*, and **319* show the real reason which lies behind those who judge people by their outer self, rather than by their inner self, i.e., establishing the strength connection and hierarchy between him and the respective person. Pascal tells us in *Thoughts 326–328* and **335* some of the bets of the peoples concerning laws and traditions. Thus, he thinks that the truth lies in tradition-laws, their oldness being the proof of their authenticity and not their authority. When they are shown that the laws are not fair, they rebel instantly. The author shows first that the opinions of the peoples are healthy being founded, but the peoples are superficial managing at the same time to remove the contrary affirmation that the respective opinions are always profoundly false and very frail. A partial explanation is offered in *Thought *335* where it is stated that after all everybody lives in an illusion. He also states that the peoples are weak and on this basis relies the power of Kings, on the reason and madness of the peoples. Having so much power at hand, in many kings or influential persons from other groups was born an incontrollable desire of domination which led to tyranny. According to Pascal in *Thought *332*, their mistake is that "they want to master everything," which nobody can do, not even force. The last thought of this section, *Thought 338*, speaks at last about Christians, not about the world and the peoples who, out of submission to God's order must sometimes submit to madness too, giving as reference the Ecclesiastes 3, 19 and the Epistle to the Romans 8, 21, see Note 5, page 294.

This section presents the conceptions of the peoples and the world on justice and their inner motivations that lead them to the respective concepts. God, the Bible and the New Testament concerning this topic are not mentioned very much, actually, only about the world, the peoples and justice, offering thus a quite fade image on life. It is easy to see that the Bible speaks in the same manner about the world and its things, in the New Testament and the Old Testament as well (Isaiah 48:20): "If you were of the world, the world would love its own." (John 15:19); "Do not love the world nor the things in the world." (1 John 2:15); "Therefore, 'come out from their midst and be separated,' says the Lord, 'and do not touch what is unclean; and I will welcome you.' " (2 Corinthians 6:17)

In many places in the Holy Scripture, the crowds, the multitudes are symbolically presented, even by Jesus in the Prophecies and the Apocalypse, like waters that flows not being steadfast and trustworthy. They are also easy to manipulate.

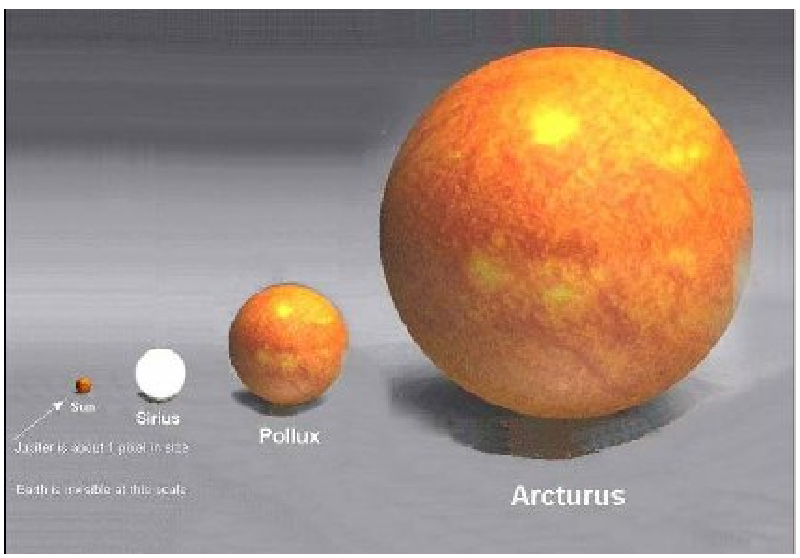
2.5. Philosophies

The following section, entitled "Philosophies" includes the most famous *Thought *347*, together with two more, *346* and **350* in which the author continues explaining more in depth the idea presented in the first one. The frailty of human thought and his weakness are presented in *Thought 336* as well through the fly that buzzes now in his ear and makes him incapable of judging correctly.

> Man is but a reed, the most feeble thing in nature, but he is a thinking reed. The entire universe need not arm itself to crush him. A vapor, a drop of water, suffices to kill him. But if the universe were to crush him, man would still be more noble than that which killed him, because he knows that he dies and the advantage which the universe has over him; the universe knows nothing of this. All our dignity lies in thought. It is from this that we must raise ourselves, and not by space and time, which we cannot fill. Let us labor, then, to think well. This is the principle of morality. (*Thought *347*)

And it continues: "It is not in space that I must seek my human dignity, but in the ordering of my thought. It will do me no good to own land. Through space the universe grasps me and swallows me up like a speck; through thought I grasp it." (*Thought 348*) and

> The mind of this sovereign judge of the world is not so independent that it is not liable to be disturbed by the first din about it. The noise of a cannon is not necessary to hinder its thoughts; it needs only the creaking of a weathercock or pulley. Do not wonder if at present it does not reason well; a fly is buzzing in its ears; that is enough to render it incapable of good judgment. If you wish it to be able to reach the truth, chase away that animal which holds its reason in check and disturbs that powerful intellect which rules towns and kingdoms. (*Thought 366*)

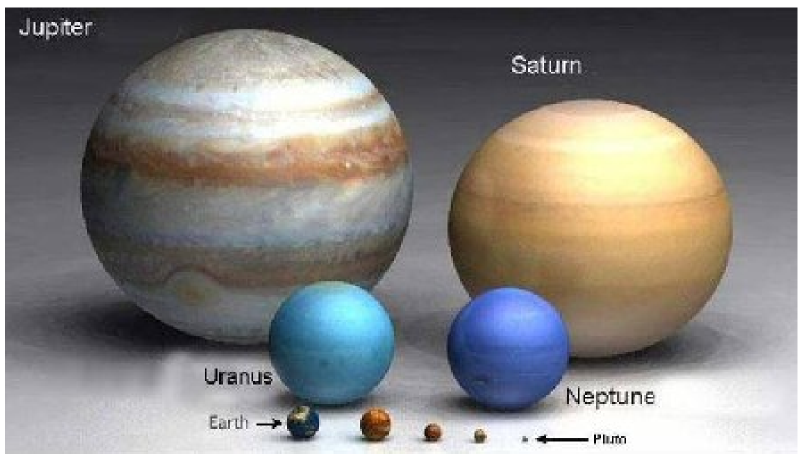


Certain connections are established between thought, human consciousness, his greatness and his weakness showing that the real force helping man understand and accept his purpose in the universe lies in though, this latter being a principle of morality which can give humans their greatness by admitting their wickedness and incapability (*Thought *397*) and can remember, can see and long for the state of happiness and freedom in which he was before being black. The Bible also mentions the power that comes from weakness: "...for My power is made perfect in weakness." (2 Corinthians 12:9). Maybe Pascal was inspired by the Bible because there too, when the word reed is used, it refers to human or peoples' frailty before God (Isaiah 42:3, in the Old Testament "A bruised reed he will not break, and a smoldering wick he will not snuff out"), and before death (see Psalm 103, the grass), and where it is said that the emperor of

Egypt and Egypt were for Israel like a reed on the Nile that broke and stung them when they wanted to lean on them (see Isaiah 36:6). *Thought *347* also resembles the verse in the Bible that says that human soul is more important than the entire universe (Mark 8:36 and Matthew 16:26). "For what shall a man be profited if he gains the whole world, but forfeits his soul-life? Or what shall a man give in exchange for his soul-life?"

This verse was probably a biblical foundation for people in old times, when they started to consider the man the center of things. On human wickedness, the Bible mentions in some places that humans are like "the worms of Israel" and about his greatness "... I have made you a little lower than the angels." The balance was made by the Lord, Jesus Christ has redeemed and restored man on the inside. Another biblical reference for the word reed that illustrates human weakness can be found in 1 Kings 14:15, and in Isaiah 40:15, 40:16, "For the Lord shall smite Israel, as a reed is shaken in the water..."

"Behold, the nations are like a drop from a bucket, and are accounted as the dust on the scales; behold, he takes up the coastlands like fine dust. Lebanon would not suffice for fuel, nor are its beasts enough for a burnt offering."



Certainly, another starting point for this thought can be found in the statement of Descartes in *Discourse on Method*, part IV, "I think, therefore I am," this statement being strengthened in *Thought *339*, which states that this action is specific to humans (see also *Thought 342*).

"One of the Apology's main strategies to use the contradictory philosophies of skepticism and stoicism, personalized by Montaigne on the hand, and Epictetus on the other, in order to bring the unbeliever to such despair and confusion that he would embrace God" (see [6]).

A feature of the skeptics in *Thought *347, 348* is emphasized by our physical vulnerability and, for example, a feature of stoicism is man's capacity of great and remarkable deeds and thinking. The two memorable reflections, *Thought 233* (Pascal's wager) and *Thought *347* (so called reed thinking) might be compared because there is a strong link between these two thoughts. First involves free will ("to do") and the second consciousness ("to be"). The author shows the natural insignificance of individual human lives, but he does not conclude that human existence is absurd. Just as Christian existentialists, Pascal points to a source of meaning that would transcend the limitations of our thought, emphasizing that beyond philosophy is God.

According to John Cruickshank, see [6], *Philosophy Dictionary*,

> It is clear, however, that human nature is investigated in the Pensées at the psychological, social, metaphysical, and theological levels. In moral-psychological terms, Pascal finds in human beings a series of dramatic contradictions which, he argues, only the Christian doctrine of original sin can properly explain. At the social-political level, he points to the fragile nature of many social relationships and the unsatisfactoriness of the legal and political concepts of his day. Rather startlingly for the period, he holds social hierarchy to be based on arbitrariness rather than justice. As with his moral and social being, man's metaphysical nature is a source of dissatisfaction...The Pensées, therefore, use an analysis of the problem of human nature in order to interest the reader in the Christian solution. They seek to convince him or her further with evidence from the scriptures, miracles, Church history, etc. Above all, they insist than only faith which responds to God's grace, not purely intellectual inquiry, will explain human life properly and bring knowledge of God and true happiness.

Some of the techniques used in the 18 *Lettres Provinciales* (one of the great polemical works of French literature, admired even by Voltaire who was not a friend of Pascal's or of the Jansenists) such as using humor, ironical mockery, and vicious satire in argumentation, bears the mark of his mathematical genius. This genius combined with the talent inherited from his father as a pleading lawyer make the reading attractive even for the unwittingly reader, and it influenced the prose of the French writers after him such as Voltaire and J.J. Rousseau and of another moralist La Bruyere in *Characters*. Pascal's influence in the people of his time and later is not restricted to literature. It starts, obviously, with mathematics and physics, but continues with philosophy, religion, and informatics. The impact it had on them can be seen by analyzing their writings and actions.

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The Search for Hamilton

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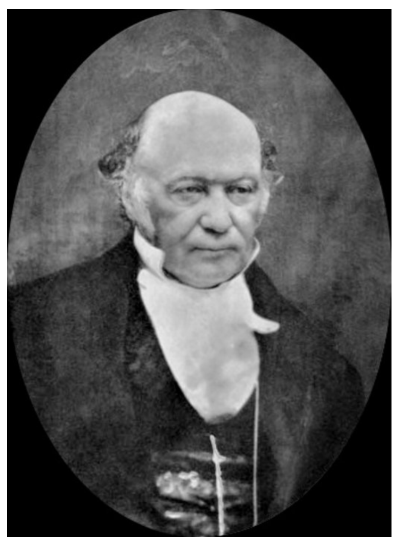
Eric Gossett earned his M.S., and Ph.D. in mathematics from the University of Arizona while specializing in combinatorics. He has pioneered the use of Mathematica within the curriculum and published the textbook *Discrete Mathematics with Proof*, Second Edition. He developed Bethel's first generation of web services. He is an avid racquetball player and is the faculty sponsor of the Bethel Anime Club.

1. Introduction

In July and August of 2009 my wife and I went on a four-week trip to Scotland and Ireland. We would be visiting Dublin, so I decided that we should visit the famous bridge where William Rowan Hamilton carved the equations for the quaternions.



Hamilton had been struggling to find a good way to multiply and divide points in three dimensions but was making no progress. On October 16, 1843 he was walking with his wife along the Royal Canal in the outskirts of Dublin. He had a flash of insight which lead to the now-famous equations that defined the quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Hamilton was evidently so excited that he carved the equations in the stone of the Broom Street bridge which crossed the canal. (The carving has long since vanished, but a commemorative plaque was added in 1958.

I teach a course in Discrete Mathematics so my interest in Hamilton is actually related to the topic of Hamiltonian cycles in graphs. A Hamiltonion cycle is a walk in the graph that visits every vertex exactly once (except that it starts and ends with the same vertex).

2. The Journey

I had the following useful information from Wikipedia:

> Broom Bridge, also known as Brougham Bridge, is a bridge along Broombridge Road which crosses the Royal Canal in Cabra, Dublin, Ireland: http://en.wikipedia.org/wiki/Broom_Bridge.

When we arrived at our hotel in Dublin, I asked the desk clerks if they knew where the famous bridge was located. They had never heard of the bridge, In fact, they had never heard of a famous Irish mathematician named Hamilton. Our tour guide, bus driver, and city guide were also not very helpful. The bus driver did think that he knew the bridge, but it was in a suburb that was too far to walk to and the tour was not going near that area.

We did have a nice walking map of Dublin which contained the cryptic message:

> Site of No. 36 Sir William Rowan Hamilton World-Famous Mathematician.

The site was part of a row house. We asked several people in the area about the house but none of them had heard of Hamilton. The best I can figure is that Hamilton lived there at one time. The house has been a teacher's club for many years.

Our hotel was next to Trinity College, so we decided to see if the math department there had a display about Hamilton. We started on one end of campus and walked a very long way to get to the math department. It is a small, old building which is surrounded by large, new buildings for departments such as chemistry and pharmacology. It was a weekend and nobody was in the math building. We were unable to find any displays about Hamilton.

We left Dublin in defeat. However, we did find the famous statue of Molly Malone,

saw the Book of Kells,

and attended a Sunday Evensong service at St. Patrick's Cathedral. The service featured a guest choir from America.

We were also able to visit Glendalough which is the site of a famous 6th century Irish monastic community. The tower on the left was for protection from Viking raiders - entry was through the hole about 12 feet from the bottom using a rope ladder which could be retracted.

We wandered around Ireland for several days, then crossed into Northern Ireland. The border really surprised me. The American-Canadian border has armed guards on the American side to keep out attacking hoards of Canadians and to stop foreign terrorists from sneaking into the country. On the Canadian side are armed guards who are desperately trying to keep out American culture. The border between the Republic of Ireland and Northern Ireland has no guards (not even the remains of guard houses), no lines on the road, not even a sign saying “Welcome to the UK” or ”now leaving the Republic of Ireland”. The only way to tell you have crossed the border is that signs in Ireland use kilometers and kilometers per hour and list places using Gaelic followed by English. In Northern Ireland they use miles and miles per hour and only list places in English.

The image is of a wall mural in Derry/Londonderry Northern Ireland. (The Catholics call the town Derry, the protestants call it Londonderry, and tourists who are unsure about who they are talking with refer to it as “your fair city.”

We eventually crossed into Scotland and wandered around there for several days. The image is of Brig O'Doon (the bridge over the river Doon). The bridge is famous for the Broadway musical and also because of the Robert Burns poem "Tam O' Shanter." Tam and his horse escape a group of angry witches by crossing the river via the bridge. Remember the Broom street bridge? We were having a good time, but at the back of my mind I was still disappointed that I missed it.

We eventually ended in Edinburgh, said goodbye to one tour group, then a day later joined a second tour group. After wandering around to more places in Scotland, Northern Ireland, and Ireland, we ended up back in Dublin. Here is yet another bridge: a new bridge across the river Lethe shaped like a celtic harp (the national symbol).

In Dublin we connected with another city guide. This time the city guide and bus driver were confident that they knew how to get to the Broom bridge, The next morning was a free time, so we walked 2 minutes from our hotel and got on a commuter train heading to the suburbs. One train change later we exited onto the Broombridge Station—a bare platform surrounded by weeds and some liter.

Up ahead was a graffiti-covered bridge over the railroad tracks. We walked up the ramp and asked a pedestrian where the Broom bridge was. He said "this is it." I told him about the Hamilton plaque and he said he thought this was the bridge some people were interested in.

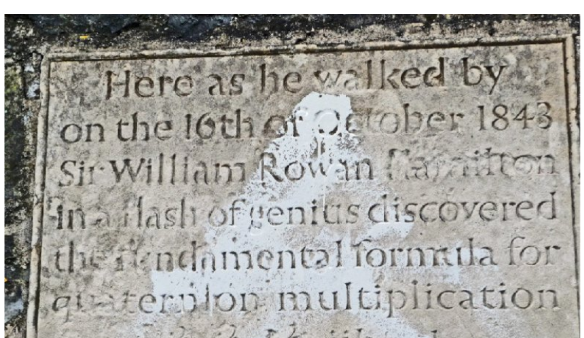
> Enter the following coordinates into Google Maps to see a satellite image of the bridge and station: 53.373023,-6.299992.

We walked across the bridge and noticed the canal running parallel to the train tracks. There was a walking path next to the canal, so we exited the road and went under the bridge, And there it was: the long-lost plaque.

There was no graffiti on this side of the bridge, but there was some white paint that had been thrown onto the wall and plaque.

The following heavily photo-manipulated image shows the plaque. The contents read:

> Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication
>
> $$i^2 = j^2 = k^2 = ijk = -1$$
>
> and cut it on a stone of this bridge.



This bridge, so dearly loved by mathematicians, is clearly not on the Dublin Tourist Board's list of places to visit. The site is pretty much off the radar of both the city of Dublin and the nation of Ireland. However, there is one group that is attempting to keep the memory alive: the Mathematics Department at the National University of Ireland at Maynooth.

> In 1990, we (the Department of Mathematics at NUI Maynooth) initiated the annual commemoration on the anniversary of the discovery, in the form of a walk from Dunsink observatory to Broom Bridge. Since then, a growing number of people have been participating. In 1993, the sesquicentenary, the first New Yorker appeared. He had flown across just for the day. http://www.maths.may.ie/hamiltonwalk

3. Epilogue

There is an even happier ending to the story. Two months after our visit, a restored plaque was installed during the 2009 annual walk.

All images except the portrait of Hamilton and the image from the Book of Kells were taken by Eric Gossett and can be used with permission. The exceptions are already in the public domain and can be found on Wikipedia.

The Need for a Graphics Programming Course in CS

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Nathan Gossett (B.A., B.S., Bethel University, St. Paul, MN; Ph.D., University of Minnesota Twin Cities) has been teaching in the Mathematics and Computer Science department at Bethel University since 2007. His research interests include Computer Graphics and Visualization, specifically Information Visualization. These days, he justifies his continued interest in video games as a professional necessity.

Abstract

A discussion of the benefits of offering a course on programming Computer Graphics in an undergraduate Computer Science curriculum. A sample course outline is provided, as well as a discussion of ways to conduct lectures, labs and a list of suggested assignments. A discussion of “dos and don’t s will also be presented, including a list of required prerequisite courses and skills that students would need in order for the course to be a success.

1. Introduction

Not every undergraduate Computer Science program includes a course focused specifically on Computer Graphics Programming. Many CS programs are already quite full of required courses, leaving little room for specialized courses. In addition, many smaller departments may lack the enrollment to sustain an elective course that non-majors are not qualified to take. Smaller departments are also less likely to have faculty who specialize in an area like Computer Graphics. However, offering an undergraduate course in Computer Graphics Programming is not only feasible, it can be a powerful student motivator and cross-disciplinary exercise. Based on my experience offering a mixed undergraduate/graduate course in Computer Graphics Programming at the University of St. Thomas, as well as an undergraduate course at Bethel University, I can offer the following observations and guidelines for those who wish to introduce such a course at their institutions.

2. Student Motivation

Many students are initially attracted to the Computer Science major due to interest in video games and movie special effects. The students that I see in my Graphics courses are often the most motivated to show up to labs and lectures, and are generally happy to spend long hours on their homework assignments. The prospect of a course in Computer Graphics at the end of their degree can help keep students who are in an early stage of a Computer Science program motivated to continue in the program.

3. Cross-disciplinary Application of Theory

A Computer Graphics Programming course is a great opportunity for a hands-on, applications focused implementation of material learned from a wide variety of prior courses. Besides the obvious computer programming content, the performance and efficiency requirements inherent to the material will drive students to use math concepts that they have learned in Linear Algebra and Numerical Methods. Getting objects to appear and move correctly on screen will require students to apply knowledge of light and reflection from their Physics courses. Understanding how a two-dimensional, discretized screen with only three distinct wavelengths of light can show convincing three-dimensional objects in a wide variety of colors will require an understanding of human sensation and perception from biology and psychology. Aesthetically pleasing depictions will require knowledge of art and drafting theory. If a student has previously been unconvinced of the usefulness of these areas, they will find the motivation that they need in a Graphics course.

4. Instant Feedback

A Graphics course also offers one of the best feedback mechanisms of any Computer Science course. Students who are wondering if they truly understand the concept of the day only need to look at their screens to discover the answer. A Graphics course held in an environment where each student has their own computer can be taught by alternating frequently between short lecture portions and lab portions throughout the class period. If students are provided with skeleton implementations of a small program, they can fill in the relevant code in 5-10 minutes to quickly close the learning loop started by the immediately preceding lecture portion. A Graphics course also offers a chance for students to build a substantial project incrementally over the course of several weeks or months while still having clear and complete products at each intermediate stage by focusing on one stage of the rendering pipeline at a time.

5. Student Prerequisites

So what type of students are required to make a Computer Graphics course a success? Students who have a passion for learning and a hardworking attitude are of course desired. However, for the course to move as quickly as it needs to, students must arrive at the beginning of the semester with a strong background in both programming and math. I have found that students with less than three semester of prior programming experience (or equivalent out-of-classroom experience) struggle to keep up with the programming requirements and have a hard time keeping up with the pace of the projects and exercises.

It is also helpful for students to have taken Discrete Math, Linear Algebra, and a Numerical Methods course. The most useful portions of those courses can be addressed as part of the Graphics course, but students who already understand common matrix and vector operations, parametric equations and numerical derivatives and integrals will be much more able to keep up with the pace of the material.

I would advise offering a Graphics course using a title such as “Computer Graphics Programming" rather than just "Computer Graphics" to avoid confusion from students looking for an art or graphic design course. An undergraduate graphics course should probably be a 400-level course and should have appropriate and clearly listed prerequisites. Make sure interested students know about the requirements well in advance of the course so they are able to prepare, and be firm about your listed prerequisites.

6. Languages, APIs and Toolkits

In my experience, the most appropriate language, API and UI toolkit for an intro level graphics course are C/C++, OpenGL and GLUT (OpenGL Utility Toolkit). This combination has the benefits of being free, cross-platform, and relatively easy to start using for simple programs. Mac and Linux users will probably find all of the tools and libraries they need already installed on their machines, and Windows users will probably only need to acquire one of the free GLUT libraries [2,5]. OpenGL is widely accepted as an industry standard, and plenty of documentation and tutorials are available in print and online. GLUT is not officially part of OpenGL, but it is so simple and widespread that it is often treated as part of the core API.

If your department focuses more on Java than C/C++, an alternative would be to use JOGL [4] (an implementation of JSR-231). JOGL is implemented closely enough to OpenGL and GLUT that the differences in syntax and capabilities will be minor (mostly dealing with the differences between Java and C syntax).

An alternative to OpenGL would be Microsoft's Direct3D library which is similar (and in some places superior) in capability, but not in syntax. This option would also have the drawback of not being cross-platform, so if you wish to target any non-Microsoft platforms this option would not be desirable.

There are also many UI toolkit alternatives if you wish to go beyond the limited feature set of GLUT. However, for an intro-level course GLUT is often sufficient.

7. Books

I have had good luck using the textbook "Interactive Computer Graphics" by Edward Angel [1]. This book provides a good coverage of intro-level material with approachable writing and examples in OpenGL. Many other books are also available, but I would offer three cautions: First, avoid texts that focus too much on a particular API or toolkit. Pick a text that provides a good coverage of algorithms and concepts that are applicable to a wide variety of APIs. Second, make sure that your textbook offers example code in the language you are using to teach the course, or else provide a secondary source that students can go to for examples of good code. Finally, while there are some excellent books on advanced rendering techniques that may even be the text of choice for a graduate level graphics course, these are not appropriate books for an intro-level undergraduate course.

If you choose to offer your course using OpenGL, an optional secondary text that you might consider using would be "The OpenGL Programming Guide" (commonly referred to as "the red book") [6]. While all of the information contained in this manual can be located scattered across many websites, it can be convenient to have all of the information gathered together into one place. However, I would not consider this to be an appropriate primary text for the course.

8. Sample Course and Homework Schedule

The ACM does include a Computer Graphics section in their 2008 update to their curriculum guide [3]. The topics list for "Graphics and Visual Computing (Core)" should serve as a basic list for your course, and should leave you with enough time to include at least a few of the elective topics at the end of the semester. A course schedule that I have used successfully for a 15 week semester is:

- Weeks 1-3
  - Intro to Graphics Programming
  - Human Visual System background
  - Common structures, user input
- Weeks 3-5
  - Geometry
  - Linear Algebra refresher
  - Affine Transformations
- Week 6
  - Viewing (Orthographic first, then Perspective)
- Weeks 7-8
  - Lighting
  - Shading
  - Midterm Exam
- Weeks 9-10
  - Rasterization
  - Fragment operations
  - Buffers and texture mapping
- Weeks 11-12
  - Procedural Rendering
  - Curves and surfaces
- Week 13-15
  - Final Project assignments

- Survey of advanced graphics topics
  * Ideas for further study
  * Inspiration for grad school
  * Perhaps watch and discuss selected SIGGRAPH videos
- Extensions
  * GLSL
  * OpenGL ES, WebGL
  * GPGPU (CUDA, CTM, OpenCL)
- Final Exam + Project Presentations

9. Sample Project Schedule

In addition to small coding exercises during the class period, I usually assign a series of 7 longer term projects to be completed as homework.

- Assignment 0 (1-2 days)
  - Make a white square appear in a window with a black background. Submit a screen shot of this program running on your chosen development machine
  - The source code for this assignment can be found as hello.c in the red book. This assignment is to make sure everyone has a working development environment
- Assignment 1 (1-2 weeks)
  - Model some object (chosen by the professor) composed of simple geometric shapes
    * A person with a body, legs, arms, head
    * Car with body, wheels, etc
- Assignment 2 (2 weeks)
  - Make the model from Assignment 1 move
    * Walk, drive, fly around screen
    * Swing legs, rotate wheels, flap wings, etc
    * User input to control motion
  - Focus on hierarchical movement using ModelView stack
  - Students may need to redo their modeling from Assignment 1
- Assignment 3 (1-2 weeks)
  - Add camera controls to Assignment 2
    * Chase cam, "through the eyes"
    * Zooming, panning, dollying, etc
- Assignment 4 (2 weeks)

  - Add lighting to Assignment 3
    * Spot lights that follow walker, headlights that light up environment, etc

- Assignment 5
  - Texture mapping
  - Skyboxes, planets with translucent cloud layer, etc

- Final Project
  - Advanced topic of students choice (with professor approval)

10. Suggestions For Extra Topics

If you find yourself with an extra week or two at the end of your course, there are a few extensions that you can make to the basic curriculum. One topic that would be appropriate to add would be making use of the programmable graphics cards that your lab and student machines are likely to have. If you have been using OpenGL as your API, you could start transitioning from the traditional fixed-function pipeline to custom GLSL shaders. If you know in advance that you will have extra time during the semester, you may even wish to start introducing GLSL right away in the semester since it will allow you to have students re-implement sections of the pipeline themselves rather than just talking about what the API is doing behind the scenes.

GPGPU (General Purpose Graphical Processing Units) research is also becoming more popular. For many workstations, the graphics card (GPU) is likely to have the most powerful processor in the machine. If the GPU is a modern stream processor unit, you could include a 2-3 week unit on High Performance Computing. You could attempt to convert non-graphics problems into a graphics context to make use of GLSL, or you could use a general purpose computing API such as CUDA (Nvidia), CTM (ATI) or OpenCL (cross-platform).

Finally, there are alternative versions of OpenGL such as OpenGL ES for mobile devices and WebGL for web browser rendering. If your students are interested in mobile app development or web development, you could devote a week or two to the highlights of these APIs.

11. Conclusion

If your school does not currently have a course in Computer Graphics Programming, I would encourage you to consider implementing one. This is an exciting, topical area of the Computer Science curriculum that should not be skipped. If you have further questions about offering such a course, please feel free to contact me at gosnat@bethel.edu

12. Acknowledgements

Special thanks to Tim Urness who provided me with invaluable advice and examples before I taught a graphics course for the first time. I would also like to thank the Math and Computer

Science departments at Bethel University and the University of St. Thomas for supporting me in offering Computer Graphics Programming courses and in offering me significant leeway in course design.

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Real Simulations and Simulated Reality

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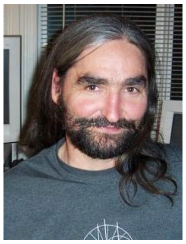

Prior to joining Westmont in 2003, Wayne Iba worked as a Research Scientist in the Artificial Intelligence group at NASA Ames Research Center and on the Automated Fingerprint Identification Sytem operated by the FBI's Criminal Justice Information Services. Wayne received his M.S. and Ph.D. in Information and Computer Science from UC Irvine, and a Bachelor's degree from UC Santa Cruz. His interests focus on Artificial Intelligence and Machine Learning.

Abstract

Movies such as *The Matrix* have stimulated popular interest in "brain in a vat" scenarios. Amidst the traditional questions of mind, we tend to overlook an integral enabling component—the world simulation—which merits consideration in its own right. When facing the simulations in these imagined scenarios, we struggle with conceptual muddles regarding what is real and not. In this paper, I argue that simulated worlds are every bit as real as the one we inhabit. This turns out to be important when considering the possibility, as suggested by Nick Bostrom, that the world we experience as "real" is actually a simulation. Can such a prospect be reconciled with an Orthodox Christian perspective? While the metaphysical status of simulations that I posit moves us towards an integration, significant obstacles remain to be addressed. I consider some of these remaining challenges and explore the associated stakes.

1. Introduction

The general population has encountered simulated worlds through many books ranging from *True Names* (Vinge, 1984) to *Idlewild* (Sagan, 2003) and movies ranging from *The Matrix* to *The Thirteenth Floor*. Hundreds of years ago, Descartes anticipated the brain in a vat scenario, and we find a related situation thousands of years ago in Plato's cave. In the contemporary brain in a vat scenarios such as *The Matrix*, we find human brains connected directly to computer simulations which provide sensory stimulation to the brain and incorporate the brain's volitional motor acts into the simulation's state.

One thing such treatments of simulated reality have in common is an implicit and under-appreciated reliance on the computer simulation. The connection between the brain and simulation and the predicament of the brain itself provide an intriguing scenario in which we can explore conundrums of the philosophy of mind. But these thought experiments do not get started without the simulation itself. When simulations are explicitly considered, they are invariably dismissed as unreal or even deceitful.

So we find two problems at work in treatments of virtual realities. First, the simulation itself is overlooked, minimized or dismissed. Second, the standard view of simulations relegates their ontological status to the unreal. In this paper, I want to redress these two problems, arguing that simulated worlds are "real" worlds.

However, responding to these two problems opens the door on a third problem: What are the implications for Christians if it turns out that the world we experience is in fact a simulation? After addressing the ontological status of simulations, I consider these implications and explore how, if at all, we can preserve a traditional Christian worldview in the face of such an eventuality.

2. Importance of Simulations

Traditional brain in a vat thought experiments carry with them an important but usually overlooked implicit assumption. They provide insight only in so far as the simulations they rely upon are adequate to the task of mimicking the "real" world. This assumption requires both the necessary computing platform and the ability to program a simulation of the physical world with fidelity sufficient to fool the brain that is actually floating in a vat.

Scientists and entertainers have discovered the challenges involved with simulating the physical world. Whether attempting to model the aerodynamics of a bicycle or to provide a compelling game experience, simulations have proved to be extremely computationally demanding. The simulation must compute the consequences of interactions between entities within the simulation, and, in the case of a brain in a vat scenario, must perform these computations fast enough to stay ahead of the brain's own processing.

The creator of the simulation must not only have the necessary computing hardware, but must also understand the properties of the environment that is to be simulated. Every simulation embeds a *physics* that determines the nature of interactions between entities. When a simulated world is intended to model our physical world, as in the case of computational fluid dynamics or many computer game experiences, the correspondence between the simulated physics and that of our physical world determines the success of the exercise. This requires an understanding of the physics to be simulated and the coding skill to adequately implement the corresponding simulated physics. Common experience indicates that both of these requirements present significant challenges. Thus, simulations play a much more important role in the brain in a vat thought experiments than we tend to realize.

However, we should note that the problems discussed here (computing hardware and simulation design and implementation) are somewhat mitigated for brains in vats if we allow that they never knew another environment. That is, if the hypothetical brain in a vat has only ever received sensory impressions from, and triggered changes through volitional actions in, the simulated environment, then only the speed of the simulation remains a problem. Since such a brain would have no comparison environment by which to judge the simulated one it is experiencing, the fidelity of the simulation loses some relative importance. Although the simulation must still keep up with the brain's processing, the designer may arbitrarily simplify the environment and its physics in order to allow the available computing hardware to perform its tasks in a timely manner since the brain knows no better.

Even allowing the mitigation of the challenges faced by the successful simulation, we see that the role of the simulation is more significant than typically thought. Furthermore, brains in a vat are not the only situations in which simulations play an important role. Suppose that we replace the brain with another computer. Although this introduces its own problems that merit consideration, the point for our present purposes is that the simulation continues to play a crucial role. Thus, we now turn to consider what we mean by "real" and the ontological status of simulations and simulated entities.

3. Ontological Status of Simulations

Informally, we make the distinction between "real" things and a variety of alternatives including those that are: artificial, imagined, or simulated. In each case, we have a slightly different meaning in mind when we use the word "real". While a meaningful distinction between simulated and real entities *can* be made, we need to reconsider the distinction that we often have in mind when we use these terms. Ultimately, I claim that the ontological status of a simulated object is the same as an object in the physical world and thus we should consider such a simulated object to be real.

First, we observe that a simulation is real in our world. That is, if someone implements a computational simulation of some physical process, we say that the resulting product is a *real* artifact—namely, the simulation itself. If we distinguish between the simulation as a whole and the entities simulated therein, we still find that the simulated entities are real in our world. We attach the "simulated" adjective in order to distinguish entities that exist within the simulation from those that exist in our world. But we see that the simulated object has a reality in the physical universe; we can point to certain memory locations and their interrelationships in the context of the computer hardware running the simulation in question.

Second, we claim that the entities within a simulation are real within the simulation *in the same way that we say objects in our physical world are real.* On what basis do we grant the ontological status of real to, say, a chair in our physical world? We can see and touch chairs; they are accessible to our senses. If we apply the appropriate measurements, we find that a chair is made of the same kinds of "stuff" of which we are made. And finally, if we sit in a chair it will support our weight. The same line of reasoning must be applied to a simulated chair. A simulated humanoid agent in our simulation observes the chair with its senses. If the agent were to examine the chair closely enough, it would find the chair to be constituted of the same stuff as the humanoid itself. It makes no difference what that stuff would look like to the humanoid. We might design the simulation such that the humanoid discovers particles much like those we find in the physical world; or it might encounter access to the bits and bytes of the simulation. In either case the important factor is that the chair is made of the same stuff as everything else in the simulated world. Thus, a simulated chair is as real to a simulated humanoid as a physical chair is to us.

Let us examine the intuitive reluctance to grant simulated objects a full ontological status of real. Although our hypothetical humanoid might view the simulated chair as real, none of us would be able to sit in the chair. I claim that it is this difference that fuels or props up the distinction even in the face of agreement on the previous two claims. However, this intuitive objection needs to be challenged. If we hypothesize an interface to the simulation analogous to the one used by Neo and the others in *The Matrix*, then we can easily imagine ourselves sitting in the simulated chair. If queried during his audience with the Oracle, Neo might report that he is reclining on board the ship, Nebuchadnezzar; but he would be much more likely to say that he is sitting in the kitchen chair while talking with the Oracle and eating cookies. Indeed, human experience with gaming in immersive simulated environments confirms that we place ourselves outside our bodies with relative ease.

All this is not to say that there is *no* distinction between objects in our physical world and those in a simulation that we create. The thrust of my claim is that the distinction is not what is typically taken for granted. Instead, I claim the distinction reflects a *sustaining character* between the physical world and a simulation. That is, a simulated chair obtains its ontological status of existence by way of the simulation running in the physical world. In this light, in may be helpful to think about the physical universe and what its source of existence might be. For anyone adopting at least a deistic position, we can think of the physical universe as having its origin and as being sustained in the mind of God. In this sense, the chairs we think of as "real" are analogous to simulated chairs in God's view. That the chair is real to God is uncontroversial; that it is also the case that the chair is essentially simulated from God's view does not take away from the chair's reality (either in our or God's views). Thus, without loss of significance, simulated objects should be granted the same ontological status as objects in the world we experience every day. This conclusion turns out to bear import on our responses to the hypothesis that the physical universe we currently experience is actually simulated.

4. Are We Simulated?

Expanding our thinking about simulations opens the door to considering hypotheses previously unimaginable. For example, Nick Bostrom (2003) has suggested that we may be living in a simulation. Based on his model and analysis, he claims a one-third chance that the world we experience and call the "real" world is actually a simulation. The details of his argument are unimportant to our present discussion; we are concerned here with the possibility and consequences if it turns out to be the case that we actually are living in a simulation.

If we follow the reasoning presented above regarding the ontological status of simulated worlds, then discovering that our physical universe is a simulation need not be catastrophic. The disorientation and identity crisis that attends such an eventuality is based on unexamined senses of real and simulated. If simulated objects are real objects, then if the physical world is in fact simulated, only the underlying makeup of the world is different from what we thought—not the fundamental status of what is real. The situation is analogous to the replacement of the theory of phlogiston with oxygen and the periodic table. Things formerly thought to consist of phlogiston and other stuff were not suddenly dismissed as unreal; instead, our model of their composition changed.

Be that as it may, discovering the world to be a simulation could require revisions of a significant nature for those holding anything stronger than a deistic worldview. For those within the Christian tradition, the understanding of scriptural revelation, miracles, and the life and works of Jesus would have to be reconsidered and perhaps modified. Proactively, we now turn to this exercise with an eye toward orthodox Christianity.

5. Exercise: What if

As an attempt to anticipate the relationships between simulations, reality and a Christian worldview, we advance the following thought experiment. Start by assuming that the physical world we experience around us is in fact a simulation being run in some *containing* environment, outside and independent of our universe. The plausibility of this assumption is already addressed by Bostrom (2003) and we will simply make it without further consideration.

Next, we define a zero-level world to be one that exists in and of itself. For the naturalist under our initial assumption, a zero-level world would be a physical universe much like, but other than, our own. For a deist, the zero-level world would be God's existence. We then define a nth-level world to be a world that is created (i.e., simulated) and sustained by a world at level $n-1$. In these terms, our initial assumption states that the world we typically think of as the real world is one at level i, where $i > 1$.

5.1. Approach

Our discussion in the previous section above concludes that our basic premise need not drastically impact the way we understand the universe or the way we relate to it. However, it could be the case that a traditional Christian faith would be incompatible with the assumption; if that is the case, we want to know it. Our approach to exploring this question uses the Nicene Creed as a rough and ready proxy for a statement of orthodox Christian commitments. If we can work through this creedal statement making sense of the claims within the context of our initial assumption, we can conclude that Christianity is compatible with the possibility that we are living in a simulation.

> "We believe in one God, the Father Almighty, Maker of heaven and earth, and of all things visible and invisible."

The first concern we encounter in this exercise revolves around the scope of "heaven" and "*all* things ... invisible." If heaven refers to the heavenly bodies—moons, planets, stars, etc.—then we have no problem. The creator of the simulation clearly is responsible for creating everything that exists within the simulation. This naturally includes things which may be invisible to agents within the simulation. For example, objects or events outside the range of our senses would be invisible although we might create instruments to detect such things. But this could also include the environment's *hidden state*, that no instrument could be devised to measure or detect; yet that inaccessible state is established by the simulation's creator. In both cases, our initial assumption of living in an ith-level simulation can be reconciled with traditional Christian beliefs by referring to the creator of our world, living at level $i-1$, as "God".

If instead, "*all* things" refers to all simulations at levels $j \geq 1$, then we let "God" refer to the creator of all level 1 simulations. The creedal claims about this god do not preclude any number of intervening or encapsulating worlds. This second explanation preserves our traditional picture of God at the expense of requiring God to jump through levels (at least conceptually) in order to interact with us. The first explanation makes our connection to God more direct but requires a more radical revisioning of that particular god and its place in the bigger scheme of things. Despite the greater revision of our current conceptual framework, that approach provides the more parsimonious approach. At the end of the day, either explanation could work.

> "And in one Lord Jesus Christ, the only-begotten Son of God, begotten of the Father before all worlds, Light of Light, very God of very God, begotten, not made, being of one substance with the Father;"

We again have two ways to approach this portion of the creed. First, we can let the "before all worlds", which refers to all ages, refer to the space-time of our physical universe. This allows "one Lord Jesus" to be particular to our universe and leaves open the possibility of other Jesus-like persons appearing in other simulations that are parallel to our own. Alternatively, we can again refer to God at level zero, the creator of all level one simulations. Either explanation suffices for our purposes here, with the same qualifications mentioned above.

> "by whom all things were made; who for us men, and for our salvation, came down from heaven, and was incarnate by the Holy Ghost of the Virgin Mary, and was made man;"

For this portion of the creed, we do not encounter any serious issues to explain. We note that coming "down from heaven" must necessarily refer to a kind of movement from a level $i-1$ world to our own world. As the creator of the simulation, Jesus can arrange for his own immaculate conception and subsequently "jack in" to our world. But the simulated nature of our assumption presents no difficulties that do not already exist with our traditional understanding of these mysteries.

> "he was crucified for us under Pontius Pilate, and suffered, and was buried, and the third day he rose again, according to the Scriptures, and ascended into heaven, and sitteth on the right hand of the Father; from thence he shall come again, with glory, to judge the quick and the dead; whose kingdom shall have no end."

First we note that, based on our earlier discussion of the ontological status of simulations, Jesus' passion was "real" and not to be diminished in any way on account of its having been simulated (under our working premise). This reflects a erroneous understanding of the distinction between simulations and the worlds that host them, rather than a creedal concern arising from the supposed simulated nature of our world.

But we also encounter a problem at the end of this portion of the creed with the claim "whose kingdom shall have no end."Clearly, if we are living in a level i world with $i > 1$, then the level $i-1$ world will come to an end. Unless we significantly alter our understanding of time (which might be called for independently of the arguments put forth in this paper), this creedal claim seems to rule out the "many Jesus" approach to explaining things introduced above. Thus, we would fall back to the second approach in which we refer to God at the zero-level who creates and sustains all level one worlds (as well as all higher-level worlds including our level i world).

> "And in the Holy Ghost, the Lord and Giver of life, who proceedeth from the Father, who with the Father and the Son together is worshipped and glorified, who spake by the prophets."

This part of the creed presents no serious difficulties to explain. We note that two-way communication between the creator of a simulation and the simulated entities is straightforward to imagine.

> "In one holy catholic and apostolic Church; we acknowledge one baptism for the remission of sins; we look for the resurrection of the dead, and the life of the world to come. Amen."

This final portion of the creed presents no problematic issues for our premise, but in it we encounter a tool or a model for understanding the resurrection and the afterlife. The brain in a vat scenario provides numerous opportunities for thinking about a person at level i taking on a body and interacting with a world at level $i+1$. But thinking in terms of nested simulations provides a way for understanding how a person could be "promoted" from level i to level $i-1$. To see this, imagine a simulation in our world. This simulation models our physical world with high fidelity. Now, suppose we build a robot body that can sense, move about, and manipulate our physical world; we design this robot body with the same sensory-effector interface as exists within the simulation. Finally, imagine taking an individual from the simulation and placing it in control of the robot body. The "person" would interact with our world as we do and would gain a perspective of its prior existence within the simulation. If we imagine two successive simulations at levels i and $i+1$, both with a foundational substrate of information, then the robotic body that is provided for the resurrected person can be indistinguishable from the bodies of persons living at level i. This presents one way to make sense of a resurrection and afterlife.

5.2. Results

In this exercise, we started out by pursuing two seemingly parallel approaches to making sense of the Nicene Creed based on our initial premise that the physical world around us is actually a simulation. The first referred to God as the creator who lives in a level $i-1$ world and created the world we experience. The second referred to God as the creator at level zero, who interacts with us through any number of levels of worlds. Although we ended up abandoning the first approach based on problems it raised with our understanding of the temporal continuum, we note that it could turn out to provide a better explanation than the second contingent on an appropriate reconceptualization of time. But our purpose at the outset was merely to examine the consequences for a Christian belief system in the case where our premise holds.

As such, there are three possible results that could obtain. First, we might find that there is no problem to be resolved whatsoever. That pretty clearly is not the case, as we have had to stretch our understanding of creedal claims on several points. In particular, our preferred explanation treats God as our *indirect* creator who created all level one worlds, and then relies on agents in one of those worlds to develop level two worlds, and so on, until our particular level i world is created.

As a second alternative, we might have found that there is no possible reconciliation with the premise. This does not seem to be the case either. The core doctrines are preserved in so far as Jesus could have *really* lived, died, rose again and ascended into heaven. Their reality and significance are not diminished because they were merely simulated events.

So our third alternative seems to be that some theory revision is necessary. To respond to a discovery along the lines of our initial premise we need to modify our understandings of "real" and "simulation" as well as of some of the creedal doctrines. The real question here is: Does such a theory revision amount to a move from geocentric to heliocentric models of our world, or is it analogous to a move from orthodox Christianity to Gnosticism? It is certainly the case that the view of simulations that I am proposing places great value on the body, and thus does not lead to Gnosticism in particular. However, whether it leads to some other heretical position in a question for others to decide.

6. Discussion

In this paper, I have argued that we need to enlarge our view of simulations with respect to what is real and what is not. In particular, I have attempted to reconcile a traditional Christian worldview with the possibility that the world we experience is actually a simulation. In conclusion, I want to consider what we, as Christians, gain through this exercise, what we might lose, and further implications.

I suggest that the analysis and the exercise above provide at least three benefits. One of these is better precision and accuracy in our use of the word "real" and a richer understanding of simulations. Through these reflections, we see that a simulation only makes sense with respect to some reference that is being simulated. Given such a reference, we can talk about how faithful or accurate or useful a given simulation is. Instead of focusing on simulations, we should talk about creations. Some creations will simulate processes or phenomena found in other creations and we can rightly call these simulations. But other creations may establish environments with novel physics populated by agents that learn to make sense of their peculiar world.

A second benefit takes the form of a plausible Christian response if we discover that our world, the creation in which we live, is implemented on some type of computer system. This holds whether or not our world is a simulation of some reference world. Third, even in the case where our universe is not computationally sustained, the nested simulation model provides a concrete model for thinking about resurrection and the afterlife.

Although we may obtain these and other benefits, we would do well to consider what we give up in the process. As I hope to have shown above, we do not give up anything along the lines of human dignity, the goodness of God, or the meaning of life. Of course, the way we think about the universe changes significantly, but only in an analogous manner to the changes accompanying the Copernican revolution.

Finally, there are further implications to consider. Thinking more broadly about simulations may lead us to reconsider ethical issues. If simulated agents are real agents, then we may need to exercise restraint with respect to some types of experiments we might like to run in simulations. For example, if there are things we would not do to living rats on ethical grounds, then those things would also be suspect in a simulation with simulated rats. Generally, this expanded view of simulations as computational creations leads us to think and act more responsibly with respect to simulations as well as with respect to our own universe.

For ages, humans have sensed that the world we see around us is not all there is. As real as this world is, many have the sense that there is something "more real" out there. Regardless of the underlying makeup of our universe—whether or not it is simulated—as Christians "we look for the resurrection of the dead, and the life of the world to come. Amen."

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Bringing Undergraduate Research into the Classroom

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Stephen Lovett is a professor in the Department of Mathematics and Computer Science at Wheaton College. He has a passion for doing research with undergraduate students. His current mathematical interests include algebra, cryptography, and differential geometry. For extracurricular activities, he enjoys music, travel, and writing fiction. He attends Trinity Church of the Nazarene in Naperville, Illinois.

Abstract

Mathematics graduate programs and companies that employ math majors often want to ascertain an applicant's potential for research. However, in many undergraduate courses, assessments consist only of regular exercise sets, quizzes, and in-class tests. Without doing a senior research thesis or landing an official REUs, students do not regularly gain experience in or an appreciation for research. Courses in the humanities regularly require students to write in the discipline, progressively preparing them methodologically for "writing in the field." This begs the question: could math departments do a little more to prepare our students to use mathematics beyond college?

In this talk, we explore options for incorporating undergraduate mathematical research and writing into regular course assignments. The speaker will present a number of topic questions used in a variety of courses and illustrate the results with actual student work.

1. The Problem

A few years of teaching a the college level has shown me a number of pedagogical problems that appear to be particular to undergraduate mathematics education.

- Many graduate schools would like to see student research. But REUs are very competitive.
- During interviews, employers ask about accomplishments. Problem sets do not count toward accomplishments.
- In other disciplines, students learn to write/perform in their field. In math, that is usually reserved for honors requirements.

This leads us to the following pedagogical question: Can we adjust teaching practices in math classes to address or fix these problems?

Dr. Aparna Higgins at the University of Dayton has led many mini-courses on leading undergraduate research at the Joint Mathematics Meetings. She outlines six different levels of what educators may see as undergraduate research.

1. Publishable results.
2. Results that extend/generalize a theorem in the literature, but probably not publishable.
3. Results obtained during some investigation, new to the student but not to the math community.
4. Results well established in the literature. The student reads and understand, but does not discover.
5. Results obtained in a group project or a team competition.
6. Problem solving.

It is also useful to draw out the reasons to promote undergraduate research:

1. it offers a good introduction to our profession;
2. it teaches communication;
3. it connects various areas of mathematics;
4. it looks good on resumes;
5. (for a college) it sounds good as a recruiting tool;
6. (for a college) it creates a selling point for hiring faculty or getting grants.

We might not be able to solve all of the above values in the classroom setting, but why not try?

2. The Experiment

2.1. Investigative Projects

For a few years now, I have experimented with a new form of assessed assignment, which I call "investigative projects." In the classes where I require IPs, I insist on the following parameters.

- Two to three projects per semester. Students should not face this assignment just once. They often need to get used to this new form of assessment in a math class.
- Teams of 2 to 3, not more, not less. This requirement promotes teamwork. Students say they appreciate this aspect of the project and regularly state that they would not have done as well as they did if they had been required to do it alone.
- Give at least two weeks. Similar to a paper in a humanities or social science course.
- The project counts for a sizeable portion of the course grade. This communicates to students that though it is a new kind of assignment, I take their effort very seriously.
- Offer a variety of topics; students are encouraged to modify or add questions.

2.2. Assessment

With many mathematics assignments (e.g., tests and exercises) it is relatively easy to assess student work. With investigative projects, the assessment carries more dimensions than just correct mathematical reasoning. I assess the project according to the following guidelines.

- I grade according to 4 Cs:
  - Clarity (Style)
  - Correctness (Mathematical accuracy)
  - Completeness (Scope of the project)
  - Creativity (Adding their own work, inventiveness)
- I give detailed feedback in each of those categories.
- Sometimes I offer a rewrite.

2.3. Creating the Questions

As the professor assigning the investigative projects, it is not always easy to imagine questions that will lead students toward the objectives that I wish students to reach. Staying on top of articles and project ideas in textbooks is helpful. Currently, I use the following principles to shape my project questions.

- Never a direct application of course content.
- Always have some form of open-ended aspect (some more; some less).
- May be guided, like some textbook projects.
- Some projects may not have clear cut solutions.

2.4. Specific Project Questions

Following are the investigative project assignment sheets that I used this past year.

1. Calculus II:
   - Project 1: http://cs.wheaton.edu/~slovett/projects/Calc2InvProj1.pdf;
   - Project 2: http://cs.wheaton.edu/~slovett/projects/Calc2InvProj2.pdf;
   - Project 3: http://cs.wheaton.edu/~slovett/projects/Calc2InvProj3.pdf;
   - Project 4: http://cs.wheaton.edu/~slovett/projects/Calc2InvProj4.pdf
2. Linear Algebra:
   - Project 1: http://cs.wheaton.edu/~slovett/projects/LAInvProj1.pdf;

- Project 2: http://cs.wheaton.edu/~slovett/projects/LAInvProj2.pdf

3. Differential Equations:
   - Project 1: http://cs.wheaton.edu/~slovett/projects/DiffEqInvProj4.pdf
4. Modern Algebra:
   - Project 1: http://cs.wheaton.edu/~slovett/projects/ModAlgInvProj1.pdf;
   - Project 2: http://cs.wheaton.edu/~slovett/projects/ModAlgInvProj2.pdf

3. The Results

The reader is welcome to peruse some specific student work.

- Differential Equations—Population Dynamics and War:
  http://cs.wheaton.edu/~slovett/projects/PopDynNWar.pdf
- Modern Algebra—When Does $x^2 + 1 \equiv 0 \pmod{n}$ Have Roots?:
  http://cs.wheaton.edu/~slovett/projects/OrewilerHsu.pdf
- Modern Algebra—The Ring $P(S)[x]$:
  http://cs.wheaton.edu/~slovett/projects/PSPolyRings.pdf
- Modern Algebra—The Ring $P(S)[[x]]$:
  http://cs.wheaton.edu/~slovett/projects/PSFormalSeriesRing.pdf
- Linear Algebra—Dating Habits of Wheaties:
  http://cs.wheaton.edu/~slovett/projects/WheatonDating.pdf
- One student project impressed me so much that we were able to write an article based on that student's work. The article, "Painting by Parametric Curves and Van Gogh's Starry Night," appeared in the November 2010 edition of *Math Horizons*.

As a professor, it is invigorating to see students excited about doing original work in mathematics. In the past few years, students have discovered results on their own, proved new theorems, and published at least one article (some more may be forthcoming).

3.1. What Students have Learned

Students can learn different things at different levels. Here a few themes that my students learn through the investigative projects:

1. Math problems are not just exercises.
2. How to write a math paper: prose, organization, LaTeX, ...
3. How to think creatively and reflectively using math.
4. How to find real data.
5. How to find what other people have already done.

3.2. Additional Benefits

I have found that the investigative projects produce a few additional benefits.

1. Students sometimes continue working on their project afterwards.
2. Projects produce many ongoing math conversations.
3. I get rich material for when I write letters of recommendation.
4. The projects begin to offer students a research experience.
5. It's exciting!

Connecting Mathematics Students to Philosophy and Faith

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1. Introduction

Many Christian colleges and universities place a high value on the continuing pursuit of integrating faith and learning. Faculty are encouraged and supported to seek out meaningful connections between their Christian faith and their scholarly discipline. The motivation behind such an effort is clear: to eliminate the ingrained notion that academia and faith provide mutually exclusive pathways to travel in the pursuit of truth. Exploring the rich supply of integrative questions that arise in each discipline is certainly a noble calling, but also brings its challenges. This is especially true in the area of mathematics.

One of the ongoing challenges that we face as educators is how to bring our Christian faith into the mathematics classroom to engage students in this important conversation. I propose that quality, practical examples of successful classroom integration projects will serve to stimulate innovation and creativity within the body of Christian mathematics educators. Toward that goal, I will discuss a class project that I implemented in a Real Analysis course. The idea for this project grew from my involvement in a workshop called *The Vocation of the Christian Professor* in the summer of 2010 at Whitworth University. This workshop challenged me to look deeper at how my faith shines light on my foundational assumptions of mathematics. Thus, the class project makes use of mathematical philosophy as a means of facilitating faith integration.

2. Project Motivation

The types of questions one asks about mathematics determines the degree to which meaningful integration can take place. I suggest there are three main questions that represent different levels of engagement with mathematics.

- *How* do we do mathematics?
- *Why* does mathematics work?
- *What* is the nature of mathematics?

The vast majority of people are satisfied to engage mathematics at the level of the *how* question. How do I get a common denominator? How do I factor a polynomial? How do I take a derivative? At this surface level, there is very little, if any, significant faith integration that can take place. Mathematics is viewed as a tool for science and a means of understanding the world. Thus, the development of specific skills is the emphasis of mathematics education from kindergarten through high school.

It is not until college when mathematics majors have the opportunity to delve into the *why* question. Here, the engagement with mathematics is at the level of proof, where students come to understand the rigorous and logical process by which mathematics is produced and justified. A large and increasing skill set must also be accompanied by a deeper conceptual understanding of mathematical objects and the interrelations that exist between them. It is in this understanding that students can begin to probe the deeper integrative questions. For this reason, the project was implemented in a proof-based upper-division course.

The most fruitful integration discourse can occur only at the deepest level of engagement with mathematics: the *what* question. What is the nature of mathematical objects? Is mathematical truth created or discovered? Is mathematics a part of the structure of the universe or the product of the human mind? How do we account for the effectiveness of mathematics in describing the physical universe? These questions, among others, serve as a gateway between mathematical, philosophical, and even theological concepts. This is fertile ground for students to begin to ask themselves the tough questions and grapple with the interplay between their faith and their view of mathematics.

3. Class Project

The project itself is not intended to be an exhaustive in-depth study of mathematical philosophy. Its purpose, however, is to provided students an *overview* of philosophical questions to stimulate conversation and reflection. The project took place during the first semester of a two-semester sequence of Real Analysis consisting of 25 students; however, I believe it would fit well within any proof-based upper-division mathematics course or a senior capstone course. It covered roughly three weeks of the entire semester, but only two class days were taken up with discussion time. There were three primary goals I sought to achieve through this project. By the end of the semester, I wanted students to:

1. Be exposed to deeper questions about the nature and foundational assumptions of mathematics
2. Gain a basic understanding of the four main philosophies of mathematics from a historical perspective
3. Reflect on these questions in light of their own Christian faith or worldview

The first goal is intended to simply broaden the students view of mathematics. That is, to help them see and appreciate an area of mathematical understanding beyond the standard undergraduate curriculum. I believe that wrestling with these deep questions is a worthwhile goal, especially at a liberal arts institution where a high value is placed on developing students with strong intellectual capacity and curiosity.

The second goal is necessary in order for students to grasp how others throughout history have addressed these questions. They have the opportunity to examine the mathematical philosophies of logicism, intuitionism, formalism, and platonism as well as the mathematicians who espoused them. This provides students a framework of terminology and concepts to help process many of the deeper issues addressed.

The third goal gets at the heart of the faith integration component by allowing students to ground their philosophical viewpoints upon their personal faith or deep convictions. I was intentional in my use of the word "worldview" in this goal for two reasons. First, like many Christian institutions, Whitworth University's student body consists of both Christians and non-Christians. Therefore, the concept of a worldview is broad enough to apply universally to all students. Second, prior to this class each student had already spent a significant amount of time defining and reflecting upon his/her own personal worldview. Whitworth has a worldview studies program that is part of the general education curriculum. It requires that students take a sequence of three courses (Core 150, Core 250, and Core 350) that furnish them with the basic categories of worldview thinking, including the nature of God, the nature of humanity, how we know, the nature of reality, and how we should live individually and corporately. The students are also equipped to explore the parameters of their own worldviews. I utilized Whitworth's Core program to provide a foundation for each student to view the deeper questions of mathematics.

The project itself is broken up into three separate assignments. The first two consist of article readings, a short response paper, a time of small group discussion, and a brief whole class discussion. The third assignment, a culminating paper, requires students to personally reflect on the first two assignments.

3.1. Assignment 1

Students read *Mathematics as an Objective Science* (Goodman, 1979), *Mathematics: A Concise History and Philosophy* (Anglin, 1996, chapter 39), and *Mathematics and the Myth of Neutrality* (Bishop, 2001). The first two articles provide a historical context for the development of philosophies regarding the foundations of mathematics. They explore how formalism, intuitionism, logicism, and platonism developed, define their core tenets, and discuss the inadequacies of each in providing a comprehensive foundation for all mathematical thought. The third article's thesis is that beliefs are integral to mathematics and that ultimately mathematics is not neutral, but shaped by worldviews. This viewpoint is articulated in the article by the following quote from Paul Ernest (Bishop, 2001):

> Mathematical truth ultimately depends on an irreducible set of assumptions, which are adopted without demonstration. But to qualify as true knowledge, the assumptions require a warrant for their assertion. There is no valid warrant for mathematical knowledge other than demonstration or proof. Therefore assumptions are beliefs, not knowledge, and remain open to doubt.

After reading the articles, the students wrote a short response in which they summarized and briefly reflected upon the content presented. Then, they had time during class to get in groups of three or four to discuss what they had read. To guide the discussion forward they were given a list of thought-provoking and reflective questions. The following are a sample of some of the questions posed:

- Do you agree or disagree with Ernest's quote above? Explain.
- To which of the four philosophies would a statistician, geometer, or physicist likely belong?
- How would each of these philosophies likely view the Fundamental Theorem of Calculus?
- We have discussed the four ways of knowing: intuition, empirical senses, innate reason, and authority. To what extent has each of these played in your understanding of mathematics?

The purpose of this assignment is to begin to move students into a deeper conversation about mathematics, one which many students had never taken part. It addressed the goal of exposing students to the larger philosophical nature of mathematics. While there was no explicit Christian focus to this assignment, it laid the groundwork for what was to come. It forced students to confront the idea that one's philosophical view of mathematics may be shaped by a particular set of beliefs and have an impact on the ways in which one engages with mathematical concepts.

3.2. Assignment 2

This assignment is primarily designed to explore how a Christian perspective can inform this philosophical discussion. With this understanding, the students read two articles in the Journal of ACMS entitled *An Augustinian Perspective on the Philosophy of Mathematics* (Bradley, 2007) and *Mathematics as Worship* (Stucki, 2001). The first article examines how Augustine addresses four basic philosophical questions of mathematics based upon his view of God and scripture. What is the nature of mathematical objects? How do we obtain knowledge of them? What is the meaning of "truth" in mathematics? How do we account for the effectiveness of mathematics in describing the physical universe? Augustine's views are presented in a very accessible way for students to grasp. This provides them a genuine example of relating the Christian faith to mathematics. As Bradley states,

> The Augustinian view of mathematics has much to commend it. It's inspiring—from this perspective the capacity to do mathematics is a gift from God, its content originates with God, a mathematical career is a calling to discover God's wonders, and it leads both to service of his kingdom and to worship.

Bradley goes on to describe the historical movement toward secularization and the diminishing role of the Augustinian perspective. In Stucki's article, worship is defined broadly as "a lifestyle of submission, in obedience to God." In this context, he explores how mathematics can be viewed as part of worship, the primary and necessary response of man to his creator.

Similar to Assignment 1, the students are given class time to converse about the articles in small groups. Here is a sampling of the discussion questions posed:

- What can you infer about the worldview of Descartes, Kant, Russell, and Augustine based on their view of mathematics?
- Do you agree with the quote "All knowledge is contingent upon faith"?
- What was the relationship between theology and mathematics before the Renaissance and Scientific Revolution? After?
- How does the author define worship in this article? How does he claim that mathematics can be seen as worship? Do you agree?

The students seemed quite engaged during this discussion time. I saw more lively debates and open conversations among students when matters of faith are interjected due to the more personal nature of the topic. This is where many of them began to see some meaningful connections between their faith and their view of mathematics. The goal of the next assignment is to convert that insight from the verbal discussions into a reflection paper.

3.3. Assignment 3

The project culminates in a two-part paper that brings together many of the concepts developed in the articles and discussions. First, in a two-page paper the students were to articulate a personal answer to one or several of the philosophical questions introduced in class regarding the nature of mathematics. Their response was not simply to draw from what they thought or felt, but from their deep convictions about the nature of God, humanity, and the world. Toward that aim, they were prompted to continually seek out connections between their view of mathematics and their personal faith or worldview. They were encouraged to use scripture, if they wished, as they reflected. I also made available some supplementary articles which were not required reading but encouraged for additional resources. Those articles are given in the list of references below (Bishop, 1996) (Fackerell, 2002) (Hampton, 1977).

The paper prompt was intentionally vague in order to provide freedom for the students to explore what they wanted. I assured my students that there were no right or wrong answers. These questions had been pondered for centuries with no tangible, concrete conclusions. They were being graded not heavily on content, but rather on the depth of reflection and the connections between their worldview and mathematics. This is why the foundation of Whitworth's Core program was vital to the students' ability to engage in this way. Already confident of their personal view of God, humanity, and the world, they were to examine the nature of mathematics in light of their beliefs.

The second piece of the paper was designed to bring some practical application to the philosophical rumination. I wanted students to discuss how any of this makes a difference in their lives. Thus, they wrote a one-page paper in response to the question: how does your view of mathematics, as described in part one of the paper, affect the way you do mathematics. They were to reflect on their own motivation for being a mathematics major. They were also prompted to consider what differences might exist in mathematics from a Christian perspective and from other perspectives.

This part of the paper was primarily intended to move students away from the standard response of "I like math because I'm good at it." After Whitworth, many of the students would go on to graduate school, teach, or work in industry in some field related to mathematics. This required them to dig deeper and examine how God might use mathematics in their lives and their future vocation.

4. Student Feedback

At the beginning of the semester I asked each student if they had previously given any thought to the philosophy or origins of mathematics. All but one student said that they had given very little or no thought to this subject. So the material covered in the article readings and discussion time was uncharted territory for nearly all of the students. Overall, they responded very positively to this project. I received feedback from the students both informally through conversations and formally through a survey I conducted at the end of the semester. Here are some of the response they wrote:

> It was surprising that there were so many different views of the origins of mathematics. Before this assignment I believed that everyone had the same view about where mathematics came from.
>
> I feel as if I have more tools to articulate my view of mathematics. My view hasn't changed, but I feel better prepared to explain and/or defend my view. It is more defined for me now.
>
> Reading those articles really caused me to think more about what I agree with and what I disagree with and how I would fit into different categories.
>
> I do mathematics because its something I enjoy and its a way for me to achieve my higher purpose of reaching the youth by building personal relationships in the classroom and showing them the beauty of mathematics. I will present some of these same things we talked about to my high school classes to get them thinking about what math really is.
>
> I don't think my worldview impacts how I do math at all. However, it definitely impacts why. The reasons I do math derive straight from my reasons for doing anything which boil down to worldview.

At a minimum, this project served to expand the view of mathematics in the minds of the students. It provided a categorical framework for them to define a view of mathematics that they had never considered. It confronted them with the prospect that their beliefs may meaningfully interact with their view of mathematics. It required them to introspectively consider their motivation and calling as it relates to engagement with mathematics.

From my experience with this project, students are hungry to discuss these foundational areas of mathematics given the opportunity. In fact one of the most frequent comments I received from students was the lack of time committed to discussion. They wanted to hear the viewpoints of others in class beyond their small discussion groups. Although I allotted the last five minutes of the period to a full class conversation, it was not enough. In the future, I will either provide more scheduled time for a large group discussion or have the students engage with several different small groups to allow for more sharing of a diversity of opinions. Another idea that may be beneficial is utilizing an online discussion forum where students are encouraged to post their thoughts.

5. Discussion

I believe that mathematical philosophy can be a tremendous facilitator of integrating faith and learning in the mathematics classroom. Understanding the underlying assumptions of mathematics in light of a Christian worldview provides a depth of education that a standard undergraduate curriculum may not afford. This serves to educate the entire person, mind and heart, as is consistent with the mission of Whitworth and many Christian liberal arts institutions.

As mentioned earlier, this project relied heavily on the previous work students had done reflecting on their personal worldviews in Whitworth's Core classes. It is vital for them to know exactly what they believe and why they believe it. Yet, this project can still function well at institutions that do not have a similar general education requirement. I would suggest spending a day in class or assigning some additional outside reading that introduces the students to the basic categories of worldview thinking. Once students have shored up their own views on the nature of God, humanity, and the world, they will be better prepared to approach the philosophical questions of mathematics.

Although I felt that my students were prepared to take part in this project, I was realistic in what I could expect from their papers. I was not looking for fully developed philosophical arguments about the nature of mathematics. Since this was the first time my students had ever confronted these issues, I was satisfied with evidence of self-reflection, critical thinking, and an effort to make worldview connections. The questions that arise from a deep philosophical examination of mathematics may never have satisfactory answers. Yet, we should not let that deter our students from wrestling with them. Their search for answers still has the potential to yield much fruit, including an awareness and clarity of personal vocation. This is vital in moving students beyond the idea that mathematics is simply a tool necessary for a successful career. It also helps students view their mathematical skills and abilities for greater purposes in light of their calling.

6. Conclusion

Mathematics is a discipline that, on the surface, yields little or no prospects for a meaningful discussion of faith integration in the classroom. It works universally and consistently without regard to one's religious, moral, or ethical predispositions. Therefore, it takes some focused effort to dig down to a place where mathematics and faith come together. Fortunately, through the ACMS we are blessed with a community of fellow diggers and a tremendous supply of resources. In this article I provide an example of how to effectively utilize those resources in the classroom. The project described in this article is simply a starting point for me. I am sure this is only one of many ways to successfully bring students to a place of meaningful interaction with these important philosophical issues.

I believe there is currently a lack of awareness and acknowledgement of mathematical philosophy among undergraduate mathematics majors. Gaining insight into these philosophical questions may not help students get into graduate school or find a job, but it certainly enlarges their view of mathematics. It also provides a means of deepening the important discussion of the interactions of faith and mathematics. At Whitworth University, and many other Christian liberal arts institutions, this type of discussion is highly valued as a key component of a larger goal of integrating faith and learning.

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Calculus Communication Circle

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Judith Palagallo is Professor Emeritus in Mathematics from the University of Akron where she taught for 28 years. Since her retirement, she taught at the Vienna Christian Center in Vienna, Austria, co-authored *Curious Curves* (World Scientific 2009), and became a certified Precepts Bible teacher. She is a member of the ACMS Board and continues to serve on the Mathematical Association of America editorial board for Classroom Notes.

1. Introduction

Calculus Communication Circle is a network for the professional development of Advanced Placement Calculus[1] teachers. In Northeast Ohio the Circle provides a forum where teachers meet to share ideas about mathematics and the teaching of calculus. This article describes the creation of the Circle and the progress it has made over its three year existence.

2. History of the Circle

Teaching AP courses is challenging and can be a daunting experience. Conversations with former students who are now Advanced Placement Calculus teachers indicated that an AP teacher can feel isolated. Usually he/she is the only calculus teacher in the school, and often in the entire school district. Nobody is available to discuss common teaching issues, like test length or difficulty, time spent on a topic, or presentation techniques. AP Central offers many helpful materials to the teachers [1], but communication through national forums can be overwhelming and a bit intimidating. So I saw a need to develop a local communication network among area AP Calculus teachers and mathematics professors at nearby universities.

Starting in 2008 with some of my former students, I put together a group of teachers from Northeast Ohio who were interested in knowing each other and sharing ideas. Since then we have been meeting two or three times per academic year for three hours on a Saturday morning. A similar group has existed in the Houston, TX area since 2003 and they have generously shared ideas with us [2].

3. Goals of the Circle

The primary goal of the Circle is to develop more successful calculus teachers. The Circle provides the opportunity for high school teachers to interact with area university professors, who then serve as pedagogical and mathematical resources. Improved communication between

[1] AP Calculus is a registered trademark of the College Board.

university mathematics teachers and nearby AP Calculus teachers benefits the whole mathematical community. This interaction provides the teachers with additional tools that enhance the students' understanding of calculus, and ultimately improve their performance on the AP exam. Colleges and universities profit by getting students who are well-prepared for college-level mathematics.

4. Format of a Circle Session

Each session contains a mathematical component, a technology presentation, a group discussion of a topic suggested by the teachers, and an exchange of materials.

4.1. Mathematics

Usually a volunteer mathematics professor from a Northeast Ohio college or university gives a presentation on a calculus topic. Most of the presenters have served as AP Calculus readers, table leaders or exam leaders. Their insights are valuable in the preparation of students for taking the exam. Presentation topics from previous meetings include the fundamental theorem of calculus, volumes of revolution, recognizing functions from graphs, unexpected limits, and particle motion. These topics are all familiar to the teachers, but speakers try to show how the topic is used or presented in a different way. We ensure that teachers take away examples that can be used in the classroom.

4.2. Technology

Technology demonstrations have included sessions on the TI-NSpire, Geometer's Sketchpad, and Geogebra for calculus lessons. The presenters demonstrate the software, emphasizing teaching objectives that are better accomplished using the technology. Lively discussion follows when the teachers share their experiences with implementing various tools in the classroom.

4.3. Discussion

Teachers assist in setting the agenda for a workshop by suggesting topics for group discussion. Topics of interest include "How do you use the weeks following the AP exam?" and "How do you ensure that the students are prepared for AP Calculus?" The latter topic addresses the issues of Pre-AP vertical programs which encourage the preparation of the students, starting in the middle school years. Teachers have also shared units they designed for the students to complete during the summer before taking calculus.

Discussion in the Spring often centers around the actual preparation of students for taking the AP exam. Experienced and successful teachers have valuable suggestions for new teachers who are just beginning to prepare their students. We also spent a session analyzing results after the exam and discussing how the outcomes could influence the course outline in future years.

4.4. Materials exchange

At each session participants bring materials to share. These materials have included copies of exams, examples of interesting applications, and projects for students. Course outlines are particularly valuable to new teachers for determining the pace of the course. We also shared these outlines via our website.

5. Forming the Circle

I discussed the idea of forming a local network of teachers with numerous colleagues at various mathematics meetings. They agreed that the plan to start such a communication group seemed like a timely idea. The task of actually forming the Circle was challenging. I started by contacting former students who were AP Calculus teachers and allowed them to add other names to the list. I scoured the Internet for references to calculus teachers in Northeast Ohio. Eventually, I sent email letters to about sixty teachers, asking if they were currently teaching AP Calculus and if so, would they be interested in a communication network. Nearly all the teachers responded, and they indicated strong support. The communication had begun.

Along with a few other colleagues, I planned the first workshop to be held in the Fall of 2008. To entice the teachers to attend, I promised that people involved with the AP reading would be there to discuss previous exams and their evaluation. I invited teachers to present an interesting project they had used with their students. Three collegiate instructors prepared mathematical talks we thought would be of interest to the teachers. Ten enthusiastic teachers attended the first workshop; one agreed to talk. They returned to their schools with new ideas about a few calculus topics, free calculus books, and certificates of attendance to include in their professional portfolios. They enjoyed meeting each other and lingered after the workshop to get better acquainted and exchange contact information. The first workshop established the basis for others to follow. The teachers added their opinions about future sessions. They also suggested we establish a website for exchanging materials.

The Circle now has about thirty active members. Of course, not all of them are present at any one workshop. The group has grown because teachers invite other calculus teachers. Experienced teachers often tell new, or prospective, AP teachers about the Circle and encourage them to attend. Teachers come from a mix of urban, suburban and rural schools, reflecting the variety of schools in Northeast Ohio. They are a collection of dedicated, talented and creative calculus teachers. They often come to the sessions with a list of questions they want to discuss with other teachers. After the sessions, they linger to discuss other topics of concern to teachers that may not involve calculus.

The teachers were initially hesitant to make presentations at the sessions, but they are now comfortable with one another and at least one presents at each meeting. Their talks about new approaches to material, interesting projects designed for students, or a clever implementation of technology are always well-received.

6. Conclusion

The three year existence of the Calculus Communication Circle demonstrates the value of a local communication network in the professional development of AP Calculus teachers. The teachers profit from interaction with each other and with nearby university mathematicians. As the teachers meet, they realize that they share similar experiences, whether they teach in rural, urban or suburban districts. The time seems to pass by quickly as discussion and exchange of ideas flow. They leave each session with ideas, materials and contacts for communication.

I am indebted to my colleagues in Ohio who volunteer a Saturday morning to share calculus ideas with the high school teachers. The enthusiasm shown by all participants at each workshop is contagious. We hope that other Circles will continue to develop throughout the mathematics community.

References

[1] http://apcentral.collegeboard.com/apc

[2] http://www.houstonact.org

The Study of Mathematics:
A Text from a Christian Perspective

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Doug Phillippy (B.A., Shippensburg University; M.S., Ph.D., Lehigh University) has been at Messiah College since 1993. Some of his early work includes the construction of a mathematical model that measures the stresses that act on the feet of a runner. Not surprisingly, he is a runner, and enjoys participating in a variety of other sports. He is married (to Deb) and has four children (Michael, Lauren, Matthew, and Mark) ranging in age from 12 to 19.

1. Introduction

I teach at a Christian college that makes the claim "Christ is preeminent." The mission statement of this college includes the following proclamation: "Our mission is to educate men and women toward maturity of intellect, character and Christian faith in preparation for lives of service, leadership and reconciliation in church and society." In addition, the department in which I teach, the Information and Mathematical Sciences Department, has its own mission statement which includes the following objective: "to challenge students to live out their faith in their vocation as they become servant-leaders in society, church, and the world." These statements suggest that both the college and the department for which I teach take seriously the importance of pursuing a career in light of the Christian faith and its principles. A natural question to ask then is how do the goals and objectives that are alluded to in these mission statements make their way into the classroom. That is, how are the students in my classroom challenged to "live out their faith in their vocation", or to "become servant-leaders in the world" by the instruction I give them?

It should be noted that we cannot assume that this type of instruction naturally takes place just because there is a Christian professor at the front of the classroom. I attended a state school as an undergraduate student and a private institution as a graduate student. At both schools the only faith-related thought that entered into my pursuit of a career in mathematics was self-generated. I received no such instruction from those who taught me. I would suggest that for a professor like me, who has had no formal instruction in this type of thought, and who has had no real mentor to help guide him in the process, the task of challenging students to think about their career in light of the Christian faith is not an easy thing to do.

Moreover, we cannot assume that just because the setting is a Christian college with fine-sounding mission statements that this type of learning is actually taking place. David Claerbaut would agree. In his book, *Faith and Learning on the Edge*, he recounts his experience at a Christian college, an undergraduate institution that claimed to teach its courses from a "Christian Perspective". According to Claerbaut, "apparently, that teaching occurred in classes I cut or slept through, because I recall scarcely a single class devoted entirely to providing an overtly Christian perspective from which to view the material studied". Instead of professors who taught from a Christian perspective, he encountered "rebellious, agnostic students–many of whom who had been forced by their parents to attend a Christian college–boldly proclaiming their unbelieving views in dormitory bull sessions". Claerbaut suggests that his education at a Christian college left him unprepared to answer some of the questions that were raised by these agnostic students. In the end, he says his college education left him "intellectually unarmed, devoid of any ammunition" to confront the examples of unbelief that he encountered even on his Christian college campus.

If a Christian college does not prepare its students to confront unbelief and also to recognize erroneous beliefs within the academic disciplines, then what is the advantage of a Christian education? Can a Christian college or university expect its graduates to challenge secular thought that contradicts a Christian world-view if it fails to actively include faith-related topics in its curriculum? Several years ago, before my department even had a mission statement, the only place in our curriculum that formally attempted to address the idea of faith and learning as it relates to pursuing a career in mathematics was the capstone course for our majors. This meant that our mathematics majors had to wait until their last semester of college before they were required to deal with these issues and the questions they might raise. Other than the capstone course, faith related issues were intermittently dispersed throughout the curriculum and usually were only devotional in nature. In many ways then, mathematics majors at my college had a similar experience (at least in terms of their major courses) to the experience David Claerbaut had at his college.

2. The Text

My text, *The Study of Mathematics: Developing a mature understanding of mathematical thought, with consideration of Christian faith and vocation*, is meant to be a tool to help first-year mathematics majors begin their study of mathematics from a Christian perspective. Currently there are few if any texts that target first-year mathematics majors with a theme of integrating faith and learning. Three published works in the field include Mathematics: Is God Silent? (James Nickel, 1990), *Mathematics in a Postmodern Age* (Howell & Bradley, 2001), and *Mathematics, a Christian Perspective* (The Kuyers Institute, 2007). My text is unlike any of these three texts. It differs from the first two texts in its goal to develop a maturing understanding of the discipline by assigning significant mathematical work. Although Nickel's book is appropriate for first-year students, both it and Howell and Bradley's book are expository in nature and discussion is limited to issues of faith in the mathematical sciences. Both texts require no significant mathematical work from the reader. Moreover, Howell and Bradley's book is more appropriate for upper-division undergraduate students or graduate students. The third text is closest in nature to *The Study of Mathematics*, but it has very little narrative and the mathematics that is presented is more appropriate for either a junior or senior high school student. It is my hope that my text, when fully developed will fill a curricular need at Christian colleges and universities that have a mission similar to that of my own college.

This text, which is now in its third draft, has been a seven-year project to this point. The original draft consisted of three chapters and was co-authored with Angel Hare during the summer of 2005 as a result of a summer grant from "The Collaboratory for Strategic Partnerships and Applied Research." A second grant from the same organization enabled me to write three more chapters in 2009. As a result of that work I was awarded a sabbatical during the spring of 2011. During that time, the third draft of the text was written. The third draft is a significant revision of the second draft of the text and when finished will be a text ready for publication. It is more than double the size of the second draft of the text with the addition of significant new mathematical material.

The main body of this text is divided into four sections. Chapters 1 and 2 form the introduction to this text. In chapter 1, I use my love of baseball and mathematics and their resulting interaction in my life as an illustration of how faith and the study of mathematics ought to interact. I argue that faith should be the tie that binds all of a Christian's activities together, especially a pursuit of a career in mathematics. In chapter 2 I will begin to consider how that might look by asking the question, "Why have I chosen to study mathematics?" In answering that question, I highlight two major branches of mathematics, the applied and the theoretical. The second section (chapters 3-8) of this text focuses on applied mathematics covering topics such as check digits, graph theory, number theory, and problem-solving. The third section (chapters 9-13) of the text focuses on theoretical mathematics covering topics such as logic, proof and small axiom systems. The fourth and final section of the text (chapters 14-19) consists of what I will call faith integration projects. These projects will cover a variety of mathematical topics and are designed to promote discussion in the area of faith integration.

The Table of Contents for the text is listed below:

There are three main themes woven throughout this text. Obviously, one of those themes is mathematics. I have included topics in this text that I believe are of interest to first-year mathematics majors. Some of these topics are foundational to the study of mathematics, and others, though they are not typically found in the curriculum of a first-year student, were chosen because they are thought-provoking. The second major theme interwoven throughout the text is that of the Christian perspective. The study of mathematics is considered in light of the Christian faith, asking how faith influences the study of mathematics and in turn how the study of mathematics can impact faith. The third theme is that of my life story. While introducing mathematical topics and asking faith-related questions, I share my journey as a Christian mathematician, noting some of my struggles and highlighting what I have learned along the way. I have included material from the text related to each of these three themes.

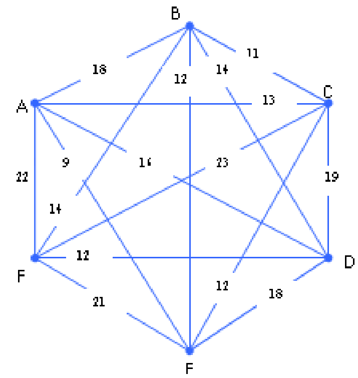
2.1. Theme 1: Introductory Mathematics

Although this book is written primarily for aspiring mathematicians, it is not meant to be a typical college mathematics text book. There are several reasons why this is so. First, the format of my text is different than most other mathematical texts. While most mathematical texts focus on one particular mathematical topic such as Calculus, my text does not. Instead, my book will present a variety of mathematical topics as a backdrop for developing a mature understanding of mathematical thought. That is not to say the topics covered are unimportant, just that they are secondary. For this reason, discussion of particular mathematical topics within my book will often be less comprehensive than corresponding discussions found in most other texts.

One of the most significant changes to the third draft of this text is the inclusion of a variety of new mathematical topics. New topics include check digits, graph theory, number theory, problem solving, and small axiom systems. The new material complements the previously included topics of logic, proof, and infinite series. In addition to a number of examples worked out in the text, there are approximately 100 exercises for students to complete on their own after reading the text. A sample of exercises is included below.

1. **Check digits:** The Canadian province of Quebec uses mod 10 arithmetic and the weight vector (12,11,10,9,8,7,6,5,4,3,2,1) to assign a check digit to its 12-digit license numbers (the last digit is the check digit).
   (a) Identify all undetected single position errors. What percentage of these errors does the scheme detect?
   (b) Identify all the undetected transposition errors (consecutive digits). What percentage of the errors does the scheme detect?
   (c) How could the weight vector be changed so as to accomplish the same results and at the same time simplify the calculations? (hint: change some of the larger weights)

2. **Graph Theory:** The weights in the graph pictured to the right represent distances for a TSP (traveling salesperson problem).



   (a) Use the Sorted Edges Algorithm to determine an efficient route for the traveling salesperson.

   (b) Use the Nearest-Neighbor Algorithm to determine an efficient route for the traveling salesperson. (start the algorithm at A and then at D).

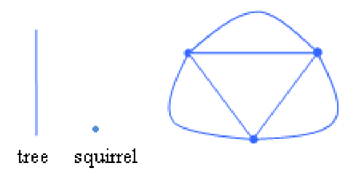
3. **Number Theory:** Convert "primes" to ciphertext using RSN encoding with the public key of $n = 437$ and $r = 5$. Encrypt the data in blocks of 3 digits each.

4. **Problem solving:** Show that if 16 positive integers (not necessarily distinct) are chosen that sum to 30, then a subset of those integers can be found that sum to any positive integer less than 30 (1 through 29).

5. **Small Axiom Systems:** Consider the following axiom system with primitive terms squirrel, tree, and climb.

   Axiom A1: Any two distinct squirrels climb exactly two trees in common.
   Axiom A2: There are at least two squirrels.
   Axiom A3: For each tree, there exists at most one squirrel which does not climb it.
   Axiom A4: Every squirrel climbs exactly four trees.

   (a) Show that this system is consistent by verifying that the axioms are satisfied by the model shown to the right.



   (b) Find a second model, different than the one given in part a).
   (c) Show that this system is independent.
   (d) Prove the following theorems (note that the theorems hold for both the given model in part a) and the second model you found in part b).
      i. There are exactly six trees.
      ii. Every tree has at least one squirrel that climbs it.
      iii. For each squirrel, there are exactly two trees that it does not climb.

6. State one observation that can be made from the model given in part a) that is not consistent with the model that you produced in part b).

7. Include the observation that you listed in part e) as the fifth axiom for this system. Is the resulting system of five axioms independent? If not, identify which axioms are dependent on the others.

2.2. Theme 2: My Life Story

A significant part of the narrative in the text relates my life story. My story is woven throughout the text, emphasizing my faith and my career. I talk candidly about my career path, including many of the struggles that I have faced as I have traveled that path. I also talk about how God's hand has been on me throughout the journey even though at times I had little or no concern for his direction in my life. In addition, I talk about how faith integration has come to be such an important part of what I do as a college professor. Sometimes I simply share how mathematics has been useful to me. The rest of this section is taken directly from my text and is an example of how I use my story to share my love for mathematics. It is taken from the introduction to the topic of "check digits."

> Before we turn to graph theory and to doing mathematics in general, I would like to share one more story from my life in which the application of mathematics could have saved my wife some heartache. These events took place while I was a graduate student at Lehigh University in Bethlehem, Pennsylvania. While living in Bethlehem, I began to develop an interest in cycling. My interest in cycling grew primarily out of necessity. My wife and I were trying to survive on her salary and the small stipend that I received while serving as a teaching assistant. To make ends meet, we only owned one car, a problem given that each day we had to travel in opposite directions to get to school/work. In order to overcome this problem, I began to make the five-mile journey to Lehigh's campus on my bicycle. I soon developed an interest in cycling and to remove some of the stress from my aging knees, from time to time I would replace my daily run with a bicycle ride.
>
> On one occasion, during the summer months when my wife did not have to work, I decided to go for a bike ride early in the morning. I slipped out of bed and said goodbye to my wife, who soon returned to her sleep. As I neared completion of my hour-long trip, I passed a police officer sitting on a motorcycle at the side of the road. The officer had positioned himself about fifty yards ahead of a T-intersection that had a three-way stop sign. Since I was traveling along the far-side of the road opposite the intersection of the two roads that formed the T, I reasoned that I did not have to stop at the stop sign, something that I had done many times before and was not about to change just because a police officer was present. That was a mistake. The officer hopped on his motorcycle, turned on his flashing lights, and pulled me over.
>
> Unfortunately the officer pulled in front of me rather abruptly on a slight grade covered with some loose gravel and I ran my bike into the back of his motorcycle. He did not appreciate that and I knew I had gotten off to a bad start as far as he was concerned. But things would only get worse. He got off his motorcycle and asked for my driver's license, which I did not have. Next, he asked for my name. I gave him my name, spelling it nice and slowly, trying to let him know that I was not happy that he had pulled me over. After he had written my name down, he pulled out his two-way radio and began having a conversation with someone back at his main office. After what seemed to be a long time, he finally put the radio down and to my dismay told me that I did not exist. He told me that the state had no record of a Douglas Phillippy having a driver's license in the state of Pennsylvania. Now the officer was beginning to doubt my honesty. He asked me where I lived and for the phone number of someone who could verify that I was who I said I was. I gave him my address and home phone number telling him that he could talk to my wife.
>
> The ringing of the phone startled my wife, who was still sleeping. She picked up

the phone, and the first thing she heard was, "Ma'am, I need you to identify your husband." Now it is important to know that several months prior to this event, as I was riding to school, I was involved in an accident with a car while on my bicycle. As a result of that accident I broke my hand and was taken by ambulance to the hospital. I had to call my wife, who was at work at the time, and tell her to meet me at the emergency room. This was still fresh in her mind when she was awakened by the officer's call and for an instant she imagined the worst. I knew that the officer's choice of words were not the best and immediately plead with him to let her know that I was alright.

After my wife had calmed down, she was able to identify me to the officer's satisfaction. The officer then handed me a $120 ticket and told me that I would receive 3 points against my driver's license. But because there was no record with the state of Pennsylvania indicating that I had a driver's license he told me that he would be stopping by my apartment later that day to finish up the paperwork and get an actual look at my license. Sometime that evening, the officer made good on his promise and showed up at our apartment. I handed him my license and after staring at it for what seemed to be an eternity, he announced that he had discovered the problem. Showing me the license, he pointed out that the spelling of my last name was different on the license than the spelling I had given him earlier that day. On the license, my name was spelled incorrectly, with only one L, not two as it should have been. Being the observant person that I am, I had not noticed the mistake and to this day I am not sure when or how it got there. On the bright side, the fact that it was only discovered as I was approaching thirty years of age because of an infraction I had committed on my bicycle goes to show that I am essentially a law-abiding citizen.

I soon got over the fact that I had received points against my driver's license for an infraction that occurred while riding my bicycle, but it would be years before the mathematical significance of what had taken place on that morning would be clear to me. Shortly after this incident, I graduated from Lehigh University and began a career in teaching. It took a couple of years, but eventually I found at my niche as a professor at Messiah College, where each year I helped to take some undergraduate students to an annual math conference held in the Lehigh Valley. At one of those conferences, Joseph Gallian was the invited speaker, and on that particular day he spoke about the mathematics that lurks behind some of the numbers we use on a regular basis: credit card numbers, UPC numbers, and yes even some driver's license numbers. While listening to Dr. Gallian speak that day, I was taken back to my summer morning encounter with the motorcycle cop and once again I was reminded of the usefulness and importance of mathematics.

The focus of Dr. Gallian's talk was on check digits. Although I had learned about the idea of a check digit years before as an undergraduate student, most of the schemes described by Dr. Gallian on that day were new to me. His presentation reminded me that the mathematics used in these check-digit schemes, though very simple, often catch common mistakes in the transfer of important numbers from one individual to another. During his talk he also described several different methods that various states use to assign driver's license numbers some of which include schemes to catch common misspellings of names. In the next section we will consider several check-digit schemes as well as some of the mathematics that underlies those schemes. In addition we will describe the sound-ex system, a scheme that several states use to assign license numbers to drivers that are unaffected by most common misspellings of a name.

Unfortunately, Pennsylvania does not use the sound-ex system. Although there was nothing that the state of Pennsylvania could have done to save me from getting a ticket on that summer morning, I now realize that if the state had employed an appropriate checking scheme in its assignment of licenses, my wife may have been spared from a troubling early morning conversation with the motorcycle cop. Thus, the sound-ex system, with its foundations in mathematical thought is just another example of the problem-solving power of mathematics.

2.3. Theme 3: Integration of the Christian Faith

Within Christian higher education, faith integration is defined in several different ways. Some would limit it to a "scholarly project." The definition I am using however is a bit broader, with faith integration being *any* attempt at relating an academic discipline and a biblical worldview. This definition allows for attempts as simple as devotional thoughts for which the academic discipline merely illustrates a biblical principle and others that are as complex as published papers in refereed journals which intricately relate discipline and faith. Faith integration can focus on the attitude of the student and not be unique to the discipline of study, or it can focus on the very assumptions that underlie a particular discipline. Faith integration can draw from a biblical worldview to shape a discipline, or it can use the academic discipline to gain insight into a proper biblical worldview.

Faith integration then comes in all shapes and sizes, and in my text I present it in a variety of ways. My goal is for the reader to not only read about faith integration from my perspective, but to begin to practice it. I hope that the reader will learn to just do it, not in any particular way, but to in some way make attempts at relating mathematics and a biblical worldview. Moreover, I hope that the reader will make those attempts in the company of other believers recognizing that the goal of the attempt is not so much a "polished faith integration project", but the development of Christian thinking skills by those who participate in the discussion generated by the attempt.

Integration in all disciplines of study should be a two-way process. William Hasker describes faith-learning integration as "a scholarly project whose goal is to ascertain and to develop integral relationships which exist between the Christian faith and human knowledge, particularly as expressed in the various academic disciplines." In describing those relationships, Hasker refers *both* to the insights of a Christian worldview that are relevant to the discipline, and contributions of the discipline to the Christian view of reality. Likewise, in describing integration, Arthur Holmes states, "Integration is concerned . . . with the positive contributions of human learning to an understanding of the faith and to the development of the Christian worldview, and with the positive contribution of the Christian faith to all the arts and sciences." It is clear from this statement that Holmes recognizes integration requires contributions from both learning to faith and from faith to learning, making it a two way process.

My text offers the reader a number of ways to practice faith integration. First of all, the reader has the opportunity to read about faith integration because it is a theme that is woven throughout the text with several chapters devoted entirely to that theme. Although most of the mathematical topics are presented without much direct reference to faith-related issues, sometimes an exercise related to a particular topic may attempt to get students to think about such issues. For the most part, faith-related discussion is contained within an entire chapter. Typically such a chapter is related to some of the relevant mathematical topics that were discussed in earlier chapters. For example, after discussing the problem-solving power of mathematics in chapters 3-6, the text asks what that might look like from a Christian perspective in chapters 7 and 8 by focusing on an attitudinal or incarnational approach to faith integration. Finally, the fourth and final section of the text consists of a number of faith integration projects. These projects not only serve as examples of what a "scholarly" faith integration project might look like, but they also provide the reader with opportunities to participate in discussion relevant to the project. Several examples of faith integration from the text are included below:

As stated above, one way that faith integration takes place is through exercises assigned at the end of a chapter. Here is an example that follows the chapter on Number Theory: In the text, the author hints at the individual and corporate nature of the study of mathematics when he says "an individual's understanding of mathematics depends on that individual's own work within the discipline as well as the current state of the discipline itself."

1. Identify several ways in which your understanding of mathematics is a reflection of who you are. Identify several ways in which your understanding of mathematics is a reflection of the community of mathematicians with whom you have had personal contact (not the global mathematical community, but only individuals with whom you have had personal contact).

2. Read Ephesians 2:19–22. In this passage Paul seems to emphasize both a communal growth of the body of believers and an individual's growth in the faith. According to this passage, what is the context in which growth takes place? Identify several ways in which your understanding of faith is a reflection of the community of believers with whom you interact.

3. Identify differences between the way the community of mathematics grows and the way the body of Christ grows. Can and should a Christian apply any principles from the growth of Christ's body to his or her own involvement in the mathematics community?

Entire chapters devoted to connecting faith and learning are a second way that faith integration occurs in the text. To illustrate this, I have included two excerpts from Chapter 8, Experiential Learning and the Collaboratory, which emphasize the importance of experience in a Christian education. The first segment is introductory and uses Job to illustrate the importance of experiential learning. The second segment describes one of the projects that make up the Collaboratory, a Messiah College organization committed to enabling students to contribute to the kingdom of God by applying their academic discipline. The chapter emphasizes the importance of practicing both faith and discipline in the education process.

> **Segment 1:** The Old Testament prophet Job understood the importance of supplementing knowledge with experience. Consider several statements made by Job during his time of suffering and restoration. During his time of suffering, Job said

this about God: “If only I knew where to find him; if only I could go to his dwelling! I would state my case before him and fill my mouth with arguments. I would find out what he would answer me and consider what he would say. Would he oppose me with great power? No, he would not press charges against me. There an **upright man** could present his case before him, and I would be delivered forever from my judge” (Job 23:3–7). Later, when Job found God, of maybe it would be more accurate to say when God showed himself to Job, God demanded of Job, “Will the one who contends with the Almighty correct him? Let him who accuses God answer him!” (Job 40:2) Job had this to say in response to God’s question, “I am **unworthy**, how can I reply to you? I spoke once, but had no answer—twice, but I will say no more” (Job 40:4,5).

Consider Job’s two statements again. They clearly reflect a difference in Job’s worldview and in particular Job’s understanding of who God is and what that in turn implied about Job himself. In his first statement, Job boldly proclaimed that if he could find God, he would state his case before God and get answers from him. He sees himself as an upright man set to be vindicated by God. But when Job finally gets the chance to argue his case before God, he no longer has such a view. He now sees himself as unworthy and has nothing to say to God. What caused this change in Job’s worldview? To answer this question, we need to understand what took place between Job’s two statements.

Job’s understanding of God greatly increased as God revealed himself to Job in a personal encounter that is recorded at the end of the book of Job (see Job 38–41). This increase in understanding occurred even though God did not answer any of the questions posed by Job earlier in the book. In fact, not only did God leave Job’s question of “Why me?” unanswered, he turned the tables on Job and demanded of him, “Brace yourself like a man; I will question you, and you shall answer me” (Job 38:3). God then fired question after question at Job: “Where were you when I laid the foundations of the earth (Job 38:4)? What is the way to the abode of light? And where does darkness reside (Job 38:19)? Do you know when the mountain goats give birth? Do you watch when the doe bears her fawn (Job 39:1)?” For four chapters, God asks questions just like these. God does not give Job answers, he simply asks questions. Nevertheless, this encounter with God left Job a changed man.

Job himself gives the reason for this change. He says “Surely I have spoke of things I did not understand; things too wonderful for me to know . . . My ears had heard of you, but now my eyes have seen you, therefore I despise myself and repent in dust and ashes” (Job 42:3,5). In making this statement, Job distinguishes between an understanding that is based on theory (my ears had heard of you) and one that is based on experience (my eyes have seen you). Job credits his change in understanding to his experience. Years of living and learning and discussing with others had given Job what he thought was a proper understanding of God. But in the end, Job’s personal encounter with God proved much more valuable to his understanding of God than all the head-knowledge he had accumulated over the years.

Segment 2: *The Burkina Faso Literacy Project* originated with the Education group. This group typically has about 8 to 12 student members, many of whom are math majors. The group describes itself as existing to "promote literacy and the development of abstract thinking skills among children and families in communities where lack of education is an issue of justice." Dr. Angela Hare, trained in mathematics education is the founder and advisor of this group. *The Burkina Faso Literacy Project* was the group's initial project. The primary goal of this project was to design and write a counting book that teaches basic numeracy and strengthens abstract thinking skills. The objectives of the project also included the production of printable materials for disabled children, in particular the blind.

The project was a result of a trip that Dr. Hare made in 2006 to Burkina Faso, a country in West Africa. In Burkina Faso, the mission agency SIM (Servants in Mission) operates a physical therapy center for both children and adults and an elementary school for children with disabilities. Approximately 100 children attend the school, most of who have some degree of cerebral palsy. A number of the schoolchildren are deaf and or blind. Dr. Hare had the opportunity to observe several classes during her three weeks in Burkina Faso. Her interaction with students in the classroom as well as with various other members of the local community including some of the teachers and missionaries led to this literacy project.

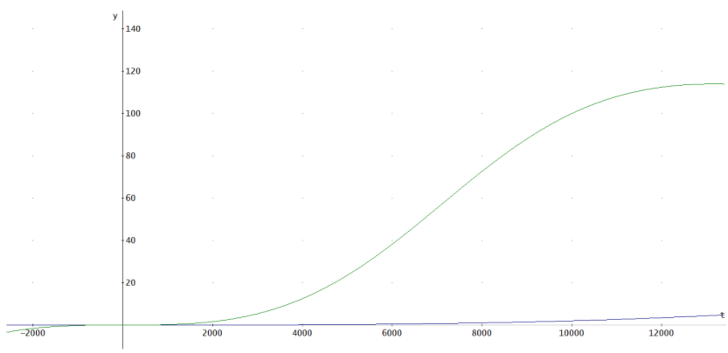
Dr. Hare observed that while the children can count, chant, and memorize math facts, they have difficulty representing this information in abstract ways (like using tally marks to represent the sentence $2 + 4 = 6$). She concluded that most of the children are very concrete in their thinking, especially with regard to mathematics. Her conversation with teachers and tutors revealed that some of these outcomes are a result of the children's environment, the very point that I have been trying to emphasize in this chapter. She noted that many children live in homes without any symbols or pictures to view, and they generally don't see signs on the road, read books, or play with games that have printed pieces. To complicate matters, multiple disabilities limit the children's learning ability. For example, many of the blind children have callused fingers from work or cerebral palsy, and therefore they cannot easily learn and read Braille. Dr. Hare's experience in Burkina Faso proved to be foundational in launching the Education group and its initial project of developing literacy among the disabled children of the community.

As a result, this two-year project which ended in May of 2009 has had an impact on the mathematics community at Messiah College. Multiple trips involving about a half a dozen students and three faculty members have been taken to Burkina Faso where participants have had the opportunity to interact first-hand with the people of that culture. Moreover, in this setting, students working under the supervision of faculty members have had the opportunity to apply what they have learned in the classroom.

An outcome of this particular project was a published book entitled *Un, Deux, Trois*, a counting book written in French for the disabled schoolchildren of the community. The book tells the story of a hungry African frog that wants to fill its stomach with insects. The twenty-page book seeks to connect concrete quantities with abstract numerals. On each successive page one more insect is added to the frog's tongue. The final version of the book included three-dimensional foam bugs to incorporate the sense of touch. This particular project also exposed team members to collaboration with students from other departments. Illustrations for the book were done by an art major and translation into the French language was handled by several French majors. Finally, team members were challenged to take what they had learned to the broader community, with the results of this particular project appearing in *The Journal of Mathematics and Culture.* This project is one of several undertaken by the Education group and is a great example of how an education in mathematics can be enhanced by experiences that seek to connect faith and discipline.

Faith Integration Projects are the third and final way that faith integration occurs in the text. This discussion is based on chapter 14, *The infinite and Time.* In that chapter I give my answer to the question, "What role (if any) should a Christian perspective of time have in discussing the solution to a problem that involves the infinitude of time?" Before answering this question, the reader is asked to complete the following four exercises.

Exercise 1 Consider the function $f(t) = \frac{2000t^3+1000}{0.001t^4+10,000,000,000,000}$ and its graph.



Evaluate $\lim\limits_{t\to\infty} \frac{2000t^3+1000}{0.001t^4+10,000,000,000,000}$. Was the graph helpful in evaluating the limit? Explain.

Exercise 2 Evaluate the following limit: $\lim\limits_{t\to\infty} \frac{\sqrt{16t^2-3t+1}}{2t+1}$.

Exercise 3 Evaluate the following limit: $\lim\limits_{t\to\infty} \frac{\sqrt{4-t^2}}{3t^3+1}$.

Exercise 4 Does the men's 200 meter freestyle world record have a mathematical limit? Explain.

Each of the first three exercises is given to illustrate why care needs to be taken when evaluating an infinite limit. Example 3 in particular illustrates the importance of the domain of the function. In the text, I argue that example 4 is similar to example 3, suggesting that a Christian perspective of time does not allow for an infinitude of time. Moreover, I argue that in the same way that the first three examples are beneficial to understanding infinite limits, example 4 is important to the pursuit of knowledge if that knowledge is to be understood from a Christian perspective. The reader is then given the opportunity to respond to my thoughts on this subject.

3. Conclusion

My text, *The Study of Mathematics* is a unique approach to the integration of faith and mathematics. It is unique for several reasons. First, it is unique in its targeted audience. It targets first-year mathematics majors and encourages them to participate in the practice of faith integration. Participation is not limited to reading about faith integration, but it occurs as students complete short-answer discussion questions and more in-depth faith integration projects or writing assignments. It is also unique in its approach, not focusing solely on the topic of faith integration. The text encourages the practice of faith integration in the context of learning mathematics. Students are introduced to pertinent mathematical topics and encouraged to think about those topics and their future careers in light of their faith. Moreover, all of this is done in a personal way, sharing my interest in the subject as well as my development as a Christian mathematician.

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Math History Study Abroad Program:
Learning Math History in a Cultural Context

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1. Introduction

In January 2011 fifteen Whitworth University mathematics students and I, their professor, traveled through Europe to study the history of mathematics. The goal was to gain an understanding of how mathematical ideas have developed over time; how social, cultural and historical factors have influenced the development of mathematics and conversely, how mathematics contributed to society and human culture.Over a course of three weeks we traveled to three countries and over a half dozen cities, viewing the tools, papers and workbooks of these mathematicians, seeing their engineering and artistic creations, and learning from local experts as they guided us through science museums and archives. The experience helped us to understand the broad roles mathematics and mathematicians played in the world around them.

The journey actually began several months earlier during a Fall Prep Course. Readings from the textbook, *Mathematics in Historical Context*, by Jeff Suzuki provided a good foundation for understanding how mathematics is an integral part of culture. The book gives a broad overview of the history of mathematics from the Ancient World (Prehistory, Egypt and Mesopotamia) to post-WWII. Though we concentrated mainly on the last seven centuries, we also studied some Greek, Egyptian, Babylonian and Islamic mathematics, in anticipation of seeing some of the artifacts from these civilizations in museums during our travels. Movies about Roman engineering and biographical films on Galileo, Newton and Einstein helped us to both visualize what we would see, but also what the world must have looked like when these mathematicians were making their important contributions. Each student chose a particular mathematician to research. The students wrote a term paper describing their mathematician in terms of historical and cultural context in which he/she lived, as well as their contribution to mathematics. Each of these mathematicians was tied to a particular site we would be visiting during our travels. When we were at that site in Europe, the student would then give an oral presentation regarding their mathematician to the rest of our group. This gave the students the opportunity to learn in-depth about a mathematician both through their library research and then through actually seeing where that mathematician lived and worked.

2. Rome: Engineering, Art, and Politics

Our travels abroad began in Rome where one is surrounded with evidence of the Roman engineering that played such a large role in Rome's ability to grow as an empire. Water was abundant in Rome thanks to an aqueduct system designed using arches, a slight grade and gravitational pull. Eleven aqueducts provided Rome with enough water to sustain a population of one million people. The aqueducts emptied into three holding tanks: one for public drinking fountains, one for public baths and one reserved for the emperor and wealthy Romans who had their own running water. The abundance of water made keeping clean easier; it is said that Romans felt superior because they were cleaner than other cultures.

(a) Roman fountain

(b) Santa Maria degil Angeli, central hall of the former Baths of Diocletian

Figure 1: Abundance of water in Rome

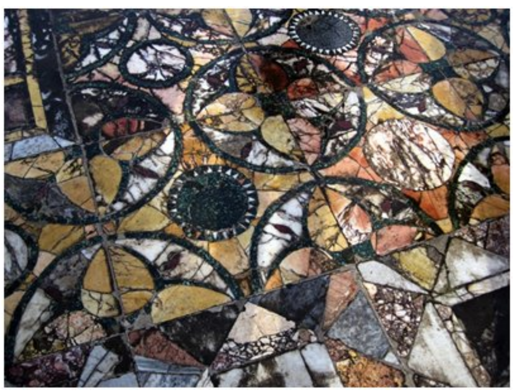
Using his access to water, Nero built a huge lake in the center of Rome as part of his palace complex to demonstrate his wealth and power. After Nero's death, the Emperor Vespasian drained the massive lake Nero had built (for his personal pleasure) and replaced it with the Coliseum built for the people of Rome. Even though we had watched a film about the building of the Coliseum, we were still in wonder when we actually saw the ingenuity and intricacies of its design which included assigned seating, an awning/weather protection system, and layers of subfloor with elevators that allowed wild animals to seemingly spring out of nowhere onto the amphitheater floor. Next to the Coliseum was the Roman Forum, which for centuries had been the center of Roman public life.

The Forum was where commerce, the administration of justice, and religious activities of Rome were conducted. It was the site of triumphal processions and a venue for public speeches. Students marveled at the size and grandeur of the buildings in the Roman Forum, just as the captives of Rome must have as they were paraded through the Forum as part of a Roman victory celebration. Visiting the Forum and Coliseum gave us a real sense of the grandeur and power of Rome and the role engineering played in making it possible. We would continue to see the influence of Roman architecture and engineering in the other cities we visited, especially Florence and the sites in England.

We spent four days in Rome. We learned about life in an ancient commercial city through our visit to the ruins of Ostia Antica, a Roman port town, where we could walk through the streets

Figure 2: Roman forum

of the city, see the design of amphitheaters and temples, public latrines, burial grounds, homes and marketplaces. Mathematical patterns were reflected in the geometric designs of many of the floor tiles as well as the layout of the city.



(a) Ostia Antica floor

(b) Amphitheater in Ostia Antica

Figure 3: Ruins of Ostia Antica

Mathematics played a large role in the aesthetics of design of plazas (such as Michelangelo's Plaza) and architecture, especially in the construction of the domes of the Pantheon and St. Peter's. The engineers, architects and artists were mathematicians as well mathematics was intricately woven into the everyday lives of the people. Sometimes mathematics was used in the construction of art and sometimes the art itself reflected the important role that mathematics played in the culture. For example, Raphael's painting, *The School of Athens*, in the Vatican Museum in Rome, portrays an idealized community of intellectuals from the entire classical world.Among them in the picture are Pythagoras with his perfect numbers, and Euclid demonstrating some geometric proposition with a pair of compasses upon a slate. Raphael's vision of Humanism pays tribute in *The School of Athens* to the importance of mathematics in human learning and understanding.

Figure 4: Euclid demonstrating a mathematical proposition in Raphael's *School of Athens*

3. Florence: Renaissance Art and Science

From Rome we journeyed to Florence, the heart of the Italian Renaissance, home to Galileo, Leonardo DaVinci, Michelangelo and many other Renaissance artists and scientists. Florence provided the opportunity to learn about these key figures in the context of where they lived, worshiped and did some of their greatest work. Mathematics played a major role in architecture and art during the Renaissance, especially with the invention of mathematical linear perspective. This innovation is credited to Brunelleschi, sculptor, architect and artisan-engineer. We saw many examples of linear perspective applied to art and architecture in the church of Santa Maria Novella including Masaccio's 1428 fresco of the Trinity, which is credited as the first painting in the history of art to use perfect linear mathematical perspective.

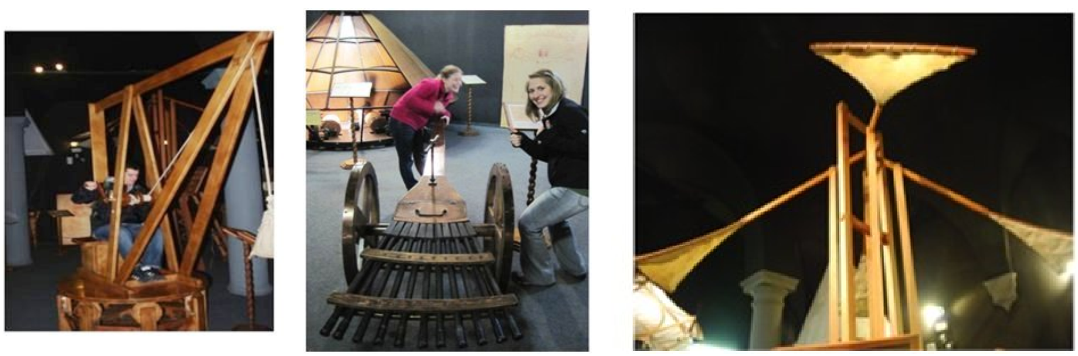
(a) Masaccio's *Trinity* (b) Brunelleschi's dome (c) Giotto's Tower relief

Figure 5: Sample of Florence art

Brunelleschi is also considered the father of Renaissance architecture with its emphasis on symmetry, proportion, geometry and the regularity of parts. Two examples of Brunelleschi's architectural achievements that we saw in Florence were the church of San Lorenzo with its Corinthian columns and geometric balance and harmony, and the dome of Florence's cathedral. This cathedral, known as the Duomo,was begun in 1296 to the design of Cambio who had envisioned a dome for the church even though neither he, nor anyone in Italy at that time had any idea of how to construct it. But the Duomo's early planners believed that by the time building of the cathedral had progressed to the point that the dome was to be built, God would provide a man with the mathematical skill to build such a dome. That man was

Brunelleschi and the students had read the story of the building of this dome in Ross King's book, *Brunelleschi's Dome*, during their prep course. Now they had a chance to climb the largest masonry dome ever built and view the unique herringbone pattern construction of the bricks that allowed it to be built without a support system during construction. Across the street from the Duomo is Giotto's Bell Tower with allegorical reliefs depicting astronomy, medicine, the building art, weaving, navigation, geometry and arithmetic. These decorations recount the destiny of man, from his creation to his dominating the world by learning technology. One relief depicts Gionitus, the mythical inventor of Astronomy, observing the height of celestial bodies using a quadrant.

The Italian region of Tuscany, which includes Pisa and Florence, was Galileo's home, and his work is celebrated and memorialized in the Museo Galileo in Florence. A science guide led us through the museum, explaining how Galileo's instruments worked (the museum houses Galileo's original instruments), how science and mathematics led to the Scientific Revolution and changed the way people viewed and understood the universe. Barometers, telescopes, quadrants, chemical flasks, an all-in-one laboratory table, were just some of the exhibits our guide showed to us. We came away with an appreciation of the variety and extensiveness of the scientific explosion that was taking place across Europe in the 17th and 18th centuries. A day trip out to Pisa gave us the chance to see the Tower of Pisa where Galileo supposedly conducted his experiments on gravity and the chandelier in the Duomo of Pisa where he began to understand pendulum motion.



(a) Crane (b) Machine gun (c) Flying machine

Figure 6: Models of Leonardo da Vinci's inventions

Art, architecture, science, inventions, and mathematics were closely intertwined in the Italian Renaissance and no one reflected this collaboration of disciplines more so than Leonardo Da Vinci, the archetypical Renaissance man. At the Leonardo Museum in Florence we had an opportunity to try out some of Da Vinci's inventions. Many of these inventions were very practical work machines. Many were war machines and reflected Da Vinci's understanding that rich rulers were willing to support mathematicians and inventors if these inventions served their military interests. Yet, other of Leonardo's inventions also reflected his visionary and somewhat fanciful designs for flying machines, floats for walking on water and even diving suits for underwater exploration.

4. London: British Royal Society and the Scientific Revolution

The Scientific Revolution may have begun in Italy with Galileo's astronomical observations that supported the Copernican heliocentric universe, but it really took off in England in the scientific community of the British Royal Society, early promoters of the scientific method. A visit to the Royal Society headquarters in London, with a guided tour by the head librarian there, introduced the students to this influential group of individuals who had changed the way science was done: from talk and conjecture to verification by experiments. We saw original scientific papers written by Royal Society members, including Newton's Principia (accompanied by Newton's drawings and cartoon sketches. Other Royal Society items on display were Boyle's air pump, Newton's telescope and Davy's mining lantern.

Figure 7: Newton artifacts at the British Royal Society in London

Many members of the Royal Society had worked at Oxford as a part of a scientific research group there. A science tour of Oxford along with a visit to its Museum of Science provided many examples of their work. As part of a workshop at the Museum of Science the students were taught how to use an astrolabe and got an appreciation for how people made sense of their universe. The students also saw how scientific instruments were used to tell time and determine navigational position, and how the development and improved design of these instruments allowed for a better understanding of how the universe worked.

Oxford also provided an opportunity to see one of Christopher Wren's earliest applications of mathematics to architecture in his design of the Sheldonian Theatre. More examples of the influence of Royal Society members in the history of 17th and 18th century science, mathematics and navigation were seen at the Royal Observatory and Maritime Museum in Greenwich and the Science Museum in London. St. Paul's Cathedral in London, Wren's greatest architectural achievement, was full of mathematical elements, from its geometrical staircase to the different mathematical curves that determine each of its three domes. The outer dome is spherical in shape, appealing from a distance but also tapping into the idea of the church representing the spherical shape of the cosmos. The innermost dome, seen from inside, is based on a catenary curve, which serves the dual purpose of an optical illusion—it draws the eye upward, making the dome seem higher than it is, but also provides a real structural advantage as a upside down catenary arch supports its own weight with pure compression and no bending. Finally, middle dome is conical, based on the curve $y = x^3$, which Wren and his fellow BRS friend, Robert Hooke, thought to be the perfect shape for a dome.

Figure 8: The three domes of St. Paul's Cathedral in London

The insides of Wren's churches also reflect the changing worldview of the Reformation with regard to the role of the church building. Wren designed his churches to reflect his view that a congregation should be able to see and hear all that went on in the church. His churches are airy, with windows containing clear glass, so people could have light to read the Scriptures. He also often brought the altars in his churches forward towards the congregation so they could see and hear the liturgy better.

Figure 9: Wren church in London

London provided opportunities to view mathematical artifacts from many different periods of history. At the British Museum in London we were given a behind the scenes viewing of the Rhind Papyrus, which is too delicate to be put on display in the museum galleries. At this museum we also saw examples of Greek architecture, Babylonian mathematical texts, and clocks (important for telling time in a city as well as for navigation) from various centuries. There was also a whole wing devoted to the 18th century Enlightenment period, containing exhibits of mathematical and scientific discoveries of this period.

5. Berlin and Göttingen: Mathematics Centers

Berlin and Göttingen became centers of European mathematics in the 18th and 19th century and Euler, Gauss, Leibniz, Cantor and Riemann were some of the mathematicians that the students learned about during their time in these two cities. One outcome of studying mathematics in a historical context was a deeper awareness of the interplay between mathematics and politics. We saw this in Rome with the way Roman engineering and architecture reflected the worldviews of their rulers. Florence brought further examples with Leonardo da Vinci's war machines, and Galileo's political and religious problems. The members of the Royal Society were caught up in the politics of the English Civil War and the society itself was formed under the Restoration of Charles II. However, the students were surprised to learn that Euler himself got involved in politics, especially during the turmoil during the occupation of Berlin by Russian troops during the Seven Years War. Thus when some of his horses were "borrowed" from his Charlottenburg estate by Russian troops, Euler complained to Frederick of Prussia, at whose request Euler had come to Berlin. Euler demanded compensation, and apparently was generously rewarded. As informal director of the Berlin Academy, Euler assumed many of its administrative duties, which ranged from juggling budgets to overseeing greenhouses. These, and other interesting facts about Euler's daily life, were revealed in some of papers in the Euler's archives, which were translated for us by the academy professor of history and science at the Berlin Academy of Sciences.

Figure 10: Euler's papers and Leibniz's calculating machine (Berlin Academy of Sciences)

Our visit to the Berlin Academy of Sciences was followed by a day trip out to Göttingen. The students learned about the mathematicians who lived and worked in both these cities and the reasons behind the rivalries between these two centers of mathematics. Markers with the names of famous mathematicians and scientists were on buildings all over Göttingen, denoting the mathematician that had lived or worked there. Göttingen has a small university town feel to it, and Gauss had enjoyed the opportunity to mix with colleagues from variety of disciplines, or keep his distance, as he pleased. He spent much of his time in his observatory, and we had an opportunity to tour this observatory and to see where Gauss did his observations, including the chair in which he sat and the telescope he used. A statue of Gauss and Weber in Göttingen commemorates their collaboration on the invention of the electromagnetic telegraph (1833). They put the system into operation to link their places of work. The statue shows the scientists talking about their joint work with Gauss holding a wire (not preserved) in his right hand, its coil at his feet, while Weber's left hand rests on the telegraphic transmitter. The Göttingen locals like to think of Weber as saying: "Go on Carl, let me sit, too."

Euler's life in Berlin, where he was involved in all aspects of life around him, including household accounts and everyday housekeeping needs of the Berlin Academy, contrasted with Gauss's "quieter" life in Göttingen, where he could choose the degree of his involvement in matters

(a) Gauss's telescope (b) Weber and Gauss statue

Figure 11: Treasures of Göttingen

outside his mathematical interests. Students saw that they were two very different personalities, but each a pillar in German mathematics.

6. WWII Changes the Mathematical Landscape

In the 20th century politics again played an integral role in the mathematical landscape. The persecution of the Jews in the 1930s and 1940's resulted in the migration of many prominent Jewish mathematicians from Germany to the United States. Many of them were applied mathematicians and a new era of applied mathematics resulted in the U.S. Those Jewish mathematicians who remained in Germany lost their jobs, and some, their lives. Readings that students did about this persecution and its ensuing hardships took on new meaning when they visited the Jewish Museum in Berlin and when they saw the many memorials around the city, testifying "Never again."

(a) Jewish museum (b) Holocaust memorial

Figure 12: Jewish museum and holocaust memorial (Berlin)

While many mathematicians were being displaced in Germany during the 1930s and 1940s, in England, mathematicians, along with linguists, classicists and "anyone good with puzzles" were being recruited to the British code-breaking facility in Bletchley Park. A Bletchley Park Educational Staff member gave us a tour of the park including the rooms where Alan Turing's Bombe machines were busy working on deciphering the Enigma code. The mathematics behind the Enigma code was explained and students had an opportunity to use a real Enigma machine to encrypt and decrypt a message. Turing, himself often the outcast during his school years, found a home in Bletchley Park, where he thrived on the intellectual camaraderie he found there. His ability to think outside the box, challenging previous assumptions about machine learning, led to his invention of the Bombe and the Turing Machine. A computer science museum on the grounds of Bletchley Park commemorates his, and other computer scientists' work and contains many examples of earliest computers. For the computer science students on the trip, this was a highlight!

(a) Main building

(b) Enigma encryption

(c) Turing bombe

Figure 13: Bletchley Park: England's WWII Cryptography Centre

7. Students' Work and Reflections

In conjunction with our travels, students had daily reading assignments, which gave further background to the time periods, and sites we would be seeing. Many of these readings were original source readings, writings of the mathematicians whose legacies we would be viewing. As we visited the site corresponding to a student's biographical research, that student would give an oral presentation about his/her mathematician. Hearing about the mathematician and then seeing where they lived and instruments they used, contributed to a fuller appreciation of their lives and mathematical contributions. Students reflected on their experiences through daily journaling. They shared about their experiences with family and friends back home through a blog set up on our university website. Each student was assigned a day to blog and was expected to give not only an account of that day's activities, but a historical context to the mathematics encountered. These blogs had an active following back home and generated interest among students back on campus to become part of the next Math History Study Abroad program. You can read the students' blogs and hear about our adventures from their perspectives at http://ma396.blogspot.com/.

When we returned to Whitworth I asked the students to reflect on how their worldview had changed as a result of their experiences abroad. Their comments reflected their deeper appreciation for history and culture. Typical student comments were: "Traveling for this study abroad trip opened my eyes to different cultures and brought the mathematicians to life. As we traveled we also learned a lot about the places we were traveling that helped me to see history from a different perspective;" "I was amazed at how much history is in other cultures. It was cool to walk around a corner and see ruins or an ancient statue or a huge cathedral. There is a lot more to the world than our (American) history and my perspective and worldview;" "Study abroad really opened my eyes to different cultures and how other people's lives are. I got to experience new food, new languages, and met new people. I have a new profound respect for other's cultures." Along with their increased understanding of the historical roots of mathematics the students gained an appreciation of other cultures. All the students expressed a desire to travel abroad again, to continue to grow in their appreciation of the history and worldview perspectives of different cultures. This was as much of a goal for trip as the mathematical component, since it is only by understanding and appreciation a culture that you can appreciate the mathematics that came out of it.

8. Conclusion

Our experiences on this Math History Study Abroad Program confirmed that math history is best learned in a cultural context and that on site experiences add to understanding and appreciation of mathematicians' world and worldviews. A math history study abroad program has benefits for all math, computer science and science students. Future teachers would especially benefit. It makes mathematics "come alive"; imparts the view of mathematics as a continually developing human activity occurring within the framework of the surrounding culture. A Math History Study Abroad program can change the way your students view mathematics. Consider instituting such a program at your university.

Google and the Mathematics of Web Search

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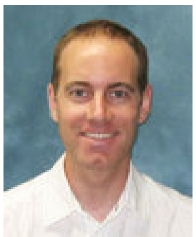

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1. Background

1.1. Web Search Before Google

When I was a junior in high school I began to think about where I wanted to go to college. I decided to use the Internet to learn more about the schools I was interested in. However, I did not know how to find the homepages for the Universities. As this was 1994 and the Internet was relatively young, I did not know that one could type in something like `www.colorado.edu` to find the homepage of the University of Colorado, for instance.

My Father informed me that I could use AltaVista to search the Internet for the university web sites. When I would type the name of a university into AltaVista the search was slow and the results would be less than helpful. For instance, typing in the name of a university would often return results for particular student groups, departments, or announcements at that university, rather than the main university homepage. My technique was to click on one of these links and try to follow links from page to page until I reached the main university web page. This was a frustrating endeavor, however, and I found myself wondering why AltaVista didn't know that I wanted the main university webpage, instead of more specific, less important web pages that were affiliated with the university, but clearly had a much smaller intended audience.

But this raises an interesting question: how do we determine which web pages are more "important" than others? If AltaVista knew which web pages were more important than others, it could list my search results in the order of importance, with the most relevant pages at the top of the list.

In what follows I will describe how Google solves this problem of determining the relative importance of web pages. One of the algorithms Google uses to rank the results of searches is called PageRank. The main concept behind PageRank is that the connected structure of the internet gives information about which web pages are relatively important. Each web page is assigned its own PageRank score and when Google returns search results it sorts the list according to each page's PageRank score.

1.2. What Happens After You Type a Search into Google

PageRank is only one part of what Google does after a user enters some search terms. When search terms are entered into Google, that text is sent to the Google web server. The web server sends the query terms to an index server. The index server contains a large list telling which pages contain those particular search terms. After the index server has found a list of the appropriate pages, the query travels to document servers which retrieve the actual contents of the relevant web pages, not just their addresses. A small part of the content of each page is displayed when Google lists the results. Finally, before Google shows the list of relevant pages (along with snippets), the results are ordered such that the most relevant pages are at the top of the list. PageRank is part of this final step: ordering the search results, but it is not the only part. The other portions of the sorting algorithm are proprietary. A nice summary of the entire search process can be found in [3].

1.3. Connectivity Implies Relative Importance

A crucial component of the internet is not only the pages themselves, but also the connections between the pages, called hyperlinks. Mathematically speaking, the internet can be thought of as a large directed graph where the web pages are nodes, and the edges are hyperlinks.

Part of the brilliance of the PageRank algorithm is the idea that the connected structure of the web can tell us about relative importance of individual web pages regardless of the actual content in those pages. A helpful analogy to help us understand how connections can tell us about relative importance is to think of each link as a recommendation. If there is a link from page A to page B, it's as if page A is endorsing or recommending page B. If a particular page has a lot of "inlinks" (pages linking to it) this suggests that that particular page is important. And PageRank takes this into account. More inlinks to a particular page typically results in a higher ranking. However, on the internet, as in life, not every recommendation is of the same worth. A recommendation from a famous person is surely more useful than a lot of recommendations from obscure people. So, PageRank takes this into account too. A link to page A from an important page helps raise page A's rank more than a link from an obscure page. So page A's rank depends on the ranks of the pages that link to page A, not just the sheer number of inlinks it has. Also, the relative value of each inlink or recommendation a page has needs to be considered in terms of how many outlinks that page makes. For instance, if page A gets a recommendation from page B, but it turns out that page B has many outgoing links (recommendations) then the recommendation from page B is not worth as much as it would have been if page B only made a few recommendations. This is consistent with the letter of recommendation idea: a letter from a famous person is no longer as valuable if it turns out that the recommender actually writes hundreds of recommendations every year. This idea of thinking of links as letters of recommendations is nicely summarized in [4].

A concise way to summarize these ideas is to say that a webpage is important if it is pointed to by other important web pages. This is circular-sounding argument, but it turns out that it works and actually works well if it is formulated mathematically. This is the central thesis of the PageRank algorithm.

2. PageRank

To formulate the PageRank algorithm mathematically, we can begin with the following conceptual equation:

$$\text{ranking of page } i = \sum \frac{\text{rankings of page } j}{\text{number of outlinks from page } j}, \text{ where all the pages } j \text{ link to page } i.$$

Being a bit more precise with our notation, we can write out the PageRank idea in the following manner:

$$r(P_i) = \sum_{P_j \in B_{P_i}} \frac{r(P_j)}{|P_j|}, \tag{1}$$

where $r(P_i)$ = PageRank of page P_i, B_{P_i} = the set of all pages linking to P_i, and $|P_j|$ = the number of outlinks from page P_j.

Let's illustrate this concept with the simple web structure shown in Figure 1.

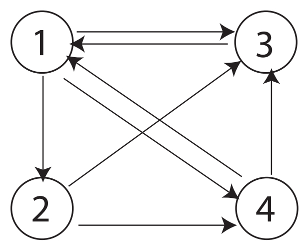

Figure 1: A simple connected web structure to illustrate PageRank

Writing out equations (1) for this particular system yields the following:

$$\begin{aligned} r_1 &= \frac{r_3}{1} + \frac{r_4}{2}, \\ r_2 &= \frac{r_1}{3}, \\ r_3 &= \frac{r_1}{3} + \frac{r_2}{2} + \frac{r_4}{2}, \\ r_4 &= \frac{r_1}{3} + \frac{r_2}{2}. \end{aligned}$$

This system of four equations can be written using matrix-vector notation:

$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix},$$

which is simply an equation of the form $Ar = \lambda r$, a classic eigenvalue-eigenvector equation. So, the PageRank of each page in the sample web can be found by solving for the eigenvector corresponding to the eigenvalue 1. In this case

$$\mathbf{r} = \begin{pmatrix} 12 \\ 4 \\ 9 \\ 6 \end{pmatrix}.$$

If the eigenvector is normalized so its elements sum to 1 we are left with the following:

$$\mathbf{r} = \begin{pmatrix} 0.387 \\ 0.129 \\ 0.290 \\ 0.194 \end{pmatrix}.$$

So for this sample web, the most "important" page is page 1, followed by pages 3, 4 and 2. Note that the matrix $\mathbf{A}$ has entries between 0 and 1 and its columns sum to 1. Such matrices are said to be column-stochastic.

Another way to formulate this problem is to imagine a random web surfer who constantly follows links from one page to another [1]. Thinking of this as a stochastic process, the more "important" pages will be the ones that the random surfer spends most of his time visiting. This "random surfer" formulation is actually equivalent to the way the PageRank algorithm is set up above [4]. The PageRank of each page is the limiting probability that the surfer will be visiting that page at any particular time.

2.1. Existence and Uniqueness of Solutions

Mathematically speaking, this formulation raises a lot of questions, like how do we know that the matrix A above has an eigenvalue of 1? Also, how do we know that there is a unique eigenvector corresponding to the eigenvalue of 1, thereby giving a unique set of rankings for the pages in the web?

Much has been written on this topic, so here I will simply mention two of the important theorems. First of all it is easy to prove that every column stochastic matrix has 1 as an eigenvalue. And secondly, if the matrix $\mathbf{A}$ is positive and column-stochastic then the eigenspace for eigenvalue 1 has dimension 1. These theorems are nicely addressed in [2].

3. Problems in Real Webs

The theory developed above works well for simple webs like the one in Figure 1, but real networks of web pages are extremely large and they often have characteristics that cause fundamental problems with this approach. Here I will focus on two of the most common problems: dangling nodes and disconnected subwebs.

A "dangling node" refers to a web page that does not have any outgoing links. Thinking of the random surfer model, if the surfer makes it to this page, there will not be any links to follow to connect to other pages in the network. Since those other pages will not be visited, the random surfer won't get a clear picture of the connected structure of the web. This is quite common on the World Wide Web since many pages are documents or images that have inlinks, but don't have any outlinks. An example is shown in Figure 2.

In this case, if a random surfer follows a link to page 2, it will get stuck there. The corresponding

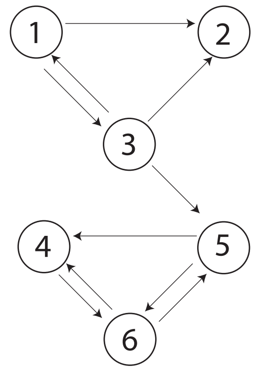

Figure 2: A simple web with a dangling node. If a random surfer gets to page 2, there isn't an outlink to connect it to any other page in the network.

connectivity matrix for this network is as follows:

$$\begin{pmatrix} 0 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}.$$

Note that the second column contains all zeros due to the fact that page 2 doesn't have any outgoing links. To remedy this situation, the creators of PageRank proposed the following solution: if the surfer reaches a page that does not have any outlinks, the surfer chooses another page in the web randomly and transports there. This keeps the surfer from getting stuck on a dangling node and it changes the connectivity matrix as follows:

$$\begin{pmatrix} 0 & 1/6 & 1/3 & 0 & 0 & 0 \\ 1/2 & 1/6 & 1/3 & 0 & 0 & 0 \\ 1/2 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1/6 & 1/3 & 0 & 0 & 1/2 \\ 0 & 1/6 & 0 & 1 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix}.$$

We will refer to this new matrix as $\mathbf{M}$. All entries in the second column are $1/6$ since the surfer is equally likely to visit any page in the network if it is currently visiting page 2. After making this correction, the matrix $\mathbf{M}$ satisfies the criteria for the Perron-Frobenious Theorem and therefore has an eigenvalue of 1 with a unique corresponding eigenvector.

The second major problem that arises when applying this algorithm to the World Wide Web is that not every group of web pages is connected to every other group. Frequently there are "cliques" that form: groups of web pages that are connected to each other, but not to any other pages outside the clique. An example of a web with a disconnected subweb is shown in Figure 3, which is the same web as in Figure 2 with the connection from page 3 to 5 removed.

Continuing with the random surfer analogy, if the surfer were to start on page 1, it would be unable to reach pages 4, 5, or 6 simply by following links. This problem is again relatively

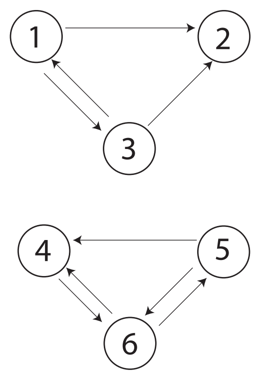

Figure 3: A simple network with a disconnected subweb.

easily overcome if instead of always following links, the random surfer occasionally teleports to another page in the network. To modify our linear equation to reflect this new strategy, we set up the matrix $\mathbf{G} = (1-\alpha)\mathbf{M} + \alpha\mathbf{S}$, where α is a constant between 0 and 1, and the matrix $\mathbf{S}$ contains the value $1/N$ in each position and N is the total number of pages in the web. The matrix $\mathbf{G}$ represents a balance between following the link structure of the web and randomly teleporting to another page. The parameter α can be chosen in such a way that the random surfer is mostly following links, and only occasionally teleporting to another site. PageRank is thought to use a value of $\alpha = 0.15$ [5].

For this particular example,

$$\mathbf{G} = (1-\alpha)\begin{pmatrix} 0 & 1/6 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/6 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1/6 & 0 & 0 & 0 & 1/2 \\ 0 & 1/6 & 0 & 1 & 1/2 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}.$$

Now we can write

$$\mathbf{G}\,\mathbf{r} = \mathbf{r},$$

and $\mathbf{G}$ is now column stochastic, so using the theorems in Section 2.1 it can be proven that $\mathbf{G}\,\mathbf{r} = \mathbf{r}$ has a unique solution. Thus, a unique ranking of pages exists.

4. A Consequence of Ranking by Popularity: Google Bombs

Since the PageRank algorithm uses the connected structure of the internet to determine the relative importance of pages, the linking structure can be manipulated to alter the PageRank scores of a particular webpage. Called a *Google Bomb*, one of the most famous examples was in 2006 when the search term "miserable failure" listed the Biography of president George W. Bush as the first result. This happened because many bogus web pages were created that contained the keywords "miserable failure" and pointed to the biography of President Bush. Interestingly, MichaelMoore.com was the second hit, apparently the result of a Google Bomb created by Bush supporters. There have been many humorous examples of Google Bombs, although they are much less common now as it seems that Google has changed their ranking algorithm to be less prone to Google Bombs.

5. Google as a Teaching Tool for Linear Algebra

One of the most common questions I receive as a Mathematics instructor is "When are we ever going to use this?" One of the main reasons that I include a discussion and a homework assignment on PageRank is to answer that question. It helps the students make a real connection to something we all use every day.

I require my Linear Algebra students to complete one assignment related to PageRank. It consists of computing the PageRank of each website in a few small webs, some of which have dangling nodes and/or disconnected subwebs. I have my students use MATLAB and I spend most of one class period teaching them the basics. I also provide some sample code that they can modify instead of writing their own code from scratch.

Since the internet is so large, PageRank cannot rely on something like MATLAB's "eig" command to compute the eigenvector. Instead, Google most likely uses an iterative algorithm called the Power Method to approximate the eigenvector with corresponding eigenvalue of one. Since an exact eigenvector is not needed, but an approximation is normally sufficient, relatively few iterations of the Power Method are required. This is another reason PageRank is so fast.

Even though the PageRank of the websites in the simple web structures in the assignment could easily be computing using MATLAB's "eig" command, I taught my students the Power Method and showed them how to implement it in MATLAB. I wanted the students to get a feel for how many iterations it takes to get a good estimate of the eigenvector with corresponding eigenvalue 1.

6. Concluding Remarks

For Linear Algebra instruction there are many examples of how the topic is used in many different applications. Including some material on PageRank is an excellent way to connect the idea of an eigenvector and eigenvalue to something that the students use every day. Also, since the PageRank concept is quite simple, it does not require a lot of class time to derive.

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The History of the Area between a Line and a Parabola

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1. Introduction

Finding the area between a line and a parabola is a typical first-semester Calculus problem, usually the application beyond finding the area under a curve. The same problem, asked for a general line and parabola, is actually pretty messy and a challenge for impatient students, even though it does lead to a formula for the area as a function of the coefficients appearing in the equations of the line and parabola. This approach actually obscures the once traditional viewpoint of area as *quadrature*: relating a region to a square or equally easy polygonal figure. It was this problem that was taken up by Archimedes, the genius geometer, scientist, inventor of antiquity. In the introduction to his treatise *Quadrature of the Parabola*, he writes to a friend, Dositheus,

> ...I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, ... [6, p. 233]

22 giving a very tidy general solution. Figure 1 demonstrates what this result says; the third vertex of the inscribed triangle is on the line parallel to the axis of the parabola which also contains the midpoint of the chord. Equivalently, the region is two-thirds of the area of the

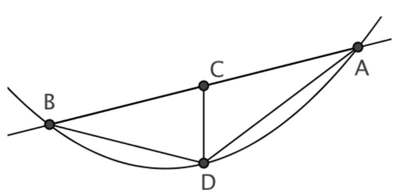

Figure 1: The inscribed triangle

parallelogram which includes the chord and sides parallel to the axis, and the area outside the parabola is one-third of the same parallelogram.

It appears that this area problem has persisted in the imagination of mathematicians over the years since Archimedes, as one worth trying their hand at. Whether they weren't aware of how Archimedes solved it, or thought that a different method might be more elegant, or just that it was worth trying a new area method on a simpler problem with known solution, many have taken the time to write down their proofs of the area between a line and a parabola. Our purpose in this survey, historical and mathematical, is to briefly introduce some of those who have given proofs (this collection is certainly incomplete), and to describe a few of those proofs which we find most interesting and diverse. Unless otherwise stated, biographical material on each mathematician is culled from the excellent MacTutor History of Mathematics archive hosted by the University of St. Andrews [7].

Before we start, it will be useful to agree on some terminology and also introduce a few geometric results that are used frequently in the proofs. The region enclosed by a parabola and an intersecting line will be called a *parabolic region* or a *segment* of the parabola. This last term makes sense from the viewpoint that any two points on the parabola determine a segment of the curve and, since the parabola is convex, when the points are connected with a chord a unique parabolic region is formed. The chord is called the *base* of the segment, while the line from the midpoint of the chord, parallel to the axis of the parabola, to the curve is the *diameter* of the segment. The point of intersection of the diameter and the parabola is the *vertex* of the segment. The *height* of the segment is the length of the diameter. Line segments from the diameter to the curve that are parallel to the chord are called *ordinates*. We will assume the following properties—known or shown by Archimedes—illustrated in Figure 2:

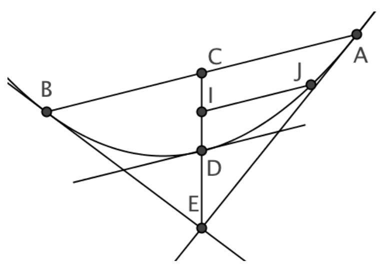

Figure 2: Geometric properties of the segment

Property 1. *A point on the parabola is the vertex of a segment if and only if the tangent line to the parabola at that point is parallel to the base of the segment.*

D is the vertex of the segment and the tangent to the parabola there is parallel to chord $\overline{AB}$.

Property 2. *The tangent lines to the parabola at both endpoints of the base intersect the extension of the diameter at a point whose distance from the vertex is equal to the height of the segment.*

The tangent lines to the parabola at endpoints A and B meet at E such that the vertex D is the midpoint of $\overline{CE}$.

Property 3. *The ratio of the length of any ordinate to half the length of the base is equal to the ratio of the squares of the part cut from the diameter by the ordinate (from the vertex) and the whole height.*

$\overline{IJ}$ is an ordinate and $CD/ID = CA^2/IJ^2$.

For the proofs that we describe in detail, we will try to follow the original style, but use modern notation for the sake of the reader. The figures that accompany the proofs are, in most circumstances, copies of those that appeared in the original treatises. The order in which we present the mathematicians is chronological by the date in which their proof was written.

2. The Greek World

In the tradition of Euclid, Archimedes relied on a rigorous approach to geometric proofs. He is known for his many results on areas, volumes, and centers of mass, as well as his scientific advancements, and engineering inventions. He was born 287 BC in Syracuse, Sicily (present Italy), and died there also, in 212. He may have visited Alexandria, the contemporary epicenter of knowledge, but certainly knew and corresponded with many of the mathematicians there. Few reliable facts are known about him; there are an abundance of far-fetched stories about his life and genius.

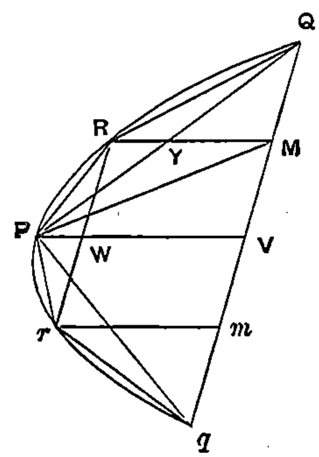

Figure 3: Archimedes' Proof [6, p. 248]

2.1. Archimedes' proof

Archimedes must have valued his success at finding the quadrature of the parabola, as he devoted a whole treatise, *Quadrature of the Parabola*, to it. Since it is the first proof of the area of the parabolic region, we will explain Archimedes' proof in some detail, following [6]. In Figure 3, suppose V is the midpoint of base $\overline{qQ}$, $\overline{PV}$ is the diameter of the segment, and that M, m are the midpoints of the halves of the base. With $\overline{RM}$ parallel to $\overline{PV}$, then $\overline{RY}$ is the diameter of the segment determined by the chord $\overline{PQ}$. By Property 3, $PV = 4PW$; also, using similar triangles, $PV = 2YM$. Thus $YM = 2RY$ and the area of $\triangle QPM$ is twice $\triangle RPQ$. $\triangle PqQ$ is four times $\triangle QPM$, hence each of $\triangle RPQ$ and $\triangle rqP$ (constructed similarly) is an eighth of $\triangle PqQ$. We can see that for each of these smaller triangles, there will be two even smaller triangles with the same proportional area. Thus, a polygon can be inscribed in the parabolic region by a sequence of added triangles, such that at each stage the area added is a quarter of the area added at the previous stage. At each step the added triangles constitute more than half of the remaining difference between the area of the segment and the area of the inscribed polygon; this is what the Greeks would term the *method of exhaustion*.

If we write this series as $a+b+c+\ldots+m$, with each term being a fourth of the previous term, then we can say that $a+b+c+\ldots+m <$ area of segment, where the difference can be made smaller than any specific quantity. Archimedes then shows that under the same circumstances, $a+b+c+\ldots+m+\frac{1}{3}m = \frac{4}{3}a$. To show that in fact the area of the segment is exactly $\frac{4}{3}\triangle PqQ$, he assumes it is not equal and carries out a double *reductio ad absurdum*. If the segment is greater, by a specific quantity, then at some step $a+b+c+\ldots+m$ is larger than $\frac{4}{3}a$, a contradiction. Similarly, if it is smaller, then at some step $a+b+c+\ldots+m$ is larger than the area of the segment, also a contradiction. We note how ingeniously this proof avoids a limit or other reference to infinity. In the same treatise, Archimedes provides a second 'mechanical' proof which makes use of results involving centroids of triangles and balancing areas suspended from a lever.

As a corollary to this proof, we have

Property 4. *The diameter of a segment divides it into equal areas.*

This is important to know since many of the proofs by later authors find the area relative to only half the parabola, with the division at the diameter.

2.2. Archimedes' Method

In 1906 a manuscript attributed to Archimedes was examined by J. L. Heiberg in Constantinople, but is wasn't until the manuscript resurfaced in 1998 that it was thoroughly examined and read (see [10]). Indeed, among its contents was a previously lost treatise of Archimedes, referred to as *The Method*, that had been saidby others to contain a technique that could lead to accurate formulas for areas and volumes of various geometric objects. Archimedes himself describes this method as an exploration that led to correct results, but not proofs; those would have to be provided by geometric means. As an example of his technique, he derives the area of a segment of a parabola. The basic method compares the lengths of corresponding cross sections of the segment and a circumscribed triangle, and shows that they balance, in the sense of moments, when placed on a lever arm. He generalizes to get the triangle and segment to balance, from which he can conclude what their relative areas are.

Let a segment of a parabola have base $\overline{AC}$ and diameter $\overline{DB}$, as in Figure 4, and form a triangle with the line tangent to the parabola at C and the line parallel to the diameter through A meeting at F. By Property 2, ED is double BD, hence FA is four times the height of the segment, and the area of $\triangle ACF$ is four times $\triangle ACB$ inscribed in the segment. Let $\overline{HC}$ have midpoint K which is also the midpoint of $\overline{AF}$. The centroid W of $\triangle ACF$ is a third of the distance from K toward C on $\overline{HC}$. Archimedes shows, based on Property 3, that for each cross section $\overline{MO}$ of the triangle (and segment) parallel to $\overline{FA}$, $PO \cdot AC = MO \cdot AO$, which is equivalent to, using similar triangles, $PO \cdot KC = MO \cdot KN$. In Figure 4 this is demonstrated visually by placing $\overline{TG}$, equal to $\overline{PO}$ at H, which is then in equilibrium with $\overline{MO}$ about K. Since this is true for every cross section of the triangle, Archimedes concludes that $\triangle ACF$ with centroid at W is in equilibrium with the whole segment located at H. Comparing lever arms, the area of the segment must be a third of $\triangle ACF$ and four-thirds of $\triangle ACB$, as expected.

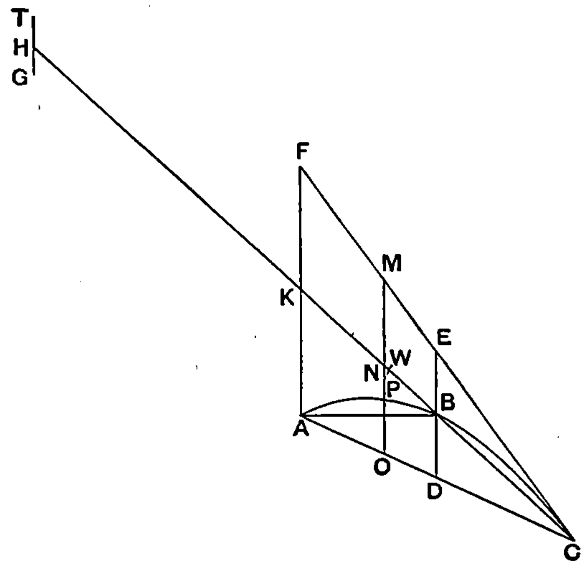

Figure 4: Archimedes' Method [6, p. 16, supp.]

Today it would not be difficult to legitimize the leap from cross sections to the whole area, but Archimedes was justified in not treating this method as a proof, but only a way to find correct formulas for areas and volumes.

3. The Muslim World

We shift our focus from the ancient Greek world to the Arab world about 1000 years later. Often there is an impression that the major Arab contribution was in preserving and translating ancient Greek learning, but their own discoveries and developments are considerable, and very clearly paved the way for the European renaissance in mathematics. The Muslim Caliphs, beginning in the late 8th century, were great supporters of scientific and mathematical scholarship, and they set up research centers, primarily in Baghdad. Scholars were recruited from all over to join these centers.

3.1. Thabit ibn Qurra

Al-Sabi Thabit ibn Qurra al-Harrani, as his name implies, was born in 836 in Harran, Mesopotamia (present Turkey) and was a member of the Sabian religious sect. As a young man he went to Baghdad to train in mathematics and medicine. Though he returned to Harran, he eventually ended up in Baghdad as court astronomer under the Caliph al-Mu'tadid. He was a language expert and produced a revision of an earlier Arabic translation of Euclid's *Elements* which became the definitive version. His work in mathematics ranged from number theory (amicable numbers), mathematization of astronomy, mechanics (equilibrium of levers), even the philosophy of mathematics, in addition to his treatise on the quadrature of parabolas that is of interest to us. He died in Baghdad in 901.

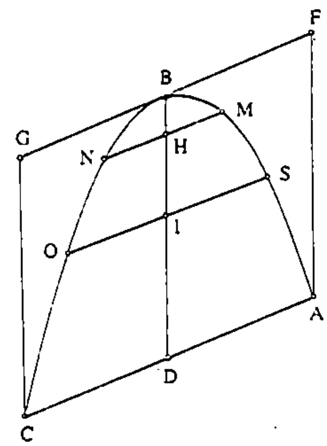

Figure 5: Thabit's Proof [12, p. 264]

Thabit's quadrature (see [12]) involves an interesting application of his developments in number theory (sums and products of integers) to a geometric problem. For the segment of the parabola with chord $\overline{CA}$, diameter $\overline{BD}$, and circumscribed parallelogram $CAFG$, as in Figure 5, Thabit divided the diameter into parts such that BH, HI, ID, etc., were in proportion to each other as are the odd integers starting from 1. This key choice was so that their sums, BH, BI, BD, etc., were in proportion to each other as are the squares of the integers starting at 1, and hence, using Proposition 3, the corresponding double-ordinate lengths NM, OS, CA, etc., would be in proportion as are the even integers starting at 2. With this set-up, he inscribes a polygon in the segment made up of trapezoids such as $CASO$, $OSMA$, and compares it, through proportionality to various products of odd and even integers, to the whole area of the circumscribed parallelogram. From this point his proof is very similar to Archimedes' "triangle" proof, in using the method of exhaustion and double *reductio ad absurdum*, again avoiding limits or infinite sums.

3.2. Muhammad ibn Isa al-Mahani

Abu Abd Allah Muhammad ibn Isa al-Mahani was born around 820 in Mahan, Kerman in Persia (present Iran). We don't know much about his life, but he worked in astronomy, wrote commentaries on Euclid's *Elements*, and worked to solve some of Archimedes' geometric problems by reducing them to problems in algebra, an important new approach. Through a reference we know that he provided another proof of the quadrature of the parabola, which improved on Thabit's, but we have no idea the nature of that proof. Al-Mahani died in Baghdad in 880.

3.3. Ibrahim ibn Sinan

Ibrahim ibn Sinan ibn Thabit ibn Qurra, 908–946, was the grandson of Thabit ibn Qurra and lived in Baghdad his whole life. His father, Sinan, was known for his work in medicine, but Ibrahim followed his grandfather's steps in focusing on mathematics and astronomy. It is said that had Ibrahim not died relatively young, he had the potential to supersede his grandfather's accomplishments. It is from his treatise *On the measurement of the parabola* (see [12]) that we learn of al-Mahani's proof, and Ibrahim's motivation to improve on that in order to bring the "bragging rights" back to the family. Ibrahim's proof is quite different from earlier methods . He first proved that if the parts of certain geometric figures are proportional (even though the figures are not similar), then their areas will also be proportional. He applies this to polygons inscribed both in a segment of a parabola and in the smaller segments of the same parabola left outside the inscribed triangle, showing, using exhaustion and *reductio ad absurdum*, that each smaller segment is an eighth of the whole segment. From this the appropriate conclusion is a simple algebra exercise. He was able to use this same idea of a geometric transformation that preserves ratios of areas to good effect in other area and volume problems.

3.4. al-Mu'taman ibn Hud

Not much is known about Abu Amir Yusuf ibn Ahmad ibn Hud al-Mu'taman, except that he was the king of Saragossa (located in what is now northern Spain) from 1082-1085, where his father had been king before him, and that he was interested in science and mathematics. He wrote and compiled a comprehensive mathematical encyclopedia called *Kitab al-Istikmal* (*Book of Perfection*). Contained in this book is another proof of the area of the segment of a parabola. It follows the same idea as that of Ibrahim ibn Sinan, though it was intended to be more general and to apply also to the other conic sections. (See [8, 12])

4. Europe in the 1600s

With the rebirth of serious mathematical scholarship in Europe during the centuries leading up to the seventeenth, interest in finding areas and developing general methods of quadrature came to the forefront. Transmission of Greek and Arab treatises gave an impetus, but it was new developments in algebra that allowed problems to be formulated in new ways. Shortly into this century we see the quadrature of the parabola surface, both in a classical sense, but also in the context of the newly developed *method of indivisibles*. Later, as the use of limits and infinite sequences became more common, they appear in quadrature also. As numerous mathematicians were making contributions toward the development of formal methods of calculus, the parabola area problem appears in the work of many of them. In some cases we know that a particular mathematician considered this problem, but don't know the details of the proof (usually because of lack of translation of obscure material, but occasionally because the notation and vocabulary is uncommon).

4.1. Luca Valerio

Luca Valerio was born in Naples (present Italy) in 1552, but studied, lived, and worked in Rome until his death in 1618. He is most known for his book *De centro gravitatis solidorum* in which he found centers of gravity of solid bodies using methods from Archimedes. He was a colleague of Galileo, and they expressed much respect for each other, with Galileo calling Valerio "the new Archimedes of our age" after his death. His treatise *Quadratura parabolae*, published in

1606, found the area and center of gravity of a segment of a parabola using classical geometry as well as new notions of limits of ratios.

4.2. Bonaventura Cavalieri

Born in 1598 in Milan (present Italy), Bonaventura Francesco Cavalieri studied in Pisa, and eventually ended up in the chair of mathematics at Bologna, which he held until he died in 1647. Cavalieri corresponded with Galileo, Marsenne, Torricelli and other mathematicians of his time. He is best known for developing his theory of indivisibles, which extended the Greek method of exhaustion to include infinitely small geometric quantities, building on Kepler's work. Cavalieri presented his methods in his 1635 book *Geometria indivisibilibus continuorum nova quadam ratione promota*, which includes the quadrature of the parabola among the many examples. Cavalieri's methods were very effective, but not rigorously developed, so he received much criticism while others worked to set the methods on a solid footing.

4.3. Galileo Galilei

Galileo Galilei was born in 1564 in Pisa (presently Italy), spending his youth between there and Florence. At his father's wish he studied medicine in Pisa, but eventually abandoned that for his true love, mathematics. Galileo held various teaching positions, at Pisa, Padua, and eventually in the court of the Grand Duke of Tuscany (the region around Florence). As his teaching duties included astronomy, he became interested in modern theories, and made significant discoveries using telescopes he constructed. He was held in high regard among his colleagues but eventually ran afoul of the Church after his 1632 publication of *Dialogue Concerning the Two Chief Systems of the World - Ptolemaic and Copernican*. In later life he was to continue his studies in motion, gravity, geometry, and mechanics, and published *Discourses and mathematical demonstrations concerning the two new sciences* [4] in 1638. He died at Arcetri, near Florence, in 1642.

At one point in his *Discourses*, the characters are discussing the strength of beams, and Salviati states that a beam with profile in the shape of a parabola (as opposed to a rectangle) will have uniform strength, and hence the material may be reduced by a third. He then proceeds to demonstrate that the area outside a parabola is a third of the corresponding rectangle.

In Figure 6, A is the vertex, $\overline{AC}$ the diameter, and $\overline{BC}$ half of the chord. With $\overline{BC}$ divided into an arbitrary number of equal parts, Then $\overline{AK}$, $\overline{AL}$, $\overline{AM}$, $\ldots$, $\overline{AP}$, are in proportion to each other as are the integers $1, 2, 3, \ldots, n$. By Property 3, $\overline{KG}$, $\overline{LE}$, $\ldots$, $\overline{PB}$, are in proportion as are the squares $1, 4, \ldots, n^2$. The region between the parabola and lines $\overline{AP}$ and $\overline{PB}$, has inscribed in it a polygon $PVITHSFREQGK$, and a similarly circumscribed polygon. In the same proportion that the area of parallelogram $APBC$ is proportional to n^3, the inscribed and circumscribed polygons are proportional to $1 + 4 + \ldots + (n-1)^2$ and $1 + 4 + \ldots + n^2$, respectively. Galileo then uses a result from Archimedes' treatise *On Spirals* (see [6, p. 162]) that is equivalent to

$$1 + 4 + \ldots + (n-1)^2 < \frac{1}{3}n^3 < 1 + 4 + \ldots + n^2,$$

along with *reductio ad absurdum*, to conclude that the area outside the segment is a third of whole.

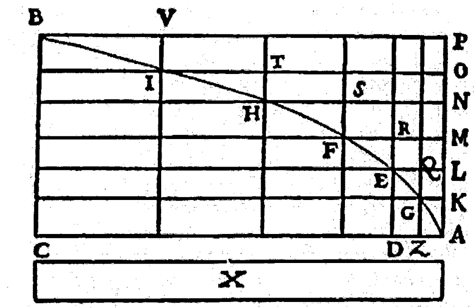

Figure 6: Galileo's Proof [4, p. 145]

4.4. Evangelista Torricelli

Evangelista Torricelli, born in Faenza, Romagna (northern Italy) in 1608, studied with, then became secretary to, Benedetto Castelli, who was at the University of Sapienza in Rome. Torricelli's work included studying projectile motion, hydrodynamics, the theory of atmospheric pressure and vacuums, as well as finding formulas for volumes and centroids. While at first suspicious of the method of indivisibles, he developed them further and used them, along with traditional geometric proofs, to find many new results. In 1644 he published *Opera geometrica* which collected his significant discoveries in a diversity of fields. Included in this tome was a treatise *De dimensione parabolae* which includes a collection of different proofs of the area of a segment of a parabola, some using classical geometric methods and some the new methods. (We have not found a translation of this yet, so we are not able to elaborate on his proofs.)

Torricelli's work came to the attention of Galileo, who brought him to his house near Florence as an assistant. Torricelli continued there through Galileo's death in 1642, when he was appointed to replace Galileo as court mathematician for Grand Duke Ferdinando II. Alas, Torricelli was only 39 when he died in Florence of typhoid in 1647.

4.5. Gilles Personne de Roberval

Gilles Personne was born in 1602 near the village of Roberval, in Oise, France. From a poor family, Gilles was taught mathematics by the town priest until he could become a tutor himself, which allowed him to travel around France meeting other mathematicians and furthering his knowledge. By 1628 he was settled in Paris, where he became part of a circle of mathematician that included Marin Marsenne, Etienne and Blaise Pascal, and had appended 'de Roberval' to his name. He eventually held two chairs in mathematics at the College Royale in Paris, which he kept until his death in 1675. Roberval was very active in all the current fields of study, including mechanics, geometry, calculus (area, volume, arc length, tangents), astronomy, though he published very little during his lifetime and did not get credit for many discoveries. He was a founding member of the Academie Royale des Sciences in 1666. He corresponded regularly with Torricelli, Fermat, and Descartes. In 1693 the Academie, in a publication of works by various members, included Roberval's *Traité des indivisibiles*. In this work, likely written around 1644, he develops general methods of integration for higher parabolas ($y = x^n$), and specifically shows the derivation for the parabola.

In Figure 7, the half-segment has vertex A, diameter $\overline{AC}$, and half-chord $\overline{CB}$. The segment of the curve is divided into infinitely many equal parts (shown as a finite partition in the diagram). As is typical of arguments involving indivisibles or infinitesimal parts, the argument

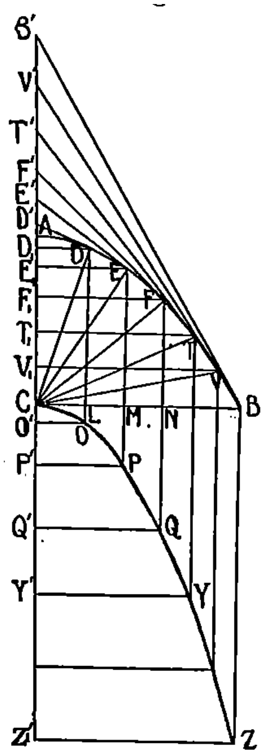

Figure 7: Roberval's Proof [14, p. 254]

is generalized or idealized from a finite division. A new curve from C to Z is constructed as follows: for example, by Property 2, the tangent line to the parabola at E intersects the diameter at E' such that $E_{\prime}E'$ is double $E_{\prime}A$. Then $\overline{EM}$ is extended to P a length equal to $E_{\prime}E'$. Thus Roberval concludes that the curve from C to Z is a parabola with vertex C and diameter $\overline{CZ'}$, and the area of region $Z'ZC$ will be double that of parabolic region CBA. Next he argues that, for example, each triangle $\triangle CFE$ is half of "parallelogram" $EPQF$, since $\overline{CP}$ and the tangent at E are parallel. Therefore region ACB is half of $ACZB$, or equal to region CZB. The required result comes from noting that $\triangle CBA$ is a fourth of parallelogram $Z'ZBC$. We note that while the final formula is correct, there are many assumptions made as a result of the infinite subdivision of the curve; this reflects a cultural move from a stress on rigorous proof to practical derivation.

4.6. Thomas Hobbes

Thomas Hobbes, born in 1588 in Westport, Wiltshire, England, is recognized more for his philosophy than his mathematics, but at one point he was considered among the best. However, his attempts to square the circle and arguments with other prominent mathematicians tarnished his reputation. Hobbes continued a long-standing argument with John Wallis over techniques, notation, and philosophy in mathematics. His attempts to reform mathematics by putting it on a footing in line with his philosophical ideas of human perception and motion make his mathematical writings hard to understand. A large part of his 1655 *De Corpore* is devoted to geometry, mechanics, and calculus, and a quadrature of the parabola, among other types of curves, is included there. By his death in 1679, his stubborn defense of obvious errors in his mathematics had given him the reputation of a crank. Nevertheless his interest and respect for rigor in mathematics served as a foundation for his very influential philosophical and political contributions.

4.7. John Wallis

John Wallis was born in 1616 at Ashford, Kent, England. He finished a BA in 1637 and a Master's degree in 1640 at Emanuel College, Cambridge, where he studied a range of classical topics, which included ethics, astronomy, and medicine, but not mathematics. He used mathematics as a pass time, and didn't pursue it seriously until he encountered Oughtred's *Clavis Mathematicae* in 1647, and started publishing his own mathematical explorations. He was a part of a group of scholars who discussed theoretical and practical science, and eventually became the Royal Society of London. Possibly with political motivation, but definitely through his increasing reputation, he was appointed to the Savilian Chair of geometry at Oxford in 1649, which he held until his death in 1703. He was en excellent expositor and was instrumental in clarifying and developing the algebraic notation and analytical methods that were forming the foundation for the discovery of the Calculus. His most famous work, *Arithmetica infinitorum* (*Arithmetic of infinitesimals* [15]), published in 1656, included results in infinite sums and products, limits, and a quadrature of the parabola, among many other things.

Showing that

$$\frac{0+1+4+9+\ldots+n^2}{n^2+n^2+n^2+n^2+\ldots+n^2}=\frac{1}{3}+\frac{1}{6n},$$

which converges to $\frac{1}{3}$ as n becomes infinite, he argues (using Property 3) that the same will hold for the proportional lengths of lines drawn, parallel to the diameter, in the region outside half of the parabola segment, as shown in Figure 8. He concludes "therefore the whole figure

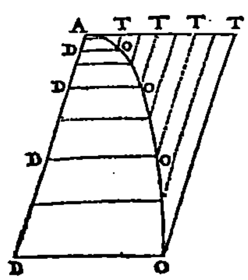

Figure 8: Lines in the Parallelogram [15, p. 28]

(consisting of an infinite number of straight lines) will be, to the parallelogram of equal height, as 3 to 1." [15, p. 28] While this assumes erroneously that a countably infinite number of lines fills a planar region, we can respect the progress made toward what later we would be called a Riemann sum.

4.8. Pierre de Fermat

Pierre de Fermat, born in 1601 in Beaumont-de-Lomagne, France, is best known for his work in number theory and probability, but he also worked in geometry and calculus (area, centroids, arc length, tangents). He was a distinguished lawyer in Toulouse, but also was in contact with the circle of mathematicians working in Paris at the time. His impatience in carefully writing down his work resulted in not much of his work being published - though much of his correspondence has survived. In his 1658 treatise *Method of Quadrature* [2], he demonstrated a method that could be used to integrate curves of the form $y^m = x^n$. He shows, as a particular example, how to apply results on sums of infinite geometric series to the quadrature of the parabola.

Fig. 143.

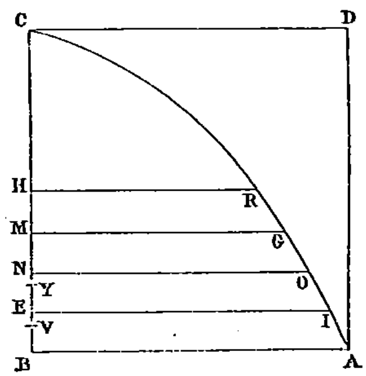

Figure 9: Fermat's Method [2, p. 5]

In Figure 9, $\overline{CB}$ is the diameter and $\overline{AB}$ the half-chord. He divides the diameter into infinitely many parts so that CB, CE, CN, CM, etc., form a geometric sequence. Then BE, EN, NM, etc., are also a geometric sequence with the same ratio. From Property 3, AB^2, IE^2, ON^2, etc., are also a geometric sequence with the same ratio. Fermat then shows that, if VC, YC, etc., are the geometric means for the first sequence then the geometric sequences of products $AB \cdot BE$, $IE \cdot EN$, etc., and $AB \cdot BY$, $AB \cdot YM$, etc., both have the same ratio, hence their sums will be in the same proportion as their first terms. Now, as the ratio that was originally used approaches 1, the ratio of the leading terms approach $\frac{2}{3}$, and the sums approach the area inside the parabola and of the whole parallelogram, respectively. Therefore, the area of the segment of the parabola is two-thirds the area of the corresponding parallelogram. We can see how this method can be adjusted, by changing the number of inserted geometric means, to find quadratures for higher parabolas and even rational powers.

4.9. Pietro Mengoli

Pietro Mengoli, born in Bologna, Italy, in 1626, was a student of Cavalieri at the University of Bologna and continued on the faculty there, eventually being appointed to Cavalieri's chair. His mathematical contributions, while not well known, were in infinite series, their applications, and methods of formally computing definite integrals, as demonstrated in his 1659 treatise *Geometria speciosae elementa*. This includes a quadrature of the parabola. Mengoli's work directly influenced Leibniz, and Newton through Wallis. He also was interested in astronomy and the theory of music. He died in 1686.

5. Post-calculus Musings

This brings us up to the 1670s and the development of Calculus methods. Explicit quadratures of the parabola would not be present in the major works of Isaac Newton and Gottfried Leibniz because by this time the parabola would be equated with an algebraic formula which had a well known antiderivative. The real test cases for the new Calculus were more challenging curves; nevertheless, there are specific parabola quadratures presented in the later literature, and some are worth comment.

5.1. Maria Agnesi

Maria Gaetana Agnesi was born in Milan, Italy, in 1718. From a wealthy family, she was tutored at home in classical languages, philosophy, mathematics and natural science. Her father often had her present essays in Latin, on many subjects, to audiences of visiting dignitaries and important friends of the family. In particular she was tutored in mathematics by Ramiro Rampanelli, a monk, who introduced her to calculus. In 1748 she published a Calculus textbook in Italian, *Instituzioni analitiche ad uso della gioventu italiana*, which was praised for its order, clarity, and precision in organizing and elaborating the different calculus methods and discoveries. As a result of the success of this book, Agenesi was offered a chair in Mathematics at the University of Bologna, but she never assumed the position. After her father's death in 1750, she devoted herself to religious study and charity work, dying in 1799 in a poorhouse.

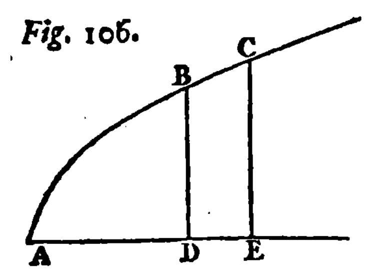

Figure 10: Agnesi's Calculus [1, p. 176]

Book III, Section III, Example 1 (#92) in her text (translated in 1760 as *Analytical Institutions* [1]), gives the quadrature of the parabola. Starting with the parabola $ax = yy$ or $y = \sqrt{ax}$, shown in Figure 10 with x representing AD and y for BD, and $\overline{EC}$ drawn parallel to, and infinitely close to, $\overline{DE}$. Writing $\dot{x}$ for infinitesimal length DE, the fluxion, region $DECB$, is approximately $\dot{x}y = \dot{x}\sqrt{ax}$. By integration we get $\frac{2}{3}x\sqrt{ax} + b$, which should be 0 when $x = 0$, hence $b = 0$. So the integral is $\frac{2}{3}x\sqrt{ax}$ or, substituting again, $\frac{2}{3}xy$. Therefore region ADB is two-thirds the rectangle formed by the abscissa and ordinate (x, y respectively). While the terminology is different, the argument and the mathematics are essentially what we see in a Calculus course today!

5.2. Percival Frost and the *Principia*

In [5], we read that Percival Frost was born at Kingston-on-Hull, England, in 1817. He studied mathematics at St. John's College, Cambridge, and later had lectureships at Jesus College and King's College, in addition to private tutoring. He was admitted as a fellow of the Royal Society of London in 1883, and died at Cambridge in 1898. He is best known for his textbooks, which include *A Treatise on Solid Geometry*, *An Elementary Treatise on Curve Tracing*, and *Newton's Principia, First Book Sections I, II, III, with notes and illustration and a collection of problems, principally intended as examples of Newton's methods.* This last, as the title suggests, contains many examples, among which is a quadrature of the parabola. Newton's Lemma IV in Book 1 Section I, states:

> If in two figures there be inscribed two series of parallelograms, the number in each series being the same, and if, when the breadths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other be the same, each to each, then the two figures will be to one another in the same ratio. [3, p. 32]

Frost applies this as follows. In Figure 11, $\overline{AB}$ is the diameter of a segment, and $\overline{CB}$ is the

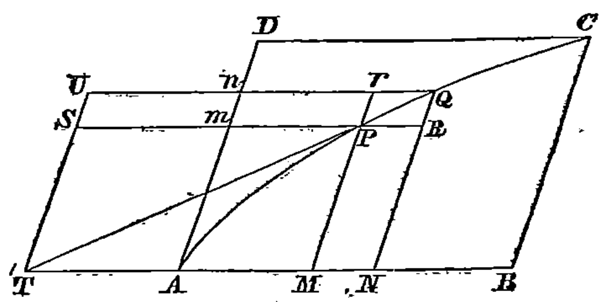

Figure 11: Inscribed Parallelograms [3, p. 35]

half-chord. For any points M, N, along $\overline{AB}$, the line through P, Q, the corresponding points on the parabola, meets $\overline{AB}$ at T. Since $\overline{QT}$ is close to tangent to the parabola at P, then, by Property 2, TM is close to twice AM. As $\overline{QT}$ is the diagonal of parallelograms $TNQU$ and $TMPS$, then parallelograms $MNRP$ and $SPrU$ have the same area. Thus $mPrn$ is close to half of $MNRP$. Sub-dividing $\overline{AB}$ into parts generates two series of parallelograms in regions ABC and ACD, to which Lemma IV can be applied, to conclude that their area will be in ratio 2 to 1, respectively, as expected.

5.3. Conclusion

After seeing so many different proofs we are amazed at the diversity, the ingenuity, and the persistence of mathematicians. We have found it interesting to trace the path from geometry to algebra, from finite to infinite, from quadrature to calculus through this simple problem.

In a Calculus class today, when asked to find the area between a line and a parabola, we typically expect to see a number as the answer; however, with a bit of algebra we can do it in some generality. The line $y = mx + b$ ($b > 0$) intersects the parabola $y = ax^2$ ($a > 0$) at $x_1 = \frac{m-\sqrt{m^2+4ab}}{2a}$, $x_2 = \frac{m+\sqrt{m^2+4ab}}{2a}$, and

$$\text{area} = \int_{x_1}^{x_2} (mx + b - ax^2)\,\mathrm{d}x = \frac{\sqrt{m^2+4ab}}{a}\left(\frac{m^2}{6a} + \frac{2b}{3}\right).$$

This last does not appear to make things clear until it is rewritten as

$$\text{area} = \frac{2}{3}\left(2\frac{\sqrt{m^2+4ab}}{2a}\right)\left(m\left(\frac{m}{2a}\right) + b - a\left(\frac{m}{2a}\right)^2\right)$$

and we recognize in the factors the width and height of the parallelogram circumscribed on the segment of the parabola. So this is still the same quadrature, just disguised by coordinatization and equation representation. A few more approaches to this quadrature can be found in [9], [11], and [13]. We are sure there are other mathematicians who have written or attempted their own quadrature of the parabola over the ages, and we would welcome references, to help complete the record.

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History of Mathematics: An Exercise in Strengths
Consistency–Harmany–Achiever–Intellection–Responsibility

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Mary B. Walkins (B.S., Lee University; M.S., Ph.D., Northeastern University) teaches general education courses, trains students for teacher licensure, and serves as a strengths vocational advisor. She volunteers at the local Learning Center for Adults and Families, and at long-term care facilities where her ministry involves songs, prayers, and encouraging words. Her motto is, "If I can help somebody as I pass along, then my labor shall not be in vain."

1. Introduction

As a leader in strengths-based education, Lee University encourages each new student, since fall 2003, to take the Gallup StrengthsFinder to determine their top 5 signature themes (out of a possible 34). At Lee, the syllabus for the History of Mathematics course calls for students to write a paper on a mathematician. In the fall 2009, as an added dimension, students were asked to critically think about and incorporate the strengths they believe that mathematician may have. Each student was required to compare and contrast his or her strengths with those of the mathematician. This was done with the hope that, as aspiring mathematicians, they may be inspired to persevere to make their mark in the history of mathematics, since math is still evolving. In this presentation, through an exercise in strengths, I share 3 examples of how students were inspired by each mathematician.

2. Example 1: Laura German on Euclid

Laura's top 5 strengths themes are: Discipline, Analytical, Deliberative, Significance, and Restorative. She was a Mathematics Education Major who graduated from Lee University in May 2011.

Laura felt there was something in Euclid's personality that pushed him toward his great achievements. Euclid would not strive so hard to accomplish so much without having such a truly amazing work ethic, which was probably developed through the strengths of arranger, discipline, significance, intellection, and learner.

His top strength was without a doubt arranger because of the immaculate way he was able to organize particularly the Elements, as well as all his other writings. Euclid figured out how the enormous amount of information that comprised his most prestigious work fit together to produce the "maximum productivity" of his book (Gallup, Inc.). He must have done a pretty great job too, because he controlled the teachings of geometry for more than 2200 years (Bell 299).

Not only was Euclid a wonderful organizer, he also had a great deal of discipline. There could not be any other way to explain his ability to focus on the same task for the amount of time it would have taken to complete a tome such as the Elements. Structure and a routine would have been a definite requirement (Gallup, Inc.). "This has been something I find myself needing as well, because without designed structure my plans fall apart and nothing gets finished."

Another equally important strong suit for Euclid would have been significance. Why else would he have worked so diligently on all his volumes? He had a burning desire to be remembered for his work (Gallup, Inc.), which "in a way I also share. I want to really make a difference, even if only in one person's life, but not necessarily by facts alone."

Two additional strengths, which were likely the bottom two, probably still had an effect on Euclid. They were intellection and learner. It would have been impossible to compose all the books he did had he not been able to understand the material. He was likely pensive and pleased with intelligent dialog. Moreover, the progression of obtaining knowledge was more exhilarating for him than simply producing an outcome (Gallup, Inc.). Euclid's strengths revealed him to be an exceptional scholar.

Laura concluded that Euclid was a complex man and much information could be inferred about him simply from all his written contributions. It was understood why he was important enough for the history books. His Elements alone would have been more than sufficient, but there would be such an inadequate amount of mathematical knowledge without him! He was responsible for organizing and helping preserve all the previous information recorded by those who came before him, but that was not enough for him. He even managed to contribute some of his own original ideas too. "It was inspiring to see just how much one person was able to do, especially one that shared a couple of my strengths!"

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3. Example 2: Carrie Ivester on Euler

Carrie's top 5 strengths themes are: Developer, Includer, Adaptability, Positivity, and Activator. She was a Chemistry Major and a Mathematics Minor who graduated from Lee University in May 2010. She now teaches in High School while pursuing a Masters in Education.

"I believe Euler and I share the strength of being an activator." Euler has written so many works and I think it is due to the strength of being an activator. Euler had to know that action is the best advice for learning and that is a characteristic for this strength. He put his name out there and followed his dream from such a young age. "Restorative, according to me, would be Euler's first strength." Anyone who has read about Euler would know that he loved to solve problems. Problems were Euler's gift even outside of mathematics; he helped the King in Prussia with problems dealing with the government. Euler had to have felt an adrenalin rush when it came to solving problems, because when reading about his life it seemed that all he was concerned with was solving problems. “Euler also without a doubt has the analytical strength.” This strength comes from all the problems he has solved like the Königsberg bridge problem. He digs deeply to find the root to the problems and determines the right questions. “Competition is also a strength I see in Euler.” He entered so many competitions and even though he was denied because he was young, it did not stop him from entering in the following year. No one could even compare Euler to another mathematician at the time except Bernoulli which Euler succeeded. This is due to Euler’s competitive spirit; he needed the Bernoulli family to get where he was going to follow his dream. Lastly, learner had to be one of his strengths, because plainly he loved to learn. Bradley states in his book, “One of his [Euler’s] most admirable qualities was a willingness to explain how he did mathematics, how he made discoveries” (22). He wanted others to learn and see what he had seen. At a young age he took all the classes he could and this was before he became interested in mathematics. Thereafter, he would read and be interested in just learning and the knowledge he could gain from other people. His whole life, seventy-six years, was all about learning and writing about all of what he learned. “Euler was clearly having fun, pursuing the game for its own enjoyment, and exhibiting a pervasive confidence that his quest would be successful” (Dunham xvi). Learning for Euler was not about receiving the prize or the best position; learning for Euler was about gaining and sharing knowledge and gaining more knowledge. “Although Euler did not take the StrengthsFinder Quest, I know without a doubt these would be his top five strengths.” Carrie concludes that “Euler is one of the greatest mathematicians, and without his works, I would not be able to explore the science of mathematics. My interest is in science and even though Euler was a mathematician, I have dealt with his works on a regular basis. The thought of not having ideas that Euler has observed or calculated is not imaginable.”

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4. Example 3: Michael Yokosuk on Cantor

Michael’s top 5 strengths themes are: Competition, Includer, Restorative, Learner, and Discipline. Michael is a Mathematics Major. He is still attending Lee University. “I believe that Georg Cantor had several strengths that were in his favor. I feel that he had several strengths that I also share. First, I feel as if Cantor was a type of “includer”.” Cantor often wanted to include people in his mathematical findings. He continually had correspondents to whom he would write about his innovations and papers. He had several correspondents, including Dedekind, Mittag-Leffler, and Jourdain.

“I believe that Cantor was also a ‘learner’.” He was a person who had a great desire to learn and wanted to continuously improve. Cantor was very accepting of new challenges. For example, his contemporaries Dirichlet, Lipschitz, Riemann, and Heine proposed a problem to Cantor. The problem was to prove or disprove the uniqueness of the representation of a function by trigonometric series. He solved this problem in 1869. He also worked for years on certain problems. He would often think he had the correct proof, but he would often times find an error or mistake in that proof the very next day. He also worked continuously on the Continuum Hypothesis, though he never found a proof for it.

"I also believe that Cantor had a "restorative" quality. I do believe this only applies to certain aspects of his life." He was given a problem and wouldn't stop working on it until he solved it. He also resolved many problems in his personal life, along with in his mathematical endeavors. At one time, he and Dedekind were good friends. They had a falling out when Dedekind rejected his offer of a job. Eventually, however, they rekindled their friendship. Cantor also had a stressful relationship with Kronecker. Kronecker rebuked Cantor's works on set theory. Cantor eventually invited Kronecker to a meeting, in which they settled their differences. However, other parts of his life were not represented by the restorative quality. He never truly could get over his troubles of depression, though he was later thought to be diagnosed with bi-polar disorder.

"As an aspiring mathematician, these three similarities between Cantor and me are very motivating. As an includer, I hope to surround myself with brilliant minds of our generation. By engaging in mathematical discussions and conversations, I can only become more interested in finding solutions. I am also a competitor, as I am an athlete. I always want to win. Therefore, I will try and find a solution to each problem presented before everyone else. As a learner, I would hope to find problems to solve and use my competitiveness to beat everyone else to their solutions. As a restorative person, I would look over my work many times in order to ensure clarity and correct work."

A Bayesian Secondary Analysis in an Asthma Study

Samuel P. Wilcock, PhD[2]; Vernon M. Chinchilli, PhD[3]; Stephen P. Peters, MD, PhD[4]

Sam Wilcock (B.A., Mathematics, Messiah College; M.S., Ph.D., Statistics, Virginia Tech) has been on the faculty at Messiah College since 2001. He loves teaching statistics, but has also taught general introductory courses. He enjoys the chance to do interdisciplinary consulting. In his spare time he enjoys playing volleyball and working with the worship team and youth at his church. He is also a life-long Philadelphia sports fan.

Abstract

A recent study published in the New England Journal of Medicine by the Asthma Clinical Research Network (ACRN) compared three different treatments for their effectiveness in treating adults with uncontrolled asthma.[5] This paper will describe the study design and its results, then detail the beginnings of a secondary analysis using Bayesian methods to estimate the parameters of interest. The methods will be explained, and the preliminary estimates given and contextualized. The paper will conclude with a discussion of the next steps and the goals for further analysis of the data in this study.

Keywords: Statistics, Bayesian Analysis, Asthma, Clinical Trials, Cross-over Design, Mixed Models

1. Acknowledgements

This work is the result of a sabbatical semester spent working with Vernon M. Chinchilli, PhD, at the Penn State Milton S. Hershey College of Medicine in the Department of Public Health Sciences, Division of Biostatistics. The clinical trial considered in this presentation was titled the Tiotropium Bromide as an Alternative to Increased Inhaled Glucocorticoid in Patients Inadequately Controlled on Lower Dose of Inhaled Corticosteroid (TALC). The TALC trial was funded by the National Heart, Lung, and Blood Institute Asthma Clinical Research Network (ACRN) and was led scientifically by Stephen P. Peters, MD, PhD. The background section of this talk is based on the paper *Tiotropium Bromide Step-Up Therapy for Adults with Uncontrolled Asthma* by Stephen P. Peters, et. al. (The New England Journal of Medicine 363;18, October 2010, pp. 1715–1726) I am grateful to Dr. Chinchilli for his collaboration on this project, and to Dr. Peters for his permission (and that of the ACRN steering committee) to share the results of this analysis in a professional setting.

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2. Introduction to the Setting

2.1. Background

The TALC study was motivated by observations about the state of treatment options available to doctors treating patients with moderate to severe asthma. Asthma, in these cases, is generally not well controlled by the use of an inhaled corticosteroid. One of the current treatments for patients whose asthma is poorly controlled is to use a long-acting beta-agonist (LABA). However, there is significant concern about the known side effects of this treatment.[6] This motivated doctors to look for alternative treatments for patients with uncontrolled asthma who have adverse side effects when using the LABA. The purpose of this study was to consider the use of tiotropium bromide, which is a long-acting anticholinergic agent approved for use in treating chronic obstructive pulmonary disorder (COPD).[7]

The study was designed as a three-way, double-blind, crossover trial. The three treatments were:

1. The "control" treatment of a double dose of the inhaled glucocorticoid
2. The current treatment of the LABA salmeterol in addition to the standard dose of inhaled glucocorticoid
3. The tiotropium bromide in addition to the standard dose of inhaled glucocorticoid

It should be noted that there was also a trial for mild-to-moderate asthma conducted at the same time, which we are not discussing here. Of the 826 patients which were enrolled in the common run-in, 625 were eligible for assignment to studies. Only 289 had severe enough asthma to be in the TALC study. 79 of these had complications which led to their exclusion before the randomization process for this trial, resulting in 210 of patients participating in the randomization for this study, with 174 of these completing the entire duration of the study. Data for each patient were used for whichever portion(s) of the trial they completed before dropping out of the study. (See Figure 1.[8]) The study began in July 2007 and concluded in May 2010. During the trial a dose of an inhaled glucocorticoid is used as a standard run-in to clear any previous treatment. At the beginning of the trial, a four week run-in was used. Then the same dosed is used for a two week run-out/run-in between treatment periods 1 and 2, a two week run-out/run-in between treatment periods 2 and 3, and a two week run-out after treatment period 3. Randomization is conducted into each of the six possible orderings of the three treatments. The design is called a cross-over trial since (roughly) half of the patients starting with any treatment (say the control) will follow this treatment with each of the other treatments. Then the "leftover" treatment is used for treatment period 3. This is illustrated in a simplified way (only three of the six orderings are shown) in Figure 2.[9] Estimates of the effect of a treatment for a particular patient can be calculated by subtracting the measurement at the beginning of a cycle (e.g. the end of the run-in: week 4/visit 3) from the measurement at the end of a cycle (e.g. Period 1: week 18/visit 6). This difference measures the change in the

[6]A list of the side effects can be found on the Mayo Clinic website at:
http://www.mayoclinic.com/health/drug-information/DR600459/DSECTION=side-effects

[7]A list of the side effects for this drug can also be found on the Mayo Clinic website at:
http://www.mayoclinic.com/health/drug-information/DR601347/DSECTION=side-effects

[8]Figure 1 from Peters, et.al., used with permission.

[9]Figure 2 from Peters, et.al., used with permission.

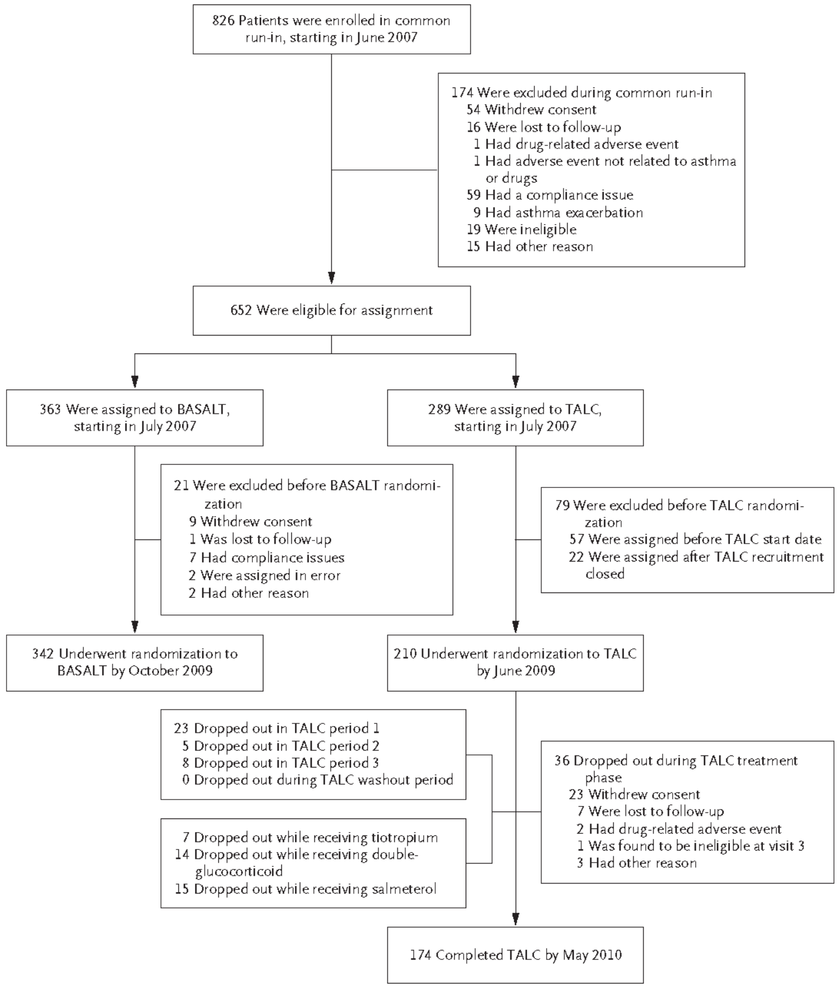

Figure 1: Enrollment and Outcome figures for the TALC and BASALT Trials

patient's condition over that period. For the parameters, we would look at a similar difference. Then we can subtract these values to find estimates of the differences between the treatments.

For reference and context, it may be helpful to note the following demographic/baseline summaries (where SD represents the usual standard deviation calculation):

- 32.9% of participants were male
- 28.1% of participants were black, 11.4% Hispanic
- Age at first visit: Mean 42.2 years, SD 12.3 years
- Duration of asthma: Mean 26.1 years, SD 14.1 years
- Weight at first visit: Mean 88.3 kg, SD 25.3 kg
- BMI at first visit: Mean 31.4, SD 8.8

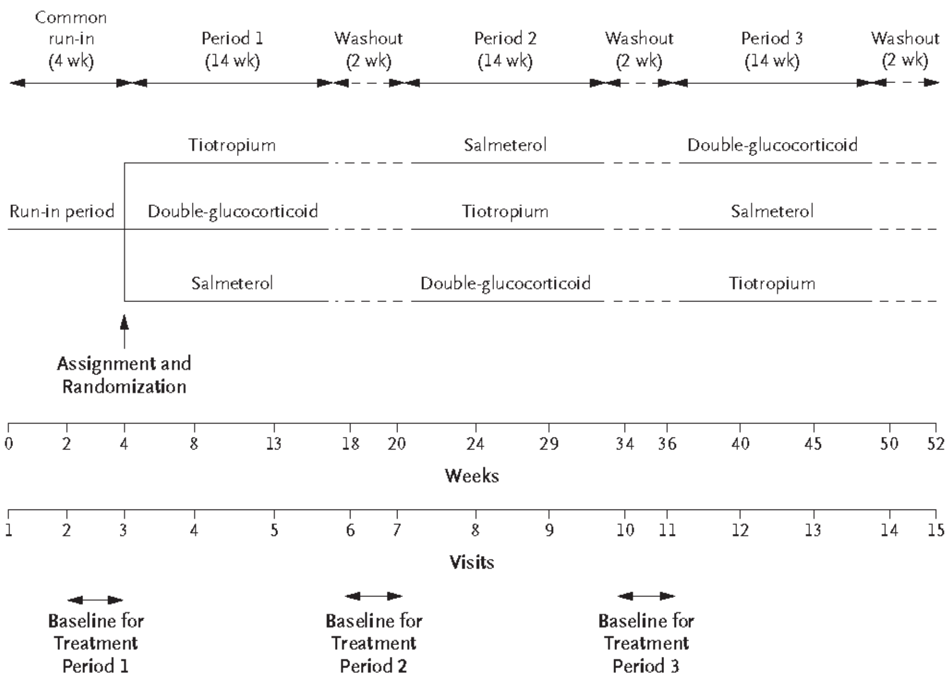

Figure 2: Treatment Schedule for Three of the Six Groups

Baselines were also taken for each of the key measures of interest in the original study. For the purposes of the secondary analysis presented in this paper, it will suffice to consider three main measurements:

- FEV_1: Forced expiratory volume in 1 second
- PEF: Peak expiratory flow (Measured morning and evening during the study)
- Asthma-control days: days without symptoms or use of the rescue inhaler (measured as a proportion)

Next we will look at the results of the primary analysis.

2.2. The Primary Analysis

The primary comparison of interest is to show that the tiotropium treatment is superior to the control. That is, the main purpose of the study is to provide a better alternative for patients whose reactions to the LABA would preclude its continued use. The estimates of the difference between tiotropium bromide and the control (tiotropium bromide minus control) that resulted from this analysis are summarized in Table 1 for the measures of interest to this report. We can note that in each of these cases, the analysis points to the superiority of the tiotropium bromide when compared to the control. Statistically, it can be concluded that for these variables tiotropium bromide is a better alternative than the control for patients whose asthma is poorly controlled, and for whom the LABA treatment is not available due to side effects.

Variable	Estimate of Difference	95% CI	P-value
Morning PEF	25.8 liters/min	(14.4,37.1)	$P < 0.001$
Evening PEF	35.3 liters/min	(24.6,46.0)	$P < 0.001$
FEV_1	0.10 liters	(0.03,0.17)	P=0.004
Asthma Control Days	0.079	(0.019,0.140)	P=0.01

Table 1: Summary of Results for Tiotropium vs. Control

Since tiotropium has now been shown to be superior to the control, a secondary analysis becomes logically interesting. The secondary comparison of interest is to show that the tiotropium treatment is non-inferior to the LABA treatment. If this statement can be shown to be true, then it may make sense for doctors to prescribe tiotropium bromide treatment instead of the LABA treatment, in light of its non-inferiority and preferable side effect profile. The estimates of the difference between tiotropium bromide and the LABA salmeterol (tiotropium bromide minus salmeterol) that resulted from this analysis are summarized in Table 2 for the measures of interest to this report. Note that the only clearly significant case (using $\alpha = 0.05$) is for the

Variable	Estimate of Difference	95% CI	P-value
Morning PEF	6.4 liters/min	(-4.8,17.5)	P=0.26
Evening PEF	10.6 liters/min	(-0.1,21.3)	P=0.05
FEV_1	0.11 liters	(0.04,0.18)	P=0.003
Asthma Control Days	-0.009	(-0.070,0.053)	P=0.78

Table 2: Summary of Results for Tiotropium vs. Salmeterol

variable FEV_1, and that the significance here actually favors the tiotropium treatment over the LABA salmeterol treatment. In the cases of the two PEF variables, while neither is statistically significant, we cannot use this to conclude equality. Still, it is helpful to note that if the failure to conclude a difference is in fact an error, it is most likely due to a difference in favor of tiotropium, rather than the reverse. This gives good reason for confidence in concluding that tiotropium is at worst non-inferior to salmeterol for these variables. The only case in which the data point toward favoring the LABA is the case of the proportion of asthma control days, and then the estimate is so nearly zero that it is also reasonable in this case to conclude that the tiotropium is non-inferior to salmeterol.

3. The Present Secondary Analysis

3.1. The Remaining Question

Since the initial analysis concluded that the tiotropium treatment was superior to the control, and pointed to the non-inferiority of the tiotropium treatment when compared to the LABA salmeterol. The question for the secondary analysis is to get a better handle on estimates of the differences between the treatments. It was decided to do a Bayesian analysis to gain these point and interval estimates, based on Gibbs sampling. This paper will detail the progress made to this point.

3.2. The Methodology

The methodology for obtaining the preliminary (or prior) estimates was based on a model from T.W. Anderson, *An Introduction to Multivariate Statistical Analysis* (2003). The model relies on certain assumptions about the nature of the setting, and the underlying distributions. We will need to specify prior distributions at this stage in order to obtain Bayesian estimates. These estimates will in turn be used as priors for the Bayesian Gibbs sampling technique (see the discussion below). To ensure that no biases or expectations from the researchers influence the outcome of the estimates, noninformative priors are used. The mixed effect model, in matrix form is $Y = X\beta + \mathrm{Diag}(Z)\gamma + \Sigma$, where β contains the fixed effects parameters in the model and γ contains the random effects parameters. Each of the parameter vectors, and the error structure (Σ) are mutually independent, and have known distributions.

The distributions are as follows:

$$\begin{aligned}
\beta_i &\sim MVN\left(\beta_{0,i}, \Lambda\right) \\
\gamma_i &\sim MVN\left(0, \Gamma\right) \\
\epsilon_i &\sim MVN\left(0, \Sigma\right)
\end{aligned} \tag{2}$$

where MVN implies a multivariate normal (or Gaussian) distribution is assumed, and $\beta_{0,i}$ is the mean of the prior distribution of the i^{th} element of the fixed effects parameter vector. Anderson shows that this results in the following conditional distribution:

$$Y \mid \beta \sim MVN\left(X\beta, Z\Gamma Z' + \Sigma\right). \tag{3}$$

Knowing this, we need the joint distribution of Y and β:

$$\begin{bmatrix} Y \\ \beta \end{bmatrix} \sim MVN\left(\begin{bmatrix} X\beta_0 \\ \beta_0 \end{bmatrix}, \begin{bmatrix} X\Lambda X' + Z\Gamma Z' + \Sigma & X\Lambda \\ \Lambda X' & \Lambda \end{bmatrix}\right) \tag{4}$$

Now, given the conditional distribution of Y given β and the joint distribution of Y and β we can develop the conditional distribution for $\beta \mid Y$. The result is a *MVN* posterior distribution for $\beta \mid Y$ with mean

$$\beta_0 + \Lambda X'\left(X\Lambda X' + Z\Gamma Z' + \Sigma\right)^{-1}\left(Y - X\beta_0\right) \tag{5}$$

and variance/covariance matrix

$$\Lambda - \Lambda X'\left(X\Lambda X' + Z\Gamma Z' + \Sigma\right)^{-1} X\Lambda. \tag{6}$$

To obtain values for the mean and variance/covariance matrix we need initial values for β_0 and each of the variance/covariance matrices $(\Lambda, \Gamma, \Sigma)$. For each of these, a noninformative prior was used. β_0 was set to the zero vector and each of the variance/covariance matrices was set to Diag(1000000). This diagonal variance/covariance matrix results in an initial assumption of independence, but the large diagonal values allows for the data to overwhelm this "assumption". This should result in estimates from this model that are primarily data-driven. Also, as discussed later, the estimates found via this process will be used as preliminary estimates for a process based on Gibbs-sampling. The point of the process is to obtain final estimates of the parameters and variance/covariance structure. This should result in an even further weakening of the importance of the initial non-informative priors.

In our case, the dimensions of the problem required a lot of computational time to do the matrix multiplication/inversion. β is a 98 by 1 vector, so Λ is 98 by 98. The dimension of β is determined by the 15 visits, crossed with the six possible orderings of the cross-over treatment structure. This accounts for 90 of the parameters. The other eight parameters come from the eight different centers which participated in the study. It is presumed that these centers are a potential source of variability. Because of this, it was decided to allow for this possible effect in the model, though we are not particularly interested in studying the significance of this, nor is there interest in obtaining an estimate of the size of this effect. For the random terms of the model, only the weekly visits are modeled, so γ is 15 by 1, and Γ is the associated 15 by 15 variance/covariance matrix. Σ, the variance/covariance matrix at the observation level, is square, with size based on the number of observations. For the FEV_1 measure, there were 2747 observations. For the AM PEF and the proportion of asthma control days there were 2600 observations. (This is due to the number of patients who responded, and varied since some measures were more consistently reported than others.)

The prior estimates discussed above, and the original data were read into SAS. The SAS IML module was used to perform the necessary matrix multiplication. As mentioned earlier, the computational time was significant, but there were no major issues with finding the inverses needed and computing the estimates of the mean and variance of β given Y (See Equations (5) and (6)).

3.3. Proviso About the Results to Date

This brings us to the results to date. It is important to remember when discussing the results shown here, that they are only preliminary. We will discuss the plans to improve upon these later in this paper. It is also important to note that no attempt has been made here to account for the variance/covariance matrices when obtaining these estimates. That is, no interval estimate is presented, and no measure of significance is implied or intended in the discussion. Based on the non-informative priors and large variances, it is likely that these estimates are rough at best. Interval estimates and statistical significance will be a matter of future investigation.

3.4. Results for FEV_1

We will first consider the updated parameter estimate for FEV_1. The estimated difference between tiotropium and the control is 0.67 liters, which points toward the superiority of the tiotropium treatment over the control. The estimated difference between the tiotropium treatment and the LABA salmeterol is 1.62 liters, which points toward the superiority of the tiotropium treatment over the LABA. The latter result, especially, is extremely surprising in its scale, though both results mirror the direction found in the original analyses. (See Tables 1 and 2 for the earlier results for comparison.)

3.5. Results for AM PEF

The second variable we will consider is the morning PEF. The estimated difference between tiotropium and the control is 7.97 liters/min, which points toward the superiority of the tiotropium treatment over the control. The estimated difference between the tiotropium treatment and the LABA is 13.47 liters/min, which points toward the superiority of the tiotropium treatment over the LABA. The latter result is surprising in its scale, though it is within the confidence bounds found in the original analysis. (Once again, see Tables 1 and 2 for the earlier results for comparison.)

3.6. Results for Proportion of Asthma Control Days

Finally, we will consider the proportion of asthma control days. The estimated difference between tiotropium and the control is 0.04, which points toward the superiority of the tiotropium treatment over the control. The estimated difference between the tiotropium treatment and the LABA is 0.04, which points toward the superiority of the tiotropium treatment over the LABA. These results both mirror the direction found in the original analysis, and are both within the confidence bounds of the original analysis. (Tables 1 and 2 contain the intervals for this variable, for reference.)

4. What Remains

As stated several times above, these results are only the first step in the process of obtaining fully Bayesian estimates (including intervals) for the parameters of interest (or linear combinations of them). The results we have discussed here will be used as prior distributions for a fully Bayesian analysis. The analysis will be completed using WinBUGS, a package of Bayesian methods built for the R platform which includes methods for mixed models, such as those we are considering here.

The end goal is to be able to use the results of this analysis to further inform recommendations to doctors in the treatment of asthma patients.

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What we can Learn from Process Theology:
Integrating Faith and Mathematics

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1. Introduction

In the inaugural issue of The *Journal of the Association of Christians in the Mathematical Sciences*, James Bradley, the founding editor, suggests fourteen areas that need to be addressed by Christian mathematicians who are serious about integrating their faith and their work. One of those areas is the topic of this paper. Bradley frames the question: "Some thinkers (perhaps influenced by process theology) have asserted the idea that God's creation is not a finished work but that he creates new mathematical objects through mathematicians. Is this idea theologically sound? Is it helpful for our understanding of mathematics?"[10] I copy Bradley's exact phrasing because I believe he poses the questions in the appropriate order: first determining the theological validity of the process movement and then secondarily examining its influence on an understanding of mathematics.

There are numerous examples of great thinkers attempting to harmonize mathematical advances with the canons of the historical Christian faith in an attempt to make Christianity relevant to modern, intellectual society. Process theology arose because its adherents believed it to be the best of such attempts. Upon close inspection, however, process theology can only be labeled as a departure from Christian Orthodoxy. Yet the process perspective still has something to offer for the construction of a framework within which a distinctly Christian perspective of mathematics might be developed. As contemporary Christian mathematicians wrestle with integrating their faith and their discipline, it is the contention of this paper that they will benefit greatly from studying process theology and, in particular, from critically examining the ways in which it departs from orthodoxy.

In the first section of this paper I will summarize the tenets of process theology and examine the deep interplay between this school of thought and developments in the field of mathematics. A brief introduction to Open theism and its relationship to process thought will be presented. By comparison, Open theism finds a much more prevalent place in contemporary scholarship and a greater number of adherents than process theology. In this sense it might seem a more relevant focal point for discussion in developing a philosophy of mathematics, however there

[10] James Bradley, "Letter from the Founding Editor," Journal of the ACMS, http://www.acmsonline.org/journal/letterfromeditor.htm (accessed April 4, 2010).

is no explicit connection between Open theism and the practice of mathematics.[11] Process theology on the other hand will be presented as a clear historical example of how theological foundations have significant impact on the practice of mathematics. It is my argument that any Christian mathematician who wishes to devote themselves to the serious integration of their faith in their work finds themselves in much the same position as the founder of process theology (and perhaps subject to similar temptations). In the next section, I will critique the tenets of process theology in light of scripture and the historical teaching of the Christian church. This critique will focus specifically on the doctrines of divine revelation, God as Trinity, and the person and work of Jesus Christ. From this analysis the paper will conclude that process theology radically departs from Christian orthodoxy and therefore its proposal for integrating orthodox Christian faith and mathematics cannot be accepted.

There are two specific ways in which the process attempt fails. Its first fault lies in the presuppositions that are brought to the task of integrating mathematics with Christian faith. And second, even if the process assumptions are granted, flaws remain in the implementation of those beliefs in both theological reflection and mathematical practice. The paper will close with several suggestions of how Christian mathematicians might refine the integration of their faith and their discipline in these areas where the process offering fails.

2. A Brief Overview of Process Thought and Theology

Process philosophy and theology, like all schools of thought, is not expressed uniformly by all of its adherents. While there do appear to be some core beliefs that are inherent in the label of "process," there are also many different variations and unique expressions of process thought present in the work of different "process" thinkers. This paper will focus on the doctrines of process faith as described in the work of Alfred North Whitehead and his contemporary disciples. The reasons for this are threefold. First, Whitehead is largely credited with the coining of the term "process philosophy" from his book *Process and Reality*.[12] Second, before he tackled more philosophical issues, Whitehead's primary claim to fame was his work in the field of mathematics, most notably his collaboration with Bertrand Russell on the *Principia Mathematica*. Third, as a nominal and nonconformist Christian, Whitehead also suggests some religious consequences of process thinking.[13] Let us briefly examine the tenets of process thought first as they are expressed in a general philosophical worldview and second, as they are applied to the doctrines of Christian theology.

[11] Perhaps one exception may be David J. Bartholomew, *God, Chance and Purpose*, (New York: Cambridge University Press, 2008). The thesis of this book is that chance is neither unreal (because of the sovereign plans of a Designer) nor non-existent (because of a deterministic God) but an integral part of God's creation. This view, written by an Open theist, is expounded, illustrated and defended by drawing on the resources of probability theory and numerous examples from the natural and social worlds.

[12] John B. Cobb, Jr., "Process Theology," Process and Faith Resources, http://www.processandfaith.org/resources/Cobb%20On%20Process%20Theology.shtml (accessed April 10, 2010).

[13] Andre Gounelle, "Process Theology," In *Encyclopedia of Christian Theology*, Volume 3, ed. Jean-Yves Lacoste (New York: Routledge, 2005), 1288.

3. Process Philosophy and General Theology

Though process philosophy rose to prominence during the middle of the twentieth century, its roots can be traced as far back as the ancient Greek philosopher Heraclitus.[14] Heraclitus suggested that the fundamental thing out of which everything is made is change. That is, everything is constantly in flux. Several of the key terms used in the philosophy of Heraclitus are found in contemporary process philosophy. Heraclitus famously made the observation that you cannot step in the same river twice. The reason is that the river is constantly changing, it is constantly *becoming*. The problem with this view is that it seems to go against our life experiences: there appears constancy. Heraclitus claimed that *Logos* (word, reason, language) is the principle of stability which gives the *appearance* of permanence. This concept of *becoming* forms the foundation for process philosophy. And as we will examine later, the stability of *Logos* plays an important role in the theological applications of process of thought.

In process philosophy, reality is characterized by becoming rather than metaphysics based on substance and essence.[15] The basic building blocks of reality are understood as *events* of extremely brief duration which reach fruition, or "a peak of satisfaction," and then perish, to be succeeded by other entities.[16] Reality is constantly developing and changing. Stability, inertia, and fixity are illusions—the world and every being in it constitute a flux, a continuous movement that changes ceaselessly.[17] Events are given primacy over substance, and therefore how we define who we are and the world in which we live is necessarily stated in terms of the affects of the culminated events. Every being is born of an interconnection of encounters and relationships, a network of conjunctions that give rise to persons and objects.[18] Cobb summarizes this philosophy in the outworking of Whitehead's thought:

> Whitehead is a radical empiricist who understands human experience as a unity of largely unconscious feelings of the body and its environment. Out of this unconscious physical experience, sensation and thought arise. Emotions, purposes, values, memories, and anticipations are more fundamental than sense experience and thought.... Each occasion of experience is an instance of the many becoming one and being increased by one. Whitehead cannot understand this process apart from something like unconscious purpose, an aim to be and to be as much as is possible under the circumstances.... Whitehead sees the ground or source of purpose, value, order, and novelty—and in human beings of moral and religious feeling—as divine. He calls it God.[19]

When the concept of God is introduced into this mode of thinking the theological ramifications of process thought begin to become a little clearer. Process theology describes itself as ultimately an effort to make sense of the basic Christian understanding that God is love.[20] Because of its convictions, in process thought divine love must mean being affected and changed

[14]Ronald H. Nash, "Introduction," In *Process Theology*, ed. Ronald H. Nash (Grand Rapids, MI: Baker Book House, 1987), ix.

[15]Paul S. Fides, "Process Theology," In *The Blackwell Encyclopedia of Modern Christian Thought*, ed. Alister E. McGrath (Oxford: Blackwell Publishers, 1993), 472.

[16]Ibid., 472-73.

[17]Gounelle, 1288.

[18]Ibid.

[19]Cobb, "Process Theology."

[20]Robert C. Mesle, *Process Theology, a Basic Introduction*, (St. Louis, MO: Chalice Press, 1993), 1.

by those who are loved.[21] Process theology conceives the world to be a social organism, an interdependent and interrelated whole, growing towards its satisfaction through a network of mutual influences, among which are the persuasive aims of God; in this process, God is affected by the world as well as affecting it.[22] A key tenet of process theology is that God's action in the world is persuasive and never coercive. Causation is a matter of influence.[23] God is viewed as the one keeping the rules of the process (the organizing principle of growth itself).[24] In the philosophy of Whitehead, God envisages all the possibilities there are for the world. These he sorts into values, graded by their relevance to any particular situation and from them he presents each actual entity with an initial aim as it sets forth on its path of growth towards its satisfaction.[25] However, the entity is free to accept, modify, or reject this divine influence. Through his interaction with the world, God is said to be affected and somehow caused by it. As Whitehead describes it, God absorbs the effects of worldly action and decision into himself, making him the "fellow sufferer who understands."[26] God persuades by his aims, understood by Whitehead to be the specific possibilities that will make for maximum satisfaction and beauty. And by his sufferings God offers himself to be prehended (another Whiteheadian term that can be best understood by the phrase "laid hold of" or "grasped") by the world.[27] All entities are influenced as they feel the effects of their decisions and actions upon God and as they feel his evaluation and harmonizing of these effects within the creative synthesis of his nature.[28]

Process theology concentrates on the nature of God's activity, redefining omnipotence in terms of persuasion. God carries on an action in the world through his capacity to persuade beings to listen to him and respond to his promptings—it is not possible for him to obligate them and he depends in part on their response and reaction.[29] Process theology redefines omniscience as God's perfect knowledge of both possibility and actuality without equating the two.[30] In other words, God knows all things that are actual at present and all things that are possible in the future, but he does not know which possibilities will become actuality.

4. Process Theology and Traditional Christian Doctrines

We have explained the basis for understanding God in process theology in only a broad theistic sense. Now we will examine the tenets of process theology as they attempt to interpret doctrines specific to Christianity through the relation-based worldview generated by Whitehead. From this point forward, the term "process theology" will be used to denote the specific applications of process thought to the *Christian* understanding of God. This paper will focus on the impact of process theology on three main doctrines which I believe are crucial to the Christian faith: the doctrine of Scripture as divine revelation, the doctrine of God as Trinity, and the doctrine of the person and work of Christ. It is important to recognize that a modification of any one of these doctrines influences our understanding of the other two. However, for the sake of

[21]Charles Hartshorne, *A Natural Theology for Our Time*, (La Salle, IL: Open Court, 1967), 75.
[22]Fides, 472.
[23]Ibid., 473.
[24]Ibid.
[25]Ibid.
[26]North Whitehead, *Process and Reality*, (New York: The Macmillan Company, 1929), 532.
[27]Fides, 474.
[28]Ibid.
[29]Gounelle, 1288.
[30]Fides, 474.

organization, each doctrine will be treated as independently as possible.

4.1. The Doctrine of Scripture as Revelation

Process theologians accord an important place to Scripture. However, just as the omnipotence of God is defined in terms of persuasion, the authority of Scripture is viewed as dialogical and persuasive rather than unilateral and coercive. That is, while affirming the Bible's role in fostering a genuine encounter with God and a meaningful struggle to discern God's will, they reject the notion that the Bible speaks directly and simplistically for God.[31] In process theology revelation takes place in concrete (non-supernatural) events and involves both divine activity as well as human reception. Revelation involves God's self-disclosure, but it is *not* the direction communication of propositional truth.[32]

As a radical empiricist, Whitehead believed that all knowledge is at its base participatory.[33] Applying this epistemology to Scripture yields the following result: if perception is participatory and valuational, the language that emerges from it will necessarily be incomplete and fragmentary.[34] Language then is limited and ambiguous in its description of God. Thus process thought accepts the notion that God is in some sense revealed through scripture, but it also insists on interpretation in which readers are given considerable freedom in the shaping of meaning.[35] The interpretive process must give weight to the life and world of the contemporary interpreters and communities of faith. By viewing the world as a process, an emphasis must be placed on the continual transformation of historical traditions. As Whitehead states:

> The inspiration of religion lies in the history of religion. By this I mean that it is to be found in the primary expressions of the intuitions of the finest types of religious lives. The sources of religious belief are always growing, though some supreme expressions may lie in the past. Records of theses sources are not formulae. They elicit in us an intuitive response which pierces beyond dogma.[36]

So then, for process theology, no one belief is absolutely necessary for Christians to hold. Christianity is (as everything else) a process, a socio-historical movement. Such a movement certainly requires beliefs about its origins, about its nature and mission, and about the world, but these beliefs change and develop from generation to generation. Process theologians then charge themselves with the task of helping to shape these beliefs in faithfulness to the movement's history *and* its new situation.[37]

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[31] Russell Pregeant, "Scripture and Revelation," In *Handbook of Process Theology*, ed. Jay McDaniel and Donna Bowman (St. Louis, MO: Chalice Press, 2006), 67.

[32] Ibid., 70.

[33] Ibid., 71.

[34] Ibid.

[35] Ibid., 72.

[36] Alfred North Whitehead, *Religion in the Making*, (New York: Fordham University Press, 1996), 144.

[37] John B. Cobb, Jr., "Jesus and Christ in Process Perspective," In *Handbook of Process Theology*, ed. Jay McDaniel and Donna Bowman (St. Louis: Chalice Press, 2006), 28.

4.2. The Doctrine of God as Trinity

The doctrine of Trinity has historically been defined as God existing as one essence in three persons. Whitehead's identification of "person" with "substance" led him to conclude that the Christian doctrine of trinity was really a crude tri-theism. Therefore the term "person," in process theology, was viewed in the abstract, as a mode of activity of a single concrete subject. This means that process theology really upholds a tri-unity of abstract and impersonal principles.[38] The subjectivity of Jesus cannot be identified with the second member of the Trinity and likewise, the subjectivity of the Holy Spirit cannot be identified with the third member of the Trinity.[39] These parts of the Trinity are viewed as abstract and impersonal principles within God's nature. While they may interact with the world through God's consequent nature, the process construction of Trinity is decidedly inferior to a dynamic society of three conscious, active, and loving persons.

So while there is no explicit attempt to develop a doctrine of Trinity in process theology, attempts have been made to interpret a process concept of God in terms of the triune symbol. The tradition of Whitehead has argued for a divine society of "primordial, consequent, and subjective natures."[40] As Fides states:

> The relational and social view of reality, its interpretation of substance in terms of event, and the way that subjects are constituted by their 'presence' in each other seem to make process thought and an apt partner for Trinitarian theology. However, while process thought certainly presents a unity and plurality within God's relationship to the world, it is difficult to find anything like inner mutual relationships between the different dimensions of God's own being.[41]

4.3. The Doctrine of the Person and Work of Christ

For process theology, God's being present or immanent in Jesus is a matter of historical fact, since God is believed to be immanent in *every* event whatsoever (a view we might refer to as panentheism).[42] The question then arises: how do process theologians see Jesus as a unique individual? To understand the uniqueness of Jesus in process theology we must first understand the picture of the spiritual life in process theology. People are said to embody the nature of God moment by moment, each to different extents. If in one moment they resist the possibilities that come to them, this is seen as hardening their hearts toward God, causing the possibilities in the next moment to be more limited. If on the other hand, moment by moment they respond to the fullest possibility God offers, new possibilities become open to them and they become more and more free—a person's positive response to God's call becomes increasingly spontaneous.[43] Jesus is unique in that he exhibits the *optimal response* to God's calling. He was given a unique mission and he responded with great faithfulness.

So then, what does it mean to say that Jesus was God incarnate? Process theologians respond by saying that the consequent, or interactive, aspect of God's nature was incarnate in Jesus.

[38] Bruce Demarest, "The Process Reduction of Jesus and the Trinity," In *Process Theology*, ed. Ronald H. Nash (Grand Rapids: Baker Book House, 1987), 82.

[39] Ibid.

[40] Fides, 474.

[41] Ibid.

[42] Cobb, "Jesus and Christ in Process Perspective," 29.

[43] Ibid., 31.

God's consequent nature is the way in which he both feels and is felt by the world.[44] As mentioned, in Greek thought, the *Logos* was the principle of reason embodied both in the human mind and in the order of the cosmos. Any process Christology must underline the free human response of Christ to the divine purpose (or *Logos*) as a way of understanding the nature of incarnation.[45] Process theologians have noted a remarkable affinity between a plausible account of how the "primordial" (or fixed, as opposed to the consequent or interactive aspect) nature of God functions in the world and what is said of the *Logos* at the beginning of John.[46] The divine that was present in Jesus is seen as also being present in all of us in the whole of creation. To process theologians it is quite natural to consider what Whitehead names as the primordial nature of God the same as what John refers to as the *Logos* (John 1:1, 14).[47] Process theology proposes that Jesus is an incarnation of God in many important ways, but also that God is like Jesus in equally important ways. God is the great companion, a fellow sufferer who understands, absorbing the world's sins and sufferings and who guides the world, not by violence or blind decree, but rather by love.[48] All of this is revealed in Jesus.

Another term that is important to define is title of "Christ." Process theology uses this term not to mean "Messiah," or "Anointed One," but rather "Divine Reality Incarnate." "Christ" is not a label unique to Jesus of Nazareth; it can be applied to any person or instance in which "creative transformation" occurs. A "creative transformation" is defined as a move to appreciate and appropriate the spiritual attainments of other religious leaders and communities (of all faiths). Christ is considered God's power of creative transformation. Because Jesus brought about and continues to bring about changes and because he mobilizes us for God's plan, he is the supreme Christ, but other people endowed with analogous powers and other Christ-like actions manifest themselves in the world, in several different religions.[49] A faith in Jesus that prompts such occurrences is seen as a transformative faith, and in such movements process theologians see Christ (that is God) at work in the world.[50]

A final point needs to be mentioned on the process understanding of evil. I group this under the doctrine of Christ because in traditional Christian theology, an understanding of human sin and spiritual lostness is vital for appreciating the salvation that is found through Christ and his work on the cross. Process theology denies the traditional doctrine of original sin—that due to the failure of Adam and Eve, the first humans, all of humanity has inherited a corrupt and sinful nature. Process theology cannot follow this view because process adherents believe that all humans are part of a great evolutionary process, and that God creates in and through this process.[51] The natural evils that result in the world are not seen as the rebellion of the created order resulting from the rebellion of the first humanity, but rather as simply necessary violence that is inherent within any evolutionary system. The moral aspect of sin is described as the refusal of love from and to God and from and to neighbor and even from and to oneself.[52] Because process thought views the world as an interdependent whole, one must talk

[44] Ibid., 35.
[45] Fides, 474.
[46] , "Jesus and Christ in Process Perspective," 30.
[47] Ibid.
[48] Ibid., 36.
[49] Gounelle, 1288.
[50] Cobb, "Jesus and Christ in Process Perspective," 33.
[51] Marjorie Hewitt Suchoki, "What is Process Theology?," Process and Faith, http://www.processandfaith.org/publications/RedBook/What%20Is%20Process%20Theology.pdf (accessed April 10, 2010).
[52] Ibid.

about communal as well as individual sin. Process adherents believe that God is always calling humanity toward the good, and when we fail to heed God's call, we fail to contribute as best we can to the commonwealth of all. This failure is sin and it has ill effects that spiral beyond its origins in the interdependent world, *including God himself* who is in the world relating to it.[53]

The crucifixion of Jesus in process theology is then *not* about a substitutionary atonement for sin. To the extent that process theologians view unnecessary violence as sin, violence cannot be that which saves us from sin.[54] Process theologians prefer to believe that Jesus reveals who God is to us and for us. The cross does not represent vicarious sacrifice, but the revelation that God is with us even in our deepest pain.[55] Jesus reveals that the sins of all humans affect God. At the crucifixion Jesus died *because* of sin, which is different than saying that Jesus died *for* sin.

5. A Word on Open Theism

At this point it may be instructive to discuss briefly the relationship between process theology and Open theism. Open theists deny that the view of God they present is shaped by the principles of process philosophy. However, to the extent that process theology is used broadly to describe any theology that emphasizes an active, ongoing, and dynamic relationship between 'God' and creation, yet limits God in one way or another, the difference between process theology and open theism seems to be blurred. Technically, however, process philosophy as pioneered by Alfred North Whitehead, employs an empirical (naturalistic) methodology. From this perspective, process theology may be distinguished from open theism. Whereas process theology tends to deny the value and authority of scripture, Open theism's rallying cry is that they above all else derive their theology solely from the Bible. Open theists believe that traditional Christian beliefs were determined by an unnecessary influence of Greek philosophy, rather than a straightforward reading of the divine text. Open theists then view doctrines such as divine immutability as being inherently flawed (akin to process theologians) in the face of passages such as Gen. 6:5–7, 2 Kings 20:1–6, and Jonah 3, which describe God as changing his mind. The goal of Open theism is to remove these Greek philosophical biases from the Christian faith. However in reality, the choice between the traditional and open views of God hinges not upon *whether* one reads scripture in light of philosophical assumptions but rather upon *which* philosophical assumptions one employs in reading scripture.[56]

6. Summary of Process Theology

Before moving on to then next section, I wish to briefly summarize the preceding one. Process thought emphasizes becoming over being, events over substance, and relationships over essence. God and the world are an evolving process. When this line of thought is applied to basic doctrines of the Christian faith, the result is an extreme departure from the historical teaching of the church. The Bible is no longer seen as truth and authoritative, it is rather a historical religious account that we can learn from. God is no longer presented as Trinity; at best he

[53] Ibid.
[54] Ibid.
[55] Ibid.
[56] Doug Blount, "TOGAS, TULIPS, and the Philosophy of Openness," *Southwestern Journal of Theology* 47, no. 2 (Spring 2005): 177–189.

is presented as duality: having a nature which effects change and a nature which experiences change. This God is no longer immutable, impassable, omnipotent, or omniscient. Christ is no longer fully God and fully man, but rather a man who demonstrated ideally what it is like to be in tune with the desires of God. His crucifixion and resurrection only have salvific effects in that they demonstrate how much God is with us in our sufferings and how he will always provide a way out. A fuller critique of these process modifications of Christian theology along with the repercussions of these conclusions will be offered below, prior to developing a proper Christian understanding of mathematics. Let us now examine the relationship of process theology to the field of mathematics in order to demonstrate that these questions of faith have serious implications for scholarly mathematical endeavor.

7. Process Theology and a Philosophy of Mathematics

As mentioned in the opening of this paper, Whitehead's initial claim to prominence came in the field of mathematics. The most notable of his achievements in this field was his work with Bertrand Russell on the *Principia Mathematica*. In this work Whitehead and Russell sought a unifying theory of mathematics based on logic and arithmetic—that is, they sought to secure all mathematical truths from a few assumptions of logic. Kurt Gödel's Incompleteness Theorem was a damaging blow to Whitehead and Russell's undertaking because it brought to light the limitations of both logic and arithmetic.[57] Gödel's theorem states that there is no set of consistent axioms, finite or infinite, from which all the true theorems of arithmetic can be derived. Gödel revealed that no rational system, or well-defined procedure, can ever present all truth, for then it would have to generate all the truths of arithmetic.[58] When translated to religion, this implies that there is no rational theology or philosophy by which we can understand the full truth about God or any other matter.[59]

In order to overcome the problem of the lack of a single all-encompassing formal system and to preserve the close relationship of mathematics and *Logos*, as well as the idea that God establishes and guarantees the unity of *Logos*, process theologians look for solutions in terms of "plurality and increased potentiality."[60] As stated by Henry: "It may be appropriate to understand the realm (however understood) of mathematical relationships and hence of potential relationships as evolving—a rather radical departure from modal western philosophy or at least from its Christian adaptation."[61]

Any traditionally conceived understanding of God has as a consequence, by and large, a platonic understanding of mathematics (mathematical structures and relationships exist independently of man's construction of them and are there existing in some way or some form to be discovered)—if nothing else than because of the assumption that God knows and understands mathematical relations, thereby giving them some kind of existence independent of

[57]Teun Koetsier and Luc Bergmans, "Introduction," In *Mathematics and the Divine: A Historical Study*, ed. Teun Koetsier and Luc Bergmans (Boston, MA: Elsevier, 2005), 36.

[58]Granville C. Henry, Jr., *Christianity and the Images of Science*, (Macon, GA: Smyth and Helwys Publishing, Inc., 1998), 164.

[59]Ibod., 165.

[60]Koetsier and Bergmans, 36.

[61]Granville C. Henry, Jr., LOGOS: Mathematics and Christian Theology, (Cranbury, NJ: Associated University Presses, Inc., 1976), 112.

man's creation.[62] It has been proposed by process thinkers that this understanding of standard mathematics conditioned the doctrine of God's immutability. Therefore the interpretation of contemporary nonstandard mathematics (Gödel's theorem points out that the structures of knowing cannot all be formalized mathematically) relaxes any restrictions, at least from mathematics itself, of requiring God to be strictly immutable.[63] Process thinkers maintain that platonic mathematical structures do not change, existing in the primordial nature of God, as affirmed by Whitehead, but they relax the requirement that no new potentials or structures be added to the realm of these eternal objects.[64] This relaxation is based on the observation that it has been primarily the axiomatic method that has given mathematicians and philosophers the authority for stabilizing the mathematical realm—for claiming it to be complete as related logically to a few unquestionable assumptions. But the axiomatic method cannot adequately characterize the nature of mathematical structures that are presently known. As Henry states elsewhere:

> The new developments in mathematics seem to me to allow a better understanding of what it might mean for God to have the freedom to change the totality of potentials—both in terms of the structure of knowing and among human consciousness and in terms of objects known. This would mean that not only could man's consciousness, as well as other structures of the world, evolve in ways hitherto unknown, and in ways impossible to know, but in ways that might be even a surprise to God—a surprise in the sense that the potential mathematical structure that could characterize (in part) such consciousness might not even *be* at present.[65]

Process theology, as developed by Whitehead and others, is therefore at least partly to be understood as an attempt at taking into account the new insights that resulted from Gödel's publication.[66] In *Process and Reality*, Whitehead expresses not only his mathematical and scientific interests, but combines them religious, aesthetic and metaphysical ones to present a unified cosmology.[67] To the proponents of process theology it is the most important development in Christian thought since the first century. It is significant, they think, because the movement gives sophisticated moderns an intellectually and emotionally satisfying reinterpretation of Christianity that is compatible with late-twentieth century ways of thinking.[68] One major attraction of Whitehead's thought is that it seems to offer a way in which the disparate branches of modern learning can be reintegrated, such as the areas of science and religion (though as we will point out shortly, this may not be the best way to view these branches in the first place).[69] Whitehead's work was closely related to mathematical physics, and it offered an integration of the findings of the sciences with the evidence of religious experience that had come to seem almost impossible.[70] Process theology was (and still is) seen as a systematic philosophical perspective that can solve problems in science and theology and relate them in an integrated manner. As described by Henry:

[62] Granville C. Henry, Jr., *"Nonstandard Mathematics and a Doctrine of God," Process Studies* 3, no. 1 (Spring 1973): 9.

[63] Ibid., 14.

[64] Ibid., 12.

[65] Ibid., 11.

[66] Koetsier and Bergmans, 36.

[67] Cobb and Griffin, 163.

[68] Nash, ix.

[69] Cobb and Griffin, 173.

[70] Cobb, "Process Theology."

> The source of science is the real world that exists objectively on its own independently of our observation. We are part of this world and experience it. Science is understood in terms of its abstract, normally mathematical, structure which we create. Because this abstract structure is necessarily a partial description of the real world by virtue of the limitative theorems, theoretical science changes because of our refined experience in the world. We discover new things about the world that challenge old theory and precipitate new theory.
>
> The source of religion is God, who exists objectively and independently of our observation. The religions of Judaism and Christianity claim that God reveals God's self to humans and that humans experience God. A deposit of this revelation and human experience is contained in the Bible, which does not have mathematical or theoretical structure. It has historical, narrative, and mythological nature. Like scientific theory, classical theology which is an attempt to describe God by abstract logical structure, is also created by human invention. Although it seeks loyalty to the biblical revelation, it is necessarily only a partial description of any divine reality because of the limitative theorems. Theology changes as religious experience becomes richer or different.[71]

The work of Gödel is key in understanding the development of Scripture and revelation in process thought. The revelation contained in the Bible is seen as being necessarily incomplete. If the Bible is a systematic structure and it is inerrant, then it cannot be general.[72] In other words, if the Bible is viewed as the set of axioms, then there are true theorems that cannot be arrived at by way of those axioms. There are truths to which the Bible offers no insight. Rather than admitting this point, process theologians prefer to claim that the Bible is not a system but rather a window to the primary events of God's causal and historical encounter with humankind.[73] Once the doctrine of Scripture has been modified, it is not hard to see how the modified doctrines of God, Christ, Sin, and Salvation follow.

8. A Brief Critique of Process Theology

Many critiques have been leveled against process theology. Much of this work has focused on the process theologians' redefinition of divine omniscience. This approach would seem in keeping with the question that was posed at the beginning of this paper and much of the material presented so far. The question, "Do mathematicians create new mathematical objects that are surprising to God?" seems to be rooted in the discussion of God's knowledge. Extensive work has been done in the area of divine foreknowledge and responses to the tension that exists between God's omniscience, omnipotence and human freedom have been given that keep orthodox Christians from necessarily making a move to process theology. One such option is to recognize God's existence outside of time, therefore to speak of his future knowledge is a very different thing than to speak of human future knowledge.[74]

[71] Henry, *Images*, 216.
[72] Ibid., 166.
[73] Ibid.
[74] Jonathan L. Kvanvig, *The Possibility of an All-Knowing God*, (New York: St. Martins Press, 1986), 166. See also Doug Blount, "On the Incarnation of a Timeless God," in *God and Time: Essays on the Divine Nature*, eds. Gregory E. Ganssle and David M. Woodruff (New York: Oxford University Press, 2002).

It is my contention, however, that the conversation need not go that far for the purposes of this paper. In other words it is not simply the fact that viable, orthodox, philosophical arguments exist that keep us from accepting process theology and a process perspective of Christian mathematics. Rather it is because process theology greatly damages foundational Christian doctrines that it cannot be accepted. If we are to maintain an orthodox faith and practice the discipline of mathematics in a distinctly Christian way there are core tenets that must be maintained. Philosophical and mathematical results that develop from (or necessarily involve) the forfeiture of any of these tenets cannot be accepted. That is not to say that they cannot be discussed, so long as that discussion brings us back to the core beliefs that Scripture is authoritative, God is Trinity, and Jesus as the God-man was crucified and resurrected as atonement for our sins. To be clear that process theology simply does give us a viable option in these doctrines, the implications of process thought in each will now be briefly discussed.

8.1. Revelation and the Trinity

As mentioned above, process thought accepts the notion that God is revealed through scripture in some sense, but because of the evolving nature of any tradition, process theology allows for considerable freedom on the part of the interpreter. In process theology, reference to the historical Jesus or the apostolic tradition is only one way that a critical element can be brought to bear on the question of the authority of Scripture.[75] Rather than viewing the Bible through this traditionally orthodox lens, process theologians prefer to approach the Scripture from the viewpoint of scientific modernity.[76] This excludes a belief in miracles; supernatural intrusions by God into the natural order. This is discarded for a trust in the evidence of science that all events have a natural sequence of cause and effect in the evolutionary process. The process school comes to the theological task with a low view of scripture—modern study precludes accepting all that is written therein as true. If God does not supernaturally intrude from without the natural order, divine revelation is by definition excluded. With the elimination of divine revelation, process theology takes its stand as a natural theology shaped from human materials alone.[77] This runs contradictory to 2 Timothy 3:16: "All Scripture is God-breathed and is useful for teaching, rebuking, correcting and training in righteousness." Once the authority of Scripture is discarded, the door is open to modify any other doctrine in a way we see fit. That a doctrine appears in the Bible or that it was faithfully upheld by the historic Christian church provides no basis for its acceptance by modern, empirically minded process thinkers.[78] Because of the process belief that entities are constituted by relatedness to other entities or persons, and thus their repudiation of a substantial ontology, the creedal formations of Nicaea and Chalcedon are no longer relevant.[79] These formulations were worked out within a world view that is alien to the modern mindset. This conclusion is unacceptable to historically orthodox Christians.

Once the Bible and the creeds have been discarded, the doctrine of God as Trinity is easily modified. In the historical Christian faith God exists as one essence in three persons. This can perhaps best be seen in Ephesians 1:3–14 where each person of the Godhead is individually

[75] Pregeant, 72.

[76] Bruce Demarest, "The Process Reduction of Jesus and the Trinity," In *Process Theology*, ed. Ronald H. Nash (Grand Rapids: Baker Book House, 1987), 64.

[77] Ibid.

[78] Ibid., 65.

[79] Ibid., 64–65.

described and praised, and at the same time the interrelatedness of the three is beautifully stated and the uniqueness of a single divinity is clear. The Christian doctrine of trinity, viewed as three related persons in one substantial unity, provides demonstration of the classical philosophical problems of the relation of the one to the many and of being to becoming.[80] It is unacceptable for Christians to dismantle the traditional view of Trinity in order to philosophically resolve these lingering tensions.

8.2. The Person and Work of Christ

Following from the process understanding of Trinity, we find a misguided approach to understanding the pre-existence of the second person of the Trinity, God the Son, Jesus Christ. Process theology regards the *Logos* as an impersonal principle—the totality of the divine aims. Since the Whiteheadian *Logos* is not a discrete person within the Godhead, it hardly can be reconciled with the historic Christian explication of the eternal pre-existence of the second person of the Trinity (Isa 9:6; John 1:1–2; Rev 21:6).[81]

The process understanding of the incarnation is also unacceptable. Process theology views the *Logos* being immanent as the Christ in the whole of creation. The incarnation connotes that the impersonal *Logos*, the power for creative transformation, was maximally immanent and operative in the man Jesus of Nazareth. This denies that the incarnation involves the eternal second person of the godhead entering space and time and becoming man for us and for our salvation (John 1:1, 14; 2 Cor 8:9; Phil 2:6–8; 1 Tim 3:16). The process model compromises the decisiveness and singularity of the incarnation by affirming that the *Logos* is immanent in all entities.[82] The process claim that two natures cannot relate except by displacement prompts Whiteheadians to insist that the orthodox belief in Jesus' deity necessarily vitiates his authentic humanity.[83] Orthodox Christians cannot accept such a heretical, docetic Christology. John makes the results of this view clear when he states: "For many deceivers have gone out into the world, people who do not confess Jesus as Christ coming in the flesh. This person is the deceiver and the antichrist!" (2 John 7). The full humanity of Christ must be maintained (Matt 13:55; John 1:14, 19:5; 1 Tim 2:5; Heb 2:14).

The process belief that Jesus was simply the supreme example of responding to God's calling moves in the other direction and seems to deny his complete deity. At best the process view can be considered on par with the heresies of Adoptionism (the spirit of God came upon the human Jesus at some point in his life) and Arianism (the Son of God was the first created being). Jesus is both fully man and fully God (John 1:1, 18; 10:30–133; 20:28; Rom 1:3–4; 9:5; 1 Cor 15:45–49; Phil 2:6–8; Titus 2:13; Peter 1:1). Scripture accords Jesus the same attributes as deity. Jesus is omnipotent (Isa 9:6; Matt 28:18; John 10:18), omnipresent (Matt 18:20; Eph 1:23), omniscient (Matt 9:4; John 4:16–19; 16:30; 21:17), and eternal (Isa 9:6; John 1:1; 8:58; Col 1:17; Heb 1:10-12; Rev 1:8). Belief in the divinity of Christ is a prerequisite of salvation (Rom 10:9; 2 Peter 1:3). If Jesus is not God, then he does not have the power to fully reveal the Father, and he does not have the power to save sinners. Soteriology demands that he be both true god and true man in order to redeem (1 Tim 2:5). He must be man to represent us

[80] Ibid.
[81] Ibid., 78.
[82] Ibid.
[83] Ibid., 79.

(Heb 4:15) and he must be God to save us (Heb 7:24–25).

Ultimately, we as Christians must confess the mystery of the hypostatic union (1 Tim 3:16). We as Christians are also called to do one thing that process theology does not allow for, and that is the worship of Jesus Christ (Matt 2:2, 11; 14:33; Phil 2:10–11; Heb 1:6). Whiteheadians commonly depreciate the unique character of Jesus' person and accomplishments by upholding a form of degree Christology: Jesus was a special man who may well be surpassed by another religious figure in the evolutionary future.[84] However, the Bible makes it clear that Christ is the consummation of all previous revelations in history (Heb 1:1–2) and is the final and unique agent of salvation (1 Cor 3:11; John 14:6).

As discussed above, Process theology has what we can label as a Pelagian rejection of human sinfulness and rebellion, both individual and social. Process theology sees the cross as the ultimate negative moment, diverging from the Biblical claim that in the cross there is victory (Col 2:14-15). Contrary to the self-salvation of process theology in which a person responds to the loving lures of God, Scripture shows that it is God's provision that saves. On the cross in Christ, God bore the just penalty for the world's sin, satisfied his justice, and thus made a way for reconciliation (Isa 53:4–12; John 3:15–17; Rom 3:21–26; 5:6–11; Heb 2:14–17). The process view does not adequately asses the profound depths of human perversity and therefore it follows that the full meaning of the cross and resurrection as events which deal with sin and death is not grasped.[85]

Process theology summarily rejects the personal and bodily resurrection of Jesus and believers in favor of the thesis that resurrection connotes God "taking up into his own memory the experiences of our Lord and his followers."[86] However, the Biblical account clearly ties the Christian faith to the hope of the resurrection (Job 19:25; Isa 25:8; Matt 16:21; 20:19; 26:32; John 2:19; 11:25). Resurrection was the focus of the church's missionary preaching, teaching, and worship (Acts 2:24, 31–32; 3:15; 4:10; 5:30–32; Rom 1:4; 6:4; 8:34; 1 Cor 15:4, 20).[87] Denying the resurrection leads to denying the remission of sins (1 Cor 15:17), the possibility of attaining salvation (Rom 10:9; 1 Cor 15:19), and ending all hope (1 Cor 15:32).

9. Developing a Christian Philosophy of Mathematics

Process theology thus denies, as biblically and historically understood, Christ's eternal pre-existence, incarnation, virgin birth, sinlessness, deity, atoning death, resurrection, ascension, and second coming, as well as the Trinity of God.[88] Process theology then does simply fall into a certain heretical category in its doctrine of Scripture, God, or Christ, rather it samples from *many* heretical beliefs. The claims of process theology cannot be entertained by the faithful Christian community. The philosophical assumption of process thinking is that reason working on the data of lived experience is judged competent to lead the mind into all truth.[89] This by definition is the very root of sin (Gen 3:6; Rom 1:18–32) and Christians cannot proceed with

[84]Ibid., 80.
[85]Ibid., 81.
[86]Ibid.
[87]Ibid.
[88]Ibid., 83.
[89]Ibid., 84.

this underlying presupposition. Rather, we are called to submit to the authority of God and his revelation. It is because of this that I have attempted to give the preceding analysis with an emphasis on the voice of the Scriptures, even though process theologians might not accept biblical authority. The purpose here is not to argue process thinkers out of their position, but rather to demonstrate the approach we must take in analyzing any subject (even mathematics) if our presuppositions are to be labeled as "Christian."

Faith and understanding are not one in the same, though they are often taught as if they are. Understanding is not where redemptive value lies. Preaching a true Christian gospel is scandalous to man. We are never predisposed to take God at his Word. It is our nature to be skeptical about his Word (again, Gen 3). Scholarship since the Enlightenment has searched for knowledge to satisfy reason and understanding. But faith calls us to receive and accept what God has said even if we don't get it emotionally or intellectually. The Spirit of God can make the doctrine of the cross easy to understand—but not for natural man. It is a supernatural action—but process theology rejects this notion.

Process theology's major attack on Scripture was shown to derive from the work of Gödel. If we accept this premise as Christian mathematicians, we need not draw the same conclusions as process thinkers. As Matsumoto writes:

> In light of Gödel's Incompleteness Theorem, one must accept the fact that there may be certain propositions that are undecidable. In systematic and biblical theology, the set of axioms is the biblical texts. If one accepts the infallibility of the Scripture, as most evangelical Christians hold, the original texts are the axioms, from which one is to draw theological and practical conclusions. However, we all know that the Bible addresses neither every practical issue that we face nor every theological issue we ponder. Hence, these things on which the Bible is silent should perhaps be considered undecidable propositions in this axiomatic system.[90]

While process theologians may be correct in asserting that the Bible does not concern itself with all matters of truth, this does not mean that we can reject the truth that is does contain simply arguing that it is incomplete. A more appropriate response is Christian humility to the unknowable aspects of God. Even a process thinker and mathematician such as Henry allows for this point:

> Our evidence historically, certainly in terms of what we know, is almost exclusively of a changing domain of mathematical structures, a domain that changes primarily by addition to itself. It may be claimed that this is simply the growth of knowledge of a fixed domain... but it may be the case that there is an actual ontological addition to mathematical structures.[91]

Notice that Matsumoto's point hinges on the presupposition that the Bible is infallible. Henry's point hinges on a presupposition that the field of mathematics is changing and evolving, rather than our finite minds are simply growing in knowledge knowing that it cannot be perfected. A Christian approach to mathematics necessarily involves distinctly Christian presuppositions. Scripture can, and should, be used in developing a Christian understanding of mathematics, so

[90] Saburo Matsumoto, "Call for a Non-Euclidean, Post-Cantorian Theology," *Journal of the ACMS* (2006): http://www.acmsonline.org/journal/2006/Matsumoto.pdf (accessed March 27, 2010).

[91] Henry, "Nonstandard Mathematics," 11.

long as it is used appropriately. As Timothy states: "Every scripture is inspired by God and useful for teaching, for reproof, for correction, and for training in righteousness, that the person dedicated to God may be capable and equipped for every good work" (1 Timothy 3:16–17). If the work we undertake (be it philosophical, mathematical, or anything) is to be righteous and dedicated to God, it must conform to His revelation. If we do recognize the authority of biblical revelation and the historical teachings of the church, then it is easy to reject the other process views of theology, Christology, hamartiology, and soteriology, as an orthodox understanding of these doctrines can be validly derived from the Scriptures.

It is clear that while many process beliefs arose out of a sincere wrestling with elements of the Christian faith, the results of this wrestling fall beyond the bounds of Christian orthodoxy. As Christians, it is important that in developing a philosophy of mathematics we maintain the convictions of our faith. The ideas put forward by process thinkers such as the human creation of eternal mathematical objects, must be examined through Christian lenses. These lenses see the Bible as inerrant and authoritative, God as Trinity, Jesus Christ as fully God and fully man, humanity as sinful and in need of redemption, and salvation as coming by grace through faith in Jesus.

While the mathematical concepts put forward by process thinkers may be intriguing, ultimately we cannot accept them because of the assumptions process thinkers use to make their conclusions. Even though a philosophical result is put forward that appears useful and eases tensions in a field of study, this does not warrant our overlooking the flawed methodology and thinking which led to the result. As Christians pursuing mathematical inquiry, we must always proceed with distinctly Christian presuppositions. As Plantinga charges:

> We must work at the various areas of science and scholarship in a way that is appropriate from a Christian or more broadly theistic point of view. We shouldn't assume, automatically, that it is appropriate for Christians to work at the disciplines in the same way as the rest of the academic world. . . .If it is important to our intellectual and spiritual health to understand these [disciplines], then what we must do, obviously enough, is use all that we know, not just some limited segment of it. Why should we be buffaloed (or cowed) into trying to understand these things from a naturalistic perspective? So the central argument here is simplicity itself: as Christians we need and want answers to the sorts of questions that arise in the theoretical and interpretative disciplines; in an enormous number of such cases, what we know as Christians is crucially relevant to coming to a proper understanding; therefore we Christians should pursue these disciplines from a specifically Christian perspective.[92]

10. Conclusion

In this paper the question of whether process thinking can be helpful for examining mathematics from a Christian perspective has been examined. The foundations of process thought were explicated and it was characterized as a movement that emphasizes becoming over being

[92] Alvin Plantinga, "On Christian Scholarship," Virtual Library of Christian Philosophy. http://www.calvin.edu/academic/philosophy/virtual_library/articles/plantinga_alvin/on_christian_scholarship.pdf (accessed April 11, 2010).

and events over substance. When process thought has been applied to Christian doctrines it has led to a radical restatement of traditional beliefs. Process theology can ultimately be categorized as placing authority on the individual, rather than God. Process theology also has deep ties to the philosophy of mathematics, originally being put forward as an attempt to square the Christian understanding of God with advances in the field of mathematics such as Gödel's Incompleteness Theorem. This demonstrates that an examination of process theology is an important undertaking for Christian mathematicians. Once process theology is examined from a Christian perspective it must be rejected. By denying the authority of the Bible, God as Trinity, Jesus Christ as both fully God and fully man, the depravity of the human race, a substitutionary atonement view of the crucifixion, and the bodily resurrection of Jesus, process theology denies the very essence of Christianity. Therefore the mathematical contributions made by process thinking must be rejected (or obtained by different reasoning) because we as Christians cannot accept the methodology by which these contributions are reached. When pursuing mathematical inquiry as Christians, we must proceed from distinctly Christian presuppositions. This means we cannot rethink the classical doctrine of God in such a way that gives precedence to scientific and mathematical advances. Instead, we must interpret these new advances in light of our Christian convictions. This means we must view new mathematical advances with humility, rather than dogmatically proclaiming that we now have a way of finding more complete knowledge. The words of Blaise Pascal provide us a reminder of this point:

> One must know when it is right to doubt, to affirm, to submit. Anyone who does otherwise does not understand the force of reason. Some men run counter to these three principles, either affirming that everything can be proved, because they know nothing about proof, or doubting everything, because they do not know when to submit, or always submitting, because they do not know when judgment is called for. Skeptic, mathematician, Christian; doubt, affirmation, submission (Penses, 170).

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Two Faith Integration Projects for Freshman Majors

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Abstract

Two projects will be presented that integrate faith and Mathematics in a freshman Introduction to Proofs class at George Fox University. The first project asks students to look at the life of a Christian Mathematician. The focus of this project is to show students that many great mathematicians also had immense faith. The second project asks students to take a close look at their own life. How do they plan to live a life of Christian faith in their chosen profession? Both projects are designed to encourage students to look at their careers in Mathematics as a vocation.

1. Background

George Fox University is a small, Christian, liberal arts university in Newberg, Oregon. George Fox and its faculty place a high value on the integration of faith and learning in our scholarship as well as the integration of faith in our teaching. The mission of George Fox University is *"George Fox University, a Christ-centered community, prepares students spiritually, academically, and professionally to think with clarity, act with integrity, and serve with passion."*[1] Mathematics, by nature, focuses on the education of the mind. Although a clever, beautiful proof can at times tug on my heartstrings, Mathematical proofs rarely appeal to my students' hearts. I have been motivated to create faith integration projects for Mathematics students by my belief that students need non-secular role models in both the spiritual realm and the professional realm. I want students to know that there is more to a Mathematician than a logical mind. Throughout history, faith has been the focus of the lives of many Mathematicians. My goal in this paper is to present to you two projects that could be implemented in any Mathematics course in order to open students' eyes to the men and women of faith that played a role in the development of Mathematics. These projects help students see that even in a field of abstract logical thinking, faith can and should be the focus of their lives.

2. Integration of Faith into the Mathematics Classroom

I have to admit that I find the concept of integrating faith and Mathematics extremely challenging. I want to bring issues of faith up in my classes, but I have felt in the past that I have to put anything that I would have to say about faith into a distinct subset of my lecture. After I have presented a mini-devotional of sorts, I dive into “real” Mathematics. A question that I struggle with is, “Is this concept of true and pure integration looking at Mathematics through a lens of faith, or is it looking at faith through a lens of Mathematics?” I find both methods of integration to be a challenge for Mathematics teachers. A different and more appealing option would be to somehow weave the study of our faith and the study of Mathematics together for our students. Heie says it this way: “I believe that I must seek coherence in my worldview beliefs. This requires that I seek to discover (or is it create?) connections or interrelationships between my biblical and theological understanding and academic disciplinary knowledge, for I believe that knowledge is all of one piece.” [5]

So, where do I find these connections? Where are these interrelationships in Mathematics and faith? I believe that one (certainly not the only) answer lies in the people who have joined these two seemingly different ways of looking at the world together: Mathematicians.

3. Project Goals

I designed these projects to make my students consider two things. First, that the history of Mathematics is rich in men and women who had a strong Christian faith. Second, I wanted students to take some time to think about their own lives and how they plan to incorporate faith into their future career. In the first project, I ask the students to tell us whether they would consider the Mathematician that they studied to be a good role model. This was an important piece of the project in that the students were asked to look carefully at the men and women that they studied in light of both their Mathematics and their faith. The reason for doing this is that I wanted students to look deeper into the Mathematician's biography than just their great Mathematical accomplishments. The Bible says in Mark 8:36-37, “What good is it for a man to gain the whole world, yet forfeit his soul? Or what can a man give in exchange for his soul?”[2]

This project challenges my students to think that there might be greater things in this world than our Mathematical accomplishments. In the words of Gauss, “There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science.”[4]

One of my goals for my students is for them to see that their gift of logical thinking can be used to discover new and exciting Mathematics, but we are also called to use our logical minds to think about issues of faith, good and evil, and salvation. Mathematics should not be the sole focus of our students' thoughts. They need to be thinking about their current and future lives in Jesus Christ. Cauchy once said, “Men will pass away but their deeds will abide.”[3] In response to this statement, students need to look at the scope of their lives and their accomplishments in light of what will remain. Mathematics is a wonderful tool to sharpen our minds and to let us catch a small glimpse of the order that is in God's creation, but there are bigger questions to be answered that Mathematics does not address. In the end, all of our wonderful Mathematics will pass away and our deeds alone will abide. This is a humbling thought for my students to think about.

The second part of the project gives students an opportunity to discuss and organize their thoughts on issues of faith in their own lives. How is their faith going to play a role in their future career? Does their chosen future career have any intersection with the values in their lives that are most important to them? These are difficult questions to answer for students and often questions that they have not thought about before. Many Mathematics majors become Math majors simply because they love Math. They have not given much thought to how they are going to apply Mathematics to their Christian lives. This project helps students to start thinking about the connections between their vocation and God's calling. Some of our students' vocations include being teachers, research Mathematicians, and working in business, but they have all been called to use their logical mind for God's kingdom.

4. The Projects

The student projects that I developed are laid out below. The first project is a ten to fifteen minute PowerPoint presentation students given to the class in groups, and the second part is a five-page paper that each student writes individually. I will present the projects here in a form close to what appeared in my Introduction to Proofs syllabus.

4.1. Project 1

The history of Mathematics is full of names of Mathematicians that made positive contributions to both Mathematics and to their faith. This presentation project is intended as an opportunity for you to take a look at one of these Mathematicians in detail and gain a better understanding of what it means to be a Christian Mathematician. My hope is that as you read about these men and women you will form your own opinions about how your worldview will affect and complement your future career in Mathematics.

Assignment:
In a group of three or less give a ten to fifteen minute PowerPoint presentation on the life of a Mathematician that contributed not only to the field of Mathematics, but also lived a life of faith. Subjects to cover in your presentation:

1. Give a brief history of the Mathematician's life. (Country of origin, where they studied, where they taught, their economic position in life, their accomplishments)
2. What are the major contributions that this Mathematician made to the field of Mathematics?
3. What are the major contributions that this Mathematician made in theology and/or how did this Mathematician live out their faith in God?
4. How did this Mathematician make connections between their professional life (as a Mathematician) and their religious life?
5. As a student of Mathematics, would you or would you not consider this Mathematician a good role model for you as you begin looking at a career for yourself in Mathematics? Why or why not?

6. Examples of Mathematicians to write about:
   Johannes Kepler, Isaac Newton, Gottfried Wilhelm von Leibniz, Blaise Pascal, Michael Faraday, Bernhard Riemann, Boole, Leonhard Euler, Bernhard Bolzano, Weierstrass, Cauchy, Georg Cantor. (You may write about any Mathematician you choose as long as you can find material about their lives.)

4.2. Project 2

It is not often that Mathematics and faith are talked about in the same lecture, much less the same sentence. In this paper, I will ask you to integrate both your skills in mathematics and your worldview into a statement about what is important to you in life. Your goals as a student of mathematics and your deep convictions as a person do not have to be mutually exclusive. I want you to take time to think about what your goals in life are and how mathematics will play a role in accomplishing these goals.

Assignment:
Write a five-page paper describing the following in your own life:

1. What are your goals as a math major/minor/student? In other words, why are you a Math student? (If your answer to this question is, "I just like Math," I encourage you to take a closer look at what is it about Math that excites you, and how Math can play a part in your contribution to the world.)
2. How will you use Mathematics in your career?
3. How will you integrate your worldview into your career?
4. Does your worldview and your career have anything in common? Explain.
5. How do you plan to keep a balance between your career and the values in your life that are most important to you?

5. Follow-Up Day

The day that students hand in their five-page paper, I have a follow up day in class. On this day I have students break up into groups of three or four and talk about what they have learned in these two projects. I then bring the class together again to discuss what they have talked about in groups. Lastly, I give the students my own personal testimony. I tell the students what integrating faith into my life looks like and why I chose to be a Mathematician and teach at a small Christian university. I tell students about my relationship with Jesus Christ and how my relationship with Him affects the way I teach and work. Students walk away from this day feeling like they got to know me better as a person and some of them feel like they know themselves better after talking through some of these integration issues.

6. Conclusions

The two projects discussed have been a positive experience both for me as a professor and for my students. The presentation is an excellent opportunity for students to get a glimpse into the history of Mathematics and to get to hear about the faith of some of our Mathematical fathers. The paper project has given me a glimpse of the call on my students' lives. It has also forced them to think about what is truly important to them and how their faith is going to influence their careers. Students have come away from both projects discussing Mathematics and faith in the same conversation, which is a victory in and of itself. I hope that you have found these projects interesting and will be able to implement them in some form into your Mathematics classroom.

References

[1] George Fox University Catalog

[2] The Holy Bible, New International Version, Zondervan, Grand Rapids Michigan, 2002.

[3] H. Eves Mathematical Circles Revisited, Boston: Prindle, Weber and Schmidt, 1971.

[4] J. R. Newman (ed.) The World of Mathematics, New York: Simon and Schuster, 1956. p. 314.

[5] Arlin C. Migliazzo Teaching as an Act of Faith, Fordham University Press, New York, 2002. p. 99.

Using Original Historical Mathematics Texts in the Classroom

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Abstract

Incorporating information from the history of mathematics into undergraduate mathematics courses is an effective way to help students make connections between the mathematics they are learning and their general education courses. It also helps students to see mathematics as living subject that has changed over time. One way to introduce historical topics into a variety of classes is through the use of original mathematical texts. This paper describes how non-experts can make use of original texts and provides sources for identifying candidate texts.

1. Introduction

A few years ago the faculty of the Mathematical, Information and Computer Sciences Department (MICS) at Point Loma Nazarene University (PLNU) read as a group Parker Palmer's book *To Know as We are Known* (Palmer, 1993). This book is somewhat philosophical in nature but provided the starting point for many fruitful conversations about teaching among the MICS faculty. More than one of the faculty discussions focused on the question: How well do our students read the mathematical and computer science texts that we put before them? It appears that all too often they are reading these texts in the same way that they would read a novel and are not extracting much meaning from them.

This conversation among the faculty has led to the implementation of more reading activities in MICS courses. One activity that is repeated in several classes is taking a single page of the textbook and working through it as a class with the faculty member demonstrating how he or she reads the text. This includes looking up definitions and notation, filling in gaps in proofs and adding written comments to explain what an algorithm is doing. There are two critical skills that students need to be successful in reading technical texts. First they need to read with a pencil in hand and not be afraid to underline, add notes and otherwise mark up the text (note that our students have not had success with technical books in e-readers). Second, students need to slow down when they read and to wrestle with the language until they understand the meaning of the text.

True to the liberal arts nature of our university, students earning a degree in Mathematics at PLNU must take at least one course in the history of mathematics. In addition, the MICS

Department infuses bits of mathematical history in many of the mathematics classes. For example, when looking at Green's Theorem the class will spend a short amount of time learning who Green was and the motivation for his mathematics. Students also learn about some of the episodes where there seems to be some obvious historical confusion or confounding (e.g. L' Hopital's role in the creating of L'Hopital's Rule). Based on survey data, students enjoy being able to see the connections between the mathematics they are learning and the "other" information that is contained in their general education courses.

Having students look at carefully selected early mathematical texts is a good way to help students learn more about the history of mathematics and to see it as a living and breathing discipline that changes over time. Because the language (even in English translation) and notation are usually unfamiliar to students it also forces students to slow down when they read and work to find, understand, and make notes about the applicable definitions and notation.

2. Some Examples

The main challenges in using original historical texts in mathematics classes are locating appropriate texts and finding ways to present them in class. As with any in-class discovery activity there is the need for the instructor to do a fair amount of advance preparation, and it is most effective to have students work in groups to understand the texts. Every student encounter with an original text requires a fair amount of scaffolding including warm up exercise, hints and diagrams, to provide the students with the necessary tools to understand the text. The easiest way to begin is using original text activities that have been prepared and tested by others. In this section some examples of successful class activities are presented and in the final section of the paper, resources are given.

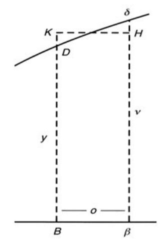
2.1. Calculus

In the three-semester calculus sequence at PLNU the students are provided with short biographies of individuals connected with the topics that they are studying. A good source for biographical information is MacTutor at St. Andrews University in Scotland:
http://www-history.mcs.st-and.ac.uk/history/BiogIndex.html.

These are well-researched short biographies, but they need to be trimmed for calculus students because the latter part of the biographies often include a summary of the advanced mathematical work of the individual that is just distracting to freshmen. For example the biography of Riemann discusses the Riemann-Zeta function.

The calculus sequence also makes use of small bits of text to illustrate the historical development of an idea. A good example of that is Newton's explanation of the Fundamental Theorem of Calculus. The students are presented D.T. Whiteside's translation of Newton's "proof" (Newton, 1967) along with the description of his proof. Newton's original work is a jumble of symbols and strange notation and is helps students to see why we are all thankful that Leibniz' notation became the standard. As you can see, Newton's "proof" was a demonstration by example. This leads to interesting conversations about the meaning of proof. Below is the description of Newton's example that is handed out in class. It was created by a PLNU research student (Butler, 2005) working from D. T. Whiteside's translation of Newton.

The fundamental theorem of Calculus says $\frac{d}{dx}\left[\int_a^x f(t)'\,dt\right] = f(x)$. In his proof that Newton uses a specific example for the area under a curve ($\frac{2}{3}x^{3/2}$), and taking the derivative of the area should give us $f(x) = x^{1/2}$, at $x = B$ (BD in the figure to the right).



Newton begins by setting $\frac{2}{3}x^{3/2} = z$. He then squares each side to get $\frac{4}{9}x^3 = z^2$.

Next Newton expands each side by adding the moments with $x + o$ for x and $z + ov$ for z (remember z is an area so we need the area of the box) and the o is a small distance not 0. These moments can be seen by looking at the figure. Making this substitution and using binomial expansion we get

$$\frac{4}{9}(x^3 + 3x^2o + 3xo^2 + o^3) = z^2 + 2zov + o^2v^2.$$

Newton now uses his method of fluxions. First he cancels $\frac{4}{9}x^3$ and z^2 from each side respectively giving

$$\frac{4}{9}(3x^2o + 3xo^2 + o^3) = 2zov + o^2v^2.$$

The second step is to divide through by the moment (o):

$$\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2.$$

Lastly Newton uses the controversial step of dropping everything being multiplied by the infinitesimal moment (o) giving the fluxion

$$\frac{4}{3}x^2 = 2zv.$$

Now we should notice that as the moment expansion went to zero, our length v was original ordinate y:

$$\frac{4}{3}x^2 = 2zy.$$

We are looking for the y by itself, so we breakdown each side by first dividing by 2:

$$\frac{2}{3}x^2 = zy.$$

Now we can separate the left side so it coincides with pieces we want on the right:

$$\left(\frac{2}{3}x^{3/2}\right)\left(x^{1/2}\right) = zy.$$

We started with $\frac{2}{3}x^{3/2} = z$, so we can drop those off from each side leaving

$$x^{1/2} = y.$$

This short text with the explanation, gives students an opportunity to wrestle with a number of the physical meanings related to integration and differentiation. The class also spends a few minutes talking about the complexities created by stating o is not 0 in one step of the computation and then treating it as a 0 in a different step. This is a nice addition to conversations about the idea of a limit and the fact that it took nearly 200 years after the work of Newton for calculus to become rigorous.

2.2. Linear Algebra

In Linear Algebra the students are exposed to additional biographical information but they also read some original texts related to the calculation of determinants. The students are given a packet of material that provides scaffolding beginning with the motivation for idea of simultaneous liner equations found in the ancient Chinese text *Nine Chapters on the Mathematical Art*. The project also makes use of Jean Borrel's system for solving equations. The translation (in italics) given below is the one that is used in class and it is taken from Jackie Steadall's very useful book *Mathematics Emerging A Sourcebook 1540–1900* (Steadall, 2008).

> One of the earliest European examples (1559) of solving a system of simultaneous linear equations can be found the *Logistica (Arithmetic)* of the French writer Jean Borrel (also known as Johannes Buteo) who was one of the first Europeans to work with notation for more than a single unknown quantity. His problem states:
>
> *To find three numbers of which the third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with the fifth part of the rest makes 8.*
>
> Borrel then goes on to say:
>
> *Put the first to be 1A, the second 1B, the third 1C. Therefore it will be that* $1A$, $\frac{1}{3}B$, $\frac{1}{3}C$ [14. *Likewise,* $1B$, $\frac{1}{4}A$, $\frac{1}{4}C$ [8. *And also* $1C$, $\frac{1}{5}A$, $\frac{1}{5}B$ [8. *Moreover, having made a second equation from these, you will have the first, second, and third, as I have put here.*
>
> $$\begin{array}{llll} 3A. & 1B. & 1C & [42 \quad 1^{ST} \\ 1A. & 4B. & 1C & [32 \quad 2^{ND} \\ 1A. & 1B. & 5C & [40 \quad 3^{RD} \end{array}$$
>
> *From these three equations others are made, my multiplication, or by adding to each other, until by subtracting the smaller from the greater there remains a quantity of only one symbol, which is done in this way. Multiply the second equation by 3, it makes 3A, 12B, 3C [96. Take away the first, there remains 11B, 2C [54.*
>
> $$\begin{array}{rrrl} 3A, & 12B, & 3C & [96 \\ 3A, & 1B. & 1C & [42 \\ \hline & 11B, & 2C & [54 \end{array}$$

$$\begin{array}{cccc}
3A, & 3B, & 15C & [120 \\
3A, & 1B. & 1C & [42 \\
\hline
 & 2B, & 14C & [78 \\
 & 22B, & 154C & [858 \\
 & 22B, & 4C & [108 \\
\hline
 & & 150C & [750
\end{array}$$

> *Again multiply the third equation by 3, it makes 3A, 3B, 15C [120. Take away the first, there remains 2B, 14C [78. Multiply by 11, it makes 22B, 154C [858, there remains 150C [750]. Divide by 150, there comes 5, which is the number of all of C. Since now you will have 1C worth 5, from the equation which is 2B, 14C [78, take 14C, that is 70, it leaves a remainder 8, which is worth 2B, therefore 4 is the second number B. Moreover, so that you have the first from the equation where the number of the total is 40, subtract 5C, and 1B, that is, 29 and it leaves a remainder of 11, which is the first number A. Therefore the three numbers are 11.4.5, which were to be found.*

This project then concludes with the students working through carefully selected excerpts from a number of works finishing with Charles Dodgson's (Lewis Carroll's) original method for finding determinants which is in *Elementary Treatise on Determinants with the Applications to Simultaneous Linear Equations and Algebraical Geometry*. Dodgson's cleaver method is computationally much easier than the traditional method for finding a determinant. The full description of this project can be found in *Computing the Determinant Through the Looking Glass* by Maria Zack (Zack, 2011).

2.3. Number Theory

In Number Theory, the students look at some original Greek texts in translation. Here is an example of one of the texts:

> *Proposition I: Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.*

This text, and all of the Greek translations used in class, are from Heath's translation Euclid's *Elements* (`http://aleph0.clarku.edu/~djoyce/java/elements/Euclid.html` or Heath, 2007). The project used in PLNU's Number Theory class is a shortened version of *Euclid's Algorithm for the Greatest Common Divisor* by Desh Ranjan (Ranjan, 2008). This project is one of a collection of projects created through an NSF grant to create curriculum for teaching discrete mathematics using primary historical texts. The principal investigators on this grant are based at New Mexico State University and the projects and classroom activities can be found on their website. All of the projects have been tested in the classroom and many have "teaching tips." Each of the projects is easily adaptable for a number of settings and faculty members are encouraged to change and modify the projects to suit their needs in both mathematics and computer science settings.

Ranjan's project begins with a set of scaffolding exercises. The students are asked to compute divisors and common divisors in a number of settings and then review some of Euclid's key definitions. Below is a sample of the definitions.

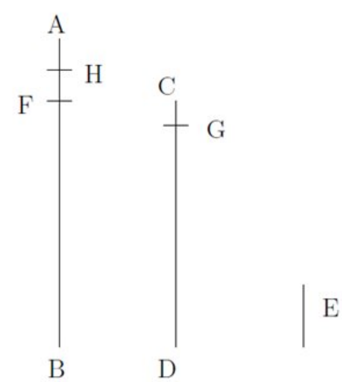
BOOK VII of *Elements* by Euclid
DEFINITIONS.

1. An unit is that by virtue of which each of the things that exist is called one.
2. A number is a multitude composed of units.
3. A number is a part of a number, the less of the greater, when it measures the greater.

These typically confusing definitions from Euclid do a very good job of forcing students to read the words carefully to extract meaning. The scaffolding exercises ask students to use these definitions and to provide examples.

Finally, the students are ready to examine Euclid's proof, the proof and not the theorem is the location of the famous algorithm. Here is the proof in translation. Note that the proof is heavily dependent on understanding the accompanying diagram.

PROPOSITION 1.
Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.



For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;

I say that AB, CD are prime to one another, that is, that an unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E; let CD, measuring BF, leave FA less than itself, let, AF measuring DG, leave GC less than itself, and let GC, measuring FH, leave an unit HA. Since, then E measures CD, and CD measure BF, therefore E also measures BF. But it also measures the whole BA; therefore it will also measure the remainder AF. But AF measures DG; therefore E also measures DG. But it also measures the whole DC; therefore it will also measure the remainder CG. But CG measures FH; therefore E also measures FH. But it also measures the whole FA; therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible. Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another.

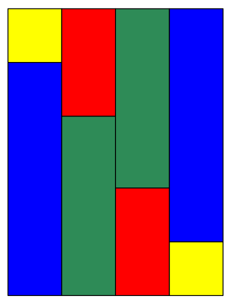
To understand this proof, students are given two strips of paper of unequal length and a pair of scissors, and are asked to follow the steps of the proof by cutting. This allows them to kinesthetically process what Euclid is doing in the proof and it also drives home for them the fact that number was a length in the ancient world, not an abstraction.

2.4. History of Mathematics

This course uses a number of original texts from various time periods. To understand a small amount of the work of Archimedes and how some of it foreshadows ideas in calculus, the class looks at a shortened version of *Sums of Numerical Powers in Discrete Mathematics: Archimedes Sums Squares in the Sand by David Pengelley* (Pengelley, 2008). Pengelley is one of the principle investigators on the NSF grant at New Mexico State University. This project is also begins with a collection of scaffolding exercises that lead to the first theorem:

> *If any number of magnitudes be given, which exceed one another by an equal amount equal to the least, and also other magnitudes, equal in number to the former, but each equal in quantity to the greatest, all the magnitudes each of which is equal to the greatest, plus the greatest will be the duplicate of those exceeding one another by an equal amount.*

The students are given a collection of Cuisenaire Rods to see if they can work through this language to obtain the modern day notation for the sum. They are encouraged to start with just three to five magnitudes and eventually they will end up with a form of the pattern below.



From this diagram, they work to generalize what they have observed in a few cases where the magnitude is small and to translate those observations into modern notation. This kinesthetic approach done in groups generally allows all groups to reach the correct sum:

$$2\sum_{i=1}^{n} i = (n+1)n.$$

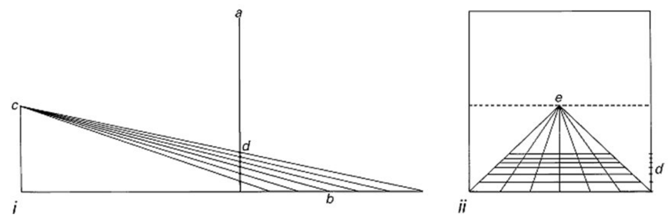
This project continues with several additional formulas of Archimedes.

2.5. Mathematics, Art and Architecture

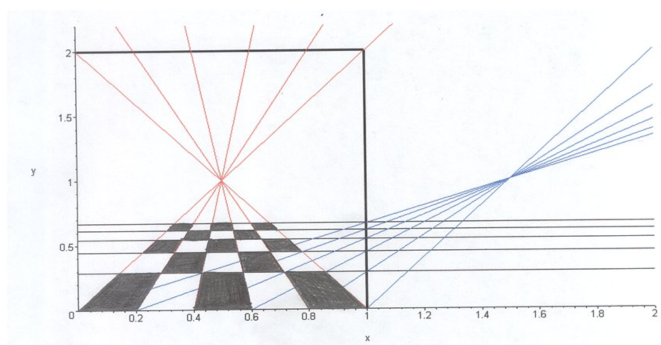
The PLNU curriculum allows for "special topic" courses and one of them taught periodically in the MICS department is on the history of the linkages between mathematics, art and architecture. One of the original texts given in this class comes from the translation of *Della pittura* (On Painting) by Leon Battista Alberti (Alberti, 1991). Alberti was a mathematician and architect who also had an interest in realistic painting and drawing. In *Della pittura* Alberti says:

> *I inscribe a quadrangle of right angles, as large as I wish, which is considered to be an open window through which I see what I want to paint .... Then within the quadrangle, where it seems best to me, I make a point which occupies the place where the central ray strikes. This point is properly placed when it is no higher from the base line of the quadrangle than the height of the man that I have to paint there ....*

The students begin by reading the text as homework. The next day in class, they receive additional notes and a few illustrations (Alberti, 1991) from *Della pittura*. The notes and the illustrations, help students to understand the mathematical principles beneath drawing in perspective.



In class the students translate this description into a drawing. The end result of this classroom activity is for each student to draw successfully a room with a tiled floor and a set of pillars along the walls. Because most of the students taking the class are not artistically talented, they are generally excited that mathematics can lead to a realistic drawing. The floor tiling below is taken from PLNU honors thesis of Sarah Littler entitled *A Linear Perspective to Art* (Littler, 2004). In her thesis Litter investigated a number of different types of mathematical perspective, and her interest in the topic had its origin in this type of activity.



The students own drawings provide a good starting point for reexamining paintings that they have seen in their general education art history classes with a specific focus on vanishing points and the mathematics behind creating a sense of depth. This discussion concludes analyzing the tiles at the feet of Christ in the painting "The Flagellation of Christ" by Piero della Francesca.

3. Resources and Teaching Tips

As with all complex classroom activities, working with original mathematical texts requires a fair amount of faculty preparation. After years of experience here are the lessons that have been learned at PLNU:

- Give the students the text in advance. Have the students familiarize themselves with the text the night before beginning any in-class activities. Make it clear that they will not fully understand the text at the first reading but that they need to arrive in class having thought about the text.
- Allow for incubation time. Have the activities and necessary work carry across two days of class. This allows students to formulate questions about the text and to develop hypotheses about the meaning of text.
- Have students work in groups. The final report for the project might be a group paper or set of computations, a group oral presentation about some aspect of the project or the submission of individual answers to specific exercises. All have been used successfully at PLNU.
- Provide "hooks" and scaffolding. The students will not be able to extract the meaning from the text without context, warm up exercise and other forms of scaffolding. Without that support they get frustrated and will stop trying to understand the complicated language particularly from periods of time when mathematics had no symbolic notation.
- Whenever possible, have the students process the information kinesthetically. Translating the words on the paper into specific drawings and activities is effective in helping the students to verify if they have accurately understood the text.

There are a number of good sources for material that can assist with introducing original texts into the classroom. These include:

University of St. Andrews in Scotland MacTutor Website:
http://www-history.mcs.st-and.ac.uk/history/index.html

New Mexico State University Projects on Using Original Texts (there are two separate projects supported by the NSF):
http://www.math.nmsu.edu/hist_projects/
http://www.cs.nmsu.edu/historical-projects/
http://www.math.nmsu.edu/~history/#Expeditions

Online books: If you are feeling more adventurous, there are many original texts available online. There are is a wealth of texts from the 1650's onward that are in English and thus require no translation. However the students will struggle with English and the inconsistent spelling of words from the Seventeenth Century but they can manage to work through it on their own.

Google Books: http://books.google.com
Early English Books Online: http://eebo.chadwyck.com/home

Printed books: A few of my favorites for English translations of some interesting texts include:

Mathematics Emerging: A Sourcebook 1540-1900, Jackie Stedall
A Sourcebook in Mathematics, David E. Smith
The History of Mathematics: A Reader, John Fauvel and Jeremy Gray (this book was created as a reader for an Open University (UK) class and is available through the MAA).

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The Mathematics of Cubic Sudoku

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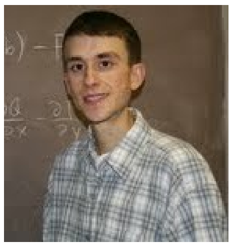

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Abstract

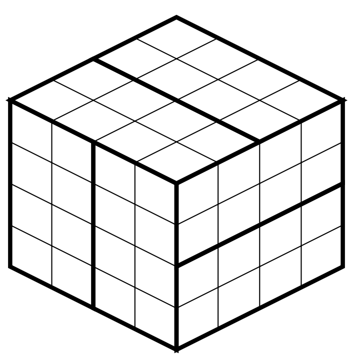
In the last decade the Sudoku puzzle has fixed itself in America's puzzle consciousness. Sudoku puzzles share space with crossword puzzles and word finds in newspaper puzzle sections, and several books have been written for the Sudoku playing community. Mathematicians are among the most dedicated Sudoku players. Although some are content with simply solving puzzle after puzzle, others have used tools from combinatorics and algebra to study its important properties.

We investigate a variant of Sudoku called Cubic Sudoku, as well as Cubic Sudoku's simpler relative, Cubic Shidoku. We successfully count the number of Cubic Shidoku puzzles in two different ways: by a direct argument and using Gröbner bases. We also make some observations about the difficulty of replicating these results for Cubic Sudoku.

1. Cubic Sudoku

Sudoku is a popular puzzle consisting of a 9×9 grid with nine clearly marked 3×3 blocks. Some of the 81 cells are filled in with the digits 1 through 9. The challenge to the puzzle solver is to fill in the blank cells so that every row, every column, and every 3×3 block contains the digits 1 through 9. Thus, a completed Sudoku puzzle is a Latin square with extra conditions.

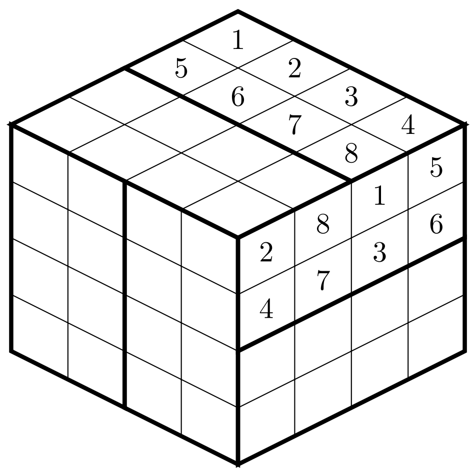
The popularity of Sudoku has spawned several variants [3]. In this paper we consider a variant called Cubic Sudoku. Here is a blank Cubic Sudoku puzzle:



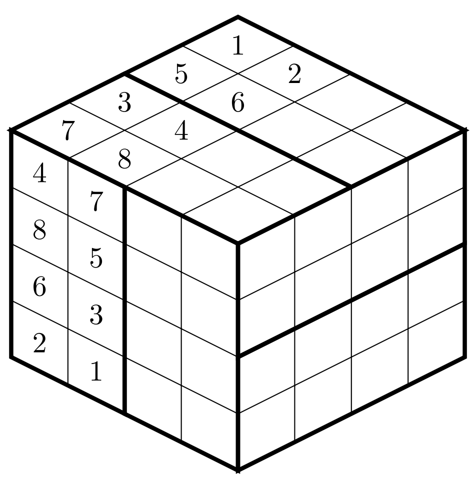
Note that the blank puzzle consists of three faces of a cube, each of which is divided into a 4×4 grid, for a total of 48 cells. We name the faces according to their orientation to the page, so we will refer to them as the *top face, left face*, and *right face.* As with regular Sudoku, a puzzle is usually presented with certain cells already filled in using the digits 1 through 8. The challenge to the solver is to fill in the blank cells with the digits 1 through 8 so that certain sub-regions of eight cells contain each of the digits 1 through 8.

1.1. Sub-Regions

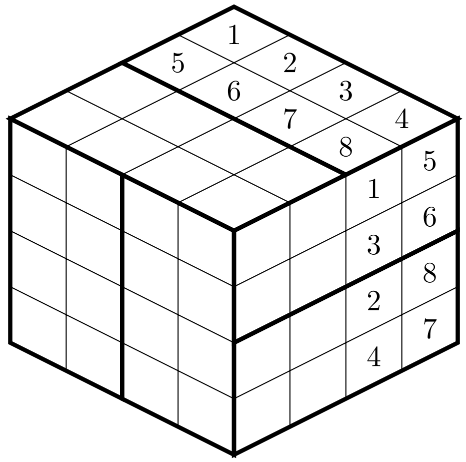
A Cubic Sudoku puzzle is divided into two distinct types of sub-regions: 2×4 blocks and L-shaped regions. There are six 2×4 blocks, two on each face of the cube. They are outlined in bold. Here are two correctly filled in 2×4 blocks:



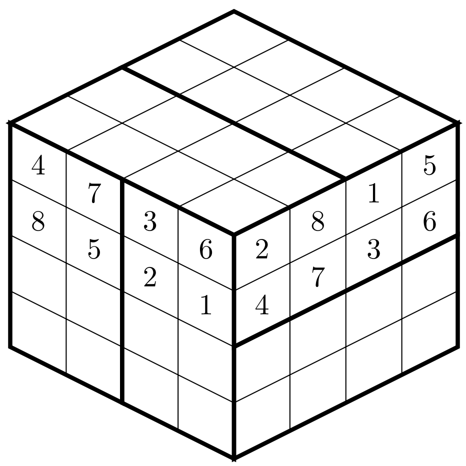
There are also twelve L-shaped regions to consider. We will call them simply *Ls.* Each L is composed of cells on adjacent faces of the cube. We divide the Ls into groups of four according to which faces they occupy. First, we present two correctly filled in *top to left Ls*:



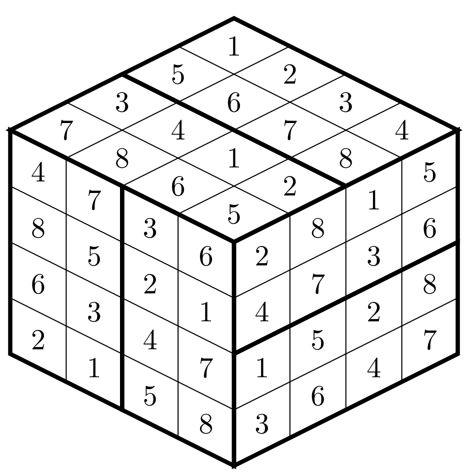
Next, we present two correctly filled in *top to right Ls*:



Finally, we present two correctly filled in *left to right Ls*:



Below is an example of a completed puzzle. Note that it follows all of the rules for Cubic Sudoku.

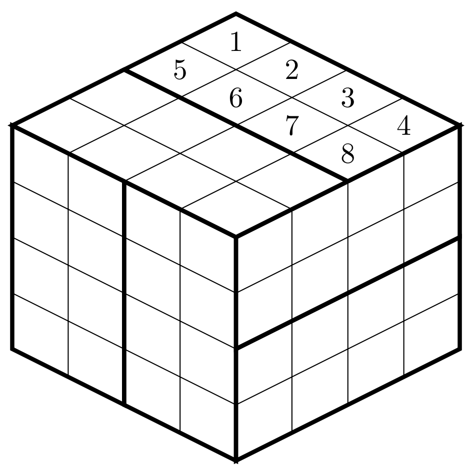


1.2. Counting Cubic Sudoku Puzzles

One question of importance in the Sudoku research community is to count the number of distinct Sudoku puzzles. Felgenhauer and Jarvis completed this task using a combination of human reasoning and brute force checking by computer programs [2]. They determined that there are $6,670,903,752,021,072,936,960 \approx 6.671 \times 10^{21}$ distinct Sudoku puzzles. Their result has been independently verified by others using computers. Nevertheless, the search continues for an enumeration of all distinct Sudoku puzzles that does not depend on brute force checking by a computer.

An attempt was made to discover if others have attempted to count the number of distinct Cubic Sudoku puzzles, but no previous work was found. We attempted to count the number of distinct Cubic Sudoku puzzles, but at the present we can only offer lower and upper bounds on this number. For a lower bound, consider the fact that if we have a completed puzzle, we may produce a new puzzle by relabeling the cells in $8! = 40320$ ways. For an upper bound, consider the six 2×4 blocks. A completed 2×4 block may be relabeled in $8!$ ways, so there are at most $(8!)^6 \approx 4.3 \times 10^{27}$ Cubic Sudoku puzzles.

This upper bound may be reduced by noting that the rules of play impose the following restriction: One cannot put the same digit in adjacent Ls on the same face if the Ls cut through adjacent 2×4 blocks. For example, consider the following partially filled in Cubic Sudoku puzzle:

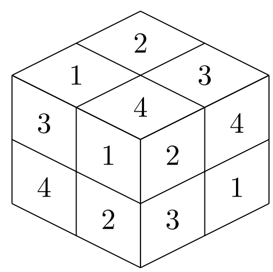


Consider the possible placements for the digit 1 in the adjacent 2×4 block on the top face of the cube. We cannot place a 1 in the top to left L that contains 1 and 5 because this placement violates the rule that each digit appears in each L exactly once. In addition, we cannot place a 1 in the top to left L that contains 2 and 6. If one were to do so, then no 1 could appear in either of the 2×4 blocks on the left face of the cube. Thus, there are only four possible placements of the digit 1 in the empty 2×4 block on the top face of the cube. Carrying out this reasoning for each of the other two faces of the cube, we find a new upper bound for the number of Cubic Sudoku puzzles: $(8!)^3(4!)^3 \approx 9.1 \times 10^{18}$.

At the present we have not successfully counted the number of Cubic Sudoku puzzles, either directly by counting symmetries or with the assistance of a computer. However, this goal was achieved for a smaller variant of Cubic Sudoku called Cubic Shidoku. We now turn our attention in that direction.

2. Cubic Shidoku

In [1] Arnold, Lucas, and Taalman introduce Shidoku, a smaller variant of Sudoku played on a 4×4 grid with four 2×2 blocks. The 16 cells in the grid must be filled in using the digits 1 through 4 so that each of the digits appears exactly once in each row, column, and 2×2 block. Following their lead, we created Cubic Shidoku, a smaller variant of Cubic Sudoku. Here is an example of a completed Cubic Shidoku puzzle:



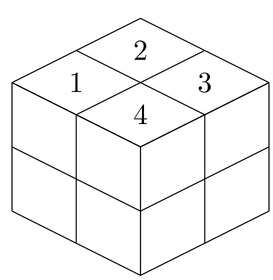
Notice that the rules of Cubic Sudoku apply to the three faces and the six Ls. In contrast with Cubic Sudoku, each face is by itself a region to which the rules of the puzzle apply.

3. Counting Cubic Shidoku Puzzles

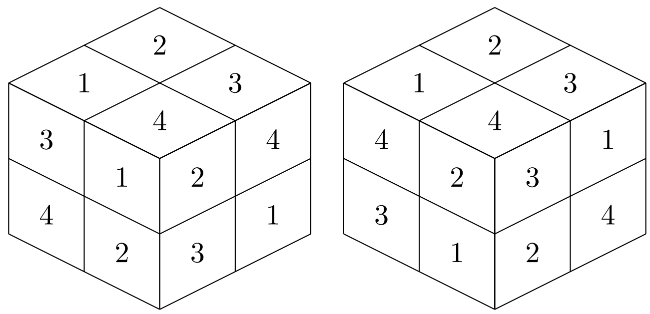
An obvious advantage of considering Cubic Shidoku is that the number of puzzles is smaller than the number of Cubic Sudoku puzzles. Arnold, Lucas, and Taalman exploit this advantage in the parallel relationship between Sudoku and Shidoku in order to count the number of Shidoku puzzles in two different ways: by counting symmetries and by using Gröbner bases.

3.1. Counting Symmetries

Consider the top face of the Cubic Shidoku puzzle. It can be filled in properly in $4! = 24$ different ways. Let us adopt the following placement of the digits 1 through 4 as the *standard* arrangement of the top face:



Next, consider the top to left L in which 1 and 2 are placed. It is completed by placing 3 and 4 in the two remaining cells. If one plays Cubic Shidoku correctly, then one discovers that there are only two ways to complete this L:

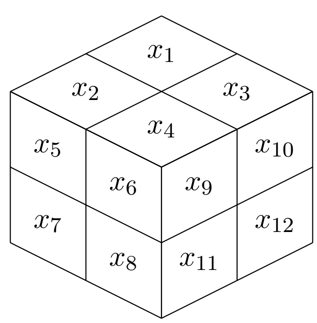


There are 24 ways of placing the digits 1 through 4 in the top face. Each of them corresponds to 2 different completed puzzles, so there are a total of $24 \cdot 2 = 48$ different Cubic Shidoku puzzles.

3.2. Using Gröbner Bases

It is often desired to find two different ways of counting the same thing in order to be more certain of the enumeration. Thus, we now present an alternative method for counting Cubic Shidoku puzzles. First, we assemble a system of polynomials to show the relationships between cells that arise from the rules of Cubic Shidoku. Then we find a simpler way of expressing this system using a Gröbner basis for the system.

To begin, we assign a variable to each of the cells in a Cubic Shidoku puzzle, as follows:



Any completed Cubic Shidoku puzzle can be expressed using these polynomials by assigning the variable for a cell to the digit appearing in that cell. For instance, in the completed Cubic Shidoku puzzle above we have $x_1 = 2$, $x_2 = 1$, $x_3 = 3$, and $x_4 = 4$.

The rules of Cubic Shidoku produce a system of polynomials that describe the structure of the puzzle. First, since each cell must contain exactly one of the digits 1 through 4, we have the 12 polynomial equations $(x_1 - 1)(x_1 - 2)(x_1 - 3)(x_1 - 4) = 0$, $(x_2 - 1)(x_2 - 2)(x_2 - 3)(x_2 - 4) = 0, \ldots, (x_{12} - 1)(x_{12} - 2)(x_{12} - 3)(x_{12} - 4) = 0$. We can model the rules concerning faces and

Ls in a similar fashion. As noted in [1], the digits 1 through 4 are the unique set of positive integers with sum 10 and product 24. Thus, the restrictions on the faces are described by six polynomial equations (two for each face), as follows:

- Top face: $x_1x_2x_3x_4 - 24 = 0$ and $x_1 + x_2 + x_3 + x_4 - 10 = 0$
- Left face: $x_5x_6x_7x_8 - 24 = 0$ and $x_5 + x_6 + x_7 + x_8 - 10 = 0$
- Right face: $x_9x_{10}x_{11}x_{12} - 24 = 0$ and $x_9 + x_{10} + x_{11} + x_{12} - 10 = 0$

Similarly, the restrictions on the Ls are described by twelve polynomial equations. Examples for each type of L are:

- Top to left Ls, e.g. $x_1x_2x_5x_7 - 24 = 0$ and $x_1 + x_2 + x_5 + x_7 - 10 = 0$
- Top to right Ls, e.g. $x_2x_4x_9x_{11} - 24 = 0$ and $x_2 + x_4 + x_9 + x_{11} - 10 = 0$
- Left to right Ls, e.g. $x_5x_6x_9x_{10} - 24 = 0$ and $x_5 + x_6 + x_9 + x_{10} - 10 = 0$

We now have $12 + 6 + 12 = 30$ polynomials that describe Cubic Shidoku. In order to count the number of Cubic Shidoku puzzles, we seek a simpler way of expressing these polynomials. If all of the polynomials were expressed in one variable, we could find a simpler system in echelon form by using Gauss-Jordan elimination. However, we are confronted with not one but twelve variables. A Gröbner basis generalizes the result of Gauss-Jordan elimination for a system of polynomials in one variable and is applicable to the multivariate setting. The resulting collection of polynomials expresses the zero set of the original system in a simpler way. The interested reader will find more details about Gröbner bases in [5] and [4].

One complication that arises is the need to choose a term ordering for the several variables that appear in the system. Following the lead of [1], we choose the reverse lexicographic ordering, i.e., $x_{12} > x_{11} > \cdots > x_2 > x_1$. A Gröbner basis produced using this ordering is in triangular form, which is similar to echelon form in the one-variable case. A system of polynomials $\{p_1, p_2, \ldots, p_m\}$, $m \geq 12$, in triangular form can be ordered in such a way that p_1 has a power of x_1 for its leading term and contains only powers of x_1; p_2 has a power of x_2 for its leading term and contains only powers of x_1, powers of x_2, and products of powers of x_1 and x_2; p_3 has a power of x_3 for its leading term and contains only powers of x_1, powers of x_2, powers of x_3, and products of powers of x_1, x_2, and x_3; etc. In our case, the triangular form of the Gröbner basis allows us to count the number of solutions by multiplying together the powers of the leading terms of the polynomials in the Gröbner basis.

We found a Gröbner basis for our system of Cubic Shidoku polynomials using Maple 14. The resulting system of polynomials has leading terms $x_1^4, x_2^3, x_3^2, x_4, x_5^2, x_6, x_7, x_8, x_9, x_{10}, x_{11}$, and x_{12}. Notice that the powers of the first four leading terms multiply together to give $4! = 24$, which is the number of ways of relabeling the top face if it is in standard form. Note also that the next leading term is x_5^2. The exponent 2 corresponds to the fact that we have two choices for the cell named by x_5 after the top face is filled. Finally, note that all of the remaining leading terms have exponent 1. This expresses the fact once a choice for x_5 is made, all other choices are completely determined. Thus, the number of different Cubic Shidoku puzzles is $4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 48$. This enumeration agrees with the one that we found above using a more elementary argument.

4. Future Work

Having witnessed the success of Gröbner bases in counting Cubic Shidoku puzzles, we attempted to use them as a means of counting the number of Cubic Sudoku puzzles. However, our attempts failed. One can begin a count of the number of Cubic Sudoku puzzles by following the argument used for Cubic Shidoku and defining a standard form for one of the 2×4 blocks on the top face. However, the analysis then branches into not two but four different cases since, as we showed in Section 1.2, there are four possible positions for 1 in the remaining 2×4 block on the top face. Keeping track of all of the branching is a tedious task. Overcoming this difficulty requires detailed record-keeping and the ability to distinguish carefully between branches that can and cannot be counted in the same ways.

We made our attempts to find a Gröbner basis for the Cubic Sudoku sum-product system of polynomials by running Maple 14 on a desktop PC with a modest amount of memory. In the future, we will try the computation again using different techniques and different software. Our model for Cubic Shidoku is what Arnold, Lucas, and Taalman call a sum-product system. In [1] they describe two other methods for using a system of polynomials to describe a Sudoku-type puzzle: a roots-of-unity system and a Boolean system. The Boolean system in particular is believed to be more computationally efficient than the sum-product system. Future work may have a greater chance of success if it uses one of these two alternative models for Cubic Sudoku. Similarly, we are not obligated to use Maple to find a Gröbner basis for a system of polynomials. Implementations of the most commonly used algorithm for finding Gröbner bases are available in Mathematica, CoCoA, GP/Pari, and other similar programs.

Despite these difficulties, we remain optimistic that the problem of counting the number of Cubic Sudoku puzzles can be solved. Once that work is complete, we may turn to related questions borrowed from the study of Sudoku puzzles, e.g. determining the minimum number of clues needed for a unique solution. It is conjectured that this number is 17 for Sudoku; it appears to be unknown for Cubic Sudoku. We may also ask the same questions for any of the other Sudoku variants presented in [3] and other locations.

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Conference Photos

Conference Delegates on Winter Hall Lawn (Mountains to the North)

Main Meeting Room (Darling Foundation Lecture Hall, aka "The UN Room")

Art Benjamin


Fred Brooks

Glen Van Brummelen

Evening Discussion Session

Santa Barbara with Westmont's Campus in the Lower Center

Morning Hikers at *Montecito Overlook*