Association of Christians in the Mathematical Sciences

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Charleston Southern University Conference Attendees, May 31 - June 2, 2017

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Editor's Introduction

Russell W. Howell (Westmont College)

The twenty-first biennial conference of the Association of Christians in the Mathematical Sciences was held at Charleston Southern University from May 31 until June 3, 2017. Thanks go to Jamie Probin and his colleagues at CSU for all the organizational efforts that went into hosting it.

Many thanks also to the three invited speakers, each of whom prepared two engaging presentations:

- Sloan Despeaux (Western Carolina University)
  - *De Morgan's Budget of Paradoxes*
  - *Fit to Print?*
    Referees' Reports of Mathematics in Nineteenth-Century London

- Dominic Klyve (Central Washington University)
  - *The Mathematics of Faith:*
    Euler's Anonymous Work on the Limits of Mathematics, Science, and Faith
  - *Translation and Betrayal: Euler's "Letters to a German Princess"*

- Derek Schuurman (Calvin College)
  - *Shaping a Digital World: Connecting Bytes and Beliefs*
  - *Responsible Automation:*
    Faith and Work in an Age of Intelligent Machines

There were a total of 66 papers presented by the 126 conference attendees. The relevant abstracts, as well as the entire conference schedule, are given in Appendix 1 of these *Proceedings*. Appendix 2 lists the participants, home institutions, and e-mail addresses.

Prior to the gathering at Charleston Southern University the ACMS board decided to initiate a peer-review procedure for the papers submitted to all conference *Proceedings*, and asked Russell Howell to serve as the editor for this first effort. Of course, not every attendee submitted an exposition of his or her talk. The following pages contain the papers that made their way through the single-blind review process, each having been scrutinized by a minimum of two reviewers. The editor expresses thanks to the authors for their cooperative spirit. Too numerous to mention are all the referees that were involved, but heartfelt thanks go to them for their diligent work.

The twenty-second biennial conference for ACMS, set to be hosted by Indiana Wesleyan University, is scheduled for May 29 – June 1, 2019. Details can be found at acmsonline.org, which is the official website for ACMS.

The Resolved and Unresolved Conjectures of R. D. Carmichael

Brian D. Beasley (Presbyterian College)

Brian Beasley (B.S., Emory University; M.S., University of North Carolina; Ph.D., University of South Carolina) has taught at Presbyterian College since 1988. He became a member of the Mathematical Association of America in 1989 and joined ACMS in 2007. Outside the classroom, Brian enjoys family time with his wife and two sons. A big fan of the various writings of C. S. Lewis, he is also an enthusiastic Scrabble player, a somewhat less than enthusiastic jogger, and a very shaky unicyclist.

Abstract

Even before heading to Princeton University to work on his doctoral degree, Robert Carmichael started influencing the path of number theory in the 20th century. From his study of Euler's totient function to his discovery of the first absolute pseudoprime, he set the stage for years of productive research. We present a brief overview of Carmichael's life, including his breadth of mathematical interests and his service on behalf of the Mathematical Association of America. The main focus is upon his two most famous conjectures - which one has been settled, and which one remains open to this day?

1 Early Years

Robert Daniel Carmichael was born in Goodwater, Alabama in 1879. Carmichael spent the first thirty years of his life in Alabama, never too far away from Goodwater. He graduated from nearby Lineville College in 1898, and three years later he married Eula Smith Narramore from Randolph (south of Birmingham). Robert and Eula settled in Hartselle with their four children, Eunice, Erdys, Gershom, and Robert Leslie. Carmichael began training to become a Presbyterian pastor, presumably with the intention of keeping his home and family and ministry in Alabama [30].

Yet starting in 1905, the situation began to change for Carmichael. Between 1905 and 1915, he submitted dozens of problems to *The American Mathematical Monthly*. For example, in 1908 Carmichael contributed the following problem [7] to the "Number Theory and Diophantine Analysis" section:

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m - q^n = 1$ may be satisfied.

Meanwhile, in October of 1906, Carmichael became professor of mathematics at Presbyterian College in Anniston, Alabama. The school was in just its second year when he arrived, and it continued as a college until 1918. As noted in [2], during its brief history, it offered both a Classical Course (B.A.) and a Scientific Course (B.S.), awarding a total of 37 degrees. Presbyterian College also somehow managed to field a football team, playing local high schools and colleges and even taking on the University of Georgia squad in 1909 and in 1911. The team's nickname? The Predestinarians.

In 1909, Carmichael moved his family to Princeton, where he started graduate work in mathematics. Under the direction of George Birkhoff, he wrote his thesis (Linear Difference Equations and their Analytic Solutions) and received his doctorate in 1911. Carmichael then accepted a professorship at Indiana University, teaching there from 1911 to 1915 [30]. Of his eventual 35 doctoral students, only the first earned the degree at Indiana, but this represented a significant milestone: In 1912, Cora B. Hennel became the first person, male or female, to receive a doctorate in mathematics at Indiana [20].

Even before heading to Princeton, Carmichael published 13 papers in various mathematical journals between 1905 and 1909. These articles ranged from shorter pieces involving *Monthly* journal problems (see [5]) to longer works on topics such as multiply perfect numbers (see [4]). The most famous of these papers, entitled "On Euler's ϕ-Function," appeared in the *Bulletin of the American Mathematical Society* in 1907 [6]. In this article, Carmichael remarked, "The object of the present note is the demonstration of certain very elementary propositions concerning Euler's ϕ-function of a number." Yet what Carmichael viewed as a basic proposition has turned out to be much more difficult to establish than he originally believed.

2 The First Conjecture

In his 1907 paper, Carmichael examined the question of whether a number could occur exactly once in the range of the Euler ϕ-function. Recall that given a positive integer n, its Euler phi-function (or totient) value is the number $\phi(n)$ of integers x with $1 \leq x \leq n$ such that $\gcd(x, n) = 1$. For example, $\phi(1) = 1 = \phi(2)$, while $\phi(n)$ is even for all values of $n > 2$. However, not every positive even integer appears in the range of ϕ; the smallest such exception is 14. The two key properties of this function are:

(i) If p is prime and k is a positive integer, then $\phi(p^k) = p^k - p^{k-1}$.

(ii) Given positive integers a and b, if $\gcd(a, b) = 1$, then $\phi(ab) = \phi(a)\phi(b)$.

Using these properties, Carmichael offered a proof of the following claim.

Proposition. The relation $\phi(m) = n$, a given number, is never uniquely satisfied for any given value of n. That is, there is always more than one value of m for every possible value of n.

We present a brief summary of Carmichael's argument [6]. For contradiction, assume there is a positive integer n such that the equation $\phi(m) = n$ has exactly one solution m. If m is odd, then $\phi(2m) = \phi(m) = n$; similarly, if $m/2$ is odd, then $\phi(m/2) = \phi(m) = n$. Thus 4 divides m, so n is even. Writing $m = 4a$ and $n = 2b$ for positive integers a and b yields $\phi(4a) = 2b$, which in turn implies $\phi(2a) = b$. Then both a and b must be even. Continuing this process, we eventually reduce to the case in which both of these integers must be powers of 2. But the equation $\phi(x) = 2^c$ has more than one solution for $c \geq 3$, since we may take $x = 2^{c+1}$ and $x = 2^{c-2} \cdot 3 \cdot 5$.

Unfortunately, there is a gap in Carmichael's proof. The argument that both a and b must be even depends on the fact that otherwise, there would not be a unique solution for the equation $\phi(2a) = b$; however, that does not necessarily lead to a contradiction with the original assumption of a unique solution for the equation $\phi(m) = n$. At first Carmichael did not recognize the error, and he included the result as a homework problem in his 1914 book *The Theory of Numbers* [10]:

(Chapter 2, Exercise 8) Show that if the equation $\phi(x) = n$ has one solution it always has a second solution, n being given and x being the unknown.

Readers of his book pointed out the gap in the proof, prompting Carmichael to publish another paper, "Note on Euler's ϕ-Function," in 1922 [11]. He noted, "Two correspondents have recently called my attention to the fact that the supposed proof of the following theorem, which I gave some years ago, is not adequate. So far I have been unable to supply a proof of the theorem, though it seems probable that it is correct. I am therefore compelled to allow it to stand in the status of a conjectured or empirical theorem."

Undaunted, in the same 1922 paper Carmichael determined a lower bound on any counterexample to the conjecture. He showed that if there is a positive integer n such that $\phi(m) = n$ is uniquely satisfied by m, then $m > 10^{37}$. This has inspired mathematicians ever since to establish improved lower bounds for such a counterexample. We present several of these results to note their progress over the years:

$m > 10^{400}$	1947 - Klee [21]
$m > 10^{10,000}$	1982 - Masai and Vallette [24]
$m > 10^{10,000,000}$	1994 - Schlafly and Wagon [34]
$m > 10^{10,000,000,000}$	1998 - Ford [17]

Such an immense lower bound is quite remarkable; in comparison, the current lower bound on the possible existence of an odd perfect number is a mere 10^{1500}, as shown by Ochem and Rao [29]. As Schlafly and Wagon [34] remarked, "We do not know of another unsolved problem in mathematics for which a lower bound on a counterexample is so high ... There can be little doubt that Carmichael's conjecture is true."

Work on Carmichael's conjecture and related topics has continued over the years. One approach has been to define the function $A(n)$ for a positive integer n to be the number of solutions of $\phi(m) = n$. For example, $A(1) = 2$ and $A(2) = 3$, while $A(24) = 10$. Also, given any odd integer $n > 1$, we have $A(n) = 0$. Erdős proved that if $A(n) = k$ for some integer n, then there exist infinitely many such n [16]. In addition, Sierpiński conjectured and Ford proved that for each integer $k \geq 2$, there is an integer n such that $A(n) = k$ [18]. This function allows us to restate Carmichael's conjecture: $A(n)$ never equals 1.

To date, Carmichael's first conjecture has not been resolved. We leave the final word in this section to Erdős [16]:

"This conjecture is still unproved and seems very deep."

3 The Second Conjecture

In order to introduce another famous conjecture by Carmichael, we recall an important result from number theory. Fermat's Little Theorem states that if p is prime, then for every integer a with $\gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Euler would later generalize this result, showing that given any positive integer n, if a is an integer with $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Fermat's Little Theorem prompts two questions:
(i) Can we find a composite n with $a^{n-1} \equiv 1 \pmod{n}$ and $\gcd(a, n) = 1$ for at least

To answer the first question, Sarrus [33] noted in 1819 that since $2^{10} \equiv 1 \pmod{341}$, we have $2^{340} \equiv 1 \pmod{341}$, even though $341 = 11 \cdot 31$ is not prime. Such composite numbers are called *pseudoprimes* (or pseudoprimes to base 2); they are also often referred to as *Sarrus numbers*.

In tackling the second question, Korselt [22] established the following result in 1899.

Korselt's Criterion. Given a composite integer $n > 1$, $a^{n-1} \equiv 1 \pmod{n}$ for every integer a with $\gcd(a, n) = 1$ if and only if n is squarefree and $p - 1$ divides $n - 1$ for every prime p that divides n.

Such composite numbers n are called *absolute pseudoprimes*. Do they exist?

In 1910, Carmichael [8] found the first example, proving that $561 = 3 \cdot 11 \cdot 17$ is an absolute pseudoprime; he also showed that every absolute pseudoprime is odd and has at least three distinct odd prime factors. Two years later, Carmichael [9] provided a list of 15 additional absolute pseudoprimes. In this paper, he offered a tantalizing footnote as well: "This list might be indefinitely extended." Carmichael's footnote has since been restated in the following form.

Conjecture. There are infinitely many absolute pseudoprimes.

To extend Carmichael's list, Chernick [13] proved in 1939 that every absolute pseudoprime with three prime factors has the form

$$(2r_1h + 1)(2r_2h + 1)(2r_3h + 1),$$

where r_1, r_2, and r_3 are pairwise relatively prime. Taking $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, and $h = 3k$, Chernick also showed that $(6k + 1)(12k + 1)(18k + 1)$ is an absolute pseudoprime if each of its three factors is prime. His list of absolute pseudoprimes included $5 \cdot 17 \cdot 29$, $5 \cdot 17 \cdot 29 \cdot 113$, and $5 \cdot 17 \cdot 29 \cdot 113 \cdot 337$.

In the spirit of Carmichael's original footnote, Chernick added, "The process may be continued to the limits of present-day factor tables."

Meanwhile, the situation for pseudoprimes had been resolved. In 1936, Lehmer [23] proved that there are infinitely many pseudoprimes. In particular, he showed that if p and q are distinct odd primes, then pq divides $2^{pq-1}-1$ if and only if the order of 2 modulo p divides $q-1$ and the order of 2 modulo q divides $p-1$; here, the order of 2 modulo r is defined as the smallest positive integer k such that $2^k \equiv 1 \pmod{r}$. Sierpiński [35] followed this in 1947 with the result that if n is a pseudoprime, then 2^n-1 is also a pseudoprime. And in 1949, Erdős [14] established that for every $k \geq 2$, there are infinitely many pseudoprimes with exactly k different prime factors.

As for absolute pseudoprimes, or *Carmichael numbers* as they had come to be known, we offer two observations from Erdős. In 1949, he commented in [14], "It seems very difficult to determine whether there are infinitely many absolute pseudoprimes." Yet by 1956, Erdős had provided a heuristic argument for the existence of infinitely many absolute pseudoprimes [15]; in particular, he conjectured that for x sufficiently large, there should be $x^{1-o(1)}$ Carmichael numbers up to x (a function $f(x)$ is said to be $o(1)$ if $\lim_{x\to\infty} f(x) = 0$). Pomerance later conjectured in [32] that the exponent in this bound could be improved to $1-\{1+o(1)\}\log\log\log x/\log\log x$.

In 1994, Carmichael's second conjecture was resolved in the affirmative. That year, Alford, Granville, and Pomerance [1] proved that there are indeed infinitely many Carmichael numbers. In their proof, they used Korselt's Criterion and modified Erdős' heuristic argument. In fact, they dedicated their paper to Erdős on the occasion of his 80th birthday. They were able to show that if $C(x)$ is the number of Carmichael numbers up to x, then for sufficiently large x.

$$C(x) > x^{2/7}.$$

In 2008, Harman [19] was able to increase this exponent from 2/7 to 1/3.

Carmichael's work on this topic continues to inspire mathematicians to this day. The search for more Carmichael numbers is ongoing, with particular interest in finding the largest known k-Carmichael (a Carmichael number with exactly k prime factors). For example, the largest known 3-Carmichael has 60,351 digits [3], and the largest known 4-Carmichael (with 30,366 digits) was just recently discovered [28]. In addition, current calculations summarized in [31] feature results such as $C(10^{16}) = 246,683$ and $C(10^{21}) = 20,138,200$. Remaining questions include:

(i) Are there infinitely many 3-Carmichaels?

(ii) Are there infinitely many k-Carmichaels for each $k > 3$?

Research also continues on the question of the existence of special types of Carmichael numbers, such as those found in arithmetic progressions. In 2012, Matomäki [27] showed that if $\gcd(a, M) = 1$ and a is a quadratic residue modulo M, then there are infinitely many Carmichael numbers m with $m \equiv a \pmod{M}$. The next year, Wright [36] proved unconditionally that if $\gcd(a, M) = 1$, then there are infinitely many Carmichael numbers m with $m \equiv a \pmod{M}$.

4 Later Years

Over the next several decades, Carmichael continued his research and service on behalf of the mathematical community. In 1915, he moved to the University of Illinois, working there as a professor of mathematics for 32 years. Carmichael served as the head of the mathematics department at Illinois from 1929 to 1934, and in his last year in that role, he also became the acting dean of the graduate school. The next year, he was elected dean, continuing in that position until his retirement in 1947 [25]. During his tenure at Illinois, he supervised 34 additional doctoral students, with nine of them completing their work after he became the dean [26].

To convey a sense of the impressive range of Carmichael's mathematical interests, we list just a few of his many publications. Having already written books on both relativity and number theory while at Indiana, Carmichael also co-authored texts on calculus and on plane and spherical trigonometry during his time at Illinois. He wrote *Diophantine Analysis* in 1915 and *Introduction to the Theory of Groups of Finite Order* in 1937. In between, he published one of his most interesting works, *The Logic of Discovery*, in 1930. This book summarized Carmichael's views on the philosophy of mathematics and received very positive reviews. In particular, the last chapter on "The Larger Human Worth of Mathematics" gave a glimpse into his theological background. Toward the end of this chapter [12], Carmichael paraphrased Isaiah 52:7, writing

> "How beautiful upon the highway are the feet of him who comes bringing
> in his hands the gift of a new truth to mankind."

While teaching students, leading his department and graduate school, and writing books on a variety of topics, Carmichael still managed to find time to serve the Mathematical Association of America in a number of roles [25]. A charter member of the MAA, he became editor-in-chief of the *Monthly* in 1918. Carmichael held the position of Vice President from 1921 to 1922 and was selected as the organization's eighth President in 1923. He also served on the MAA Board of Governors on three separate occasions, in 1920, 1924-1929, and 1939-1941.

In conclusion, when one considers the many accomplishments of Carmichael's career, along with his obvious love of mathematics and philosophy, it is no wonder that he was held in such high regard by those who knew him best. In tribute [30], his friend Harrison E. Cunningham said of Carmichael:

> "Of unyielding integrity, he loved the truth and hated sham
> and pretense. His appreciation of the beautiful,
> the true, and the good is exceptional.
> His friendship is firm, his loyalty unbreakable.
> Those who know him are fortunate beyond words."

Acknowledgment. The author would like to thank the referees, whose constructive feedback helped to improve the paper.

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"Big Idea" Reflection Assignments for Learning and Valuing Mathematics

Jeremy Case, Mark Colgan (Taylor University)

Jeremy Case holds a Ph.D. from the University of Minnesota and an M.A. from Miami University. While at Taylor University, he served as General Education Director for five years and has chaired the Mathematics Department. He points to his ACMS and Project NExT experiences as seminal moments in his professional development. In addition to family and church, he enjoys reading, playing the piano, and walking. He is currently taking his first steps into the realm of Statistical Education.

Mark Colgan earned his Ph.D. from Indiana University. Having served many years as a department chair, he is interested in ways to build a collaborative spirit among faculty members in order to improve teaching and learning. He believes teachers should give students creative opportunities to experience and value mathematics. He enjoys Bible quizzing, watching sports, hiking, and spending time with his family. He is a Fellow of Taylor's Bedi Center for Teaching and Learning Excellence.

Abstract

While participating in a Faculty Learning Community, we explored the "big questions" we wanted our students to take away from each of our mathematics courses. We called these questions the Big Ideas of the course, and we developed a Big Idea Reflection Assignment, which we continue to assign at the end of each of our courses. Students are able to demonstrate understanding and application of their learning as well as their values and appreciation of mathematics. The assignment encourages students to move beyond a focus on techniques and symbolic manipulations towards a broader and more holistic approach, including making connections between their learning and the Christian faith. We also noticed that the assignment changed the way we approach our teaching and designed our courses, so that our students are beginning to see our courses not as a collection of neatly packaged isolated chunks of material but as a unified collection of important mathematical ideas.

1 Introduction

The 2015 CUPM guide [12] suggests that

> Major programs in the mathematical sciences should present the beauty, fun, and power of mathematics. They should be designed so that all students come to see mathematics as an engaging field, rich in beauty, with powerful applications to other subjects and contemporary open questions.

Furthermore, many Christian faith-based institutions expect faculty to be engaged in integrating academic disciplines with faith commitments grounded in Christianity.

These are worthy but challenging goals. Students do not naturally associate beauty and fun with mathematics, and they have difficulty integrating all parts of life, particularly mathematics, within a Christian framework.

Mathematics instructors can fall into a trap of presenting mathematics in isolation without an overall or ultimate purpose. Most mathematics textbooks are written section by section where each section of material is neatly packaged into isolated chunks of material followed by several practice problems. While typical, the process does not always encourage us to connect the dots and present the course as a whole or relevant to Christianity. Such an approach encourages students to learn each section in isolation (and too often to quickly forget the concepts) without appreciating the context and without giving much thought to the overall themes of the course or purposes of their learning. According to Ken Bain's *What the Best College Teachers Do* [1], excellent teachers ask themselves what "big questions" a course will help students answer, and how we can motivate students to care about these questions. The best teachers stir up excitement and curiosity within students to explore the important issues and applications of the discipline. Bain's book stimulated our thinking about how we could focus our mathematics classes on the "big questions," which we called the "Big Ideas" of the course.

We (the two authors) participated in an interdisciplinary Faculty Learning Community [10] consisting of eight faculty members at our small, Christian liberal arts college in which the group developed the concept of Big Ideas and the Big Idea Reflection Assignment. The community was an outgrowth of a Teaching Squares program [4, 9] where four faculty members visit each other's classes with a group discussion at the end.

With support from our institution's Center for Teaching and Learning Excellence, our interdisciplinary community met biweekly for a year over lunch to discuss teaching and learning issues related to student reflection with the goal of developing an assignment that all participants could utilize in their courses regardless of discipline. The group was attracted to Bain's concept of "big questions" and how this concept could assess how students were engaging with their learning. How well were students learning the most important concepts of a course, which the group called the Big Ideas, and how well might they be able to remember and apply these ideas to their future studies, ministry, and personal life? Each member created a list of five to eight Big Ideas for a particular course. The accountability of the Faculty Learning Community then helped to refine the Big Ideas and to motivate their implementation [3].

What we found was that after we explicitly identified the Big Ideas of a course, we were more likely to communicate them to our students throughout the semester. We believe students better understood the purposes of the course, its design, and its assignments. The Big Idea Reflection Assignment turned out to be an effective teaching tool in itself because students were not only able to connect their learning to skills and concepts useful in the future, but the reflective process itself stimulated additional learning [13, 14].

2 Developing the Big Ideas of a Mathematics Course

Thinking about the Big Ideas for a course can be a challenging though stimulating process. Envisioning overall goals independent of "covering" specific content is exciting and freeing. If we believe mathematics courses ought to be required for all students because they help students to think or to appreciate the beauty of mathematics, how is that reflected in the structure of a culminating mathematics course? If we feel that the interaction of faith and mathematics is important, what evidence do we have that students are making these connections? How will we know if students see mathematics as an engaging field, rich in beauty, with powerful applications to other subjects and contemporary open questions? We believe generating the Big Ideas for a course is a way to begin to answer these questions.

For example, we can "cover" the definition of probability in a Finite Math course, but we are free to help students discover the Big Idea of chance and surprising results. The content we need to cover may be important for a particular course, but the Big Ideas can provide purpose and meaning with various mathematical content. Calculus 2 could be viewed as the course after Calculus 1 and before Calculus 3 filled with integration techniques, sequences, and series. Alternatively, the purpose of Calculus 2 could be coming to terms with the infinite both in a mathematical and theological sense. Various mathematical topics may be used to meet the same Big Idea. For example, the Big Idea of seeing the beauty in mathematics could be taught with a unit on fractal geometry or a unit on number theory. While the lessons and teaching methods might be completely different, the Big Idea would be the same.

To begin generating the "big questions," Bain asks what it was that excited us about our discipline. What are the historical problems that motivated the creation of the course? For example, Real Analysis could be a course centered on developing mathematical thinking through the historical issues shaping the development of analysis with its various false starts of continuity and differentiability [6]. For one author of this paper (Case), the idea that the angle sum of a triangle need not be 180 degrees motivated him to pursue mathematics since "obvious" claims may not be so obvious.

Other questions to get started on developing our Big Ideas were: "What do we want our students to say when someone asks about what they had learned from our course?" and, "How do we want our students in five years to be able to apply what they learned in our course?" We wanted students to think about the most important concepts in the course, the interesting questions, and also how they could apply the concepts now and in the future. As Ken Bain describes it, "Learning takes place when [students] evaluate how they think and behave well beyond the classroom. [The best teachers] stress the need for students to grapple with important concepts and ideas . . . and have ample opportunity to apply their learning to meaningful problems" [1]. Parker Palmer states it this way, "Perhaps the classroom should be neither teacher-centered nor student-centered but subject-centered. Modeled on the community of truth, this is a classroom in which teacher and students alike are focused on a great thing, a classroom in which the best features of teacher- and student-centered education are merged and transcended by putting not teacher, not student, but subject at the center of our attention" [11]. Our goal then was to focus our courses on a handful of Big Ideas where we as teachers and students could join together in exploring the nature of mathematics and its meaning for our Christian lives.

Our previously developed course learning objectives in our syllabi were a place to start, but the course objectives tend to be more content focused and specific to current student outcomes, whereas the Big Ideas tend to be overall meta-goals that students can apply to the course as a whole with a desire for long-term understanding. One of the benefits of the learning community was sharing and defending our Big Ideas to others because we were reminded of our commitment to the discipline. In fact, we believe the Big Ideas are different for each instructor because they are dependent on what the instructor determines as the most important for the students to gain. We believe it is helpful for each instructor to think about what he/she thinks are the Big Ideas for each course and then communicate these to the students to help motivate them to learn.

The Big Ideas seem to work particularly well for liberal arts mathematics courses where a key goal is to help students understand the nature of mathematics and where the content is open to many different preferences. (Appendix 1 gives a list of Big Ideas for a liberal arts mathematics course.) On the other hand, more content-dependent courses like Calculus and Geometry may also benefit from this approach because it forces one to identify the key ideas and confront larger issues such as how courses fit together within a major. It encourages the development of themes which transcend particular topics such as multiple representations which can benefit both pre-service teachers and those going to graduate school. (Appendix 2 gives examples of Big Ideas from courses for mathematics majors.) When the instructor articulates the Big Ideas, topics can be woven together to present an important and useful branch of mathematics.

3 Developing the Big Ideas Reflection Assignment

After developing our list of the Big Ideas for a course, the next step was to determine how to word our reflective assignment for students to complete. Instead of evaluating the class experience, we wanted the students to reflect on what they had learned in the course and give them an opportunity to communicate that learning. We explicitly told the students to focus on their own learning. The assignment is not a course evaluation since there are other instruments to evaluate the course and the instructor. For the reflection assignment, we listed all the Big Ideas of the course (usually five to eight), and we asked students to choose three of these Big Ideas that they thought they had met through the course. Giving students the choice of the three allowed for more ownership of their work. For the three they chose, they were to write a short reflection (about half a page each) explaining how the course experiences had helped them accomplish the Big Idea. For each of the three reflections we asked them to include three components: 1) describe what activities in the class helped them learn this Big Idea, 2) analyze how this Big Idea helped them think more like a mathematician, and 3) apply this Big Idea to their future career, service, ministry, family, etc. (See Appendix 1 for the complete assignment.)

While there was flexibility in how we as instructors implemented the Big Idea Reflection Assignment, we generally had students complete the reflection assignment toward the end of the semester and gave students credit as part of their homework score. The assignment was a good way to review the main concepts at the end of the course and give students an opportunity to recognize the intention and experiences of assignments to integrate the course concepts. By asking students to connect the course content to their lives, we believe they had a greater sense of engagement with the material. By asking students to give personal examples of how they were learning, we believe the reflection stimulated higher-order thinking and solidified their learning. Metacognition is difficult to measure, but our readings of their responses indicate an awareness and understanding in their thought processes throughout the semester in a way an exam could not do.

Since we knew students would be completing the reflection assignment at the end of the semester, it also encouraged us as instructors to talk about the Big Ideas throughout the semester. The assignment motivated us to discuss with students *why* we were assigning projects, reflective papers, group activities, etc., because these activities were designed to help students learn the Big Ideas. In this way the course seemed to fit together because there was a reason for everything we did. In fact, we feel like this has changed the way we teach. Instead of teaching various topics and skills, we now focus on teaching the course as an integrated whole contributing to the broader educational mission of the institution.

4 Analyzing the Results of our Big Idea Reflection Assignment

We were pleased when we evaluated the results of our students' responses to the Big Idea Reflection Assignment. In addition to the three main components of the assignment, our students demonstrated a higher level of metacognition than we had expected with evidence of deep thinking about what they had learned instead of just restating their original knowledge. The reflection helped us understand how students were thinking in their conception of Christianity and Mathematics. Somewhat surprising to us was how students expressed evidence of learning on the affective level as they discussed motivation for learning the ideas and excitement over discovering new concepts. A fellow professor in our faculty learning community crystallized what we observed, "[The Big Ideas assignment] gave affective feedback on the class – not just what they had objectively learned (I had other measuring devices for this). I was able to see where students were inspired and discouraged, and what they felt like they didn't really understand" [3].

4.1 Skills and Concepts Attained

In responding to the first prompt, students described the skills and concepts related to the Big Ideas that they believed they had attained through the course. A calculus student mentioned development in her problem-solving and communication skills:

> I'm confident that I will be able to think more creatively when it comes to problem solving. Overall, accomplishing this big idea will increase my ability to see new perspectives and make connections among different topics. . . . In this class my communication skills were sharpened . . . I now understand that different people learn different ways . . . Now, if one doesn't understand something right away, I don't just give up and say, "Ask someone else," rather I explain it with a new approach.

Several students mentioned that they developed and learned to value skills in the area of working cooperatively, a skill that they will be able to apply to their future studies and careers. One student said:

> The [group projects] were often a challenge as I tend to like to work alone . . . . I found that while working in groups I was able to gain completely new perspectives on problems that I never would have been able to come up with. I found myself over and over again being extremely surprised when a team member would come up with a brilliant solution to a problem that had never even crossed my mind. Through these experiences I have gained a value for other people's ideas that I never had before. This will help me in the future as I will have to work with others and understand that other people may have equally good if not better ideas than me and I will need to be humble and let others help me when I am stuck on a problem and need a fresh perspective.

Reading what key skills and concepts the students believed they were learning encouraged us particularly when they reported being hesitant of a Big Idea at the beginning of the semester and buying into it by the end of the semester.

4.2 Thinking Like a Mathematician

Parker Palmer [11] suggests that "we honor both the discipline and our students by teaching them how to think like historians or biologists or literary critics [or mathematicians] rather than merely how to lip-sync the conclusions others have reached." While a traditional, content-oriented exam may help in measuring how a student thinks and reasons, it has difficulty assessing how a student conceptualizes the nature of mathematics. The Big Idea reflection assignment asks how students have learned to think more like a mathematician as a result of exploring the Big Ideas in a course.

Students expressed a sense of gaining a new way of "seeing" as a mathematician. One student commented, "The ideas of calculus are incredibly useful and, once you know them, you start to see them everywhere. Knowing things such as derivatives and integrals gives one a new perspective and approach to problems, an approach far closer to one that a mathematician would use." Another student commented, "I had never thought of mathematics being beautiful. I had always thought of it as being very analytical and precise–with no room for creativity or beauty. But through this course, I have been shown many ways in which math can be beautiful.. . . In the future, I think I will be more aware of the beauty around me, and maybe even find beauty in something that others may not see." Another student agreed, "Beauty and mathematics go hand in hand, and I did not see it until this class."

Many students seemed to enjoy looking at mathematics in terms of the Big Ideas. "Prior to this course I honestly was somewhat bitter towards math and had a lot of negative experiences.. . . Now I see math as a way to have fun; though not in all respects necessarily. I can enjoy sitting and drawing a tessellation or playing with shapes." One suggested that "being able to find enjoyment in math was hard, just because none of my teachers have ever really helped me and given me the opportunity to try it for myself. So given this opportunity I took it and now can reap the benefits of it. This can help me in finding joy in all areas of my life, no matter the consequences or circumstances." Others reported new ways of thinking. "Concepts that stretch one's thinking were a constant in [the liberal arts math course]. The 4th dimensional thinking stretched my thoughts. I had to consider a reality that was beyond my normal view of the world. This is humbling as well as interesting."

Finally, students began to see precise thinking as something important that they were learning in mathematics. In an introduction to proofs course, a student commented:

> I have developed many frameworks that will help me think more like a mathematician. Relating this question back to the big idea, I am now more likely to be skeptical about what is true and not true. Rather than taking information and assuming it is right because someone told me so, I am more likely to dive into why the statement is right, or find a way to alter the statement. I also am more likely to be more careful in the way I present my proofs and to make sure what I am saying is actually true.

Another commented, "The big take away that will be applicable for the rest of my life is that there is typically more than one way to solve every problem we encounter in life." The Big Idea Reflection Assignment allowed students to articulate how they could think more like a mathematician and see mathematics all around them.

4.3 Application to Future Life

The Big Idea reflection asks students how they could apply what they had learned in their math course to their future studies, ministry, and personal life. Several students wrote how they enjoyed seeing the connections between mathematics and other subject areas and others focused on the thinking process itself. One calculus student commented:

> Now when I encounter problems in the real world I will be able to approach things with the logic and reason that this class has taught me. In my future career, family, or whatever I will be able to utilize these skills.... Whether it's in the finance world, engineering, chemistry, or numerous other areas, the problem solving learned in Calculus is vital and the actual tools that it gives you are very powerful.

One student said, "This Big Idea (learn to solve problems and experience real-world applications of mathematics) has taught me that I should not limit what I learn to what I think will be immediately relatable. By sticking with and engaging Calculus, I was able to find beauty in math and I was able to apply it to my life." This comment encouraged the instructor because it demonstrated that the students were not just learning mathematics content, but through that content they were applying the Big Ideas to their lives.

4.4 Issues of Faith

As Christian instructors at faith-based institutions, we hope we are able to integrate mathematical concepts with our Christian faith, but it is difficult to know how well our students are making those connections in our mathematics courses. We were encouraged by comments like this:

> Another topic that I have connected with through this class is dealing with issues of faith. I have seen God all throughout this course. From the Fibonacci Numbers in nature, to infinity, to the fourth dimension ... Finding ways to relate my faith to mathematics has helped me relate my faith to other aspects of my life. I want to acknowledge the Lord in everything I do.

Another student in a liberal arts math class said:

> One of the hardest parts of math for me in the past was not being able to see any real application. ... This class, however, helped me greater understand its importance, primarily by relating it to my spiritual life. ... Now I can see math in the world around me in a more real way ... and how it beautifully ties into God and His magnificence.

Calculus students often connected the study of the infinite with God's nature. One commented that "as we looked at series, sequences, and all the other things involving the concept of infinite it was interesting to realize that we will never fully understand the infinite and I now see a deep connection between the infinite attributes of God." Another said:

> I don't know what the future holds for me — one of the limitations of being finite, myself, I suppose. However, this area of math provides a reasonable allegory for the person of God: while humans (and mathematicians alike) can never grasp the width and breadth of God, we certainly benefit when we allow the Infinite into our problems (both mathematical and spiritual).

Students in an introduction to proofs course made connections between the assumptions in mathematics and the assumptions in our faith. One wrote, "Going forward, I will try to be more careful to avoid allowing my (unsubstantiated) assumptions to give me false security in knowledge. Analogously, I ought to be careful in assuming knowledge about God and biblical doctrine. We often refer to God as 'infinite.' What does that even mean?" Another student wrote:

> Similarly, my doctrine is naturally influenced by assumptions. These assumptions could come from my cultural and familial context, era in history, personality, and/or influence of the flesh. This does not disqualify my beliefs as invalid or wrong (genetic fallacy). It simply means that I ought to have high standards for absolute truths.

The Big Idea Reflection Assignment gave students an opportunity to reflect and write about the connections of mathematics and their faith while providing us instructors a glimpse into their conceptions of faith integration.

Longtime ACMS member Gene Chase [8] breaks down three areas of integration:

1. Incarnational Approach: Who am I?
2. Incompleteness Approach: What do I not know?
3. Imago Dei: What do I know?

It appears from our student reflections that most students incorporate all three areas to some extent, but most comments regarding mathematics and Christianity fall into the second category. A calculus student remarked, "Going forward, I will try to be more careful to avoid allowing my (unsubstantiated) assumptions to give me false security in knowledge" and "my doctrine is naturally influenced by assumptions. These assumptions could come from my cultural and familial context, era in history, personality, and/or influence of the flesh." Also, "I don't know what the future holds for me—one of the limitations of being finite, myself, I suppose." Gene Chase notes that the incompleteness approach is not as complete of a perspective as what we would like because it has a "God of the gaps" feel to it. On the other hand, a positive of the Incompleteness Approach is that it captures a sense of humility, awe, and mystery before a great God. In our examination of the reflections, the mathematical areas in which this awe is elicited generally involve interesting patterns, infinity and the higher dimensions.

5 Benefits of the Big Ideas

The written feedback from students solidified our expectations that they really were "getting" the Big Ideas. Before this assignment, we hoped they were learning the main ideas in the course, but after reading the student reflections, we were much more confident they were meeting the course expectations.

The responses revealed a higher level of metacognition than we had expected, and rather than resistance, most students seemed to welcome the opportunity to discuss the course's impact. Brookfield [7] suggests in three core assumptions regarding teaching: "that skillful teaching boils down to whatever helps students learn, that the best teachers adopt a critically reflective stance towards their practice, and that the most important knowledge we need to do good work is an awareness of how students are experiencing their learning and perceiving our teaching." While the assignment is inappropriate for summative assessment, the student reflections provided a rich source of assessment for us as teachers about how students believed they were learning in our courses. The reflections assisted us in discerning which assignments were effective for learning, how our students were applying the concepts, how they felt about their learning, and which Big Ideas seemed to be working, particularly those related to faith integration.

Somewhat surprisingly, we realized we were more intentional about teaching the Big Ideas because we knew that the students would be writing about what they had learned at the end of the course. The Big Ideas reminded us to regularly steer the students toward the important questions and applications of the course, sometimes writing a key question on the board before class started. We designed class activities that day to help students begin to explore the answer to this important question.

As instructors, we found that developing the Big Ideas was very energizing, as it released us from the constraints of the content that had to be covered and helped us remember why we enjoyed teaching and learning mathematics in the first place. The Big Ideas essentially changed the way we teach by forcing us to view our courses as a whole and by encouraging us to design assignments and activities that highlight the most important aspects of the course.

6 Conclusion

We found the Big Idea Reflection Assignment was a very valuable teaching tool that helped our students describe what they had learned and how they could apply that learning in the future. Students seemed to enjoy writing about what they were learning in their mathematics classes. One student said, "I will admit, I was fairly skeptical about written reflections and group projects in a math class, however, they have been pleasantly enjoyable." She goes on to discuss why they were helpful for her, "I really enjoyed writing reflections, they gave me space to explore concepts and how they apply to my current life and who I want to be in Christ in the future. Being able to write about math has helped to expand my writing skills and thus my written communication ability. Being able to write about things that seem like they are hard to describe will be a great advantage in the future." Because of the success we have seen from our students, we have now integrated the Big Idea Reflection Assignment into all of our mathematics courses, and we (and our other Faculty Learning Community members) have enjoyed opportunities to share our results with other teachers. We have appreciated the journey of developing the Big Ideas approach, and we now have more enthusiasm for our teaching.

We believe our students are benefiting from mathematics courses that are unified by important questions and themes, and we enjoy reading about what they have learned from the Big Ideas each semester. One student said the assignment was great review for the final because it "helped me to see the educational value of this course and tie everything back together." While writing assignments may be utilized in mathematics courses for various purposes [5], we have found that one of the best uses is to allow students the opportunity to reflect on what they have learned throughout a course. Through writing, students are able to put together the major themes of the course, the instructor's purposes, and various assignments, and they are able to value course applications. As mathematics teachers, this "group project" has taught us what it is like for our students when we make a group assignment. We are learning the joys and challenges of investigating and doing mathematics cooperatively, which is one of the Big Ideas in our mathematics courses. Our hope is that our students are beginning to see our courses not as a collection of neatly packaged isolated chunks of material but as a unified collection of important mathematical ideas.

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Appendix 1: An example of a Big Idea Reflection Assignment

MAT 120 Investigations in Mathematics
Reflection Assignment # 8

Choose three of the following "Big Ideas" of MAT 120 that you believe you have particularly met through this course. For each of the three "Big Ideas" you choose, write a short reflection explaining how the course experiences have helped you accomplish this "Big Idea." (Each of the three reflections should be about half a page each, for a total of at least 400 words.)

For each of your three reflections, answer the following:

1. What were some valuable course activities, assignments, projects, discussions, or topics (probability, finance, numbers, geometry, etc.) that helped you grasp this component of the course? (**Describe**–the past)
2. In what ways have you developed skills that help you "think more like a mathematician" when you now encounter problems? (**Analyze**–the present)
3. What are some ways that you can apply this component to your future studies, ministry, and personal life? (**Apply**–the future)

BIG IDEAs of the Investigations in Mathematics Course:

Big Idea #1 Begin to appreciation the Beauty and nature of Mathematics.

Big Idea #2 Issues of Faith–Explore life lessons and issues of faith, including ways to be good stewards of our resources and how to use math to make wise decisions.

Big Idea #3 Group Work–Actively investigate and do mathematics individually and cooperatively.

Big Idea #4 Investigate–Problem-Solving strategies and Critical Thinking skills (particularly using Diagrams and Similar/Simpler problems) to solve problems.

Big Idea #5 Develop and Stretch your mind and explore the consequences of Surprising ideas while being inquisitive and open-minded.

Big Idea #6 Enjoy your experience in a Fun and challenging classroom environment that develops a sense of friendly community.

Big Idea #7 Apply–experience real-world applications of mathematics that you can apply to issues in your everyday Life.

Appendix 2: Additional examples of Big Ideas for various courses

Calculus 2

Big Idea #1 Multiple Representations -using various perspectives such as symbolic, numerical, graphical, and verbal techniques to solve problems.

Big Idea #2 Slice, Approximate, Sum, Evaluate–Slicing up functions or graphs so that the varying amount is nearly constant on the subinterval and then integrating to find volumes, areas, work, etc. Extending elementary fixed ideas (rate x time) to varying quantities.

Big Idea #3 Undoing differentiation to draw conclusions from graphs and to integrate functions using w-substitution, integration by parts, partial fractions, etc.

Big Idea #4 Recognizing that finite sensibilities do not always work when dealing with the infinite such as sequences, series, domination, LHôpital's rule, and Geometric series.

Big Idea #5 Developing critical thinking skills in identifying assumptions in problem solving and translating verbal statements into mathematical statements in order to model real world situations.

Big Idea #6 Developing workforce skills in using technology, working with others to solve difficult problems and to resolve individual differences, and writing technical reports.

Big Idea #7 Developing study and learning skills to tackle difficult concepts and to reflect on one's learning.

Big Idea #8 Create your own. Be sure to articulate the big idea.

College Geometry

Big Idea #1 Experience the axiomatic development of a mathematical subject and recognize the importance of methodology, logic, and models.

Big Idea #2 Develop skills in critical thinking, deductive thinking, writing, writing proofs, recognizing flaws in arguments, and advanced mathematical reasoning.

Big Idea #3 Stretch your mind by exploring the consequences beyond the "certainty of mathematics" or beyond what appears to be self-evident or common sense thinking.

Big Idea #4 Recognize the tremendous impact non-Euclidean geometry had on the nature of mathematics.

Big Idea #5 Develop and critique teaching strategies for high school geometry.

Big Idea #6 Investigate the role of the visual and the deductive statements in solving problems and believing the validity of the results.

Big Idea #7 Hear the suspense story of the discovery of non-Euclidean geometry through the role of the parallel postulate

Big Idea #8 Determine another big idea of the course.

Modeling with Numerical Analysis

Big Idea #1 The computer is fallible–Understanding error as it inevitably occurs in order to detect, predict, and minimize error as well as identifying the shortcomings of numerical recipes.

Big Idea #2 Skeptically examining solutions–Examining assumptions, using sensitivity analysis, and other means to determine how much the answer and computer output should be trusted.

Big Idea #3 "It depends"–Understanding other criteria such as cost and speed in getting a "good enough" answer, and questioning whether a "right answer" is even feasible in solving problems.

Big Idea #4 Reality? Really?–Using various and sometimes indistinct criteria in developing and critiquing models of their inherent shortcomings which accompany trying to mirror reality. (And to what extent).

Big Idea #5 Why we do not know everything–The nature of scientific computing (adding more points is not always the answer, human solutions are not always as good as computer solutions, the rule of diminishing returns in using more terms) which limit our understanding and which areas of current research would greatly enhance our understanding if progress were made.

Big Idea #6 Our Series–Using the fundamental theoretical tools such as Taylor series to approach numerical problems and why these are often not the best way but they are important first steps.

Big Idea #7 Your own big idea. Define and describe another big idea of the course.

Start a Math Teacher Circle: Connect K-12 Teachers with Engaging, Approachable, and Meaningful Mathematical Problems

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Abstract

Many K-12 math teachers are not ready to teach from a conceptual and inquiry-oriented perspective because they have an algorithmic understanding of mathematics. One solution is to create a math teacher circle (MTC), which provides conceptual and inquiry-based learning activities and builds professionalism among the teachers. In this paper, we describe the origins of two such MTCs, highlighting the process of identifying leadership team members, submitting the grant proposal for seed money, and hosting launch events, intensive summer workshops, and monthly meetings during the academic year. We also share opportunities for professional development for college and university faculty, including research linked to shifts in in-service teacher attitudes. We finish the paper with several of this year's best activities used at our MTC meetings, including fair division, extensions and generalizations of numerical and algebraic patterns, and applications in cryptography.

1 Introduction

Close your eyes and imagine a typical middle school mathematics classroom. Are you picturing students seated at desks in rows and columns solving dozens of practice problems involving decimals, fractions, and percents? Or, were you envisioning students working collaboratively to solve mathematically rich, engaging, and open-ended tasks? While we might hope that teachers at these and similar grade levels would create classroom environments resembling the latter description, too many bear a striking resemblance to the former characterization. The result is procedure and algorithmic learning with little conceptual understanding or development of problem solving ability. To address this phenomenon, groups of K-12 mathematics teachers have collaborated with college and university faculty members to form math teachers' circles. In fact, during the past decade, nearly 100 MTCs have been formed in the United States, most with seed money grants from the American Institute of Mathematics (AIM). We hope to convince you to expand the reach of these MTCs by creating one in your local area.

2 How to Create a MTC

The necessary components of a MTC include people, problems, and pizza (or other food items). For people, it is important to form a leadership team that can apply for start-up funds often from the American Institute of Mathematics (AIM), form the circle, and hold regular meetings to make decisions about upcoming monthly meetings. To receive the seed money grant from AIM, the team must include at least two K-12 mathematics teachers along with multiple college and university faculty members. Typically, these faculty members will come from mathematics, mathematics education, or education departments. Once the team is in place, a chairperson and treasurer will need to be identified who can prepare regular reports and financial statements for AIM.

For problems, a great resource is the national website for MTCs listed in the reference section. Individual MTCs also maintain websites that include resources for activities used in recent meetings (see reference section for our two MTC websites). Leadership team members are also encouraged to develop their own mathematical activities or recruit local or national speakers to present their own activities at meetings. If your MTC agrees to host a single or multi-day workshop for local mathematics teachers, then separate funds from AIM may be available to cover the speaker costs, food costs, and materials costs for the workshops. In the case of the Southwest Chicago MTC, these funds proved effective to host an initial launch event prior to a later 3-day workshop and the start of monthly meetings during the subsequent academic year. The AIM funding also helped to bring in two national MTC leaders to help with the workshop. These speakers initially led sessions on their own but then co-led sessions with members of our own more inexperienced leadership team before handing full responsibilities over to us. The Northwest Iowa MTC kicked off with a two-day summer workshop that also offered continuing education credits for the participants. Monthly sessions during the academic year followed including a fascinating investigation of the mathematics of bicycle tracks led by James Tanton with support from the AIM seed grant.

With regard to food and other logistics, it has worked well for the Southwest Chicago MTC to ask interested teachers to complete an RSVP on our website that includes information such as dietary restrictions. Some colleges and universities place restrictions on food brought onto campus by outside vendors, but most allow pizza or similar food for a monthly meeting or workshop. In fact, as part of the seed fund application, potential MTCs must provide letters of support from the department chairs of the institutions at which the monthly meetings are held. It is expected that the hosting departments are willing to provide basic resources and spaces for the meetings, though sometimes departments will work with local school districts so that meetings can be hosted in area schools.

Goals for monthly meetings include time for an initial math problem or two, a break for dinner, and the main activity with time for reflection at the end of the session. It is ideal if the initial problems transition well into the main activity although this is not necessary. Where appropriate, we provide the K-12 math teachers with links to state or national standards (e.g., Common Core State Standards for Math, [5]) for the main activity. In some cases, teachers must modify the activity to meet the education needs of students at specific grade levels. However, the overriding goal for MTC sessions is to offer the K-12 teachers mathematically rich and engaging activities, even if some of these activities are not usable in their classrooms without major modifications, if at all. The final 15-20 minutes of a monthly meeting is an opportunity for the teachers to reflect, first individually in writing and later in conversation with the entire group, on ways that their own strategies for solving math problems and willingness to try open-ended mathematical tasks may have grown or improved as a result of the activities from the session.

We recommend hosting meetings on a monthly basis. However, the Southwest Chicago MTC met only six times during the last academic year, skipping holidays and other conflicts in December as well as state testing preparation and other conflicts in March and April. The location for the MTC can be the same for all meetings (as with the NW Iowa MTC hosted at Dordt College) or can rotate among several colleges and universities (as with the Southwest Chicago MTC).

3 Benefits of MTCs

There are numerous potential benefits of a math teacher circle. First and foremost, the attitudes of K-12 mathematics teachers often shift toward a greater enjoyment of mathematically rich and engaging activities as well as a willingness to experiment with more open-ended math problems in their own classrooms. Angela Antonou reported in [1] on data from pre- and post-surveys completed by 13 such teachers at the 3-day workshop hosted at Trinity Christian College. Among the findings were shifts to more inventiveness and more confidence in creating opportunities to develop students' conceptual understanding of mathematics. Reflection journals were equally notable in the high praise from the teachers. Here's a sample:

"Each activity continues to open possibilities I hadn't considered doing with my students."

"I've noticed that I'm becoming less afraid to try different things."

"It made me understand the vast usefulness of productive struggle. My students need to engage in this type of problem weekly! It would help them to persevere!!"

"I appreciate the opportunity to stretch my math skills. I don't often have a chance to do this."

Beyond the benefits for classroom teachers, current math education students at your own institution will benefit from participating at an MTC session. At the Southwest Chicago MTC meetings, the host institution typically invites up to five such students to attend. The NW Iowa MTC provides the opportunity for local undergraduate students to participate as well. We have found that a critical mass of classroom teachers is essential, but having a few undergraduates and/or university faculty members in attendance enhances the experience for everyone.

As illustrated by this paper, MTCs also provide opportunities for professional development among college and university faculty members. Several members of the Southwest Chicago MTC have spoken at local, regional, and national conferences. Clark recently published an article [4] in MTCircular, a national journal that solicits short articles describing original activities used at MTC meetings. Hendrickson devoted her entire dissertation [6] to an analysis of math teacher circles and published a short overview of her findings in MTCircular. Recent upcoming national and regional conferences, including the Joint Mathematical Meetings, MathFest, MAA Section meetings, and the Annual Convention of the National Council of Teachers of Mathematics, have included multiple sessions and even a full panel of presentations related to MTCs.

4 Exemplary Activities from our Recent MTC Meetings

We close the paper with brief descriptions of three original mathematical activities that were presented at our MTCs during the past year. Full descriptions and/or worksheets are available either on the MTC websites, [10] and [11], or directly from the authors. More examples of activities and related classroom resources can be found among the websites listed in the reference section.

4.1 Fair Division

In February of 2016, the NW Iowa Math Teachers Circle played with fair division questions. A fair division problem is one in which a number of participants (also called *players*) seek to divide a heterogeneous resource[1] in such a way that each player feels the resulting division is *fair*. There are at least two notions of fairness: proportionality, and envy free-ness. A proportional division is a division in which each of the N players feels they have received at least $1/N$th of the value of the resource being divided. An envy-free division is a division in which each player values his/her resulting piece at least as much as every other piece.

[1] For the purposes of the problem, it is important that the resource be heterogeneous. A homogeneous resource (such as a can of soda) can be split evenly by volume, but I may be willing to accept a slightly smaller (by volume) piece of a cookie if it has the majority of the chocolate chips.

Many are familiar with the simplest fair division problem: two people seek to divide a heterogeneous resource. A common example is two siblings seeking to divide a cookie fairly. The elegant solution is known as the "I cut, you choose" method: the first sibling cuts the cookie into two pieces she would be equally pleased with, and the second chooses the piece he wants. This leads to an *envy-free* division in which neither sibling values the other's piece above the one s/he ended up with.

Our MTC activity began by exploring this two-player fair division problem. We discussed underlying assumptions and goals, and ensured that we could clearly articulate how the "I cut, you choose" method produces an envy-free division. We then asked the natural question: what about three players? For the purposes of experimentation, brownies were provided.

The solution to the three-player fair division problem is subtler. With some time and prompting, participants were able to construct the 'moving knife' procedure. In this procedure, player 1 moves a knife left-to-right across the brownie. As soon as one of the players believes that, in his/her opinion, $1/3$ of the value of the resource is to the left of the knife, that player yells 'stop!' and player 1 cuts the brownie. The remaining two players use the "I cut, you choose" method on the remaining part of the brownie.

While the moving knife solution is elegant, it produces a division of the brownie which is proportional but not necessarily envy-free. We considered why that is the case, and tried to come up with a way around this obstacle. However, we were unable to do so. This is not terribly surprising, as an envy-free algorithm was not discovered until the twentieth century by Selfridge and Conway [3].

The feedback from the session was fairly positive. The one negative comment was that the problem did not "feel like math". This is true, but the author sees it as a feature and not a bug. While it is true that fair division problems are not generally a part of the K-12 curriculum, they are a rigorous application of mathematical/algorithmic thinking to solve real-world problems, and there are several points of contact with the Common Core State Standards for Mathematical Practice which can be made explicit.

4.2 Numerical and Algebraic Patterns

Exploring number patterns and sequences can provide interesting and accessible activities to actively engage students. Students are challenged to look for relationships and test hypotheses to determine if the patterns that they see are always true or if they describe only some but not all of the values. This exploration builds students' abilities to organize and visualize information and enables them to uncover relationships through different representations and to make generalizations. This type of thinking resonates with the Common Core State Standards. The varying levels of complexity of sequences also allow these activities to be used with students at multiple grade levels.

The May 2017 Southwest Chicago MTC challenged teachers to explore multiple number patterns and representations. We started by considering the standard locker problem [2] and modifications where only even numbered students changed lockers or only odd numbered students changed lockers.

We provided two-color disks as a means of representing closed and open lockers. It surprised us that few of the teachers chose to use the manipulatives as they solved the problems preferring instead to

[2] If, in a school of 500 lockers, one student opens every locker, a second student, beginning at the second locker, closes every second locker, a third student, beginning at locker three, changes every third locker and so on until the 500^{th} student changes the 500^{th} locker, which lockers are then open?

utilize a variety of table structures to organize their work. After some prompting, some of the groups of teachers incorporated these disks. Several interesting number theory results emerged in the final solutions including perfect squares for the original problem and two times a perfect square for the extension involving only the even-numbered students.

From there we moved to exploring sums of consecutive numbers. We had the teachers investigate the sum of the first one hundred natural numbers and then generalize the relationships that they discovered for n natural numbers. This problem is interesting because the teachers were able to utilize a variety of strategies and different representations offered different insights even though the solutions were equivalent. We found that the in-service teachers were able to explore the problem more freely and effectively than some of the college professors. Knowing that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ seemed to inhibit some of the professors from exploring the problem as a middle school student might. We provided linking cubes to enable tactile exploration and through discussion many were able to discover and justify the generalized relationship. Shown in figure 1.

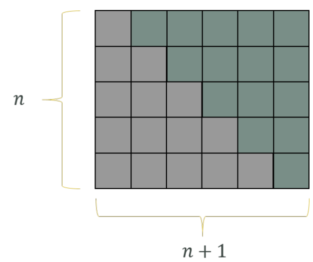

Figure 1: Number of blocks in one staircase = $\frac{n(n+1)}{2}$

The connection to visual representations provided the impetus for the remaining number patterns explored. The teachers initially were given a sequence of numbers, for example: $5, 11, 19, 29, 41, 55, \ldots$ and were asked to find the next term, then the 10^{th} term, and finally the n^{th} term for arbitrary n. The teachers were encouraged to use linking cubes to construct visual representations to supplement the numerical evidence they were generating. The representations shown in figure 2 were considered and a rich discussion of how a visual representation can provide new insight or justification for conjectured numerical relationships ensued. Teacher feedback was positive and some commented that having groups of students construct visual representations and translating them into numerical patterns to challenge their classmates could be a nice extension.

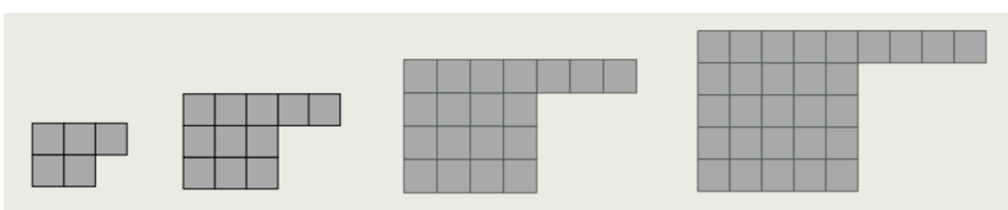

Figure 2: Visual representation of numerical patterns.

4.3 CryptoClue

Cryptography is a natural focus for mathematically rich, open-ended, and accessible activities that bridge academic disciplines and enhance professional experiences of K-12 mathematics teachers. Fun puzzles designed using historical cryptosystems allow the mathematics classroom to become an interdisciplinary learning arena. In this arena, teachers can motivate the study of measurement, frequency analysis, statistics, and modular arithmetic using the historical context and political intrigue of secret codes.

At the February 2017 meeting of the Southwest Chicago MTC, participants solved CryptoClue puzzles using three historical cryptosystems. For the first, teachers deciphered a clue with letters written on a ribbon by identifying the particular diameter of tube from among six options the sequence of letters reformed into a sensible statement. This modern version of a fifth century BC Spartan scytale (rhymes with "Italy") is an easy way to demonstrate how rod diameter can camouflage a message. The second clue was hidden using a shift cipher, where the alphabet is shifted uniformly by an unknown amount. For this variation on the Caesar cipher, teachers used a cipher wheel and were encouraged to experiment until the message was revealed. The final clue was hidden using a monoalphabetic substitution cipher where the denomination and suit of playing cards replaced letters in a random order. Since this message was designed to be too short to succumb to frequency analysis, teachers earned card-letter pairing by correctly solving a series of standard mathematical problems at the middle school level.

After the teachers solved each portion of the puzzle, the facilitator presented some cryptographic extensions, ideas for how to use cryptography in the individual classrooms of the teachers, and resources to aid in clue development. Examples included a rail fence cipher as a variation of the scytale and a Vigenère cipher as an extension of the shift cipher. A few excellent resources for those who wish to develop their own puzzles are books by Singh, Beissinger and Pless, and Lewand; a helpful online tool for both encryption and decryption is the Black Chamber website that serves as a companion to the Singh text.

The CryptoClue puzzles used in the 2017 MTC meeting were developed by Trinity Christian College students enrolled in a cryptography class for use in a junior high mathematics competition. Both MTC teacher participants and junior high competitors were initially reluctant to experiment in solving the puzzles, but all became more willing to explore and experiment as the respective events progressed. Perhaps by having the participants encode messages using each cryptosystem prior to working on decoding the puzzles, the participants might have felt more comfortable and jumped more quickly into the activities.

At the conclusion of the event, teachers were eager to bring tools home to develop more puzzles to enhance their own curriculum; at least one junior high student reported that he continued looking for cryptographic puzzles to solve the summer after the initial exposure to cryptography. Of the college students who designed the puzzles, one reported using cryptography as a basis for a mathematics classroom activity as a practicing teacher while several others reported how study of cryptography expanded their personal view of mathematics.

The puzzles used in the February 2017 MTC are available upon request.

5 Conclusion

Math Teachers' Circles offer opportunities for collaborative solving of engaging, approachable, and worthwhile mathematical tasks. Some tasks are designed to impact K-12 mathematics classrooms by introducing novel and engaging mathematics to classroom teachers as have been described above, other tasks allow teachers to explore typical content in new and interesting ways for example "Exploding Dots" [8] or "Conway's Rational Tangles" [9]. MTCs also offer professional development options for college and university faculty members. For all of these reasons, and many more, we encourage others to start a MTC too!

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[1] Antonou, Angela. (2017). "Impact of the Southwest Chicago Math Teachers' Circle on the Disposition of Teachers Toward Mathematics and Toward the Teaching and Learning of Mathematics." Presentation at *Joint Mathematical Meetings*, Atlanta, Georgia.

[2] Beissinger, Janet, and Pless, Vera. *The Cryptoclub Using Mathematics to Make and Break Secret Codes*. Wellesley, MA: AK Peters, Ltd., 2006.

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[4] Clark, Tom. (2017). Spot It! Mathematical Structure in a Children's Game. *MTCircular*, 10-12.

[5] Common Core State Standards Initiative. 2011. *Common Core State Standards for Mathematics.*

[6] Hendrickson, Katie. (2017): A Teacher and a Mathematician: The Identity Element. *MTCircular*, 4-7.

[7] `http://www.mathteacherscircle.org/resources/math-sessions/` provides a wealth of engaging, approachable, and worthwhile activities for use at your MTC.

[8] `https://www.explodingdots.org/` provides guides and worksheets to do exploding dots, an excellent exploration of place value and polynomials.

[9] `http://www.geometer.org/mathcircles/tangle.pdf` provides a paper by Tom Davis describing how to use ropes to review fractions, functions, slopes of lines, and more.

[10] `https://www.dordt.edu/events/math-teachers-circle` is the homepage for MTC sponsored by Dordt College.

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A Pre-Calculus Controversy: Infinitesimals and Why They Matter

Karl-Dieter Crisman (Gordon College)

Karl-Dieter Crisman attended his first ACMS conference in 2001, and since then has found it to be a great place to talk about connections between teaching, math, and faith. He teaches mathematics, including a senior seminar in math history, at Gordon College in Massachusetts. In addition to doing research in the mathematics of voting, his work with open source mathematics software, and service-learning, he enjoys learning the stories behind the names in our textbooks—especially Marin Mersenne.

Abstract

In teaching calculus, it is not uncommon to mention the controversy over the role of infinitesimals with Newton's and Leibniz' calculus, including Berkeley's objections. In a history of mathematics course, it is a required topic! But rancor over infinitesimals and their role in mathematics predates calculus—so much so that a popular recent book is dedicated to this topic.

In this paper, I will discuss not just the relevant controversies between Cavalieri and the Jesuits, between Thomas Hobbes and John Wallis, but also why they matter to us today, both as mathematicians and as Christians. Perhaps most importantly, I will present the book in question on these controversies as an example of overreach in the history of science, along with suggestions for how to bring the whole story into the mathematics classroom.

1 Introduction

Students often repeat colorful stories we tell them to 'spice up' our classes. And why not? I have long used the non-mathematical anecdotes about John Napier which Eli Maor [28] recounts to give them a sense of the real—and often quirky—people in mathematics. Ramanujan recognizing the taxicab number 1729 as a sum of cubes in two different ways [21] is inspiring in a whole different way, and reinforces the mathematics. What's wrong with a good story?

But there can be pernicious aspects to telling a story. The oft-told meeting between Diderot and Euler where Euler 'proves' God's existence with a random equation seems to have existed (in its usually-told form) only in the playful mind of Augustus DeMorgan[1], and doesn't really do credit to Euler's faith or Diderot's mathematical ability. Some 'liberal arts' math courses still assert the golden rectangle is the basis for art through the centuries, including the Parthenon, even though it has been years since the notion was rightfully debunked (see e.g. [24]).

When there are good true stories to tell, everyone suffers when we tell the false ones. This is especially important when the stories directly impact the matters students are studying. If one tells the story of a man named Hippasus being tossed from a ship with regard to incommensurables, better to talk about what it says about worldviews that such a story seemed credible when first written down centuries later[2]. It's good to tell about Newton versus Leibniz; it's even better to use this as an opportunity to say just

[1] See [12] on some of the history around how this became popular.
[2] See [17], p. 46 for a particularly dry dismissal of this story.

how difficult the process of coming up with calculus was, and how remarkable it is that something so challenging and controversial is now taught to college freshmen as a matter of course.

Along those lines, in this paper, I will discuss some aspects of the story of infinitesimals, and why they matter. But I will also talk about why they don't matter in quite the way a recent popular book says they do, and why that matters. Along the way, we'll see why you've probably never heard of Thomas Hobbes the scientist, and why that matters. My goal is that you would be able to incorporate both the lesson about stories for your classes, as well as to see infinitesimals in a new light that could inspire them.

2 Infinitesimals

Usually an aspiring mathematician first encounters the idea of infinitesimals when studying calculus. In a typical 'modern' curriculum, they are cloaked in the language of limits, but as educators we are not amiss in mentioning Leibniz and how he came up with the $\frac{dy}{dx}$ notation[3]; in differential equations we might even teach students how to use differentials in an formal/infinitesimal fashion. In an ambitious class, or later in real analysis, we may be made aware of Berkeley's famous objections[4] to "ghosts of departed quantities" and his taking to task those who "submit to Authority, take things on Trust" in mathematics but refuse to do so in religion.

Later, in a math history or capstone course we may see fascinating revelations (see [34], [29]) that even Archimedes used infinitely small sections to originate many of his great theorems on volumes and areas, even if he had to hide their genesis in his formal proofs. I will assume that most of us did not profit from a mathematical logic course enabling us to learn 'modern' Robinsonian [31] infinitesimals, though there is once again a movement afoot to make it useful for the freshman audience[5], as the pre-Cauchy practice was in France [6].

But let us start at the beginning; what does it mean for something to be infinitely small? Right off, we can see that such an object must be at least partly notional, since we couldn't directly see it. Nonetheless, there are two ways to conceive of an object that small. A typical analogy, most often associated with the Italian Jesuat[6] brother and mathematician Bonaventura Cavalieri, is of infinitely thin pages of a book.

1. The pages could be so thin that, no matter how (finitely) many you stacked, all of them together would not be as thick as any actual book; you would need infinitely many. Such pages are *infinitesimal*.
2. The pages could be so thin that, no matter how you tried, you could not slice them any thinner; they are *indivisible*. They are parts of the book, but if infinite in number, whether they comprise the whole book was open to question.

[3]Though note that perhaps Leibniz did know what he was doing with regard to infinitesimals; see "Is Mathematical History Written by the Victors?" [4].

[4]See any source on the history of calculus for *The Analyst*; excerpts are in V.12 of [35], though the second quote is from Query 64 at the end, which is easy to find online.

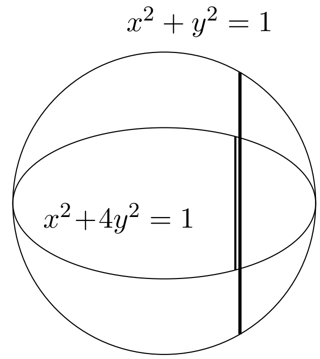
[5]There has always been a small cohort using the now-venerable Keisler text, which he has since retained copyright for and posted online [22]; and several authors of [4] have also advocated for this for some time. For a more recent venture, see publications by Bryan Dawson of Union University, such as [11].

[6]As always has to be explained regarding Cavalieri, he was a member of the Jesuat/Gesuat, not Jesuit, order. It flourished from 1361 until suppressed by papal decree in 1668.

An important distinction between these notions is that indivisibles are typically considered to have strictly lower dimension than the original object, so for instance in the book example the pages are only two-dimensional, even though the book itself is three-dimensional. In contrast, infinitesimals in principle do not have to be indivisible; one could imagine three-dimensional pages which, though already infinitely thin, could still always be sliced to be half as thin again. These would be homogeneous, of the same dimensionality as the page, just ever slimmer. We will return to this idea again.

To see how this might be used to solve 'pre-calculus' problems, consider Figure 1. Using this type of argument, one can relate the area of an ellipse to that of a circle by noting that each indivisible 'slice' of this particular ellipse is half the height of the circle. Hence the total area of the ellipse is half that of the circle as well. Cavalieri himself was both a master user of indivisibles (his book was called *Geometria Indivisibilibus Continuorum*; see [35], IV.5 for excerpts) and perforce a very careful one—cautious to the point of his work being very difficult to read—but this example would have passed muster.

Figure 1: Ellipse area via Cavalieri's principle.



Now we are in a position to state the main thesis of historian Amir Alexander's recent book [2], *Infinitesimal*. Namely, those who believed in such entities, and in using them without caution in mathematics and science, were at the forefront of "the ultimate victory ... [of] a new and dynamic science, [of] religious toleration, and [of] political freedoms." Those who put barriers in the way of their use are responsible for the general and grim intellectual darkness of post-Galilean Catholic Europe; the lack of such barriers made England a veritable beacon of prosperity and (relative) religious freedom. We will unpack this, and the infinitesimals themselves, in the succeeding sections.

3 The Jesuits and Indivisibles

In the first half of his book, Alexander tells a gripping story of Jesuits seeking to control the post-Reformation madness and anarchy in philosophy, theology, and civil order, using their effective network of schools to remove this pernicious and anti-Aristotelian notion[7] from mathematics in lands where they had the upper hand. Only Euclidean geometry in its unique perfection could be taught, and then only as a handmaiden to the notion that there is one truth, one right theology, and one approach to civil order

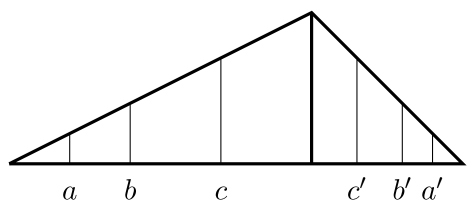
[7] An article [32] to appear in *Foundations of Science* argues that the Jesuits more objected to indivisibles (and not infinitesimals) due to implications for the nature of the Eucharist; Alexander has a response [3] which is presumably to appear with the article when it is officially published.

– headed by the Pope. Infinitesimals, which used sketchy logic, had to be suppressed, or political and religious chaos would result, because they were a "simple idea that punctured a great and beautiful dream: that the world is a perfectly rational place, governed by strict mathematical rules."

Now, there is plenty of truth in this story. First, there is no doubt as to the remarkable success of the Jesuit order in (among other things) establishing rigorous, desirable, 'safe' schools for the minor nobility[8] and nascent bourgeoisie – all in the name of the Catholic/Counter-Reformation. Their most prominent mathematician, Christopher Clavius, was a solid proponent of Euclidean geometry in these schools; under his aegis the church (and hence much of Europe) achieved the long-sought goal of calendrical reform in 1582, where October 15th followed October 4th to make up for the solar year not being evenly divisible by standard days.

Indivisibles did not seem to follow that same impeccable logic. Using similar ideas (indeed, this non-example goes back to Cavalieri), one could assert that two obviously non congruent triangles have the same area. See Figure 2.

Figure 2: Triangles which do not have the same area.



This should be thought of as two triangles, each with the same height but obviously different areas. It should be clear that any vertical segment (indivisible) in the left-hand triangle can be corresponded to a vertical segment of the same length in the right-hand triangle; three sample pairs (a and a', b and b', c and c') of segments are drawn in Figure 2. Cavalieri's principle would, improperly applied, imply that they have the same area.

Such deficiencies had no place in traditional mathematics. While some proponents found various ways to govern their use, for many geometers[9] it was better to simply avoid their use altogether. And it is certainly true that the Jesuits as an order repeatedly disallowed any use or teaching of these concepts in their schools, and many individual Jesuits wrote long screeds against those who used them for the first half of the seventeenth century. (In my view, Berkeley's later objections, though different, are analogous.)

But the overall story, alluring as it may be in our society of absolute freedom and intellectual inquiry, and as exciting as it might be to think of calculus (!) as having political importance so early on[10], is selling a bill of goods. The suppression of indivisibles[11] was surely part of a long fight over new ideas and to what extent the Church could direct or restrict many activities, but not the only one, and much of

[8]For example, Descartes attended the 'College' at La Flèche.

[9]See [25] for many interesting remarks not just on the use of the terms geometer/mathematician, but for the more subtle question of whether anyone before 1650 could properly be called a mathematician in the modern sense.

[10]This is completely separate from whether applications like the Gini coefficient or discussing unequal access to calculus might have contemporary political ramifications.

[11]Not infinitesimals, though Alexander constantly equates the terms; see again [4] as well as [15].

the hyperbolic verbiage used was par for the course at that time.

Despite Alexander's often deft handling of the distance between (post) modern secular readers and the worldview landscape fifty years on either side of Galileo, much of the political discussion is speculation, or simply wrong[12]. As just one example, while in such a political age it was possible that the Jesuits got the Pope to suppress the Jesuat order, and (barely) conceivable they would have been motivated because some of the most prominent advocates of indivisibles (notably Cavalieri) were from it, among his copious endnotes there is not one to be found about this topic.

We should teach about incommensurables, and that the Pythagoreans may have had strong feelings about this; but we should also make it clear that the death of Hippasus is probably just a story. Likewise, the example of the triangles is a good warning against setting up integrals without caution, and the very real fights waged over infinitesimals/indivisibles for some of these reasons (Galileo was an early advocate) is a wonderful topic in a Calculus II course. But let's not suggest it is really about preparing the West for a modern secularist worldview (as even his subtitle *How a Dangerous Mathematical Theory Shaped the Modern World* implies) that Alexander acknowledges no one in question was actually fighting for[13].

4 Hobbes versus Wallis

I have promised a positive statement of why infinitesimals matter, and why they should matter to mathematics educators. But to prepare us for this, we need to observe a second, somewhat more famous controversy, which the second half of Alexander's book focuses on.

Until I began teaching history of mathematics to pre-service teachers I was unaware, perhaps like many readers, that 'the Monster of Malmesbury,' Thomas Hobbes, wrote anything other than political philosophy. Yet his early fame rested as much on contributing to 'natural philosophy', in particular geometry, as anything else. This fact makes it less surprising that only three years after publishing *Leviathan*, Hobbes includes many geometric results, including no fewer than three 'squarings of the circle', in what was intended to be his magnum opus, *De Corpore* ('On Body').

In Hobbes' philosophical scheme everything was material, and all his accounts of the world followed as logically and inexorably from body in motion as geometry did from Euclid's definitions and postulates. This materialism includes mathematics, as demonstrated in the latter third of *De Corpore*. More specifically, Hobbes used methods very similar to Cavalieri's[14]. Cavalieri himself was very careful to leave open the question of whether these indivisibles actually comprise a figure. Consider the opening

[12] It is amazing how many reviewers came to this conclusion. The Chronicle of Higher Education review [14] confirms "[this] world of moribund Roman Catholic thought going nowhere" has little contemporary traction among historians of science. One blogger even went so far as to suggest the story should rather not be told than be told this way, though I hope this paper argues something more positive. In the *Mathematical Intelligencer* review (by the author of [20]) we read, "The errors in this account [are] . . . so great that it is difficult to know where to begin ..." Interestingly, Alexander then responds in the next issue that the review "is so misleading that it cries out for at least a brief correction," [1, 18]. Perhaps a war of their own over infinitesimals is brewing.

[13] In [1], he suggests that the book does not argue "the struggle over infinitesimal methods was 'actually' about politics" but rather that "one will search the pages of *Infinitesimal* in vain for any suggestion of such subterfuge." Perhaps, but even if "the implications of their stance[s] extended . . . to the proper order of the world" then one should stick to their story in their times, rather than veer toward Whiggish history.

[14] Alexander and Jesseph concur in one thing – that Hobbes may have been the only man in Europe to actually read Cavalieri's work all the way through in detail and understand it.

example; do all the leaves of such a book make up the book? Under Aristotelian principles, this would imply an actualized infinity, which is anathema. Hobbes takes a more direct, physicalist, view, that the pages are very small indeed – so small that we do not take their width into consideration – but they are still 'body' (as is everything in existence) so they do still have a width; this is decidedly not the usual viewpoint on indivisibles, but he requires it.

Now, Hobbes reasoned if he could achieve the long-desired (Euclidean) construction of a square with the same area as a circle using his techniques, then surely his entire philosophy would be accepted – including otherwise-distasteful-to-all outcomes such as the Leviathan totalitarian-yet-not-monarchist regime. (Here is where Alexander draws a direct comparison to the top-down regime promoted by the Jesuits, adherents of papal authority as their *raison d'etre*.) Hobbes alluded to his successful squaring on various occasions; when Oxford professor Seth Ward suggested he reveal it, Alexander rightly says "it was a trap" – one Hobbes was incapable of turning down, and the one that led to his downfall.

Philosopher Douglas Jesseph[15] tells the tale of the scientific controversy erupting from these attempts in the extremely scholarly *Squaring the Circle* [20]. It is a comprehensive (at times too comprehensive for the casual reader) description and analysis of the so-called 'Hobbes-Wallis controversy'. For the man primarily responsible for Hobbes' downfall was another Oxford don, erstwhile Presbyterian-party preacher and Parliamentarian-party cryptoanalyst John Wallis.

Through all the regime change of the English Civil War era, Wallis retained favor by adroit maneuvering; unlike many similar men, he seems to have been addicted to producing (and publishing) correspondence aimed at defeating anyone he disagreed with about anything. Since Hobbes was just as stubborn, and since Wallis disliked Hobbes' theology, views on the university, and (to him) inadequate mathematics, their dispute lasted through over twenty years' worth of letters, publication, and 'transactions'. They argued primarily over mathematics, but interspersed accusations of plagiarism, bad Latin[16], and proper authority of a minister of the Gospel.

Hobbes was wrong in his circle-squaring, though, and there is no way around it; even his friends found errors fairly easily, which he kept attempting to fix before publication. Still, this was not somehow an English version of the Italian controversy. In fact, Wallis was even more cavalier about his use of infinitesimals than Hobbes was about his indivisibles (see the next section for an example). In Alexander's telling, Wallis, not Hobbes, plays the role of Cavalieri's school, with repeated direct references to the Baconian ideals of experimental induction and free inquiry espoused by the Royal Society of London. All Wallis wanted, in the best experimentalist tradition, were "theorems that were sufficiently 'true' for the business at hand."

Hence the Society's (with Wallis) repudiation of Hobbes[17] (and so his philosophy) was an example of the key to English (and, by extension, later American?) pre-eminence in science being tolerance of differing opinion, where "a land of many voices. . . discover[ed] its path to wealth and power". This seems to be an awful lot to lay at the feet of the infinitesimals, or even the Royal Society, alone[18].

This is especially so since the technicalities of these questions looked so different in the dawning eigh-

[15] Who also wrote the Intelligencer review [18] of *Infinitesimal*.

[16] At which point it seems relevant to note that Alexander's translation of 'in Guldinum' as 'on Guldin' seems far more appropriately rendered 'against Guldin', like Cicero's 'In Catalinam' speeches.

[17] Despite repeated attempts, Hobbes was *not* a member of the Royal Society; according to both Jesseph and Alexander, this was 'overdetermined'. My favorite reason (Jesseph quoting Quentin Skinner) is his potential to be the 'club bore'.

[18] Economic historians might have some quibbles with it, for instance.

teenth century, where people abandoned Cavalieri or Wallis as dead ends to grapple with utilizing Leibniz' tools (and Newton's in England), with different controversies. Similarly, it is very hard to imagine the political development of Italy or England going very differently with or without infinitesimals. Even if the Royal Society's openness to work with the sort of deficiencies Wallis' had might perhaps be a token of a more general openness in English society to once-heretical ideas, it is not a main reason we should care about this dispute.

5 Taking Care

The first reason it matters is because we, too, should be circumspect with our claims, cautious like Cavalieri in our assertions. For those who believe in fallible humans and infallible God, that alone should cause us to reconsider any overreach; for those who don't, the ample history of human blundering should be sufficient – including Hobbes' 'delusions' with regard to geometry. This goes for grandiose teleological claims, whether by Hobbes or modern authors, as much as for mathematics. If even someone who really did understand geometry could mess up due to philosophical preconceptions, so can we.

Let us see how this works out in Wallis' work in a nice example from his *Arithmetica Infinitorum* (usually rendered 'Arithmetic of Infinitesimals' – see the 2004 translation [33]; Alexander gives this example on page 270) one can use in class. Here, Wallis is about to compare sizes (areas of infinitesimal parallelograms) which he needs in order to reconstruct the area under a parabola (in this particular case, already known by Archimedes). As a crucial step, he adds the perfect squares up to n^2, divided by n copies of that same square.

$$\frac{0+1}{1+1} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$
$$\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$
$$\frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$
$$\frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{1}{3} + \frac{1}{24}$$

At each step, we get one third plus one over $6n$. Wallis then asserts that this will hold forever, as well as in the infinite case, which is what he needs if he is adding up the infinitely many infinitesimally thin parallelograms he has chopped his area into. This would lead to a final answer of $\frac{1}{3} + \frac{1}{\infty} = \frac{1}{3}$, which should look familiar[19] to those who have integrated parabolas before. See [35], IV.13, for Wallis' computation of the same sums for n^3.

This is not by any means a traditional proof, and on the surface certainly has more in common with experimental induction than the mathematical variety. It is a fantastic exercise to ask students to determine what grade this argument might receive in work on infinite series today. However, even if Wallis claims this is more than sufficient, even he recognizes that the argument as stated is not complete, and he doesn't only care about sufficient truth. Wallis, on not showing every detail: "[Euclid] leaves you to supply . . . and then infers his general conclusion. Yet I have not heard any man object . . . " Here is a

[19] It is a relevant trivia tidbit that Wallis himself introduced the now-usual symbol for infinity.

perfect opportunity to lead discussion or ask for responses on what level of proof or detail is sufficient in different contexts[20]. It is interesting to note that Leibniz provides essentially the same justification for his own infinitesimal methods in replying to the less well-known criticisms of his work by Bernard Nieuwentijdt (see e.g. the end of [13], Ch. 9): "For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner . . . "

Wallis cared about this perception in much of his writing. In [19] Jesseph remarks that "Wallis was also intent upon defending the rigor of his approach" in a long correspondence with Leibniz about the relation of his calculus with Wallis' methods, going so far to claim that they are equivalent to Eudoxus' methods: "for it is shorter, nor is it less demonstrative, if it is applied with due caution." Late in his career he even took pains to critique the types of indivisibles which Clavius had used for discussing 'horn' angle measures for similar reasons (see [26]).

Now, there is no doubt that later users of calculus did often defend results partly because they just worked, especially in physical applications[21], while providing evidence of various kinds. As for Wallis, he almost certainly found proper proofs of these results tedious; he may have found that anyone who disagreed with him about it was backward-looking; he may even have been dismissive of those who would wish him to put in the time (all of which I agree with Alexander in his discussion of his correspondence with Fermat on these matters). But that is not the same as suggesting that his defenses were all ironic or crypto-heretical[22], or that he "dismissed thousands of years of tradition."

In any case, Jesseph demolishes the notion that the Hobbes-Wallis scientific controversy was solely about 'deeper sociopolitical differences', even though that may have been a contributing factor to its vehemence[23]. The dispute was largely coldly scientific in nature, even among the personal aspersions. Indeed, we can trace with students through writings of Berkeley, MacLaurin, Cauchy, and others how the same objections persisted or impacted thinking about calculus in one form or another in a slow, groping process. Books intended for classroom supplement about these matters such as [7] or [30] can be valuable in this regard. In any case, by the late 1800s persisting doubts finally led to modern treatment of calculus lacking infinitesimals entirely (and in the mid-1900s a logic-based versions thereof). As Jesseph points out, Hobbes' "objections were not the ravings of a madman."

6 The Creative Impulse

Aside from all controversy and teachable moments, I find there is a deeper reason infinitesimals matter. Now as mathematicians, we know the amazing victories of science they make possible, and teach them to our students. There is almost nothing our technological world touches that isn't somehow enabled by calculus, from modern engineering to predicting space weather to being able to download this very article. But this is not the reason.

I believe infinitesimals matter because they are an expression of the human urge to create, instilled as

[20]This is quoted in [20] in a footnote on pages 177-178, from one of Wallis' screeds, *Due correction for Mr Hobbes Or Schoole discipline, for not saying his lessons right.* Euclid's number theory proofs, including that of the infinitude of the primes, are good to ask about here.

[21]Modern physics essentially follows the same path, with mathematicians scrambling to unpack the often-prescient implications.

[22]Compare the question of whether Hobbes' statements about God were perhaps intended ironically.

[23]Again, one would have liked [2] to have read more like [1] on this point.

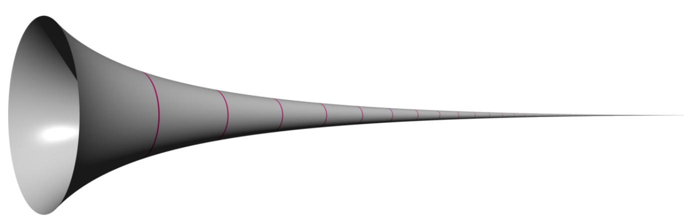
part of the 'imago Dei'. J.R.R. Tolkien expresses this beautifully in his notion of a 'sub-creation' [36]; he himself constructed magnificent worlds that captured his Roman Catholic faith as well as his love of epic saga. But computer programmers creatively make tools to analyze vast quantities of data pouring down on us daily (see 'The Tar Pit' in [8]). Musicians compose and perform and sample to make new pieces that can touch us in new ways while evoking the past (see the discussion on 'Liberating Constraint' in [5]).

Our notions of infinitesimals or indivisibles aren't physical realities, but are creatively brought into math by analogy[24]. We know we can't continually subdivide any real book forever, but we also know that someone more skilled than us could probably slice that slice of bread just once more then we can. These concepts are inherent in mathematics – you can take your pick as to whether that means the 'mind of God', as a shared social construct or something else, [16]. Exploring where those concepts go, if they can solve new problems, if they can lead our imagination to discover new worlds – that is what infinitesimals did and do. Mathematics can do this, whether in progressive situations like England under the Glorious Revolution, or in incredibly repressive ones like the Soviet Union, where mathematics was one of the sole refuges of the creative[25].

To illustrate this, let us consider one of the most beautiful and yet perplexing examples of a victory of the infinitesimal. Evangelista Torricelli was a disciple of Galileo's, like Cavalieri. He embodies the whole story here well, as one in the Baconian experimentalist mindset (he constructed the first true vacuum), devoted to indivisibles (far beyond Cavalieri's conservatism), and possibly a casualty of the Jesuits' distaste for indivisibles. The following result secured his mathematical fame, and figured prominently in the Hobbes-Wallis controversy (see [27]).

Consider the solid in Figure 3, constructed by rotating a hyperbola around its axis. This solid has infinite length but has finite volume. In fact, the volume is equal to that of a cylinder with base the same circle as the solid, and with height the radius of the base circle.

Figure 3: Gabriel's Horn – public domain graphic courtesy of Wikipedia



So something infinite can yet be contained by finitude. (Torricelli managed to give both a proof by indivisibles and one using 'conventional' methods.) Needless to say, both philosophers and mathematicians of the 17th century had a lot to say about this, including how it related to Aristotelian notions that one simply cannot compare finite and infinite magnitudes. When Hobbes discovered it[26] via Wallis' invective, he famously wrote (regarding whether Torricelli could have meant a completed infinitely long solid, rather than an indefinitely long one) that to understand it, "it is not required that a man should be a geometrician or a logician, but that he should be mad." Wallis, on the other hand, not only accepted it

[24]Though not only thus, contra [23].

[25]Just two books worth exploring about this are *Naming Infinity* and *Love and Math*.

[26]Or claims perhaps it isn't there ("I do not remember this of Toricellio"); see Hobbes' *Considerations upon the answer of Doctor Wallis to the three papers of Mr. Hobbes*, where all these quotes appear.

but posited that to understand its implications “requires more of Geometry and Logick than Mr. Hobs is Master of.”

Who could have imagined such a prodigy? It does not end there. Neither author mentions one of the most amazing ultimate products of infinitary thinking. Two centuries later Georg Cantor[27] discovered that there might even be different sizes of infinity—perhaps ironically, a notion at first only supported by Catholic theologians, not other mathematicians. Any course discussing Cantor can take this opportunity to look back on the long history of fights over infinity—and whether we might not want to be more cautious in our own pronouncements about such a challenging concept.

Cantor’s dictum, in the face of much opposition to his results, was that “the essence of mathematics lies in its freedom.” Yet this is clearly not a complete liberty, since there are results which are wrong – like squaring the circle. This expansive freedom also means that we cannot always find a correspondence between mathematical objects like Torricelli’s and ‘physical’ entities, and that we may yet find paradoxes we cannot resolve – notably regarding infinity, as Russell and Gödel helped us see.

Early 20th-century mathematician Hermann Weyl put it better in Section III, “Infinity” in Chapter 4 of [37]. “We stand in mathematics precisely at that point of intersection of limitation and freedom which is the essence of man himself.” In my view, that is precisely the intersection of limitation and freedom imposed by God.

Infinitesimals cannot do everything, and they cannot by themselves lead to political freedom or scientific discovery. But they can bring joy and beauty of discovering all the things of creation, while submitting to our inability to ever fully comprehend their implications. Even, in the words of the edition of the Bible of the very king whose progeny sponsored both Hobbes and Wallis, “that leviathan, whom thou hast made to play therein.”

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Blended Courses Across the Curriculum: What Works and What Does Not

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Abstract

Recent hype around online and blended courses touts the benefits of immediate student feedback, flexible pace, adaptive learning, and better utility of classroom space. Here we aim to summarize the results of a 3-year pilot study using blended courses across the quantitative science curriculum (Mathematics, Statistics and Computer Science), in both upper and lower division, major and GE courses. We present findings on student attitudes towards this format, most helpful course components, student perceived benefits, and how different types of students use the flexibility. This summary can be used to inform best practices in blended (also called hybrid) design, implementation and faculty expectations in the quantitative sciences.

1 Introduction

The use of online learning is increasing and becoming more common in higher education. With concerns about utilizing resources and the need to provide greater access to a variety of learners, the number of universities offering online courses, blended (also called hybrid) courses or fully online degrees is rapidly growing [10]. The 2013 Babson report [1] states that as of 2011, 32% of students has taken at least one online course compared to 9.6% in 2002. Blended courses are a popular alternative to fully online or face-to-face courses. In a recent report by Babson, it was reported that academic leaders still have concerns about the quality of online learning but are far more favorable about courses that combine elements of online instruction with those of traditional face-to-face teaching [2]. The report continues by stating that academic leaders rate the promise of blended course as superior to that of fully online [2]. While a blended course can have significant online content, this format does not eliminate face-to-face meetings. The amount of content delivered online varies from course to course but it typically ranges from 30-79%. [2]. The perceived benefits of blended courses compared to face-to-face courses are things such as flexibility for students and faculty, better use of classroom space, and increased student learning independence.

The literature on student learning gains and satisfaction with blended courses is mixed and varied. Several studies have found that, when comparing the student results on learning outcomes, blended courses are at least as good as face-to-face courses. [3, 7, 8, 14]. Another area of research in blended course design is on student satisfaction with the blended format. There are a number of studies that give mixed results on students satisfaction with blended format versus face-to-face courses [4, 6, 12]. Several of these studies report that the data supports the assumption of student preference for blended courses over face-to-face course, however there often are confounding variables. Recent research has begun to refine these results by looking at whether all types of students benefit and are satisfied with blended learning [11, 13].

However, the implementation of blended courses is not without its costs. There is substantial investment in building the online content. Technologies and the blending of different technologies sometimes require patience and often some type of learning curve. For the faculty teaching blended courses, keeping up the daily correspondence with large classes of remote students can be burdensome. There's a cost to our students- they may need to acquire technical skills to access the online content and have more accountability for their own learning.

In this study we aim to understand better which components of our blended courses the students found value in and in which type of course the bended format is perceived as most beneficial. The primary objectives of this study are: to examine the characteristics of students most satisfied with the blended format, the components in blended courses students find most helpful and how students use the flexibility provided by the blended format. To accomplish this, five blended courses in the quantitative sciences were surveyed: Problem Solving (MTH 303), Introduction to Statistics (MTH 203), Fundamentals of Elementary Mathematics I and II (MTH 213/223) and Introduction to Computer Science (CSC 143). The rest of this paper presents the implementation and results of research that was conducted over a three-year period at PLNU designed to attempt to answer the following four questions:

1. What type of student has the most favorable opinion (the most satisfied with the blended format)?
2. What particular components of the course do students find the most helpful?
3. Which students perceive the most benefit from the flexibility and self-paced nature of blended courses?
4. How do different type of students use the flexibility of the blended format?

2 Methods

2.1 Course descriptions

Point Loma Nazarene University (PLNU) is a private, Christian Liberal Arts university located in San Diego, California. There are approximately 2,500 undergraduate students, most of whom are residential. In 2013 the faculty of the department of Mathematical, Information, and Computer Sciences (MICS) of PLNU decided to test the blended format with some of their courses. We started by offering blended format of our mathematics general education course MTH 303, then the next year offered four other courses in the blended format; MTH 203, CSC 143 and a two course sequence, MTH 213/223. The mathematics general education requirement at PLNU is currently met with either Calculus or MTH 303. MTH 303 is a quantitative literacy course. The prerequisites for this course are elementary algebra and junior or senior standing by units completed. MTH 303 is a three credit course. We offer 10 to 12 sections per year with enrollment between 30 to 40 students in each section. Most students in MTH 303 are non-science majors and many of them are not comfortable with mathematics.

MTH 203 course at PLNU is a statistics course for non-mathematics majors. It is a first course in statistics for the general student and is usually taken mostly by sophomores and juniors. MTH 203 is a three unit course with a pre-requisite of elementary algebra. Majority of the students who enroll in this course are business, nursing, or psychology majors. We typically offer 8 to 9 sections per year with between 30 to 35 students in each section.

CSC 143 is the first computer science course at PLNU. It is designed to introduce students to programming and its uses in other disciplines.It is a three unit course with a pre-requisite of intermediate algebra. Every computational science minor, mathematics, computer science, information systems and software engineer majors take this course at PLNU. These students typically are more familiar with, and frequently use, technology. Because this is the first computer class for the majors of MICS most of those students are freshmen, however, because the computational science minors also take this class, they tend to be sophomores or juniors. We offer one section per year with between 30 to 50 students.

The MTH 213/223 are our mathematics courses for primary education majors. These courses cover the material necessary for the California multiple subject teaching credential (K-8). Each of these courses is three units. The prerequisite for these courses is intermediate algebra and only education majors take these courses. Most students who take these courses are sophomores or juniors. We offer one section of each course per year with between 20 to 35 students in each section.

In the PLNU MICS department, there are two different but similar models of blended courses. The first model (of blended) replaces the seat time by fifty percent, or 1.25 hours of lecture, with online content. This is the model for MTH 303, MTH 203, MTH 213 and MTH 223. All four of these courses have a weekly optional lab and a required face-to-face session. The optional lab is designed to be a resource for students who want additional help on the pre-class activities or on any other course material. Typically students who attend these sessions will independently work on the pre-class activities or homework, asking questions as they go. The single weekly class meeting for these courses typically begin with a short lecture followed by group work. The goal of the lectures is to review and clarify the online material. All four of these courses have both online and written assignments. For MTH 303, MTH 213 and MTH 223, we used the online material found in Pearson's MyMathLab [5]. It consists of videos from the publisher, online practice problems and a e-textbook. Additionally, MTH 303 also had included online timed quizzes. For MTH 203, we used the e-textbook and software called Acrobatiq [9]. It consists of textbook, interactive applets and practice problems.

The second model of blending we use for CSC 143. For this class, we replaced a third of the face-to-face time with online content and offered an additional optional lab. Since this is a programming class, there is a required weekly face-to-face class meeting and a face-to-face lab. The optional lab time is to give additional help to the students and is staffed by both the professor and a lab assistant. The online content is designed by the professor and includes readings followed by online quizzes.

2.2 Instrumentation

In order to measure student satisfaction with blended courses and its components, and with student time allocation, we considered data that falls into two main categories: time on task and student attitudinal data. This data was collected by using an in-class survey administered during class time to all students from all sections during the last two weeks of the semesters. This was done for fall and spring semesters for the years 2014-2016 for a total of 1,301 surveys (73 for CSC 143, 498 for MTH 203, 84 for MTH 213, 44 for MTH223 and 602 for MTH 303). The in-class survey contained between 25 and 28 questions depending on the semester. Eighteen of the questions required a response based on a Likert scale: 6 regarding the students attitudes towards problem solving (these questions were for MTH 303 student only), 5 regarding which aspects of the course they found helpful, 2 regarding their pre-course and post-course desire to take courses in the blended format, 2 regarding their attitude towards the blended format, 2 regarding the benefits of the blended format, and 1 regarding the technology used for the blended portion. The first 6 questions on attitudes towards problem solving (for MTH 303 students only) were on a 5 point Likert scale (not analyzed here), while questions 11-15, 17-19, 22, 23, 26 and 27 were on a 4 point Likert scale. The changes in scale were due to merging an ongoing course evaluation containing items 1-6 with the new blended course assessment.

The remaining 10 questions had varied formats. Students were asked to estimate the number of hours they had spent each week outside of class on each of the four main components of the course: online reading, online practice problems, online quizzes, and written homework. Students were also asked about their anticipated grades (before the course and currently) and their study habits. Self-reported student grades were grouped into the categories: A, B, C, and Other. The authors used the self-reported expected course grade as an estimate of student performance in the course. The self-reported expected grade prior to the course was not used as most students had unreasonably high expectations, and hence there was little variability in this measure. Additionally, the students were asked to list the classes previously taken in the blended format, which, for the purposes of this study, were simply counted.

As mentioned, the number of questions varied by semester. Items 26, 27, and 28 were added beginning in Fall of 2015, hence there are only 226 surveys containing these items. The complete text of the survey is included in Appendix B.

2.3 Analytical methods

All statistical analyses were run in R (`http://www.R-project.org`). In order to assess the practical significance of each finding, effect sizes were computed for each statistical test. In an effort to avoid assumptions on the distribution of Likert scale items, the authors opted to analyze them using nonparametric methods. In order to compare differences in Likert responses between groups, e.g. course or expected course grade, a Kruskal-Wallis H test was used instead of the parametric ANOVA. Additionally the epsilon-squared effect size was computed using the formula $E_R^2 = \frac{H}{(n^2-1)/(n+1)}$. A Wilcoxon signed-rank test, using the V statistic, and Cohen's d for effect size, was used for paired questions in the survey instead of the parametric paired t-test. Chi-squared tests and the effect size $\phi = \sqrt{\frac{\chi^2}{n}}$ were used to assess differences in work patterns between the groups of students. All effect sizes were interpreted using Cohen's suggested cutoffs of 0.2, 0.5 and 0.8 for small, medium and large effects respectively.

3 Results

Between the spring semester of 2014 and the fall semesters of 2016 the survey was completed by 1,301 students: 73 from CSC 143, 498 from MTH203, 84 MTH 123, 44 from MTH 223 and 602 from our GE course MTH303. Summaries of the student responses are given in Table 1. Below we analyze the results as they pertain to each of the research questions.

3.1 Opinion of the blended format

First, we aimed to understand student attitudes towards the blended format and what benefits they perceived. Four items, numbers 18,19, 22 and 23, assessed student opinions of the blended format in general and specifically for the course they were enrolled in. Prior to the blended course, only 40% of the students reported desiring to take a blended course, whereas after their experience 59% desired to take a course in the blended format, see Table 1. A Wilcoxon test comparing the pre- and post desire to take a course in the blended found a significant difference with a small effect size (V=19,333, $p < 0.001$, d = 0.45) showing that overall, after taking these blended courses they were more likely to want to take another blended course.

Students in the different courses varied much in their attitudes towards taking courses in this format. In Figure 1 (n.b., all figures appear in Section 5 at the end of this paper), we see that prior to the blended course, 58% of students in CSC 143 desired to take a course in the blended format, while only 23% of students in MTH 203 desired to take courses in the blended format. Students had a stronger desire to take more blended courses after their experience, with 85% of students in the CSC 143 wanting to take another course in the blended format and 42% of the students in MTH 203 reporting the same. However, it is interesting to note that 88% of students in CSC 143 say the blended format is preferable for this course, while only 29% of students in MTH 203 say the format is preferable. For MTH 203 students, at least, they still do not believe this format is particularly effective in this setting. Kruskal-Wallis tests were run comparing responses between courses on each of the four individual items regarding attitudes towards the blended course format and summarized in Table 2. In each of the four cases there was a significant difference between attitudes in students from the different courses, but the effect size was small ($H > 40.6$, $p < 0.001$, and $E_R^2 < 0.104$).

Additionally, attitudes towards the blended format varied between students at different performance levels. Figure 2 shows the opinions about the blended format broken down by expected course grade. Here we see that students earning higher grades were more likely to say that they wanted to take a course in the blended format, although still a minority of students in general, at 48%. However, after

Table 1: A summary of student responses to survey items.

Item	N	Mean	SD	Disagree(%)	Agree(%)
I found the reading helpful in learning course material	1247	2.55	1.10	38	62
I found the online quizzes helpful in learning course material	1256	2.84	1.03	26	74
I found the online practice problems helpful in learning course material	1207	3.1	0.93	17	83
I found the written homework problems or labs helpful in learning course material	1274	2.93	1.03	24	76
I found the in class activities and lectures helpful in learning course material	1276	3.26	0.93	17	83
Prior to taking this course I wanted to take a hybrid blended course	1199	2.15	1.19	60	40
After taking this course I would like to take another hybrid blended course	1219	2.67	1.10	41	59
The blended hybrid format contributed to my ability to learn	1227	2.53	1.13	45	55
For this course the blended hybrid format is preferable to traditional lecture	1223	2.56	1.2	50	50
I appreciated being able to learn at my own pace	1214	3.07	0.88	15	85
I appreciated the increased flexibility in my schedule as compared to meeting for the traditional hours	1212	3.22	0.93	15	85

Table 2: A summary of Kruskal-Wallis tests for significant differences in attitudes towards the blended format between students in different courses, and in difference in attitudes between students by grade expectation.

Item	Course Comparison			Grade Comparison		
	H	p	E_R^2	H	p	E_R^2
Prior to taking this course I wanted to take a hybrid blended course	40.6	< 0.001	0.03	80.6	< 0.001	0.07
After taking this course I would like to take another hybrid blended course	126	< 0.001	0.104	125	< 0.001	0.10
The blended hybrid format contributed to my ability to learn	122	< 0.001	0.1	124	< 0.001	0.10
For this course the blended hybrid format is preferable to traditional lecture	121	< 0.001	0.1	182	< 0.001	0.15

their experience with a blended quantitative course, a majority of high achieving students reported that they wanted to take another course in the blended format, 69% for A students and 51% for B students. Meanwhile, low achieving students did not desire to take more courses in this format. Interestingly the only group of students where a majority of students reported that the blended format is preferred in their quantitative course, are the A students with 64%. Meanwhile only 28% of the C students say this format is preferable. Lower achieving students are less likely to prefer courses in the blended format. Kruskal-Wallis tests were run comparing differences between students expecting to earn different grades and each of the four individual items regarding attitudes towards the blended course format and summarized in Table 2. Again, in each of the four cases there was a significant difference between attitudes in students at different performance levels, but the effect size was small ($H > 80.6$, $p < 0.001$, and $E_R^2 < 0.15$).

3.2 Most helpful course components

We aimed to better understand which components of a blended course students find the most helpful, and which types of students are more likely to find these components helpful. The majority of students reported that all of the course components were helpful, with the fewest reporting that the reading was helpful (62%) (See Table 1). The results of what the students found to be most helpful broken down by course are presented in Figure 3. We note that the CSC 143 students find all of the course components to be helpful ($> 89\%$ in all categories). While the MTH 203 students are more varied in what they found helpful, with the lowest finding the online quizzes and reading to be helpful (64% and 65% respectively). In nearly every course component, the MTH 223 students, reported lower helpfulness levels in each of the course components, with the lowest (38%) reporting that they found the reading helpful. It is interesting to note that, except for the in-class activities, students in MTH 223 find the course components to be less helpful than the students in MTH 213. Five Kruskal-Wallis tests were run to determine whether the perceived differences in the helpfulness of the each of the five course components were different between courses. In each of the five course components, there was a significant difference, with a small effect size, between courses ($H > 15.25$, $p < 0.002$, $E_R^2 < 0.05$) (See Table 3).

Additionally, we looked to determine whether students at different abilities found any differences in the helpfulness of the different course components, these are reported in Figure 4. The trend here is clear that students reporting a lower course grade are less likely to report any component of the course as being helpful, with the fewest (51%) reporting the reading as helpful. It is notable that all self-reported grade levels find the reading to be the least helpful course component. Again, Kruskal-Wallis tests were run to identify significant differences between how helpful each of the course components were to students at different achievement levels. Students at different achievement levels showed significant differences in how helpful they found the course components, however the effect sizes were small ($H > 46.1$, $p < 0.001$, $E_R^2 < 0.105$) (See Table 3).

3.3 Perceived strengths of the blended format

So then, what are the perceived benefits to the blended course and who holds these perceptions? For this we considered responses to the two questions regarding their appreciation of working at their own pace and the increased flexibility. Students highly appreciated learning at their own pace (85%) and the flexibility in their own schedule (85%) (Table 1).

Table 3: A summary of Kruskal-Wallis tests for significant differences in responses on the individual items between students in different courses, and between students by grade expectation.

Item	Course Comparison			Grade Comparison		
	H	p	E_R^2	H	p	E_R^2
I found the reading helpful in learning course material	15.3	0.002	0.01	67.9	< 0.001	0.06
I found the online quizzes helpful in learning course material	57.0	< 0.001	0.05	60.0	< 0.001	0.05
I found the online practice problems helpful in learning course material	66.1	< 0.001	0.06	127	< 0.001	0.105
I found the written homework problems or labs helpful in learning course material	25.7	< 0.001	0.02	96.8	< 0.001	0.08
I found the in class activities and lectures helpful in learning course material	51.8	< 0.001	0.04	46.1	< 0.001	0.04

Table 4: A summary of Kruskal-Wallis tests for significant differences in responses on the individual items between students in different courses, and between students by grade expectation.

Item	Course Comparison			Grade Comparison		
	H	p	E_R^2	H	p	E_R^2
I appreciated being able to learn at my own pace	78.8	< 0.001	0.065	93.3	< 0.001	0.08
I appreciated the increased flexibility in my schedule as compared to meeting for the traditional hours	85.0	< 0.001	0.07	93.2	< 0.001	0.07

Overwhelmingly, all courses and all levels of achievement appreciate the learning at their own pace and the course flexibility. MTH 203 appreciated these benefits the least with 74% and CSC 143 students appreciating the flexibility the most with 94%. Two Kruskal-Wallis tests were run to compare the differences between courses on their appreciation of (1) self-pacing, and (2) the flexibility of the course; both were significant, with small effect sizes ($H > 78.8$, $p < 0.001$, and $E_R^2 < 0.07$) (See Table 4). Similarly, C and D students were least likely to appreciate the self-paced learning (71% and 76% respectively) and the flexibility (74% and 70% respectively), while A and B students highly appreciating the self-pacing (94% and 82% respectively) and the flexibility (93% and 83% respectively). Kruskal-Wallis tests were run comparing the differences between grade expectation and their appreciation of (1) self-pacing and (2) the flexibility of the course and in both cases there were signifiant differences between grade levels, with small effect sizes ($H > 93.2$, $p < 0.001$, and $E_R^2 < 0.07$) (See Table 4).

Table 5: Summary of how students reported spending their time working by course.

Course	Long Sessions (%)	Short Sessions (%)
MTH 303	0.81	0.19
MTH 213	0.77	0.23
MTH 223	0.70	0.30
MTH 203	0.80	0.20
CSC 143	0.89	0.11

Table 6: Summary of how students reported spending their time working by course by self-reported expected course grade. .

Grade	Long Sessions (%)	Short Sessions (%)
A	0.86	0.14
B	0.76	0.24
C	0.78	0.22
Other	0.74	0.26

3.4 Utilization of the flexible course format

Finally, the instrument shows us that the majority of students tend to work in one long session, while lower performing students worked in shorter sessions. The majority of all students reported working in one long session (80%), versus several short sessions (20%). Table 5 shows how students from the different courses reported working on their course material outside of class. A Chi-squared test was used to assess whether there was a relationship between the work patterns and the course. This revealed that there was not a significant difference in the ways that students spent their time working on the course material in the different courses ($\chi^2(4)$ = 6.65, p = 0.16, ϕ = 0.07).

Additionally, we desired to know if the students at different performance levels studied for the course in different manners, and these results are summarized in Table 6. A Chi-squared test revealed there to be a significant difference between work patterns of students and their expected course grade ($\chi^2(3)$ = 17.5, p = 0.0005, ϕ = 0.12). We see that the highest performing (A) students most frequently reported working in long-sessions (86%), as opposed to the other categories ($< 78\%$).

The survey included questions on the number of hours students spent working on each of the course activities. As the activities across courses differed, the results are not presented here, but are summarized in Appendix A.

4 Conclusion

Online and blended learning have been very popular in recent years, with many perceived benefits to educators and students. Some of the claimed benefits include increased scheduling flexibility, more personalized instruction, self-paced learning, and more efficient use of resources [10, 13]. Despite the hype, we find that many students do not actually share the enthusiasm, although they do appreciate the benefits in flexibility and self-paced learning. Additionally, this study aimed to identify course components which students found particularly helpful, in order to aid instructors in course design. However, few patterns emerge from these results.

When initially developing these courses faculty believed many students would enthusiastically support the blended approach due to the increased scheduling flexibility and self-paced learning. However, we found that although students appreciated these features, they did not desire to take blended courses before, and although their desire to take blended courses increased after their experience, the majority of students still did not desire to take courses in the blended format. Here we find that although students perceive many benefits to the blended course format, they still tend to favor other more traditional pedagogies in quantitative courses. The only groups of students where a majority prefer the blended format for these courses are students in the upper division MTH 303, majors in the CSC 143, and students expecting to earn an A. It would appear that high-performing students and more mature students, perhaps who have busier schedules, tend to favor the blended format, whereas students who did not perform as well, or who have less higher education background prefer to spend more time in class. This finding is consistent with the results found in [6, 11], suggesting that the blended format may be less desirable for lower performing and less experienced students.

Another purpose of this study was to identify course components which students found most helpful in order to aid instructors in designing blended courses. Overall the students found all of the course components to be very helpful, although reading was the least helpful of all. Low performing students and students in MTH 223 were least likely to find the reading helpful. Reading mathematics and computer science is certainly not an easy task, and the authors have noted that even in traditional formatted courses, students frequently complain about reading mathematics. This result may be more indicative of student attitudes towards reading in general.

Finally, we find it interesting that despite instructor suggestions to work on the course material in small blocks of time, students typically used the flexibility to work in very long sessions. This behavior was fairly consistent across the curriculum, but more pronounced among high performing students. As our study provides limited understanding of this behavior, in the future it would be interesting to try to further understand why they work this way when given the opportunity.

As educators we would make several recommendations for consideration when considering implementing courses in the blended format:

1. Faculty need to spend more time discussing the reasons for the blended courses and helping students understand how the blended format can aid in their learning.
2. Faculty need to help raise student awareness of different studying strategies.
3. Faculty need to be more deliberate in teaching reading skills and emphasizing the importance of reading.

In future work, we would expand on these results to gain a better understanding of the impact of the blended format on learning gains. This would allow educators to better understand the benefits of blended format on learning and potential tradeoffs to some of the positive aspects of the blended courses.

Additionally, with the busy schedules of our students and their widespread use of technology in their daily lives, we would like to better understand why they do not have more favorable opinions towards blended pedagogies.

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5 Figures

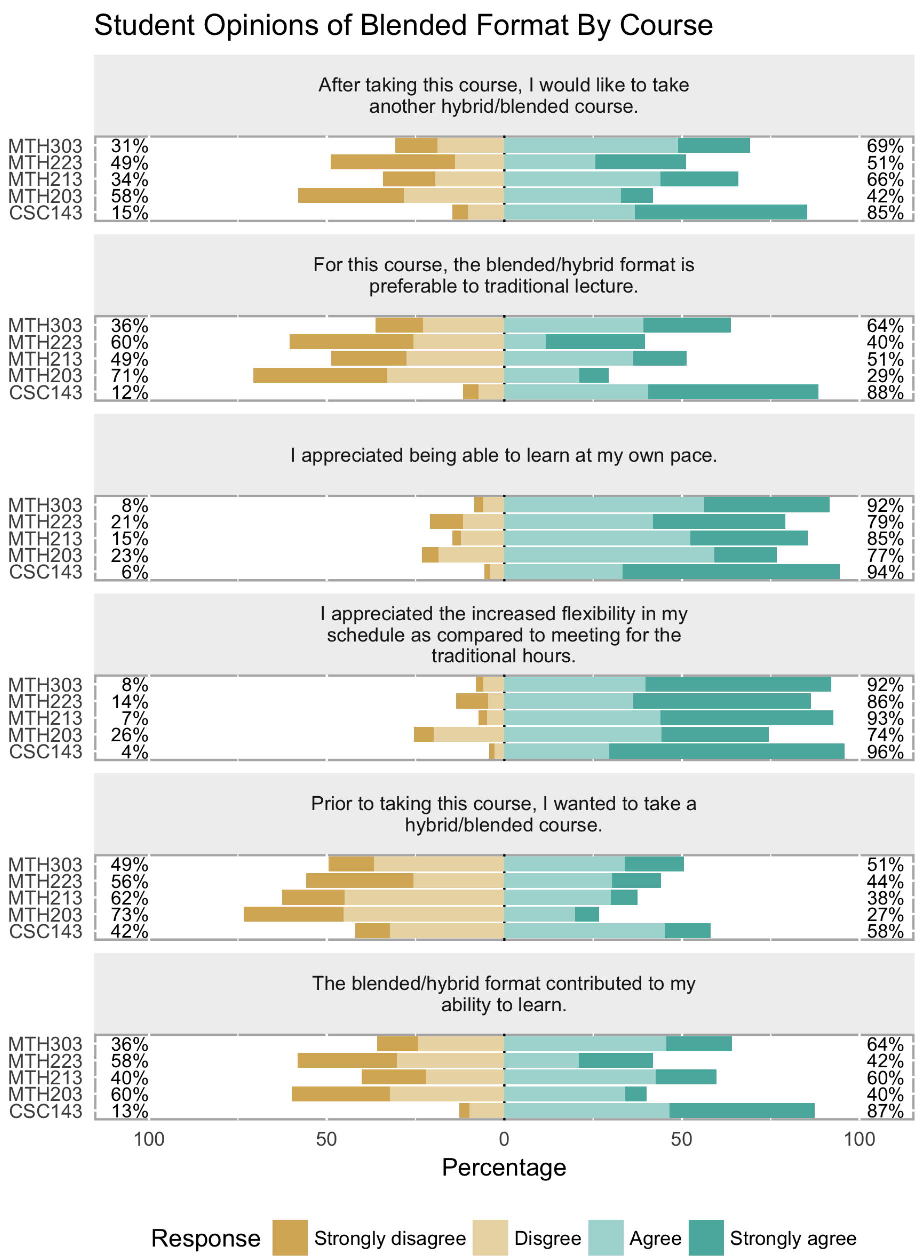

Figure 1: Summary of student attitudes towards blended learning by course. MTH303 is problem solving. MTH 213 and MTH 223 are elementary mathematics I and II, respectively. MTH203 is introduction to statistics. CSC143 is introduction to computer science.

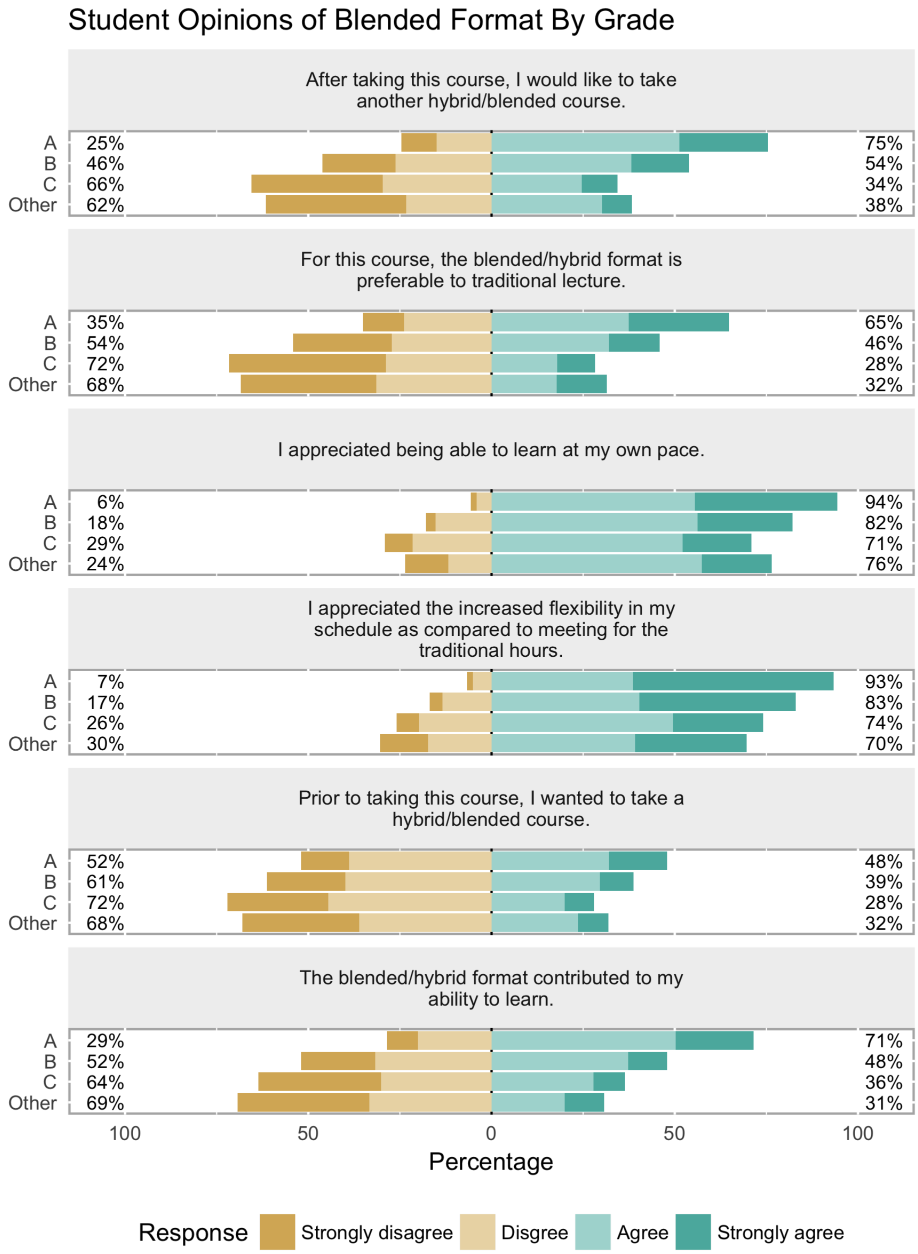

Figure 2: Attitudes towards the blended format by self-reported expected course grade. The Other category represents D, F and Unknown grades.

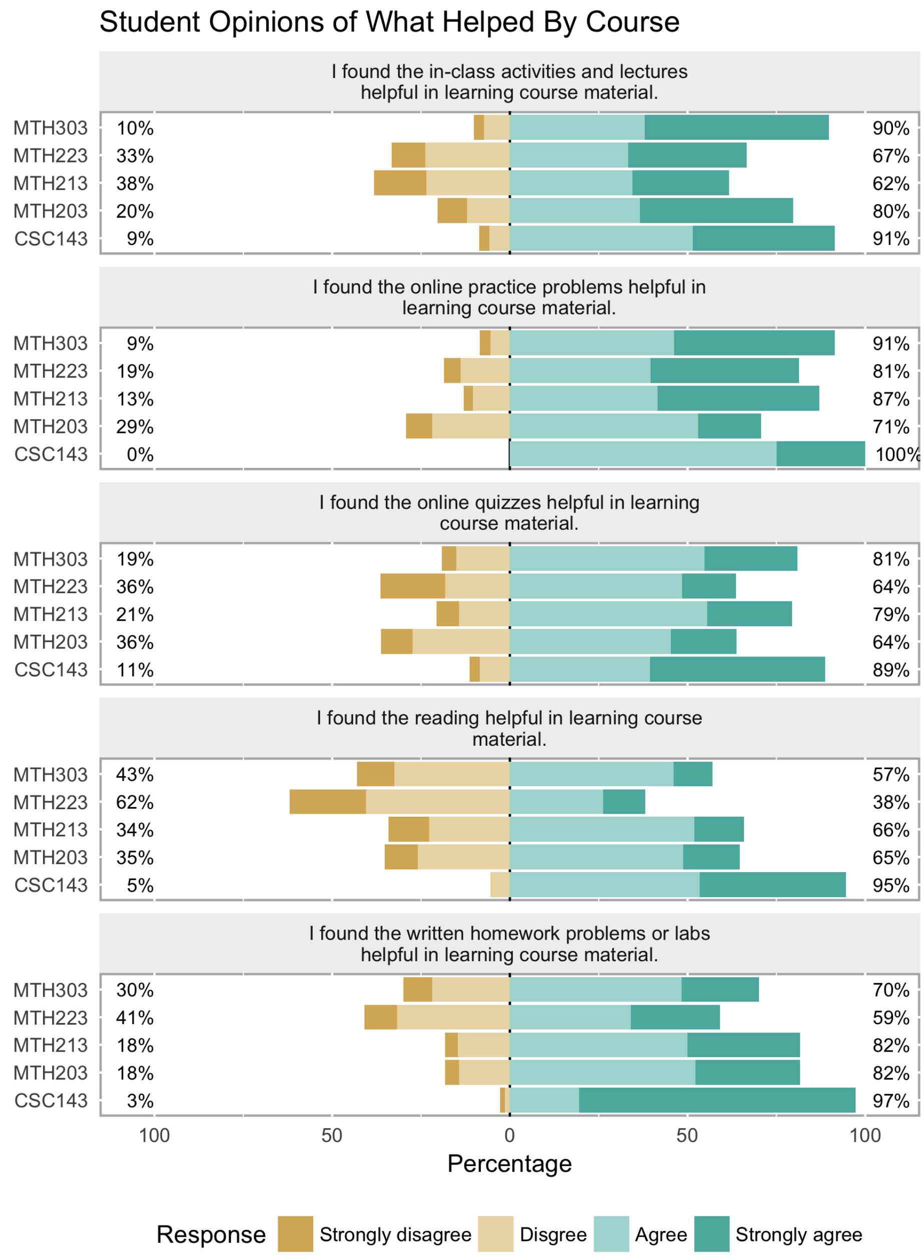

Figure 3: Summary of student responses as to what course components were most helpful in their learning by course. MTH303 is problem solving. MTH 213 and MTH 223 are elementary mathematics I and II, respectively. MTH203 is introduction to statistics. CSC143 is introduction to computer science.

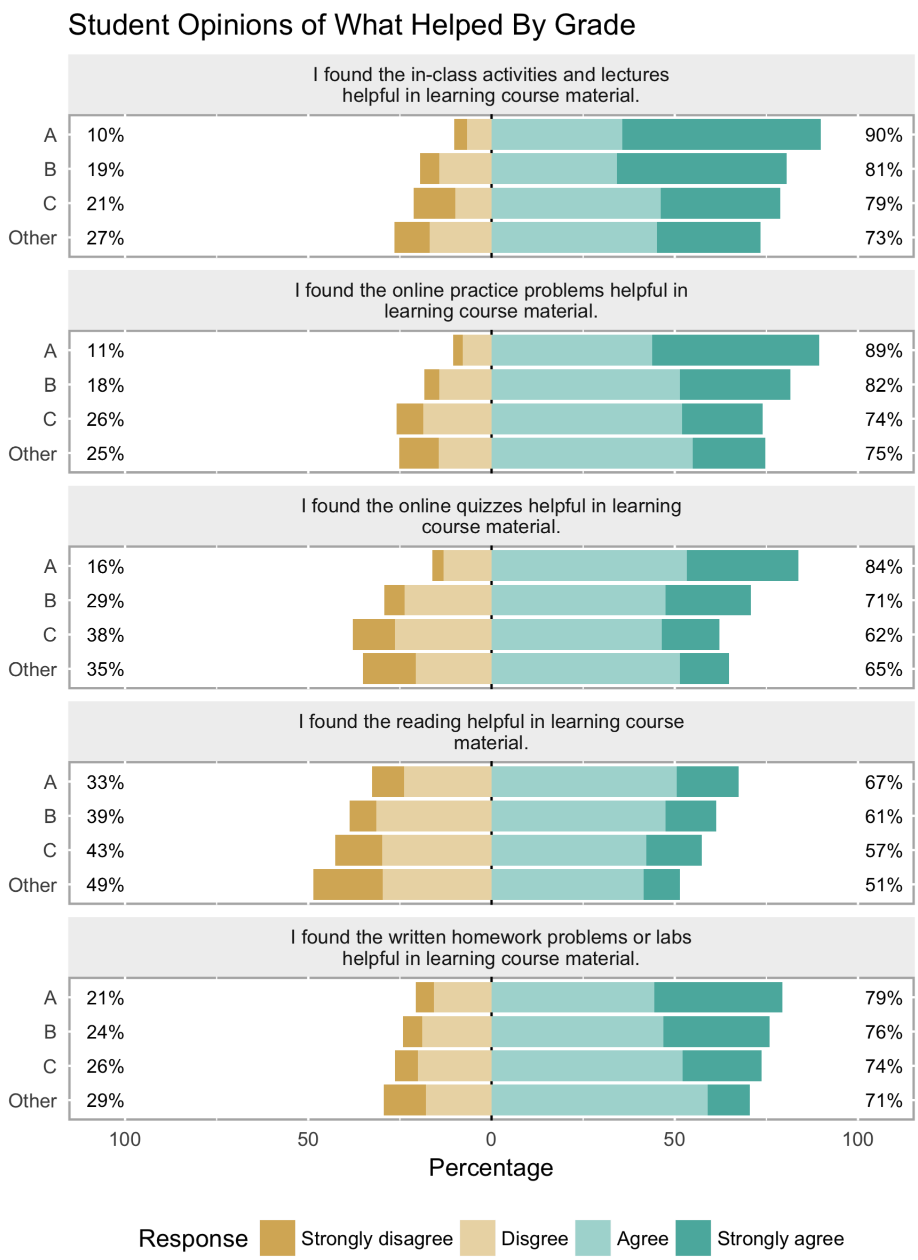

Figure 4: A summary of which course components were most helpful by self-reported expected course grade. The Other category represents D, F and Unknown grades.

Appendices

1 Summary of hours spent working on the course components

Question	N	Mean	SD
Approximately how many hours per week did you spend outside of class doing the reading (online or textbook)?	1243	2.21	3.05
Approximately how many hours per week did you spend outside of class doing the online quizzes?	1232	1.54	1.69
Approximately how many hours per week did you spend working on the online practice problems?	1228	1.63	1.87
Approximately how many hours per week did you spend on the written homework or labs?	1251	2.16	1.96

2 Blended Course Survey

		Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Does Not Apply
1	In this class, we have been directly involved in problem solving activities.	SA	A	N	D	SD	NA
2	This class has contributed to my ability to solve different types of problems.	SA	A	N	D	SD	NA
3	This class has expanded my methods of exploration in problem solving.	SA	A	N	D	SD	NA
4	This class has contributed to my ability to make educated guesses and check their correctness by analyzing their implications.	SA	A	N	D	SD	NA
5	This class has helped me to understand major concepts, methods and applications of critical thinking.	SA	A	N	D	SD	NA
6	This class has helped me to see the importance of problem solving in our modern society.	SA	A	N	D	SD	NA
7	Approximately how many hours per week did you spend outside of class doing the **reading (online or textbook)**?						NA
8	Approximately how many hours per week did you spend outside of class doing the **online quizzes**?						NA
9	Approximately how many hours per week did you spend working on the **online practice problems**?						NA
10	Approximately how many hours per week did you spend on the **written homework** or **labs**?						NA
11	I found the **reading** helpful in learning course material.	SA	A		D	SD	NA
12	I found the online **quizzes** helpful in learning course material.	SA	A		D	SD	NA
13	I found the **online practice problems** helpful in learning course material.	SA	A		D	SD	NA
14	I found the **written homework problems** or **labs** helpful in learning course material.	SA	A		D	SD	NA
15	I found the **in-class activities** and **lectures** helpful in learning course material.	SA	A		D	SD	NA
16	How did you typically work on course material?	A few long sessions			Several short sessions		
17	The course technology was easy to use.	SA	A		D	SD	
18	Prior to taking this course, I wanted to take a hybrid/blended course.	SA	A		D	SD	
19	After taking this course, I would like to take another hybrid/blended course.	SA	A		D	SD	
20	At the start of the course, what letter grade did you expect to earn?						
21	What letter grade do you currently expect to earn in this course?						
22	The blended/hybrid format contributed to my ability to learn.	SA	A		D	SD	
23	For this course, the blended/hybrid format is preferable to traditional lecture.	SA	A		D	SD	
24	List any other blended/hybrid courses that you have taken.						
25	Assuming that the class is taught in blended/hybrid format, what are some ways that we could improve this course? (If needed, use the other side of the paper for your answer to this question).						
26	I appreciated being able to learn at my own pace.	SA	A		D	SD	
27	I appreciated the increased flexibility in my schedule as compared to meeting for the traditional hours.	SA	A		D	SD	
28	Regarding the optional class (open lab sessions) check all that apply:						
	I come almost every week, but do not find it helpful.	☐					
	I come almost every week and find it very helpful.	☐					
	I do not come often because it is my only day to sleep in.	☐					
	I do not come often because I understand the material from reading alone.	☐					
	I do not come often because I have scheduled something else at that time.	☐					
	Other:	☐					

Mentoring as a Statistical Educator in a Christian College

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Abstract

In this paper, I present principles based on more than thirty years of intentional mentoring as a statistical educator in a Christian college. I believe this mentoring has been enhanced due to the setting—a Christian college, and the discipline—statistics. I review distinctives of the Christian college setting that positively impact mentoring in any discipline with respect to the mentor, the mentee, and the pervading campus atmosphere. I focus on mentoring as a statistical educator by specifically considering the following: attracting students to the discipline of statistics, preparing students for careers using statistics, and preparing students for graduate study in a statistics-related field. For each, I consider principles of successful mentoring in statistics at the undergraduate level regardless of the type of institution and how these principles can be expanded within the context of a Christian college.

1 Introduction

In this paper, I present principles based on more than thirty years of intentional mentoring as a statistical educator at Messiah College, a Christian college of the liberal and applied arts and sciences. In addition to two introductory statistics courses provided as a service to other departments, Messiah's statistics program includes eight more mathematically-based courses of which six can be chosen to form a statistics minor. This minor was initiated in 1985 with the first cycle of that program completed in 1987. One measure of the program's success is the large number of strong students who have decided to pursue graduate work in a statistics-related field. To date, more than forty-five students have done so at more than thirty schools.

As I have reflected on reasons for the success of the statistics program at Messiah College, I conclude that mentoring plays a significant role. My ***thesis*** is that ***this mentoring is efficacious due to two factors: the setting - a Christian college, and the discipline - statistics.*** Several years ago, a graduate encapsulated these two factors as follows.

> "There are just a few pivotal moments in my life that have been transformational, and you have been part of two of them - encouraging me in the field of statistics (both to take the minor and to pursue graduate studies) and helping me to reconcile the secular and the sacred in my life's pursuits. I would be foolish to think I have arrived where I am today because of myself; that's

> why I like to take the opportunity to thank those who have given me a hand up along the way, like you."

In this paper, I use *mentor* as both a noun and a verb. As a noun, a *mentor* is "someone who teaches or gives help and advice to a less experienced and often younger person" (Merriam-Webster (2016)). As a verb, *mentor* is the action of being a mentor. A *mentee* is the one being mentored. At Messiah, my mentoring has been purposeful but informal. My interactions could be scheduled (e.g., during a class or in my office) or incidental (e.g., in the library or dining hall). The mentee is always a student, and my primary goal is to help that student use their God-given gifts in a way that brings maximum satisfaction and fulfillment to the student.

I assume that the reader already has some appreciation of the role of mentoring in higher education. For those who do not, perhaps because they were never mentored themselves, I encourage consulting the rich body of literature which espouses the benefits of mentoring in higher education. I direct the interested reader to Johnson's (2007) thorough consideration of the topic which includes an extensive list of references. Zachary (2000) considers the higher education setting, as well as business and nonprofit settings, and also includes an extensive list of references.

"Good mentoring relationships (mentorships) in academic settings are dynamic, reciprocal, personal relationships in which a more experienced faculty mentor acts as a guide, role model, teacher, and sponsor of a less experienced student (protégé)" (Johnson (2007: ix)). Mentoring is much more than academic advising. Many of my most significant mentoring relationships involved students who were not my formal academic advisees.

While mentoring is done to some extent at all undergraduate institutions, religious or secular, and in all disciplines, statistics or otherwise, the nature of mentoring can depend on the type of institution and the discipline involved. For example, if the spiritual dimension is included, mentoring at a religious institution may be more comprehensive than that at a secular institution.

The intended audience for this paper consists of educators in Christian colleges, in general, and statistical educators in Christian colleges, in particular. While readers from wider audiences can benefit from the ideas presented in this paper, those from the target audiences should benefit the most.

I develop my thesis by focusing on the setting in Section 2 and the discipline in Section 3. In Section 2, I review distinctives of the Christian college setting that have a positive impact on mentoring in any discipline. I consider the mentor, the mentee, and the pervading campus atmosphere. In Section 3, I focus on mentoring as a statistical educator by specifically considering the following: attracting students to the discipline of statistics, preparing students for careers using statistics, and preparing students for graduate study in a statistics-related field. For each, I consider principles of successful mentoring in statistics at the undergraduate level regardless of the type of institution and then how these principles can be expanded within the context of a Christian college. I conclude by revisiting the thesis of this paper in Section 4.

2 The Christian College as a Setting for Mentoring

2.1 Overview

In this section, I focus on a particular type of mentoring - between a Christian educator and a Christian student within a Christian college. I look at the Christian college as a setting for mentoring by considering the key factors - the *mentor*, the *mentee*, and the *campus atmosphere* - that influence the mentoring process rather than the mentoring process itself.

In describing mentoring in Christian higher education, Penner (2001: 9) asserts that the Christian college is a natural setting for mentoring. “The Christian community has incentive to be a mentoring community based on its nature as a caring family, explicit directives for spiritual elders to equip youngers, and the mentoring examples of Barnabas and Paul. At heart, mentoring is about being concerned not only with one’s own success but also that of one’s colleagues and students. Much of such successful facilitating requires not only the transmission of information but also the care and encouragement of persons. To the extent we are able to do this well, we will achieve the end goal of education.”

2.2 The Mentor

In this paper, I focus only on mentors who are educators. In an explicitly Christian college, I will assume each educator has a personal faith commitment to Christ that can be articulated. Such a commitment can be expected to shape each part of the educator’s life - private and professional. Using Jesus Christ as their example, all Christians are urged to strive to attain qualities or virtues such as selflessness, humility, gentleness, graciousness, and patience (Ephesians 4:2, Philippians 2:3, Colossians 3:12). When these qualities are reflected in the life of the educator, that educator’s mentoring will be mentee centered. That is, mentoring will focus on what is best for the mentee, not on what is best for the mentor (e.g., developing a personal professional portfolio.)

Hopefully, Christian educators think of their position as a calling or vocation and not just a career. With this view, an educator at a Christian college is in that position because they sensed a calling to teach specifically in a Christian college. The deliberate choice of this type of teaching venue should shape how educators view their responsibilities to their students. Appropriately telling students of this choice sends them a powerful message.

A Christian mentor is more likely than a nonChristian mentor to appreciate a mentee’s other interests and motivations such as vocational Christian service or ministry as well as the fact that materialistic concerns may have little influence on one’s choice of a vocation. Recognizing the importance of the individual, the Christian mentor should not adopt a one-size-fits-all approach and can help the mentee navigate through the discipline/ministry decisions.

2.3 The Mentee

In this paper, I focus only on mentees who are students. In general, in an explicitly Christian college, one cannot assume each student has a personal faith commitment to Christ that can be articulated. However, in this paper, I assume that the students being mentored will have such a personal faith commitment. One manifestation of this commitment is that students should have a keen sense that their natural abilities are God-given, that God expects them to be good stewards of those abilities (Matthew 25:14-30), and that God has a plan for how those abilities should be used vocationally. This perspective helps them through the tough times when they might otherwise doubt the path they have chosen (e.g., while in a particularly challenging course or in graduate school).

The idealism which incoming students bring into college often has a "save-the-world" theme. If a Christian student combines this idealism with the example and call of Jesus Christ to be a servant, the student can be motivated by a mentor in additional ways. Examples in the discipline of statistics will be given in the next section.

For most students, a major factor in the choice to attend a Christian college instead of a secular one is that the college is Christian. While likely not fully understanding the nature of a mentoring relationship, or perhaps not even being familiar with the word "mentoring," these students expect to relate to faculty members in a Christian college differently than they would in a secular institution. Thus, these students enter college with an expected appreciation of at least some of the distinctives that a Christian mentor provides.

2.4 Campus Atmosphere

When educator-student mentoring is done in a Christian college, it most likely will involve a Christian mentor with a Christian mentee. As noted before, this is the nature of the mentoring relationship I assume in this paper. A Christian-with-Christian pairing can occur on a secular campus, but it will occur with much less frequency.

A significant factor that makes Christian-with-Christian mentoring on a Christian campus different from that on a secular campus is the pervasive Christian atmosphere that exists on the Christian campus. The messages conveyed in the Christian-with-Christian relationship are reinforced by those messages conveyed in most, if not all, aspects of the student's curricular and cocurricular life.

One such message is the importance of a sense of vocation. It is easy for students to confuse the concept of career with life calling or vocation. The concept of Christian vocation should be more motivating to Christian students than the concept of vocation in general because it combines the concept of vocation with the concept of Christian responsibility.

In a Christian college setting, the mentor and mentee are likely to share more core values resulting in the mentee trusting more of what the mentor says. This trust is based on something more than expertise. Shared core values can extend the mentor-mentee relationship into a friendship and/or fellowship relationship.

Closely related to the shared core values in a Christian college setting is the holistic nature of mentoring there. In a Christian college, mentoring can address the spiritual component of a student's life in addition to the usual academic and personal components. The spiritual component is crucial in the mentor's modeling of a sense of vocation. In broad terms, the academic component involves the mentor's profession or career while the personal and spiritual components involve the mentor's calling or vocation. Thus, the setting of a Christian college is more conducive to the mentor instilling a sense of vocation in the mentee.

3 Mentoring as a Statistical Educator

3.1 An Overview

A substantial part of the general statistics literature on mentoring deals with one statistician mentoring another (i.e., where both the mentor and mentee are professional statisticians). For example, Allen (2005) is recognized for his expertise in mentoring and was the first recipient of the Jeanne E. Griffith Mentoring Award which was established to encourage mentoring of junior staff in the Federal statistical system. His experience is primarily in the statistician-with-statistician context. His insights on mentoring are based on "40 years of observing individuals who achieved solid mentoring results" (Allen (2005: 9)) and the principles he presents are applicable in a wide variety of fields.

In the statistics *education* literature, mentoring is usually considered only in a limited sense, at best touching on one of the three areas in the following subsections: *Attracting Students to the Discipline of Statistics*, *Preparing Students for Careers Using Statistics*, and *Preparing Students for Graduate Study in a Statistics-related Field*. There is no attempt to integrate all three areas.

An exception is the literature describing the five-year $1.3 million National Science Foundation grant awarded to Legler, Roback, and Richey (2004) at St. Olaf College. St. Olaf already had a well-developed undergraduate statistics program prior to the awarding of this grant. The goals of this grant addressed four areas: the three areas in Subsections 3.2 through 3.4 and a fourth area which was one of two primary goals for the project, attracting statistics PhD's to faculty careers in four-year colleges. (Because of the ever-increasing demand for statisticians with graduate degrees, the other primary goal of the project was to increase the number of graduates from four-year colleges who pursue graduate study in a statistics-related field.) Legler, Roback, et al. (2010) report on one work product resulting from this grant, the creation of an interdisciplinary undergraduate research program. This program complements the already-existing robust student research programs at St. Olaf, particularly in the sciences, and builds on the relatively unique interdisciplinary collaborative nature of the field of statistics.

Mentoring by a statistical educator is facilitated when statistics is a respected discipline within the college. Rarely at the undergraduate level is statistics housed in its own department. It is often combined with mathematics with or without other disciplines. In this setting, it is not uncommon for statisticians to be regarded as somewhat "second-class citizens" by mathematicians. Some of this is due to mathematicians not understanding what makes professional endeavors (e.g., scholarly research) in statistics different from those in mathematics. This treatment can also be fueled by professional rivalry or selfishness. As can be seen in Subsection 3.2 and elsewhere, there are aspects of statistics that may make it more attractive to students as a career option than mathematics.

Hopefully, this mathematics-statistics alienation is not present, or at worst minimal, in Christian colleges. A refreshing example is my experience at Messiah College where statistics is housed in a mathematics, physics, and statistics department. Instead of alienation, I have experienced a very supportive environment. My mathematician colleagues are happy for the success of our statistics graduates and do not resent their choice of statistics over mathematics. Through their teaching of mathematics, they are an integral part of their academic preparation.

3.2 Attracting Students to the Discipline of Statistics

One may tend to think of mentoring taking place with students that are already part of one's program or discipline, but as noted by Gray (2005), mentoring can be used to recruit students into one's discipline. Landes (2009) considers problems and solutions in recruiting individuals into the profession of statistics. While he recognizes the importance of attracting students to the discipline of statistics, his paper is broader in scope than the topic addressed in this section. He deals with the public's misperception of a statistician's professional activities. He defines "public" as those outside of the statistics profession. Landes organizes the problems and solutions drawn from his extensive review of the statistics literature on these issues. He also suggests some strategies drawn from a more general body of literature, including literature specific to other disciplines, which may be applicable for statistics. Landes cites Eby (2006) as providing a helpful list of suggestions that statistical educators can use to attract students to the discipline of statistics and cites some of the suggestions specifically. Eby (2006) wrote to a secular audience, and thus does not suggest ideas unique to the context of a Christian college. I now consider such ideas in this section. However, I first consider at what point in the process mentoring should begin.

It is important to start early. One can start recruiting students even before they begin college by speaking to high school classes. This is particularly helpful in the discipline of statistics because, as Landes (2009) notes, most individuals have little or no understanding of a professional statistician's activities. One of our graduates who is a high school teacher uses Skype to have professional statisticians interact with the students in her *AP Statistics* class.

A more formal mentoring relationship begins in college. The educator must take the initiative in establishing the mentoring relationship. As Johnson (2007: 119), writing from a secular perspective, notes, "...owing to developmental immaturity and low awareness of the value of mentoring, undergraduates may be less assertive and intentional in pursuing potential mentors. In spite of these obstacles, mentoring college students can be deeply rewarding for faculty and genuinely life-altering for undergraduates. Rarely will you have the opportunity to more profoundly shape both a student's life and career path than in the context of bachelor education." Understanding the idea expressed in the last sentence should strongly motivate Christian educators.

One of the more attractive features of statistics to students is the increasing gap between the demand of professionally trained statisticians and the supply of such professionals. Lindsay, Kettenring, and Siegmund (2004: 406) address the seriousness of this shortage in their report on the future of statistics. Dixon and Legler (2003) focus on the serious shortage of statisticians needed to work on applications in the biological sciences. Such statisticians are called biostatisticians. (Biostatisticians are involved in the development of new drugs, the evaluation of the effectiveness of medical procedures, efforts in fighting disease and other public health problems, and environmental issues.) In presenting this message in the Christian college setting, the mentor's emphasis can be opportunity-driven (i.e., the many opportunities that exist to use one's God-given abilities) in addition to career- and dollar-driven.

Knowing that there is a greater likelihood of being employed in one's actual field allows students to be more focused in their preparation. It is reasonable for them to think that what they are doing now in college will likely be closely related to what they will be doing later. The Christian student who has a sense of God-given abilities and the responsibility to use them for His glory can especially appreciate this opportunity to develop their abilities in a purposeful manner.

The shortage of professionally trained statisticians also allows a statistical educator (mentor) to recruit students into the discipline with integrity, something which should be quite important to the Christian educator. One would expect all educators to be very enthusiastic about their respective disciplines. A statistical educator can say, with integrity, to a prospective student, "If you pursue this discipline and do well in your studies, there is a very great likelihood, particularly with a graduate degree, that you will find a meaningful professional position." Even if the bachelor's degree is the terminal degree, there is a greater likelihood of employment in a discipline-related field, in this technological and data-driven age, for the student with preparation in statistics.

Another attractive feature of statistics is the breadth of application. Wherever there are data with variability, there is the opportunity for the application of statistics. There is virtually no discipline devoid of data. No other scientific discipline can be applied in more discipline areas than can statistics. With the ever increasing presence of data-driven research, this supremacy is likely to be maintained. Beginning in introductory statistics classes, the educator should deliberately make the students aware of the breadth of application. Encountering real-life examples helps the students see purpose in studying the discipline. Having statistical consulting experience is particularly beneficial for an educator since an overview of some of the diverse applications encountered in those experiences can be presented in class.

Outside speakers can speak of statistical applications in their areas. Practitioners not only bring relevance to their presentations, but they also tend to be enthusiastic about their discipline. Possible speakers are graduates from your institution who are now in graduate school in a statistics-related field or have already earned a graduate degree; professional statisticians from academia, industry, or government; and current students with statistics-related internships. Choose speakers who will describe statistical applications, promote statistics as a career, and/or encourage graduate study in statistics. When the speakers are Christians (e.g., former students), have them include the concept of Christian vocation in their presentations. Former students are particularly effective recruiters because current students often perceive themselves to be more similar to former students from your institution than to faculty members. Thus, they may be more open to their encouragement and suggestions. Also, these former students likely were involved in a mentoring relationship during their undergraduate years and can help current students see the value of such relationships.

The breadth of application should be particularly attractive to Christian students who may have more diverse interests (e.g., ministry) than other students. Not only can Christian students have interest in most typical areas of application, they can also be interested in all sorts of ministry-type applications. Presentations about ministry-type applications can be well received by students in a Christian college. For example, their statistical skills can be used in church and parachurch organizations that need help in designing studies and analyzing data. Some specific examples of ministry-type applications are presented in the following sections.

As noted in Subsection 2.3, many students enter college with a desire to make a difference. Christian students can have an especially strong desire to work on "things that really matter." By its breadth of application, statistics is a discipline where that can be done. The following two examples reflect this desire and its discipline fulfillment in statistics.

One former student, who was quantitatively strong, entered college with the career goal of "helping people." He initially chose to major in one of the behavioral sciences because he did not see how he could use his quantitative strengths to achieve his goal. While taking an introductory statistics course for nonmajors, he learned otherwise. Sometime later, he switched his major to mathematics and took several statistics courses. He then pursued graduate study in statistics. Today, he is a PhD statistician in one of the major cancer research centers in the world. As a young professional, he has already made an extraordinary number of significant contributions in his field and has given invited presentations on six continents.

Another former student entered college with the goal of being a mathematics teacher. Along the way she was introduced to, and chose to minor in, statistics. Initially, she had no graduate school plans, but through several statistics-related internships, listening to professional statisticians speak in her classes, participation in a collaborative research project, and extensive mentoring, she saw the benefits of graduate study in statistics. Today, she is a PhD biostatistician in another one of the major cancer research centers in the world.

Since students often have the misperception that a statistician works alone or only with other statisticians, it is essential to communicate to students that statistics is very much a collaborative discipline. Its collaborative nature is emphasized by Brown and Kass (2009). In fact, its collaborative nature is relatively unique. A statistician can be part of virtually any research effort since most involve data. As mentioned before, real-life examples should be presented to show the diversity of statistical application. Such examples should also be used to show the discipline diversity among team members on a collaborative project. Being part of a collaborative effort in another discipline requires the statistician to learn something about that discipline. For those students with a real love of learning, the prospect of doing this professionally is very appealing. Christian students can see this as an opportunity to continue learning, using their God-given abilities. Collaborative work allows the professional impact of a statistician to be much wider. A Christian can also see this as an opportunity for the personal impact through one's profession to be widened. One graduate reports that his being a successful statistical scientist provides him opportunities to share his faith in situations that would be otherwise closed to him.

Since most real-world problems have data associated with them, students can more easily envision their involvement in the solution by using their statistical skills in a collaborative setting. For Christian students, these real world problems can be in ministry-type situations. Thus, they can see using their professional skills as a statistician to build on their Christian call and commitment to service.

Another appealing aspect of being a statistician is that one must be a holistic professional. This goes against the popular misperception of a statistician as a "nerd." Statistics is very much a people-related discipline, a feature that is attractive to the Christian student who has a strong commitment to reflect Christ in every aspect of their life and wishes to interact with others frequently. A successful statistician needs good communication skills - written, oral, and interpersonal. Being able to work with others on a team is essential.

While ethical and expertise considerations are present in all disciplines, they often take on added importance in the application of statistics because data-driven research decisions can affect every area of life. Since Christians are called to be people of integrity, they should be especially sensitive to ethical considerations, and a discipline that regards such considerations seriously could be attractive to them.

First, I consider expertise. There is something about the discipline of statistics that too often tempts individuals with insufficient statistical training to perform statistical work. Such individuals are more likely to do things inappropriately out of ignorance; hence, the importance of statistical training. That importance should be stressed to students, particularly those in beginning courses. Students in these courses should also be cautioned about independently attempting to do too much statistically. The syllabus for my introductory course contains the following paragraph: "My goal is not to make you a statistical practitioner but rather an intelligent consumer of statistical information. Knowing that valid research results are very much dependent on careful consideration of the statistical aspects will lead you to seek professional statistical help in most major research projects and, also, to review the research results of others with appropriate caution and skepticism."

Second, I consider the ethical. Individuals, regardless of level of training, can willfully do things inappropriately. These are ethical violations. It is very important to talk about these dangers in class. It should be noted that most people cannot distinguish between erroneous results based on insufficient statistical training and those based on ethical violations, but the impact of either type of result can be quite serious.

Finally, attempts to attract students to the discipline of statistics (i.e., recruiting) can be done corporately in class situations and individually with students having quantitative potential. The personal or individual contact by the statistical educator is impressive to the student. Statistical educators should not become discouraged. Successful recruiting can be a slow process. They should be persistent since there may be only a few recruits initially. However, success breeds success. Some of the most effective recruiters are students who have already been recruited into the discipline. Think of recruiting efforts as "planting the seed," a concept familiar to many Christians. I attest that one can be pleasantly surprised by the results of these efforts, not knowing where they will take effect and maybe not seeing the results for years.

3.3 Preparing Students for Careers Using Statistics

In this section, I consider preparation applicable to all statistics students - those who attend graduate school and those who do not. As much as possible, this preparation should be attentive to those features of statistics that attracted students to the discipline. A major component of the preparation is the coursework. As long as other appropriate courses are taken, a student will never be disadvantaged by taking too many statistics courses. ASA (2001a), Dixon and Legler (2003), and Eby (2007) give course suggestions. Since the theoretical foundation of statistics is mathematics (i.e., why statistical methods work), I believe strongly that a mathematics-based statistics minor provides a better preparation for a career using statistics than does a statistics major. This does not imply that all statistics minors need to be mathematics majors, but they must be quantitatively strong.

The American Statistical Association (ASA) Undergraduate Statistics Education Initiative (USEI) (ASA (2001a)) curriculum guidelines for undergraduate programs in statistical science (ASA (2001b)) recommend development of the following five skills: statistical, mathematical, computational, nonmathematical, and substantive area.

The first three skills should be primarily developed through the coursework required by the statistics program. Nonmathematical skills include written, oral, and interpersonal communication skills which are necessary for a holistic professional. These should be primarily developed through the general education requirements, if they are strong, of all undergraduate programs. They can also be further developed through the coursework required for the statistics program, statistics-related internships, and/or collaborative projects. Development of skills in a substantive area can be accomplished through a second major, minor, concentration, and/or work on an interdisciplinary collaborative project. Study in this substantive area can provide a good foundation for current or future interdisciplinary work.

In Subsection 3.1, I briefly described the work of Legler, Roback, et al. (2010) at St. Olaf College - the creation of an interdisciplinary undergraduate research program called the Center for Interdisciplinary Research (CIR). In the CIR, statistics students work collaboratively on a variety of interdisciplinary research projects. Each research team consists of a statistical educator mentor and a faculty member mentor from the primary discipline of the project. These authors see the CIR as an effective recruiting tool for the discipline of statistics, a means of providing preparation for a career in statistics, and primarily a way of encouraging students to pursue graduate study in a statistics-related field.

With respect to the transportability of the St. Olaf CIR model, it appears that relatively few colleges possess the necessary infrastructure (e.g., undergraduate research centers and faculty release-time funding) in student-involved interdisciplinary research to support the full CIR model. However, there are possibilities for more modest forms of the model. At Messiah College, the Collaboratory for Strategic Partnerships and Applied Research "adds value to classroom learning by enabling participants to apply academic knowledge and live out Christian faith through imaginative, hands on problem solving that meets needs brought ... by Christian mission, relief, and development organizations and businesses. ... projects enable students to engage classroom fundamentals in an authentic client provider environment. Student leaders run the Collaboratory organization in partnership with the educators who mentor them. Collaboratory projects connect the scholarship and service of faculty members directly to student learning." (Messiah College (2016: 1)).

Over the years, several statistics students have become involved in Collaboratory projects requiring statistical expertise. While no statistical educator was a formal member of any of these Collaboratory teams, statistical educators were consulted from time to time on all of these projects.

Two students were involved in the multiyear *Mali Water and Disabilities Project* in Mali, West Africa. The project had goals of assisting disabled individuals in three ways - accessing and using hand pumps, transporting and using water domestically, and accessing and using latrines. (A major sponsor of this project was the Conrad N. Hilton Foundation.) One student was a member of a site team that visited Mali during the project. Her role on that trip was to begin the statistical assessment of water access by conducting survey work. That student is now a PhD biostatistician working on major health issues. The other student on that project pursued graduate study in a statistics-related field dealing with public health issues in developing communities.

Another statistics student was part of the Collaboratory Education Group working on the *Strengthening Mathematics Literacy Project*, a two year project of curriculum development and teacher interaction in Burkina Faso. Her project required her to travel with a site team to Burkina Faso to conduct a statistical survey of educational practices and levels in three regions of the village. Her work was summarized in a departmental honors paper, "Educational Assessment in Burkina Faso, West Africa." This student hopes to use her statistical expertise in working with a national or international organization focusing on needs in third world countries.

Still another student provided statistical expertise for the Collaboratory Water Group on the *Village Water Ozonization System (VWOS) Project*. This project's goal was to develop and implement a simple small scale water purification system to meet the needs of partnering Honduran communities. This student went on to complete his doctorate in applied mathematics.

Whenever possible and relevant, Christian values should be incorporated with a topic. In statistics, a primary way of doing this is by considering ethical issues. In the previous section, I noted that ethical considerations are very important in the application of statistics, more so than in many other disciplines. The pressures to compromise in grant-driven and profit-motivated research environments are strong and Christians need to be able to withstand them. The ASA's ethical guidelines can be found on its web site (ASA (1999)). In a Christian college setting, educators can take the consideration of ethics to a deeper level by adding morality to the consideration. The ethical code is the code of conduct for the profession. The moral code is the code of conduct for all of life. A Christian's moral code should exceed, and include, their ethical code. It is not unusual for a statistician to serve as an expert witness. However, a Christian statistician should not be a "hired gun." The Christian should not only consider the ethical aspects of the project but also the moral aspects. For example, there may not be anything ethically wrong with a statistician providing expertise in a study of the economic viability of a proposed location for a state-sponsored gambling facility. However, a Christian statistician may feel it would be morally wrong to participate in such a study.

3.4 Preparing Students for Graduate Study in a Statistics-related Field

An obvious part of preparing students for graduate study is coursework. In an invited presentation at the 2007 Joint Statistical Meetings (JSM), Eby (2007) provided guidelines for courses that should be part of the undergraduate preparation for graduate study. Dixon and Legler (2003) and Legler, Roback, et al. (2010) also give course suggestions. If necessary and possible, departments should supplement students' programs with independent studies and/or special topics courses. To do so requires institutional support. Since directing independent studies may not be rewarding monetarily, these efforts may require professional sacrifice.

At St. Olaf, "mentoring undergraduates in the field of statistics with the aim of encouraging them to attend graduate school in statistics has been the overarching goal" (Legler, Roback, et al. (2010: 61)). I recommend a broader view of mentoring. Mentoring should include all statistics students, not just those intending to pursue graduate study. Because of the wide spectrum of graduate programs in statistics, more students are capable of pursuing graduate studies than one might realize. The goal should be to match the student with the graduate program that is most appropriate. Consideration of what is best for the student is paramount. While one might love to see students enter prestigious graduate programs, such programs are certainly not the best fit for every student with graduate school potential. Hopefully, the Christian statistical educator will focus primarily on the student's need and not so much on one's professional or undergraduate statistics program reputation.

Not all students with graduate school potential will pursue graduate study. The statistical educator must respect the student's sense of calling and unconditionally accept it if that calling leads the student somewhere other than graduate school. Because of shared core values, the Christian mentor should be better equipped to respect and accept the student's calling. The overarching goal should be what is best for the student, whether or not they go to graduate school.

Several years ago, one student who was eminently qualified to pursue graduate study in a statistics-related field chose instead to accept a position in full-time ministry. Two years later, she told me how much she appreciated my support of her call into ministry even though she had the ability to do graduate study.

Another student, the one quoted in Section 1, with a ministry-sensitive heart chose to pursue graduate study. He and his wife, who grew up on a foreign mission field, had a ministry to international students while in graduate school. Now that he is a professor, that ministry continues through their involvement in an international church. While he was an undergraduate, I mentored him about incorporating the academic (e.g., statistics) with his call to and love of ministry.

Perhaps the greatest area where a mentor needs to encourage is in convincing students that they have graduate school potential. An experienced mentor has the advantage here. If you have a record of former students achieving success in graduate study, ask current students to rely on your judgment in assessing graduate school potential. That is, you are asking them to trust you. That should be relatively easy for them to do if you have demonstrated your trustworthiness to them throughout the mentoring relationship. If you do not yet have a record of former students achieving success in graduate study, use your ability as a graduate student and that of your graduate school peers as your frame of reference.

The example of one former student summarizes well the role of mentoring particularly with respect to preparation for graduate study. She is appreciative of the opportunities to study the extra things she needed for graduate school but did not know that she needed. She regards the persistent encouragement, particularly with respect to graduate study, as a very positive factor in her experience, stating that she probably would never have considered that option if I had not continually put the thought in her head. Today, she is a biostatistics professor in an R1 university.

4 Conclusions

The title of this paper, *Mentoring as a Statistical Educator Within the Context of a Christian College*, might suggest a rather narrow focus. However, my thesis, ***this mentoring is efficacious due to two factors: the setting - a Christian college, and the discipline - statistics***, allows for the possibility of broader considerations.

To show the increased positive impact of a Christian college setting on a mentoring relationship, I also considered positive impacts of a mentoring relationship in any college setting. To show the added possibilities for mentoring as a statistical educator in a Christian college setting, I first discussed mentoring as a statistical educator in any setting. As a statistical educator in a Christian college, I believe the impact is greater than the sum of the impacts of the parts - the Christian college setting and the discipline of statistics.

I close with my vocational life verse which I embraced early in my career at Messiah College. "You yourselves are our letter of recommendation, written on our hearts, to be known and read by all" (2 Corinthians 3:2 English Standard Version). This verse shows the regard of a mentor for the mentee and the disseminating nature of mentoring. Several years ago, a former mentee affirmed my life calling by independently quoting this verse to me.

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Developing the Underutilized Mathematical Strengths of Students

Patrick Eggleton (Taylor University)

Patrick Eggleton is an Associate Professor of Mathematics at Taylor University, where he teaches mathematics and prepares future mathematics teachers. He has taught mathematics at grade levels from elementary to the university, encouraging students to explore their ability to solve problems and develop logical solutions. From 2000 to 2004 he served as the President to the Indiana Council of Teachers of Mathematics. His primary interests are in instructional strategies and motivation.

Abstract

One of the continual challenges faced by mathematics educators is the predisposition of many students with an angst toward mathematics. One student wrote, "If I could describe my high school mathematics experience, I would describe it as my worst enemy gouging my eyes out with a silver spoon, while playing the high-pitched mosquito sound. In simpler terms, math in high school for me was absolute torture." (College freshman, February 2017) As Christian educators, we not only want such students to overcome their anxiety and dread toward mathematics, but we also want them to see its value to all aspects of their lives. "Indeed, we maintain that the study of mathematics is important for every educated person, and especially for a Christian. In fact, a good argument can be made that such study will not only make people more effective in their Christian calling; it will also enrich their lives in personal ways, and in ways that will make them more effective as they work to bring about peace, wholeness, and harmony in our world." [2, p. 246] To help students confront any negative predispositions that prevent their ability to actualize their full potential - which includes the thinking habits inherent in mathematics, we need an awareness of how mathematical dispositions are developed and what might have the greatest impact on reforming those dispositions so the student's full potential can be achieved. The following study shares influences related to mathematical disposition development and provides possible direction for educators who seek to mold this disposition so their students will be more effective in their Christian calling.

1 Background

The *Curriculum and Evaluation Standards for School Mathematics* emphasized the assessment of mathematical disposition as an essential element in the mathematical development of students. [6, p. 233] This document articulates an excellent description of a positive mathematical disposition (e.g. the student has confidence using mathematics to solve problems and the student is willing to persevere in mathematical tasks) and makes suggestions on how to assess this disposition, yet in practice, disposition is often considered a byproduct to our instructional attempts. The *Principles to Action: Ensuring Mathematical Success for All* [7] and the *Common Core State Standards for Mathematics* reiterate the importance of a "productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy)". [3, p. 6] Wilkerson adds that Christian mathematics educators have a responsibility to influence students' mathematical affections, "...not merely knowing, but also loving, and practicing the truth, beauty, and goodness inherent in mathematics." [10, p. 112] Research studies have developed a number of ways to assess student dispositions toward mathematics (often referred to as attitude toward mathematics). Zan and DiMartino [11] used questionnaires and student essays titled “Me and mathematics: My relationship with maths up to now” to identify themes related to mathematical dispositions. They observed three primary themes related to disposition: (1) the emotional disposition toward mathematics (I like/dislike mathematics), (2) the perception of being/not being able to succeed in mathematics, and (3) the vision of mathematics (mathematics is ...). While this study provided valuable themes associated with mathematical dispositions, the disposition remained a byproduct with the development of those dispositions primarily implied by the themes. Haladyna [4] used a self-reporting survey on attitude toward mathematics and inferred from those surveys the primary influences toward mathematical disposition as teacher quality and the social-psychological climate (assessed by responses to items such as “How much do you like the students in your class?” and “The students would be proud to show the classroom to a visitor.”) While it seems plausible that teacher quality and the climate of the classroom would be influential in a student’s emotional disposition, perception of success, and view of the nature of mathematics, these studies could benefit from some form of replication to validate or expand upon these findings.

2 Method

Students taking Investigations in Mathematics at Taylor University begin each semester with a required one-page essay entitled, “Mathematics in my Past and my Hope for the Future.” Following the submission of their essays, students in the Spring 2017 classes were asked if they would allow their essays to be analyzed in order to identify constructs that influenced their mathematical disposition. (Approval of the use of human subjects was granted by the Taylor University Institutional Review Board in January 2017.) Seventy of the 75 students agreed. As essays were read, any information communicated that directly or indirectly related to the student’s emotional disposition toward mathematics, perception of success, or vision of mathematics was cut from the text (including context as necessary) and arranged in a spreadsheet for further analysis. Over 200 quotes were identified (See Figure 1).

Example Quotes per Category

Teacher:
Because of certain teachers I had, I started to despise math and wanted to avoid it at all costs. (#7) Then during sixth grade, I had a teacher whose method of teaching just clicked with me for some reason, and after that I didn’t hate math anymore. (#40)

Nature of Math:
The thought of spending hours just thinking about how to solve a problem, or trying unsuccessfully numerous times to do so has never been appealing to me, so I chose not to set myself up for that type of situation. (#19) During my senior year I took statistics, a senior level class. While this class was a walk in the park compared to my precalculus class, I really did enjoy it. I found a way to make it fun and to make it applicable to events that could arise in my future as an adult. I think this skill is valuable because if I get the idea in my mind that something is either boring or not useful, then I am less likely to be willing to learn it. (#25)

Success:
Mathematics has always been something that usually came naturally to me. Many times, when I would grasp a concept, I could start cranking out answers left and right. However, grasping the concepts would be something that frustrated me. If I couldn’t quite figure out a solid answer, it would irritate me so much. (#35) I know for sure I will fail. Failing is not an option. (#63)

Figure 1: Examples of categorized quotes from the analysis.

The collected quotes were then reviewed several times in order to infer any general themes related to influence within the descriptions of mathematical dispositions. Three categories of influence surfaced from this analysis: (1) mathematics teachers, (2) the perceived nature of mathematics, and (3) success or lack of success in prior classes. The quotes related to influence were then coded based on these categories (using a 4^{th} category of "other" for influences that did not fit one of these categories). At the end of the semester, students were given a brief survey (see Figure 2) where they were asked to rate their personal disposition of mathematics while also ranking the three categories that surfaced from the analysis of the data.

Thank you for helping with the research related to what has influenced your current disposition toward mathematics. To help finalize our results, would you please answer each of the following:
On a scale of 1 to 10, 1 being I dislike math to 10 being I like math, I would rate my disposition to mathematics as (circle one)
1 2 3 4 5 6 7 8 9 10
All of us have had influences in our life that affect our current dispositions. In looking through the reflection journals we generalized to 3 common influences on a person's disposition toward math: Teacher (either helping me to like or dislike math), the Nature of Math (from being useful and precise to being confusing and difficult), and Success (I was usually successful in math to I was usually not successful in math). Please rank these 3 influences based on how they have affected your disposition (my tendency to dislike or like) toward mathematics. (For instance, the following ranking would indicate that the teacher had the most influence, that Nature of Math had the second most influence, and my Success had the least influence on how I feel about math: 1 - Teacher, 2 - Nature of Math, 3 - Success)

Rank: 1st, 2nd, and 3rd
______ Teacher
______ Nature of Math
______ Success

Figure 2: Self-assessment of mathematical disposition and ranking of influences related to it.

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3 Results

Figure 3 shows the distribution of the influences as they were designated by our coding schema.

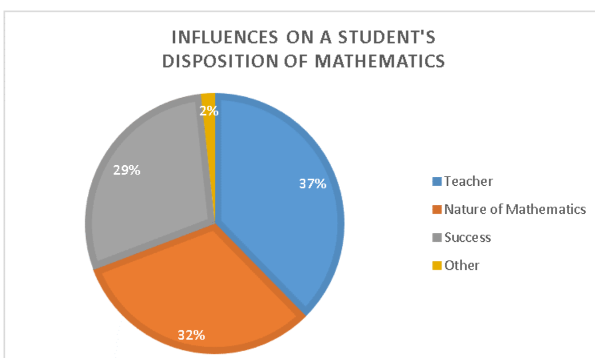

Figure 3: Distribution of influences on a student's disposition of mathematics as coded.

A Chi-Square goodness of fit test between the 3 primary categories indicated that our coding showed no significant differences between the categories (χ^2 = 1.27, p = .529). Of course, one student may have had more than one influence indicated in their essay or may have had several different quotes all related to the same influence with each quote being counted separately. Allowing students the opportunity to rank these influences yielded slightly different results (see Figure 4).

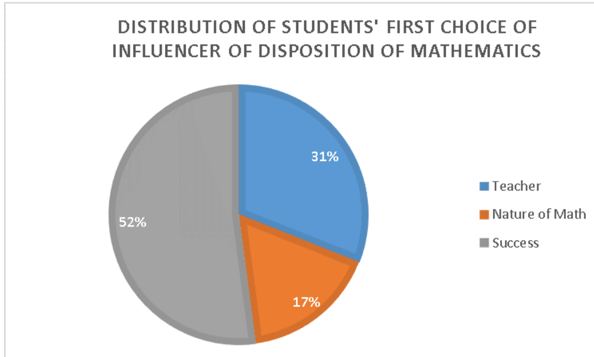

Figure 4: Distribution of students' first choice of influence on their disposition of mathematics.

These results did show significant differences between the categories ($\chi^2 = 13.4$, $p = 0.001$) with "Success" being the most significant influence on a student's disposition toward mathematics. Using the students' self-reported disposition level (1 being disliking math to 10 being liking math), "Success" or "Lack of Success" continued to dominate as the most significant influence to a student's mathematical disposition. (See Table 1). It is noted that students signifying a stronger disposition to mathematics did indicate the teacher as being more influential in developing that disposition.

		Influence		
		Teacher	Nature of Math	Success
Self-Reported Disposition Level	Lowest (1-3)	20%	30%	50%
	Highest (8-10)	39%	6%	56%

Table 1: Top ranking of influences to mathematical disposition compared to student's self-reported disposition.

4 Self-reported Disposition Level

While there are certainly concerns regarding subjectivity with the coding of the data and oversimplifying when eliciting the self-evaluations, the prevalence of the influential nature of a student's success and the influence of their mathematics teacher is significant for professional mathematics educators to note. Outcomes from other studies imply similar results. For instance, Hannula shared detailed observations of a case study, Rita. Initially Rita had a poor disposition toward mathematics. As she gained success or understanding, Rita's disposition improved. "How can we then explain this change? Why did it happen? What brought it about? Our first answer comes from Rita. Mathematics was 'more fun' because she had 'been understanding more.' In another interview she remarked that 'that must be the nicest thing exactly that one understands the topic.' " [5, p. 41] Wamsted's story of a breakfast conversation with his two children communicates a similar result. While his second grade daughter, Kira, deliberated over a mathematics problem that her father had suggested, her kindergarten brother cuts in with answers on two subsequent occasions. Following the second answer from her younger brother, Kira exclaims, "John is so much better at math than me." [8, p. 486] Wamsted further explains that Kira actually had an affinity for math and had been identified for the gifted and talented program in the district. While Wamsted attributes his concern for Kira to gender stereotypes, this episode also seems to communicate how quickly the lack of success (especially when shown up by your younger brother) can contribute to a child's mathematical disposition.

There is also a surprising connection to the influence of progress and success shared by research in the business world. Amabile and Kramer's research on what motivates workers communicated counter-intuitive results to leaders in the business world. They provided a multiyear study tracking the day-to-day activities, emotions, and motivation levels of hundreds of knowledge workers in a variety of settings and found that the top motivating factor for workers was progress. "When workers sense they're making headway, their drive to succeed is at its peak." [1, p. 44] In contrast, the managers of these workers thought that making progress was one of the least important motivators to their workers, opting for incentives or working atmosphere as more important. Progress and its eventual product success (or lack of progress and eventual failure) is not only important to the disposition of workers in the business world, but it is also of significant value to workers in the classroom.

While the importance of making progress and developing success are influential factors for developing positive mathematical dispositions, the teacher and the nature of the mathematics that is taught are also essential factors, as noted by the student essays and their survey responses. Teachers who communicated that they cared about students or really enjoyed the content tended to have the most positive impact on mathematical disposition. Similarly, teachers who communicated indifference to the challenges students faced or promoted a dry routine of drill and practice tended to have a negative impact on mathematical disposition. The following student captured both of these influences:

In the past I have not enjoyed math class. That is not to say that I do not enjoy mathematics—I believe that I would if it had been taught to me in a different manner—but I have had very negative experiences with math in school. I believe that the root of this is, as I have already suggested, the utterly prosaic way in which mathematics seems to be traditionally presented and taught. It is presented as a thing onto itself, an existence which excludes connection to all other subjects (except, perhaps, science) and only seeks to serve the practical. It had seemed far removed from the world of the imagination that I cherish and, for that matter, from the world [of] logic and thinking. This, of course, is a vast misperception. Nevertheless, this is the vague and misconstrued form of mathematics that my teachers (unknowingly) presented to me. Equations were learned by memorization, without reference to the natural world from which they were derived, and problems solved by the simple method of plugging in a number to an equation and following through on the taught steps to find the answer. To me, the pursuit seemed purposeless. (Student #54)

5 Conclusions

"God created us with a capacity to do mathematics, so using that capacity must be part of God's will for us." [2, p. 11] One of the greatest frustrations we face as Christian mathematics educators is when we work with students who fail to recognize the influence that mathematics can have on their vocation. Mathematics develops important thinking habits of all sorts: "analyzing carefully the implications of one's statements, being careful with the meanings of words, imagining alternative interpretations or possibilities for action, and recognizing that the mere declaration of your opinion on an issue does not render it true." [2, p. 248] Regardless of an individual's calling, these mathematical habits will improve the quality of participation within that vocation. To benefit from the study of mathematics, students need a disposition toward mathematics that will motivate them to value their God-given capacity for the subject. As shared by the students in this study, the teacher plays a significant role in developing a student's disposition toward mathematics either positively or negatively. To promote positive dispositions of mathematics, teachers must accept the responsibility for the important role that they play. One tool teachers need to use in developing a positive disposition toward mathematics is the use of progress to develop success. Students who don't feel they are making progress can quickly succumb to a poor disposition toward their work. Adolescence is a turbulent time when attitudes can quickly derail due to the challenges of learning. Our classrooms need to promote progress that leads to genuine success so that students will develop the ability and determination to persevere through the rigors of their studies to maximize their potential. When progress and success seem to be insurmountable goals, teachers need to break down the tasks and concepts in order to make progress and success attainable once again. Teachers can also provide opportunities to explore various facets of mathematics, helping students reflect upon the beauty and "unreasonable effectiveness" [9] with which this language communicates about creation. Scripture shares that we should exercise whatever abilities God has given us "with all the strength and energy that God supplies so that God will be glorified through Jesus Christ." (1 Peter 4: 11 TLB) As Christian mathematics educators, we make it our goal to do what is needed so that the underutilized mathematical strength of many students will be better utilized in their vocations and ultimately in glorifying God.

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The Daily Question: Building Student Trust and Interest in Undergraduate Introductory Probability and Statistics Courses

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Abstract

Introducing probability or statistics to disinterested undergraduate students is challenging. I share an unobtrusive way to build trust with students, creating a medium to both naturally create an intimate classroom atmosphere and have your students look forward to attending each class. The context is the United States Naval Academy, a four-year undergraduate institution with an emphasis on leader development. Based on a technique of daily questions suggested by Penn State University Lecturer Dr. Heather Holleman [1], I integrate daily questions with the course content. The daily question offers opportunities for expressions of faith and invitations to further conversations. Anonymous midterm assessments and end-of-term student opinion forms demonstrate initial success of this method.

1 Introduction

> *I liked how he asked the class a question before class started. It made the class a more personal environment and we got to know our classmates better.*
>
> Statistics Student

Can an introverted mathematics professor connect to humanities students a generation removed? Is it possible for non-technically oriented students taking a required mathematics course to look forward to attending class each day? Can a Christian academic at a secular institution mention his or her faith in a math class? The reader will discover the answer to all three of these questions is a resounding "yes."

In this paper, I share a technique that costs a few minutes a day and yields a bounty of trust, interest, and connection, potentially improving the quality of student learning and opening the door for faith-sharing opportunities. I express the motivation for desiring a technique, the technique itself, and the setting for its use in Section 2. I proceed to give examples of daily question types (Section 3) and present results based on anonymous student feedback (Section 4). I also outline a helpful procedure for soliciting midterm feedback known as the Formative Assessment of Classroom Techniques (FACT). Section 5 summarizes the journey and outlines future steps, while the Appendix contains a list of daily questions.

†The views expressed in this paper are those of the author and do not necessarily reflect the official policy or position of the United States Naval Academy, Department of the Navy, Department of Defense, or United States Government.

2 Motivation and Setting

The use of personal questions of the day gave the classroom a relaxed atmosphere. Which is good in a math class–because math is stressful.

Probability Student

Taking the time to poll the class on a question every day may seem an unwise indulgence. To provide some background, I describe the factors leading to my decision to adopt this approach and portray the environment in which it is used.

2.1 Desire for Connection

Early in my role as a faculty member, several circumstances culminated in adopting a new classroom approach. After a few weeks of teaching, I realized I was not connecting with my probability students as much as I felt I could. My students were mostly Operations Research majors, and as someone with a PhD in Operations Research, I felt I should be able to interest them more easily. This same week of that realization, I learned I would be teaching a required probability course to non-majors during the next term - students potentially more likely to be disinterested. On top of all of this, I was reading through Mellichamp [2] and felt that I was missing out on identifying myself as a Christian in class. Naturally an introvert, I fervently sought something to help me engage my students, or I would easily build a dead-end career in the classroom and lose my students semester after semester. Surely there wasn't a magic button I could push to allow me to build a classroom community and engender interest while sharing my Christian identity with my students, was there?

At this point of desperation, I received my first Faculty Commons Missional Moment weekly email [1]. In the missive, Penn State University Lecturer Dr. Heather Holleman discussed her method of building community in her writing courses. She would simply "ask students to answer one question before [her] teaching begins every single class day. The goal is to bond well with one another, build empathy, and overcome the fear of sharing authentically." She was using this technique as a way to encourage her students to take risks in their writing. Once I reviewed her list of questions, I thought, this is my "magic button." I also saw this tool as a way to vary my teaching practices and reach students with different learning styles [3]. Over the next several weeks, I resolved to incorporate this technique into the non-majors probability course.

It is important to note a few ground rules about the questions. First, students are always able to pass on any given day, since the questions may ask them to reveal more than they are comfortable sharing. Second, responses are only discussed in the context of the classroom, or directly with the person who made the response. Third, class policies remain in effect (adapted from Nardi [4]):

- The classroom is a place of mutual respect between instructor and student, and between students.
- We won't belittle people for asking questions or expressing opinions.
- Debates and critical analysis are good; personal attacks are not.
- We won't tolerate crude, sexually explicit or offensive jokes or remarks.
- We won't use profanity.

With these ground rules in place, the classroom can become a safe place for students to authentically speak their minds.

2.2 Naval Academy Academics

The United States Naval Academy is a four-year undergraduate institution with an emphasis on leader development. There are several ways in which the Naval Academy may be different from other institutions. The graduation ceremony also functions as a commissioning ceremony; graduates serve at least five years in the Navy or Marine Corps as commissioned officers. Classroom sections generally consist of 17 to 21 midshipmen; class attendance is mandatory. Students typically have a demanding schedule that includes 17-18 credit hours each semester as well as a physical education course. Time management is further tested through a professional knowledge training program. Students are responsible to learn and then train their peers and subordinates about Navy and Marine Corps history, programs, policies, and weapons systems. First-year students take weekly professional knowledge quizzes and engage in other projects administered by upperclassmen. Academy academic life also includes healthy peer competition; class ranking or order of merit plays a significant role in determining a student's service assignment following graduation and commissioning, such as opportunities in the more selective communities of Marine Corps aviation or Navy submarine warfare. The athletic and military training requirements on top of academic requirements mean successful time management is a chronic struggle.

Mathematics requirements for midshipmen include a three-semester calculus sequence and at least one additional math course determined by the student's major. Typically, Humanities majors enroll in one of two courses: Probability with Naval Applications or Introductory Statistics. [The fourth-course option of Differential Equations is required of majors in the Science, Technology, Engineering, and Math (STEM) fields.] At this point of the academic program, the Humanities majors are sophomores or juniors. No longer under the strict rules of first-year midshipmen, these students also do not yet have the leadership positions of seniors. In the muddy middle, these students can yearn for both more direction and more responsibility, all the while lacking individual recognition. Furthermore, they fall into one of two groups, those who have no intention of enrolling in subsequent statistics courses (English, History, Language majors), or those who will take a follow-on statistics course as part of their major (Economics, Political Science).

Probability with Naval Applications (hereafter Probability) and Introductory Statistics (hereafter Statistics) are usually seen by the faculty as thankless courses to teach, populated by students who are likely to have a negative view of mathematics. On the contrary, these potentially jaded students are ready for a different experience in a math class. I was ready to help them to see the fun in mathematics and explore the connections amongst each other.

3 Daily Question Types

> *I personally really enjoyed our daily questions; I not only got to know my classmates better, but I felt that the prof. cared about us too.*
>
> Statistics Student

Having amplified the attraction of the technique and the setting for its use, here I outline some examples of how the daily questions can be integrated with course content, used to establish trust, and even provide opportunities to introduce my faith. These categories are not mutually exclusive - the personal nature of the questions means that even those questions including course content are a means of fostering classroom community.

3.1 Integrating Questions and Course Topics

With some creativity, the daily question can incorporate topical course content. This integration has the benefit of providing the student with an additional access point to the material, a personal connection and interest. Although this interest in the material can be fleeting, Renninger and Hidi show that "triggered and maintained situational interest" can lead to the development of individual interest in the subject material [5]. Several possibilities for introducing topics are outlined here and shown in Table 1. Beyond those presented here, surely there are other questions that can be related directly to probability or statistics concepts.

Topic	Question
Set Operations (Unions and Intersections)	Get in groups of three. What is the most bizarre thing you have in common?
Fundamental Counting Principle	What is your favorite home-cooked meal?
Measures of Relative Position	What were you known for in high school?
Independent Bernoulli Trials	What was your first job?
Hypothesis Testing	Describe a run-in with law enforcement or the USNA conduct system.
Central Limit Theorem	How tall are you?

Table 1: Questions that lead to course topics

One early topic in Probability is Set Operations. To illustrate this topic, form student groups of three and ask them, **what is the most bizarre thing you have in common**? Some groups struggle to find any common oddity. Other groups happen across something interesting, such as common talents, injuries, family names, vacation spots, habits, etc. The ensuing discussion naturally includes the fact that the set of bizarre things about student A may have limited overlap with the set of bizarre things about student B, the difference between unions and intersections, and an illustration using Venn diagrams.

A combinatoric principle typically introduced in both Probability and Statistics is the Fundamental Counting Principle, also known as the Product Rule for k-tuples. This principle can be taught by using the question, **what is your favorite home-cooked meal**? Set up a white or black board with columns for entrée, side dish, dessert, and drink. Then, have students enter their meals appropriately, also instructing them to not bother writing duplicate entries. Once the items are entered, have a brief discussion about any unusual items. Count how many items are in each category and write the totals next to the column headings. Then, pose a question such as the following: suppose this were the menu of a restaurant, and you had to order a dinner consisting of one of each type of item. How many different meals could you choose? In my experience, several students will know the proper calculation. Introduce the mathematical notation, and now they have learned the Fundamental Counting Principle. Further examples could include the following: what if you had to use the restroom - you ask your friend to order fried chicken for you, but leave the other choices up to him or her. Now how many different meals are possible?

The daily question can be of further help with the Statistics topic, Measures of Relative Position (quartiles, percentiles, quantiles). One way to introduce this topic is to use the question, **what were you known for in high school**? Even if the answers do not include being near the top of the class, this question can lead to a discussion on how the top 10% of a class are determined.

The concept of independent Bernoulli trials is part of the definition of the binomial distribution in Probability, and is introduced in Statistics as a step toward estimating proportions. In both courses, the daily question, **what was your first job**? works well. The students get to hear who actually had jobs before college, usually jobs they never want to hold again, and I have the opportunity to explain my first job, a paper route. (Every week, I delivered papers on each of the five weekday mornings, while my brother, who was five years older, only had to deliver papers on the two weekend mornings. We split the proceeds 50-50. Did I mention he was five years older?) My paper route pay was docked $1 for every customer complaint. We treat the receipt of a complaint as a "success." Assuming I can receive no more than one complaint on any given day, and that days with complaints occur with some fixed probability independent of any other day, then each weekday can be considered an independent Bernoulli trial.

Hypothesis testing is fundamental in Statistics, and is sometimes covered towards the end of the Probability course. The question, **describe a run-in with law enforcement or the US Naval Academy conduct system**, can yield fruitful results. In addition to a raft of great stories, someone will usually mention a speeding incident. This example can be used to discuss the presumption of innocence (not speeding), the binary response of receiving a ticket or not, how much evidence is necessary for the police to act, what errors could happen along the way, how a ticket might be successfully challenged, and the calibration and accuracy of speedometers vs. radar guns. This discussion becomes a helpful framework for further teaching on any hypothesis testing topic.

Other topics can be discussed by using the daily question as a data-gathering tool. Several questions eliciting data sets are fun and useful for community building:

- **How many times did you take the ACT and/or SAT?**
- **How much sleep did you get in the last 24 hours?**
- **At this point, how many years will you serve in the fleet after graduation?**
- **How tall are you?**

This last question provides a great opportunity to teach the Central Limit Theorem. Dividing students into groups of four, have them record their individual heights in a table on the board, with one row per group. Ask them to determine the mean and variance of all of the heights, and the mean and variance of the group average height. Even though we do not necessarily meet the conditions required of the Central Limit Theorem, this exercise tangibly demonstrates the equality of means and the reduction in variance when considering sample means.

3.2 Questions to Establish Trust and Build Community

Mathematics can be stressful. Many of the students in the Probability and Statistics courses have one or both attributes shown to indicate risk of math anxiety: women and students who previously received low math grades [6]. I ask some daily questions to simply break the ice and allow for community building.

- **Do you have an irrational fear or strange addiction?**

  I like this question for several reasons; it (1) does not take a great deal of class time, (2) gives me surprising insight into the student's thought processes and influences, (3) allows people to laugh at themselves.

- **What is the funniest thing you did as a child that people still talk about?**

  This question seems to have one of the longer response times, but always leads to laughter and community building.

- **Tell us something quirky about you.**

  Although the question has a high percentage of "passes," memorable responses often occur. There may be strange physical feats or unusual habits that your students will disclose.

- **What is your favorite way to procrastinate?**

  Asked the last meeting before an exam, this question acknowledges that procrastination happens. Their answers give me some sense of the particular student's bent toward extroversion (walk around and start conversations with people) or introversion (go for a run, watch YouTube).

3.3 Questions to Open the Door to Conversations about Faith

A few questions can lead to direct or indirect sharing of faith. Although I teach in a secular environment, I do want my students to know that it is possible to be both Christian and an academic.

I have to be sure that I do not cross the line between sharing my faith and attempting to convert someone to my faith (proselytizing). I have found that students generally understand my intentions. If you do teach in a secular setting, be sure your personality supports maintaining this distinction. In the eight classrooms of teaching with this technique, the only adverse comment I have received was: "I appreciate the candor but the push toward religion was a bit too much for me. (He in no way tried to push his beliefs on us, it was just too often a topic of conversation.)" I do not know which specific question(s) led to this remark, or if it was due to one or more responses given by myself or other student(s). Moral development is one-third of the mission of the Naval Academy, and it is not unusual for religion to form the basis of that morality. Nonetheless, this is one comment too many, and I continue to work to create an environment characterized by both mutual respect and frank discussion.

- **Have you ever experienced something unexplained or supernatural?**

  Be prepared for some truly amazing responses. Each semester I am surprised at the number of "ghost stories" this question solicits. I come away much more informed about the spiritual background of my students and their acknowledgement of the supernatural. Unfortunately, many students express the popular view that anything spiritual can be assumed to be benevolent. For my part, I enjoy sharing a personal story that is particularly unlikely, such as the fact that my wife completed her doctorate while birthing (and subsequently caring for) five children, conducting five out-of-state moves (one characterized by a moving truck fire), and having her husband unexpectedly deployed for seven months.

- **What is your favorite quote?**

Of course, this question can elicit memorable movie lines, but most students will recall something that inspires them to steadfast leadership. Several students will share inspiring quotes from their faith tradition, including Bible verses.

- **What is something you believe that most people might not believe?**

  Some students will go the sports route with this question, e.g. "I believe the Jets will thrash the Patriots this year." Other students share core beliefs, like a belief in the afterlife, karma, aliens, etc. After prayerful consideration, I may share something about the Bible, such as that the Biblical flood actually happened, or that the universe was created in seven days. The phrasing of this question allows for considerable risk-taking, because in stating your belief, you are at the same time stating that you expect most people not to believe it.

- **What is the kindest act you have ever witnessed?**

  These stories will speak for themselves. Adoption, sheltering, financial provision, and various other forms of charity invariably come up as a result of this question.

These daily questions do not solicit conversion testimonies or proclamations of the Gospel. However, they do let the students know that spiritual topics are not taboo. I have the great advantage of being in uniform, so I do not feel a need to work to earn my students' respect militarily. As a PhD (if my students care about that) I have their academic respect. From day one, I can talk about military matters and the course material without any anxiety about justifying myself. So I strive to earn their emotional and relational respect, because softening those defenses will allow me to speak into their souls.

4 Assessment and Results

> *Of all the mathematics professors that I've encountered thus far in my academy career, this instructor is by far the best I've had. [He] very much cares about his students and their success and takes the time to help us both inside and outside of the classroom. I personally do not like math, especially statistics, but he made the class enjoyable and I would highly recommend him to other midshipmen.*
>
> Statistics Student

Unscientific as it is, there is anecdotal evidence that supports the efficacy of the daily question to increase student interest, classroom trust, and faith-sharing opportunities. I review two methods of obtaining feedback, and share some feedback. I am of the opinion that I came across the daily questions due to divine providence, and this feedback confirms that opinion. Positive student feedback in itself, however, is not my reward; I keep in mind what Paul wrote in Colossians 3:23-24: "Whatever you do, work heartily, as for the Lord and not for men, knowing that from the Lord you will receive the inheritance as your reward." [7]

4.1 Methods of Measurement

Gathering student feedback regarding classroom techniques can be difficult. I employed both midterm and end-of-term anonymous assessments. Perhaps due to the compulsory nature of most tasks at the Naval Academy, there was nearly a 100 percent response rate.

After the first test has been returned (about the 6^{th} or 7^{th} week of the course), I ask the students to consider what is helping their learning, what is hindering their learning, and what suggestions they might have. I use two instruments, a focus group style exercise and an online survey.

For instructors voluntarily requesting a focus group, The US Naval Academy's Center for Teaching and Learning administers a Formative Assessment of Classroom Teaching (FACT) [8]. I provide details here on the process to encourage replication of this formative practice at other institutions. The instructor turns the class over to a facilitator and leaves the room for the last 20 minutes of class. (The facilitator is a volunteer fellow faculty member from outside the instructor's Division [college].) The facilitator then leads the students through an exercise to identify techniques that have helped their learning, hindered their learning, or that they would suggest for their instructor. The facilitator then feeds this back to the instructor, who then debriefs the students. The debrief may include justification for current practices, changes the instructor will make, and/or explanations where changes would be detrimental to learning. FACTs are generally limited to one per instructor per semester. Due to the limited availability of FACTs, I give the other sections I teach an online survey with similar questions, and sequester class time to provide their responses.

Information from both the FACT and the online survey is collected on the same day, but the results can be divergent. The FACT provides a mitigating environment for criticism; students who initially disagree about the helpfulness of a particular method can debate the matter and reach resolution. The individual nature of the online surveys leaves conflicting views unresolved. In any case, by conducting these assessments less than halfway through the term, there is time to make changes for the current round of students, as well as send the message that it is permissible to dialogue about what is and is not working in the classroom.

During the last week of instruction and prior to final exams, departments administer end-of-term student evaluations of teaching and instruction. In the Mathematics Department, these Student Opinion Forms, or SOFs, are administered online. There are free-response course-related prompts and instructor-related prompts. Most of the responses I share here come from the instructor-related prompt, "Personal," which invites students to comment on the instructor's "relationship with students; classroom atmosphere; encouragement; willingness to help; enthusiasm; generation of interest; stimulation of thought; invites questions, discussion." [9]

4.2 Results

In all midterm focus groups (FACTs), the students listed the daily question as a factor that helped their learning. Of 98 free-response end-of-term surveys across five Probability and Statistics sections, 46 respondents (47 percent) mentioned the daily question - all positively. The following responses and the quotations opening each section come from both midterm and end-of-term surveys.

The daily question builds interest and reduces anxiety:

> I really liked how [he] had the class answer daily questions unrelated to statistics. It kept things interesting and students engaged.

> The questions about life you ask during class are a very cool way to keep people attentive.

> This instructor does a lot of work with the chalkboard and incorporates student questions and answers into his lessons. Both of which make him a very effective teacher.

> Made the classroom a very comfortable environment. I like coming to this class. I learned a lot from the life questions he'd ask us.

> I also felt more comfortable because of [him] asking philosophical questions to help break the ice.

> Created one of the better class atmospheres that I've experienced and very good at getting to know students. I actually began looking forward to his daily "question" which is not something I can say about all classes.

> Fantastic relationship with students. I looked forward to coming to class with this Professor, and hear what he has to say about recent news, or the question of the day.

> Asked a question everyday which made the class entertaining and not dreadful like most other math classes.

The daily question builds community:

> Thank you for asking us the "question of the day" before we start covering the day's material. I have never had a teacher do this before, and I think it is awesome. Consequently, I don't think I would ever know the interesting things about my classmates that I now do without you having asked the questions (like the one girl who has dreams every night before her sisters declare their pregnancies to the family, ha! neat!). I was worried about the method at first, thinking we wouldn't have enough time to cover all the material, but I think you spend an appropriate amount of time on the question of the day because we are still able to cover everything we need to before class ends. So, thank you.

> ...he fosters an atmosphere of community by asking daily questions (I was skeptical at first, but it doesn't eat up our class time–we still learn all the knowledge we need to know + we now know more about each other than we would ever have known otherwise—people have come up to me outside of class to discuss my remarks in class with me.

> The instructor had a great personal relationship with the class as a whole as he incorporated time in class to ask us as a group personal daily questions that sparked conversation amongst the students and shared stories of his family and job life with us as well as his time as a Mid here back in the early 90s. He definitely clicked with the entire class.

> Class atmosphere was very good. I have not had a technical course professor who has encouraged the class to get to know each other as much as he has through name games at the beginning of the year and questions each class period. It was a nice way to break up the traditional Academy classes, and made you think about more than just math

> [He] is professional with midshipman and is a favorite among his sections for the "question of the day" section of class where he asks a personal question to everyone in the class. I greatly enjoyed answering the question of the day and learning more about my classmates.

The daily question makes the instructor appear more approachable and caring, opening the door to outside conversations:

> He is a great guy and you can tell he cares about us and our futures. We do a thing called "question of the day" and the goal is to find out a little bit more of us and for us to find out more about him. Very approachable and genuine.

> My instructor always maintains the highest form of professionalism. However, the difference between him and other instructors is that he actually cared about where each and every students was from, including their life background. He is the only instructor that I have had who has asked a daily question. These questions were usually about our lives and experiences we have had.

The effect of the daily question on learning is not established. To design an experiment to do so would mean depriving some students of this technique. After receiving such positive feedback, I cannot bring myself to conduct such a study.

5 Summary and Next Steps

> *Best teacher I have had when it comes to atmosphere, relationship, and interest. I think the question of the day should be in every classroom on the yard. It takes two minutes, and changes the entire dynamic in the classroom. I think every professor should adopt this.*
>
> Statistics Student

The daily question, incorporated in desperation, has outperformed my expectations. It has been instrumental in raising interest in the course, building a classroom community, and providing avenues to deeper conversations - including those about faith. I will continue to incorporate this technique in future courses. In particular, I aim to find more ways to integrate the daily question with course material. I also hope to spread this practice so that those instructors desiring to build a classroom community have an effective and relatively inexpensive means of doing so.

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6 Appendix: List of Questions

There are more questions here than there are class meetings for the typical course. The first 50 questions are from Penn State University Lecturer Heather Holleman. The remaining questions are from the author, with some input from his students, Naval Academy Midshipmen.

1. What is the most interesting course you have ever taken in school?
2. What is your favorite quotation?
3. What is one item you might keep forever?
4. What were you known for in high school? Did you have any nicknames?
5. If you could have witnessed any event in sports history, what would it be?
6. What is something you consider beautiful?
7. What was your first CD or song you played over and over again?
8. What accomplishment are you most proud of?
9. If you could be an apprentice to any person, living or deceased, from whom would you want to learn?
10. What are three things that make you happy?
11. What's one movie you think everyone should see? What's a movie you think nobody should see?
12. Who inspires you?
13. What's one thing you want to do before you die?
14. Get in groups of three people. What's the most bizarre thing you have in common?
15. Whenever you are having a bad day, what is the best thing you can do to help cheer yourself up?
16. Have you ever experienced something unexplainable or supernatural?

17. What was your best Halloween costume?
18. What's the last item you purchased?
19. What was the last thing you Googled?
20. What YouTube video do you watch over and over?
21. What's the kindest act you've ever witnessed?
22. Tell us one thing you know you do well (a talent?) and one thing you know you cannot do.
23. What is your favorite way to procrastinate?
24. What is your favorite home-cooked meal?
25. What was your favorite childhood toy?
26. What do you do other than study? What clubs are you involved in?
27. What was your first job?
28. Have you met a famous person? Who?
29. What's the story behind your name?
30. Do you believe in anything that most people might not believe in?
31. I wish everyone would...
32. What's the best sound effect you can make?
33. What's the funniest thing you did as a kid that people still talk about today?
34. What idea do you think is worth arguing about?
35. Tell us something quirky about you.
36. For what reason do others often seek your help or input?
37. Share your guilty pleasure (something you enjoy that embarrasses youlike watching Disney Channel)?
38. What is one thing that's important for others to know about you?
39. Do you still do anything today that you also loved to do as a child?
40. Do you have any daily rituals?
41. What is the most misunderstood word you can think of?
42. What is the first book you remember changing you somehow?
43. Pass on one piece of wisdom to the class.
44. Do you have an irrational fear or strange addiction? Or something that started in college?
45. What's been the most surprising thing about college?
46. What is your biggest pet peeve?
47. Tell us about any animal friends you have.
48. What did you dream of becoming when you were younger?
49. What's something new you've learned this week?
50. What thought keeps you up at night?
51. What was your favorite thing about last summer?
52. Did you have a back-up plan to USNA? What was it?
53. Who would be your best man / maid of honor?
54. At this point, how many years will you serve post-commissioning?
55. How many times did you take the ACT / SAT?
56. Describe a run-in with law enforcement or the USNA conduct system.
57. How much sleep did did you get in the last 24 hours?
58. What is your superpower?
59. What is your heritage? (for example, my grandparents were from Mexico on my mother's side, and descendants of slaves on my father's side.)
60. What is your birth order?
61. (adapted from 11.) What's one Forrestal Lecture you think everyone should see?
62. What is the most memorable reaction you've had to being in uniform?

63. What is your earliest/favorite/funniest childhood memory?
64. What is one thing you are passionate about?
65. What is a pressing problem at USNA? If you could change one thing, what would it be?
66. Who is your favorite musical artist?
67. What is your favorite ice cream?
68. What topic was your funniest squad meal conversation about?
69. Where do you want to live?
70. What is your service assignment preference?
71. What is the craziest thing a teacher has ever done in the classroom?
72. What is the furthest you have traveled from home? What is the favorite place you have traveled?
73. What is the most important lesson you have learned at the Academy?
74. What are you most looking forward to in the next five years?
75. What is something you learned to do the hard way?
76. What is your favorite thing to do when you go home?
77. What was your favorite TV show growing up?
78. Where did your initial interest in the Naval Academy come from?
79. Tell us your most awkward dating story, or your most embarrassing story.
80. Do you have a favorite sport to watch? Your favorite college football team other than Navy?
81. What is your hidden talent?

The Set of Zero Divisors of a Factor Ring

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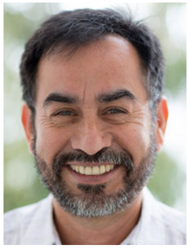

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Abstract

We show that if A is a commutative ring with unity and $\mathfrak{a}$ is an ideal of A which is a finite product of relative prime ideals $\mathfrak{b}_i$ then the factor ring $A/\mathfrak{a}$ is a direct sum of ideals $\mathfrak{a}_i/\mathfrak{a}$. Moreover, each ideal $\mathfrak{a}_i/\mathfrak{a}$ endowed with addition and multiplication modulo $\mathfrak{a}$ is a ring isomorphic to the factor ring $A/\mathfrak{b}_i$. We give examples when A is the ring of integers $\mathbb{Z}$, the Gaussian integers $\mathbb{Z}[\mathbf{i}]$, or a ring of polynomials $\mathbb{F}_q[x]$ over a finite field with q elements $\mathbb{F}_q$.

1 Introduction

Throughout this paper A will be a commutative ring with a non-zero multiplicative identity. The group of units of A will be denoted by $U(A)$ and the set of zero divisors together with the zero element will be denoted by $Z(A)$. If A is finite then the set $Z(A)$ is the complement of $U(A)$, Lemma 1. In [2] and [6], the authors show that if $A = \mathbb{Z}_a$, the ring of integers modulo a, then there exist positive integers n such that $U(\mathbb{Z}_a)$ can be mapped isomorphically to a group D contained in the set $Z(\mathbb{Z}_n)$ with the group structure of D given by multiplication modulo n. Here we will show, by means of the Chinese Remainder Theorem, that there exist positive integers n such that the ring $\mathbb{Z}_a$ can be mapped, ring isomorphically, to a ring R contained in the set $Z(\mathbb{Z}_n)$ with the ring structure of R given by addition and multiplication modulo n. This shows that the group $U(\mathbb{Z}_a)$ is isomorphic to $U(R) \subset R \subset Z(\mathbb{Z}_n)$.

2 Elements of Ring Theory

An element x of a A is *nilpotent* if and only if there exists a positive integer n such that $x^n = 0$. An element e of a A is *idempotent* if and only if $e^2 = e$. If e_1 and e_2 are nonzero idempotent elements of A, we say that they are *orthogonal* if and only if $e_1 \cdot e_2 = 0$. We remark that if an element y of A is both nilpotent and idempotent then $y = 0$. We say that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A are *relatively prime* or *coprime* if and only if $\mathfrak{a} + \mathfrak{b} = A$. A proper ideal $\mathfrak{m}$ of A is a *maximal* ideal if and only if for any ideal $\mathfrak{a}$ of A such that $\mathfrak{m} \subseteq \mathfrak{a} \subseteq A$ either $\mathfrak{m} = \mathfrak{a}$ or $\mathfrak{a} = A$. A ring A is called a *local ring* if and only if it has only one proper maximal ideal $\mathfrak{m}$. If $A_1, A_2, \ldots, A_k$ are rings then the *direct product* of these rings is the ring

$$A_1 \times A_2 \times \cdots A_k$$

with component-wise addition and multiplication.

Lemma 1. *If A is a finite ring then $Z(A) = A \setminus U(A)$.*

Proof. First, we observe that $U(A) \cap Z(A) = \emptyset$ since a unit may not be a zero divisor. Next, let $a \neq 0$ be an element of A and define the homomorphism $\mu_a : A \to A$ (A consider as a commutative group under addition) by $\mu_a(x) = ax$. If μ_a is not injective then μ_a has a nonzero kernel so there exists $b \neq 0$ in A such that $\mu_a(b) = ab = 0$ and it follows that a is a zero divisor.
Otherwise, since A is finite, if μ_a is injective μ_a must be onto. Therefore, there exists $c \neq 0$ in A such that $\mu_a(c) = ac = 1$. This implies that a is a unit. □

Lemma 2. *If A is a local ring then the only idempotent elements of A are either 0 or 1.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and e an idempotent of A. We observe that in a local ring an element x is either in $\mathfrak{m}$ or is a unit. Also, it is not possible to have both e and $1-e$ be elements of $\mathfrak{m}$ since this implies that $1 = e + (1-e)$ is in $\mathfrak{m}$ which contradicts the fact that $\mathfrak{m}$ is a proper ideal. Therefore, either e is a unit or $1-e$ is a unit. Since $e = e^2$, we have $e \cdot (1-e) = 0$. If e is a unit then $1 - e = 0$, this implies that $e = 1$. On the other hand, if $1-e$ is a unit then $e = 0$. □

Lemma 3. *Let $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k$, be ideals of a ring A. Assume that $\mathfrak{b}_i + \mathfrak{b}_j = A$ whenever $i \neq j$. Let $\mathfrak{a}_i = \prod_{j \neq i} \mathfrak{b}_j$. Then, for all $i = 1, 2, \ldots, k$, $\mathfrak{b}_i + \mathfrak{a}_i = A$.*

Proof. For every $j \neq i$ let $a_j \in \mathfrak{b}_j$ and $b_i \in \mathfrak{b}_i$ be such that $a_j + b_i = 1$. Then,

$$a_i = \prod_{j \neq i} a_j = \prod_{j \neq i} (1 - b_i) \equiv 1 \bmod \mathfrak{b}_i.$$

This implies that $a_i - 1 \in \mathfrak{b}_i$. So, there exists $b_i \in \mathfrak{b}_i$ such that $a_i + b_i = 1$. This shows that $\mathfrak{a}_i + \mathfrak{b}_i = A$. □

Lemma 4. *Let $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k$ be ideals of ring A. Assume that $\mathfrak{b}_i + \mathfrak{b}_j = A$ whenever $i \neq j$. Then,*

$$\prod_{i=1}^{k} \mathfrak{b}_i = \bigcap_{i=1}^{k} \mathfrak{b}_i.$$

Proof. We will use induction on k. If $k = 2$ then $\mathfrak{b}_1$ is relatively prime to $\mathfrak{b}_2$ by assumption. Let $b_1 \in \mathfrak{b}_1$ and $b_2 \in \mathfrak{b}_2$ be such that $b_1 + b_2 = 1$. If $x \in \mathfrak{b}_1 \cap \mathfrak{b}_2$ then $x = xb_1 + xb_2 \in \mathfrak{b}_1\mathfrak{b}_2$. Since $\mathfrak{b}_1\mathfrak{b}_2 \subseteq \mathfrak{b}_1 \cap \mathfrak{b}_2$, it follows that $\mathfrak{b}_1\mathfrak{b}_2 = \mathfrak{b}_1 \cap \mathfrak{b}_2$. Assume that $k \geq 3$. By Lemma 3, $\mathfrak{b}_1$ is relatively prime to $\prod_{i \geq 2} \mathfrak{b}_i$. By the case $k = 2$

$$\mathfrak{b}_1 \cdot \prod_{i \geq 2}^{k} \mathfrak{b}_i = \mathfrak{b}_1 \bigcap \prod_{i \geq 2}^{k} \mathfrak{b}_i.$$

By induction hypothesis

$$\prod_{i \geq 2}^{k} \mathfrak{b}_i = \bigcap_{i \geq 2}^{k} \mathfrak{b}_i.$$

The last two equations show that

$$\prod_{i=1}^{k} \mathfrak{b}_i = \bigcap_{i=1}^{k} \mathfrak{b}_i.$$

This completes the proof. $\square$

Lemma 5. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *ideals of* A *such that* $\mathfrak{a}+\mathfrak{b}=A$. *Let* $a \in \mathfrak{a}$ *and* $b \in \mathfrak{b}$ *be such that* $a+b=1$. *Let* $A/\mathfrak{a}$, $A/\mathfrak{b}$, $A/\mathfrak{ab}$ *be the factor rings of* A *by the ideals* $\mathfrak{a}$, $\mathfrak{b}$, *and* $\mathfrak{ab}$ *respectively. Then, the ring morphism*

$$\pi : A/\mathfrak{ab} \to A/\mathfrak{a} \times A/\mathfrak{b},\ (x \bmod \mathfrak{ab}) \mapsto (x \bmod \mathfrak{a}, x \bmod \mathfrak{b})$$

is an isomorphism and the ideals $\mathfrak{a}/\mathfrak{ab}$ *and* $\mathfrak{b}/\mathfrak{ab}$ *endowed with addition and multiplication modulo* $\mathfrak{ab}$ *are rings with unity. Moreover, the unity of* $\mathfrak{a}/\mathfrak{ab}$ *is* $a \bmod \mathfrak{ab}$, *the unity of* $\mathfrak{b}/\mathfrak{ab}$ *is* $b \bmod \mathfrak{ab}$ *and the ring morphisms*

$$\mathfrak{a}/\mathfrak{ab} \to A/\mathfrak{b},\ (x \bmod \mathfrak{ab}) \mapsto (x \bmod \mathfrak{b}) \quad \textit{and}$$
$$\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a},\ (y \bmod \mathfrak{ab}) \mapsto (y \bmod \mathfrak{a})$$

are isomorphisms.

Proof. The ring morphism

$$A \to A/\mathfrak{a} \times A/\mathfrak{b},\ (x \mapsto (x \bmod \mathfrak{a}, x \bmod \mathfrak{b})$$

is well defined and surjective and its kernel is $\mathfrak{a} \cap \mathfrak{b}$. By Lemma 3, $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$. This implies that π is an isomorphism. Since $a + b = 1$ we have

$$(b \bmod \mathfrak{ab}) \mapsto (b \bmod \mathfrak{a}) = (1 - a \bmod \mathfrak{a}) = (1 \bmod \mathfrak{a}).$$

Now we show that $\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a}$ is surjective. Let $(x \bmod \mathfrak{a}) \in A/\mathfrak{a}$. We have $bx \in \mathfrak{b}$ and

$$\begin{aligned}(bx \bmod \mathfrak{ab}) \mapsto (bx \bmod \mathfrak{a}) &= (b \bmod \mathfrak{a})(x \bmod \mathfrak{a}) \\ &= (1 \bmod \mathfrak{a})(x \bmod \mathfrak{a}) \\ &= (x \bmod \mathfrak{a}).\end{aligned}$$

Next we show that $\mathfrak{b}/\mathfrak{ab} \to A/\mathfrak{a}$ is injective. If y_1 and y_2 are in $\mathfrak{b}$ and

$$(y_1 \bmod \mathfrak{a}) = (y_2 \bmod \mathfrak{a})$$

then $y_1 - y_2 \in \mathfrak{a}$. It follows that $y_1 - y_2 \in \mathfrak{a} \cap \mathfrak{b} = \mathfrak{ab}$. This implies that

$$(y_1 \bmod \mathfrak{ab}) = (y_2 \bmod \mathfrak{ab}).$$

and it follows that the morphism is injective. Therefore, the morphism is an isomorphism. $\square$

Theorem 6 (Chinese Remainder Theorem)**.** *Let* $\mathfrak{b}_1, \mathfrak{b}_2, \ldots \mathfrak{b}_k$, *be ideals in* A. *Set* $\mathfrak{a} = \prod \mathfrak{b}_i$ *and assume that* $\mathfrak{b}_i + \mathfrak{b}_j = A$ *whenever* $i \neq j$. *Then, the ring morphism*

$$\pi : A/\mathfrak{a} \to A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k,\ \pi(x \bmod \mathfrak{a}) = (x \bmod \mathfrak{b}_1, x \bmod \mathfrak{b}_2, \ldots, x \bmod \mathfrak{b}_k)$$

is a ring isomorphism.

Proof. We will use induction on k. If $k = 1$ there is nothing to prove. Assume that $k = 2$. Since $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are relatively prime the result follows from Lemma 5. Suppose that $k \geq 3$. By Lemma 3, $\mathfrak{b}_1$ is relatively prime to the product $\prod_{i\geq 2}^{k} \mathfrak{b}_i$. Therefore,

$$\begin{aligned} A/\mathfrak{a} &= A/(\mathfrak{b}_1 \cdot (\mathfrak{b}_2 \cdots \mathfrak{b}_k)) \\ &\simeq A/\mathfrak{b}_1 \times A/(\mathfrak{b}_2 \cdots \mathfrak{b}_k), \text{ (by Lemma 5)} \\ &\simeq A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \mathfrak{b}_3 \times \cdots \times A/\mathfrak{b}_k \text{ (by induction hypothesis).} \end{aligned}$$

□

Corollary 7. *Let*

$$\pi : A/\mathfrak{a} \to A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k,\ \pi(x \bmod \mathfrak{a}) = (x \bmod \mathfrak{b}_1, x \bmod \mathfrak{b}_2, \ldots, x \bmod \mathfrak{b}_k)$$

be as in Theorem 6, $\pi_i : A/\mathfrak{b}_1 \times A/\mathfrak{b}_2 \times \cdots \times A/\mathfrak{b}_k \to A/\mathfrak{b}_i$ be the canonical projection, and $\mathfrak{a}_i = \prod_{j\neq i} \mathfrak{b}_j$. Then $\mathfrak{a}_i/\mathfrak{a}$ endowed with addition and multiplication modulo $\mathfrak{a}$ is a ring with unity and

$$\pi_i \circ \pi \mid_{\mathfrak{a}_i/\mathfrak{a}} : \mathfrak{a}_i/\mathfrak{a} \to A/\mathfrak{b}_i$$

is an isomorphism.

Proof. This follows from Lemma 5 since $\mathfrak{a}_i$ is relatively prime to $\mathfrak{b}_i$ and $\mathfrak{a} = \mathfrak{a}_i\mathfrak{b}_i$. □

Corollary 8. *Let $\mathfrak{a}_i$ be as in Corollary 7 and $e_i \in \mathfrak{a}_i$ be such that $e_i \equiv 1 \bmod \mathfrak{b}_i$. Then e_i is an idempotent in $\mathfrak{a}_i$ for all $i = 1, 2, \ldots, k$, e_i is orthogonal to e_j whenever $i \neq j$, and $e = e_1 + e_2 + \cdots + e_k \equiv 1 \bmod \mathfrak{a}$. Moreover, e_i is the multiplicative identity of $\mathfrak{a}_i/\mathfrak{a}$*

Proof. Since the map $\pi_i \circ \pi \mid_{\mathfrak{a}_i/\mathfrak{a}}$ is an isomorphism and

$$\pi_i \circ \pi \mid_{\mathfrak{a}_i/\mathfrak{a}} (e_i^2 \bmod \mathfrak{a}) \equiv 1 \bmod \mathfrak{a} \equiv \pi_i \circ \pi \mid_{\mathfrak{a}_i/\mathfrak{a}} (e_i \bmod \mathfrak{a})$$

we have $e_i^2 \equiv e_i \bmod \mathfrak{a}$ for all $i = 1, 2, \ldots, k$. Now, if $i \neq j$ then $e_i \cdot e_j \in \mathfrak{a}_i\mathfrak{a}_j \subseteq \mathfrak{a}$ and it follows that $e_i \cdot e_j \equiv 0 \bmod \mathfrak{a}$. Also, since

$$\pi(e \bmod \mathfrak{a}) = (1 \bmod \mathfrak{b}_1, 1 \bmod \mathfrak{b}_2, \ldots, 1 \bmod \mathfrak{b}_k) = \pi(1 \bmod \mathfrak{a})$$

we have $e \equiv 1 \bmod \mathfrak{a}$. That e_i is the multiplicative identity of $\mathfrak{a}_i/\mathfrak{a}$ follows from Lemma 5. □

Corollary 9. *Let $\mathfrak{a}_i$ and $e_i \in \mathfrak{a}_i$ be such that $e_i \equiv 1 \bmod \mathfrak{b}_i$, then $A/\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots + \mathfrak{a}_k$ and every element $x \in A$ can be uniquely written as $xe_1 + xe_2 + \cdots + xe_k \equiv x \bmod \mathfrak{a}$. That is, A is the direct sum of the ideals $\mathfrak{a}_i$.*

Proof. Since $e_1 + e_2 + \cdots + e_k \equiv 1 \bmod \mathfrak{a}$ it follows that $xe_1 + xe_2 + \cdots + xe_k \equiv x \bmod \mathfrak{a}$. To show that this representation is unique it suffices to show that if $xe_1 + xe_2 + \cdots + xe_k \equiv 0 \bmod \mathfrak{a}$ then $x \equiv 0 \bmod \mathfrak{a}$. Since

$$0 \bmod \mathfrak{a} = \pi_i \circ \pi \mid_{\mathfrak{b}_i/\mathfrak{a}} (0) = \pi_i \circ \pi \mid_{\mathfrak{b}_i/\mathfrak{a}} (xe_1 + xe_2 + \cdots + xe_k) = xe_i \bmod \mathfrak{a}_i = x \bmod \mathfrak{a}_i$$

it follows that $x \in \mathfrak{a}_i$ for all $i = 1, 2, \ldots, k$ so $x \in \cap_{i=1}^{k} \mathfrak{a}_i = \mathfrak{a}$. This shows that $x \equiv 0 \bmod \mathfrak{a}$. □

3 Factor rings of the ring of integers

Let $\mathbb{Z}_a$ be the ring of integers modulo a. The set $U(\mathbb{Z}_a)$ consists of all the elements in $\mathbb{Z}_n$ which are relatively prime to n. The number of elements of $U(\mathbb{Z}_a)$ is $\varphi(a)$, where $\varphi(a)$ is the Euler function. $U(\mathbb{Z}_a)$ is a group under multiplication modulo a and its group structure is well known. Recall that the set $Z(\mathbb{Z}_a)$ is the complement of $U(\mathbb{Z}_a)$ in $\mathbb{Z}_a$. The nonzero elements of $Z(\mathbb{Z}_a)$ are zero divisors under multiplication modulo a. The set $Z(\mathbb{Z}_a)$ is closed under multiplication modulo a, since the product of zero divisors is a zero or a zero divisor. At first sight, it seems that there is no hope of finding subsets of $Z(\mathbb{Z}_a)$ that could be groups under multiplication modulo a. However, as we will show in the example below, there are subsets of $Z(\mathbb{Z}_a)$ that are not only ideals of $\mathbb{Z}_a$ but are rings with unity when endowed with addition and multiplication modulo a and the units of these rings are groups under multiplication modulo a. See [2], [6]. The example below illustrates the central theme of this paper.

Example 10. *Consider the ring $\mathbb{Z}_{360}$. The sets*

$$\begin{aligned} Z_5 &= \{0,72,144,216,288\} = 72\mathbb{Z}_{360}, \\ Z_8 &= \{0,45,90,135,180,225,270,315\} = 45\mathbb{Z}_{360}, \text{ and} \\ Z_9 &= \{0,40,80,120,160,200,240,280,320\} = 40\mathbb{Z}_{360} \end{aligned}$$

are principal ideals of $\mathbb{Z}_{360}$. These ideals are rings with unity under addition and multiplication modulo 360. The multiplicative identities of the ideals Z_5, Z_8 and Z_9 are $e_5 = 216 \bmod 360$, $e_8 = 225 \bmod 360$, and $e_9 = 280 \bmod 360$ respectively. The element 0 is the additive identity for them. The ring isomorphisms and their inverses are given below.

$$\begin{array}{llllll} \theta_{216}: Z_5 & \to & \mathbb{Z}_5 & \theta_{216}(72x \bmod 360) & = & 72x \bmod 5 \\ \theta_{225}: Z_8 & \to & \mathbb{Z}_8 & \theta_{225}(45x \bmod 360) & = & 45x \bmod 8 \\ \theta_{280}: Z_9 & \to & \mathbb{Z}_9 & \theta_{280}(40x \bmod 360) & = & 40x \bmod 9 \end{array}$$

$$\begin{array}{llllll} \theta_{216}^{-1}: \mathbb{Z}_5 & \to & Z_5 & \theta_{216}^{-1}(x \bmod 5) & = & 216x \bmod 360 \\ \theta_{225}^{-1}: \mathbb{Z}_8 & \to & Z_8 & \theta_{225}^{-1}(x \bmod 8) & = & 225x \bmod 360 \\ \theta_{280}^{-1}: \mathbb{Z}_9 & \to & Z_9 & \theta_{280}^{-1}(x \bmod 9) & = & 280x \bmod 360 \end{array}$$

The elements, e_5, e_8, and e_9 satisfy

$$e_5 \cdot e_8 \equiv e_5 \cdot e_9 \equiv e_8 \cdot e_9 \equiv 0 \bmod 360$$

and

$$e_5 + e_8 + e_9 \equiv 1 \bmod 360.$$

Therefore, for all $x \in \mathbb{Z}_{360}$, we have

$$xe_5 + xe_8 + xe_9 \equiv x \bmod 360$$

and this representation is unique. That is,

$$\mathbb{Z}_{360} = e_5\mathbb{Z}_{360} \oplus e_8\mathbb{Z}_{360} \oplus e_9\mathbb{Z}_{360}.$$

The other nonzero proper ideals of $\mathbb{Z}_{360}$ (besides Z_5, Z_8, Z_9) that are rings with addition and multiplication modulo 360 are

Ideal	*Identity*				
$Z_5 \oplus Z_8$	$e_5 + e_8$	$\equiv$	$216 + 225$	$\equiv$	$81 \bmod 360$
$Z_5 \oplus Z_9$	$e_5 + e_9$	$\equiv$	$216 + 280$	$\equiv$	$136 \bmod 360$
$Z_8 \oplus Z_9$	$e_8 + e_9$	$\equiv$	$225 + 280$	$\equiv$	$145 \bmod 360$

We have the following general result.

Corollary 11. *Let $\mathbb{Z}$ be the ring of integers, $p_1, p_2, \ldots, p_k$, be prime integers, $r_1, r_2, \ldots, r_k$, be positive integers. Set $b_i = p_i^{r_i}$ for $i = 1, 2, \ldots, k$, $a = \prod b_i$, and $a_i = \prod_{i \neq j} b_j$. Then,*

$$\mathbb{Z}_a = e_1\mathbb{Z}_a + e_2\mathbb{Z}_a + \cdots + e_{k-1}\mathbb{Z}_a + e_k\mathbb{Z}_a,$$

where e_i is the unique nonzero idempotent in the ideal $a_i\mathbb{Z}_a \subsetneq \mathbb{Z}_a$, e_i is orthogonal to e_j whenever $i \neq j$, and the sum is a direct sum.

Proof. Denote by $\mathfrak{a}$, $\mathfrak{a}_i$, and $\mathfrak{b}_i$ the ideals generated by a, a_i, and b_i respectively. The idempotent e_i is an element of $\mathfrak{a}_i$. The orthogonality of e_i and e_j, $i \neq j$, follows from Corollary 8. By Corollary 7, the ideal $\mathfrak{a}_i/\mathfrak{a}$ is isomorphic to $\mathbb{Z}/\mathfrak{b}_i = \mathbb{Z}_{b_i}$ which is a local ring. The uniqueness of e_i follows from Lemma 2. The direct sum decomposition of $\mathbb{Z}_a$ follows from Corollary 9. □

It follows from Lemma 5, that given positive integers a, and b such that $\gcd(a, b) = 1$ then the ring

$$E_a = b\mathbb{Z}_a = \{0, b, 2b, \ldots, ab - b\}$$

is a ring under addition and multiplication modulo ab that is isomorphic to $\mathbb{Z}_a$. The ring E_a has multiplicative identity ($b^{\phi(a)} \bmod ab$). This construction can be done for an infinite number of integers b but since it could happen that ($b^{\phi(a)} \neq b \bmod ab$), b may not the multiplicative identity of E_a. However, we have the following proposition.

Proposition 12. *Let e be an integer, $e \geq 3$. Let b be a divisor of e, and $a > 1$ be a divisor of $e - 1$. Then, e is a nonzero idempotent in the ring $\mathbb{Z}_{ab}$ and is the identity of the ring E_a. Moreover, the map*

$$E_a \to \mathbb{Z}_a,\ x \bmod ab \mapsto x \bmod a,$$

is an isomorphism.

Proof. The proposition follows from Lemma 5 since $\gcd(a, b) = 1$. □

Next we observe that, if an integer $e \geq 3$ is an idempotent modulo n then $e(e-1) = 0 \bmod n$. This implies that n must be a divisor of $e(e-1)$. Since we want e to be a nonzero idempotent modulo a, n must be of the form $n = ab$ with a and b chosen as in the proposition. This proves the following corollary.

Corollary 13. *Let e be an integer, $e \geq 3$, and N_1 and N_2 be the number of divisors of e and $e - 1$ respectively. Then there are $N_1(N_2 - 1)$ integers of the form ab so that $(e \bmod ab)$ is the identity of the ring E_a.*

Example 14. *Consider the integer $e = 2016$. Since $2016 = 2^5 \cdot 3^2 \cdot 7$ and $2015 = 5 \cdot 13 \cdot 31$ we have $N_1 = 36$ and $N_2 = 8$. Therefore, there are $36 \cdot 7 = 252$ integers of the form ab (ab as in Proposition 12) such that $(2016 \bmod ab)$ is the identity of the ring*

$$E_a = \{0, e, 2e, \ldots, ae - e\}.$$

1. Let $a = 5$ *and* $b = 2$*. Then*

$$\begin{aligned} E_5 &= \{0, 2016, 4032, 6048, 8064\} \\ &= \{0, 6, 2, 8, 4\} \\ &= \{0, 2, 4, 6, 8\} \end{aligned}$$

endowed with addition and multiplication modulo $10 = 2 \cdot 5$ *is a ring with multiplicative identity* $6 = 2016 \bmod 10$*.* E_5 *is isomorphic to* $\mathbb{Z}_5$*.*

2. Let $a = 65$ *and* $b = 6$*. Then*

$$\begin{aligned} E_{65} &= \{0, 2016, 4032, \ldots, 127008, 129024\} \\ &= \{0, 66, 132, \ldots, 258, 324\} \\ &= \{0, 6, 12, \ldots, 378, 384\} \end{aligned}$$

endowed with addition and multiplication modulo $390 = 6 \cdot 65$ *is a ring with multiplicative identity* $66 = 2016 \bmod 390$*.* E_{65} *is isomorphic to* $\mathbb{Z}_{65}$*.*

3. Let $a = 31$ *and* $b = 2016$*. Then,*

$$E_{31} = \{0, 2016, 4032, \ldots, 58464, 60480\}$$

endowed with addition and multiplication modulo $62496 = 31 \cdot 2016$ *is a ring with multiplicative identity* 2016*.* E_{31} *isomorphic to the field* $\mathbb{Z}_{31}$*.*

4. Let $a = 7$ *and* $b = 288$*. Since* $288 - 1 = 287 = 7 \cdot 41$*, then*

$$E_7 = \{0, 288, 576, 864, 1152, 1440, 1728\}$$

endowed with addition and multiplication modulo $2016 = 7 \cdot 288$ *is a ring with multiplicative identity* 288*.*

4 Factor rings of the ring of Gaussian integers

In this section we will recall some properties of the Gaussian integers, define and compute an Euler function φ_G and give some examples of factor rings of the ring of Gaussian integers. First, recall that the Gaussian integers (denoted by $\mathbb{Z}[\mathbf{i}]$) is the sub-ring of the field of complex numbers given below

$$\mathbb{Z}[\mathbf{i}] = \{x + y\mathbf{i} \,|\, x \text{ and } y \text{ integers, } \mathbf{i}^2 = -1\}$$

endowed with addition and multiplication inherited from the field of complex number. The ring $\mathbb{Z}[\mathbf{i}]$ is an Euclidean domain with Euclidean function

$$\lambda : \mathbb{Z}[\mathbf{i}] \to \{0, 1, 2, 3, \ldots\}, \;\; \lambda(x + y\mathbf{i}) = x^2 + y^2.$$

Since $\mathbb{Z}[\mathbf{i}]$ is an Euclidean domain we have a division algorithm on it.

Theorem 15 (Division Algorithm for Gaussian Integers)**.** *Let* $z_1 \neq 0$ *and* z_2 *be Gaussian integers then there exist* q *and* r *Gaussian integers such that*

$$z_2 = qz_1 + r, \text{ and } r = 0 \text{ or } \lambda(r) < \lambda(z_1).$$

Proof. Let $d = \lambda(z_1) \in \mathbb{Z}$. Write $z_2\overline{z}_1 = A + B\mathbf{i}$. By the division algorithm on the ring of integers, we can write A and B as $A = q_1 d + r_1$ and $B = q_2 d + r_2$ with $-\frac{d}{2} \leq r_1 \leq \frac{d}{2}$ and $-\frac{d}{2} \leq r_2 \leq \frac{d}{2}$. Therefore,

$$z_2\overline{z}_1 = A + B\mathbf{i} = (q_1 + q_2)d + (r_1 + r_2\mathbf{i}) = (q_1 + q_2\mathbf{i})z_1\overline{z}_1 + (r_1 + r_2\mathbf{i}).$$

Since $\overline{z}_1$ divides $z_2\overline{z}_1$ and $(q_1 + q_2\mathbf{i})z_1\overline{z}_1$, it follows that $\overline{z}_1$ divides $r_1 + r_2\mathbf{i}$. That is,

$$r = \frac{r_1 + r_2\mathbf{i}}{\overline{z}_1}$$

is a Gaussian integer. This shows that

$$z_2 = (q_1 + q_2\mathbf{i})z_1 + r.$$

Since, $-\frac{d}{2} \leq r_1 \leq \frac{d}{2}$ and $-\frac{d}{2} \leq r_2 \leq \frac{d}{2}$ either $r = 0$ or

$$\lambda(r) = \lambda\left(\frac{r_1 + r_2\mathbf{i}}{\overline{z}_1}\right) = \frac{\lambda(r_1 + r_2\mathbf{i})}{\lambda(\overline{z}_1)} = \frac{r_1^2 + r_2^2}{d} \leq \frac{(d/2)^2 + (d/2)^2}{d} = \frac{d^2/2}{d} = \frac{d}{2} < \lambda(z_1). \quad \square$$

The ring $\mathbb{Z}[\mathbf{i}]$, being an Euclidean domain, is a unique factorization domain. Therefore, if a is a Gaussian integer then a can be written uniquely (up to units) as the product of prime Gaussian elements. The following theorem characterizes the prime elements of $\mathbb{Z}[\mathbf{i}]$.

Theorem 16. *Let p be a prime in $\mathbb{Z}$. Then:*

1. *If $p = 2$, then $1 + \mathbf{i}$ is prime in $\mathbb{Z}[\mathbf{i}]$ and $2 = \mathbf{i}^3(1 + \mathbf{i})^2$.*
2. *If $p \equiv 3 \bmod 4$, then p remains prime in $\mathbb{Z}[\mathbf{i}]$.*
3. *If $p \equiv 1 \bmod 4$, then there exists a prime $\pi \in \mathbb{Z}[\mathbf{i}]$ such that $p = \pi\overline{\pi}$, and the primes π and $\overline{\pi}$ are nonassociate in $\mathbb{Z}[\mathbf{i}]$. Furthermore, every prime in $\mathbb{Z}[\mathbf{i}]$ is associate to one of the primes listed in (1)-(3) above. (Two primes are associate if they differ by a unit factor.)*

Proof. See [3] page 81. $\square$

Denote by $\langle a \rangle$ the ideal of $\mathbb{Z}[\mathbf{i}]$ generated by a and by $\mathbb{Z}[\mathbf{i}]_a$ the quotient ring $\mathbb{Z}[\mathbf{i}]/\langle a \rangle$. We have the following result.

Corollary 17. *Let $\mathbb{Z}[\mathbf{i}]$ be the ring of Gaussian integers, $\pi_1, \pi_2, \ldots, \pi_k$ be prime elements in $\mathbb{Z}[\mathbf{i}]$, $r_1, r_2, \ldots, r_k$ be positive integers, $a_l = \pi_l^{r_l}$ for $l = 1, 2, \ldots, k$, $a = \prod a_l$, and $b_l = \prod_{l \neq j} a_j$. Then*

$$\mathbb{Z}[\mathbf{i}]_a = e_1\mathbb{Z}[\mathbf{i}]_{b_1} + e_2\mathbb{Z}[\mathbf{i}]_{b_2} + \cdots + e_{k-1}\mathbb{Z}[\mathbf{i}]_{b_{k-1}} + e_k\mathbb{Z}[\mathbf{i}]_{b_k}$$

where e_l is the unique nonzero idempotent in the ideal $b_l\mathbb{Z}[\mathbf{i}]_a \subsetneq \mathbb{Z}[i]_a$, e_l is orthogonal to e_j whenever $l \neq j$, and the sum is a direct sum.

Proof. The proof is similar to the proof of Corollary 11, so we omit it. $\square$

We remark that Proposition 12 and Corollary 13 remain valid in the Gaussian integers setting. We also have the following theorem about factor rings of the ring $\mathbb{Z}[\mathbf{i}]$.

Theorem 18. *Let $z = a + b\mathbf{i}$ be a Gaussian integer then*

1. $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{-a-b\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{b-a\mathbf{i}} \cong \mathbb{Z}[\mathbf{i}]_{-b+a\mathbf{i}}$
2. *If $a > 1$ and $b = 0$ then* $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} = \mathbb{Z}[\mathbf{i}]_a \cong \mathbb{Z}_a[\mathbf{i}]$
3. *If* $\gcd(a, b) = 1$ *then* $\mathbb{Z}[\mathbf{i}]_{a+b\mathbf{i}} = \mathbb{Z}_{a^2+b^2}$
4. *If $n > 0$ is an integer then*
   (a) *If $n = 2m$,* $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n} \cong \mathbb{Z}_{2^m}[\mathbf{i}]$.
   (b) *If $n = 2m + 1$, $m \geq 1$ then* $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n} \cong \mathbb{Z}[x]/\langle 2^m x, 2^{m+1}, x^2 + x + 2\rangle$. *In this case,* $\mathbb{Z}[\mathbf{i}]_{(1+\mathbf{i})^n}$ *is not isomorphic to $\mathbb{Z}_c$, $\mathbb{Z}_c[\mathbf{i}]$, or to any direct product of rings of this type.*

Proof. See [4] Fact 1 and Theorems 1, 2, and 5. □

Example 19. *Consider the ring $\mathbb{Z}[\mathbf{i}]_{360}$. Let $z_1 = 1 + 2\mathbf{i}$. Since $360 = \mathbf{i} \cdot z_1 \cdot \overline{z}_1 \cdot 3^2 \cdot (1+\mathbf{i})^6$ The sets*

$$\begin{array}{rcccl} Z_5 & = & \{x72(1+2\mathbf{i}) \mid x \in \mathbb{Z}_5\} & = & 72(1+2\mathbf{i})\mathbb{Z}[\mathbf{i}]_{360}, \\ \overline{Z}_5 & = & \{x72(1-2\mathbf{i}) \mid x \in \mathbb{Z}_5\} & = & 72(1-2\mathbf{i})\mathbb{Z}[\mathbf{i}]_{360}, \\ Z_8[\mathbf{i}] & = & \{45(x+y\mathbf{i}) \mid x, y \in \mathbb{Z}_8\} & = & 45\mathbb{Z}[\mathbf{i}]_{360}, \text{ and} \\ Z_9[\mathbf{i}] & = & \{40(x+y\mathbf{i}) \mid x, y \in \mathbb{Z}_9\} & = & 40\mathbb{Z}[\mathbf{i}]_{360} \end{array}$$

are principal ideals of $\mathbb{Z}[\mathbf{i}]_{360}$. These ideals are rings with unity under addition and multiplication modulo 360. The multiplicative identities of the ideals Z_5, $\overline{Z}_5$, Z_8 and Z_9 are $e_5 = 288 - 144\mathbf{i}$, $\overline{e}_5 = 288 + 144\mathbf{i}$, $e_8 = 225$, and $e_9 = 280$ respectively. The element 0 is the additive identity for them. The ring isomorphisms and their inverses are given below.

$$\begin{array}{rcllcl} \theta_{e_5} : Z_5 & \to & \mathbb{Z}[\mathbf{i}]_{1-2\mathbf{i}} & \theta_{e_5}(x72(1+2\mathbf{i}) \bmod 360) & = & x72(1+2\mathbf{i}) \bmod 1-2\mathbf{i} \\ \theta_{\overline{e}_5} : \overline{Z}_5 & \to & \mathbb{Z}[\mathbf{i}]_{1+2\mathbf{i}} & \theta_{\overline{e}_5}(x72(1-2\mathbf{i}) \bmod 360) & = & x72(1-2\mathbf{i}) \bmod 1+2\mathbf{i} \\ \theta_{225} : Z_8[\mathbf{i}] & \to & \mathbb{Z}_8[\mathbf{i}] & \theta_{225}(45(x+y\mathbf{i}) \bmod 360) & = & 45(x+y\mathbf{i}) \bmod 8 \\ \theta_{280} : Z_9[\mathbf{i}] & \to & \mathbb{Z}_9[\mathbf{i}] & \theta_{280}(40(x+y\mathbf{i}) \bmod 360) & = & 40(x+y\mathbf{i}) \bmod 9 \end{array}$$

$$\begin{array}{rclrcl} \theta_{e_5}^{-1} : \mathbb{Z}[\mathbf{i}]_{1-2\mathbf{i}} & \to & Z_5 & \theta_{e_5}^{-1}(x \bmod 1-2\mathbf{i}) & = & (288-144\mathbf{i})x \bmod 360 \\ \theta_{\overline{e}_5}^{-1} : \mathbb{Z}[\mathbf{i}]_{1+2\mathbf{i}} & \to & \overline{Z}_5 & \theta_{\overline{e}_5}^{-1}(x \bmod 1-2\mathbf{i}) & = & (288+144\mathbf{i})x \bmod 360 \\ \theta_{225}^{-1} : \mathbb{Z}_8[\mathbf{i}] & \to & Z_8[\mathbf{i}] & \theta_{225}^{-1}(x+y\mathbf{i}) & = & 225(x+y\mathbf{i}) \bmod 360 \\ \theta_{280}^{-1} : \mathbb{Z}_9[\mathbf{i}] & \to & Z_9[\mathbf{i}] & \theta_{280}^{-1}(x+y\mathbf{i}) & = & 280(x+y\mathbf{i}) \bmod 360 \end{array}$$

The elements, e_5, $\overline{e}_5$, e_8, and e_9 satisfy

$$e_5 \cdot \overline{e}_5 \equiv e_5 \cdot e_8 \equiv e_5 \cdot e_9 \equiv \overline{e}_5 \cdot e_8 \equiv \overline{e}_5 \cdot e_9 \equiv e_8 \cdot e_9 \equiv 0 \bmod 360$$

and

$$e_5 + \overline{e}_5 + e_8 + e_9 \equiv 1 \bmod 360$$

Therefore, for all $x \in \mathbb{Z}[\mathbf{i}]_{360}$, we have

$$xe_5 + x\overline{e}_5 + xe_8 + xe_9 \equiv x \bmod 360$$

and this representation is unique. That is,

$$\mathbb{Z}[\mathbf{i}]_{360} = e_5\mathbb{Z}[\mathbf{i}]_{360} \oplus \overline{e}_5\mathbb{Z}[\mathbf{i}]_{360} \oplus e_8\mathbb{Z}_{360} \oplus e_9\mathbb{Z}_{360}.$$

The other nonzero proper ideals of $\mathbb{Z}_{360}$ (besides Z_5, $\overline{Z}_5$, Z_8, Z_9) that are rings with addition and multiplication modulo 360 are

Ideal	*Identity*				
$Z_5 \oplus \overline{Z}_5$	$e_5 \oplus \overline{e}_5$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i}$	$\equiv$	$216 \bmod 360$
$Z_5 \oplus Z_8$	$e_5 \oplus e_8$	$\equiv$	$288 - 144\mathbf{i} + 225$	$\equiv$	$153 - 144\mathbf{i} \bmod 360$
$Z_5 \oplus Z_9$	$e_5 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 280$	$\equiv$	$218 - 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_8$	$\overline{e}_5 \oplus e_8$	$\equiv$	$288 + 144\mathbf{i} + 225$	$\equiv$	$153 + 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_9$	$\overline{e}_5 \oplus e_9$	$\equiv$	$288 + 144\mathbf{i} + 280$	$\equiv$	$218 + 144\mathbf{i} \bmod 360$
$Z_8 \oplus Z_9$	$e_8 \oplus e_9$	$\equiv$	$225 + 280$	$\equiv$	$145 \bmod 360$
$Z_5 \oplus \overline{Z}_5 \oplus Z_8$	$e_5 \oplus \overline{e}_5 \oplus e_8$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i} + 225$	$\equiv$	$81 \bmod 360$
$Z_5 \oplus \overline{Z}_5 \oplus Z_9$	$e_5 \oplus \overline{e}_5 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 288 + 144\mathbf{i} + 280$	$\equiv$	$136 \bmod 360$
$Z_5 \oplus Z_8 \oplus Z_9$	$e_5 \oplus e_8 \oplus e_9$	$\equiv$	$288 - 144\mathbf{i} + 225 + 280$	$\equiv$	$73 - 144\mathbf{i} \bmod 360$
$\overline{Z}_5 \oplus Z_8 \oplus Z_9$	$\overline{e}_5 \oplus e_8 \oplus e_9$	$\equiv$	$288 + 144\mathbf{i} + 225 + 280$	$\equiv$	$73 + 144\mathbf{i} \bmod 360$

Example 20. *Consider the integer $e = 2017$. Since we have the following prime power decomposition of 2017 and 2016 over the Gaussian integers*

$$2017 = (9 + 44\mathbf{i})(9 - 44\mathbf{i}) \text{ and } 2016 = 2^5 \cdot 3^2 \cdot 7 = \mathbf{i}^3(1 + \mathbf{i})^{10} \cdot 3^2 \cdot 7$$

we have $N_1 = 4$ and $N_2 = 66$. Therefore, there are $4 \cdot 66 = 264$ Gaussian integers of the form ab, where a is a divisor of 2017 and b is a divisor of 2016, so that $(e \bmod ab)$ is the identity of the ring $E_a = b\mathbb{Z}[\mathbf{i}]_a$ endowed with addition and multiplication modulo ab.

5 Factor rings of rings of polynomials

Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ be the ring of polynomials with coefficients in $\mathbb{F}$ and indeterminate x. The ring $\mathbb{F}[x]$ is an Euclidean domain with Euclidean function

$$\deg : \mathbb{F}[x] \to \mathbb{Z}^+ \cup \{0\}$$

where $\deg(f(x))$ is the degree of the polynomial $f(x)$.

Theorem 21 (Division Algorithm for Polynomial Rings)**.** *Let $d(x) \neq 0$ and $f(x)$ be elements of $\mathbb{F}[x]$ then there exists $q(x)$ and $r(x)$ elements of $\mathbb{F}[x]$ such that*

$$f(x) = q(x)d(x) + r(x), \text{and } r(x) = 0 \text{ or } \deg(r(x)) < \deg(d(x)).$$

Moreover, $q(x)$ and $r(x)$ are unique.

Proof. See [5] Theorem 16.2. □

Since $\mathbb{F}[x]$ is an Euclidean domain, it is a unique factorization domain. We have the following result.

Corollary 22. *Let $p_1(x), p_2(x), \ldots, p_k(x)$, be irreducible polynomials in $\mathbb{F}[x]$, $r_1, r_2, \ldots, r_k$, be positive integers. Set $b_i(x) = p_i(x)^{r_i}$ for $i = 1, 2, \ldots, k$, $a(x) = \prod b_i(x)$, and $a_i(x) = \prod_{i \neq j} b_j(x)$. Let $\mathbb{F}[x]_{a(x)} := \mathbb{F}[x]/\langle a(x) \rangle$. Then,*

$$\mathbb{F}[x]_{a(x)} = e_1(x)\mathbb{F}[x]_{a(x)} + e_2(x)\mathbb{F}[x]_{a(x)} + \cdots + e_{k-1}(x)\mathbb{F}[x]_{a(x)} + e_k(x)\mathbb{F}[x]_{a(x)},$$

where $e_i(x)$ is the unique nonzero idempotent in the ideal $a_i(x)\mathbb{F}[x]_{a(x)} \subsetneq \mathbb{F}[x]_{a(x)}$, $e_i(x)$ is orthogonal to $e_j(x)$ whenever $i \neq j$, and the sum is a direct sum.

Proof. The proof is similar to the proof of Corollary 11. □

Example 23. *Let p be a prime and $q = p^r$. Let $\mathbb{F}_q$ be the finite field with q elements. Let n be a positive integer with $\gcd(p, n) = 1$ and $a(x) = x^n - 1$ be an element of $\mathbb{F}_q[x]$. The condition $\gcd(p, n) = 1$ implies that the factorization of $a(x)$ has not repeated factors, that is,*

$$a(x) = x^n - 1 = p_1(x) \cdot p_2(x) \cdots p_{k-1}(x) \cdot p_k(x)$$

where $p_i(x)$ is irreducible of degree d_i and $p_i(x) \neq p_j(x)$ if $i \neq j$. Corollary 22 implies that the factor ring $\mathbb{F}_q[x]_{a(x)} := \mathbb{F}_q[x]/\langle a(x) \rangle$ decomposes as the direct sum

$$\mathbb{F}_q[x]_{a(x)} = e_1(x)\mathbb{F}_q[x]_{a(x)} + e_2(x)\mathbb{F}_q[x]_{a(x)} + \cdots + e_{k-1}(x)\mathbb{F}_q[x]_{a(x)} + e_k(x)\mathbb{F}_q[x]_{a(x)}.$$

The ideal $e_j(x)\mathbb{F}_q[x]_{a(x)}$ is a field under addition and multiplication modulo $a(x)$, it is isomorphic to the field with q^{d_i} elements $\mathbb{F}_q[x]_{p_i(x)}$, and the idempotent element $e_j(x)$ is given by the formula $e_j(x) = a_j(x)^{q^{d_i}-1} \bmod a(x)$. The latter statement follows from the fact that the order of the group of units $U\left(\mathbb{F}_q[x]_{p_i(x)}\right)$ of the field $\mathbb{F}_q[x]_{p_i(x)}$ is $q^{d_i} - 1$. The idempotent elements $e_j(x)$ are known as primitive idempotent elements and they generate the minimal ideals of the factor ring $\mathbb{F}_q[x]_{a(x)}$.

This example is important because the ideals of the ring $\mathbb{F}_q[x]_{a(x)}$ correspond to $q-$ary cyclic error-correcting codes and the minimal idempotent elements correspond to the $q-$ary minimal cyclic codes. See Theorem 8.1 in [1] for a proof of this correspondence.

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The Topology of Harry Potter: Exploring Higher Dimensions in Young Adult Fantasy Literature [†]

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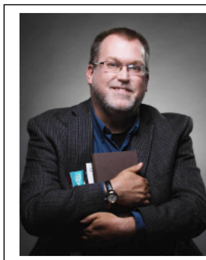
Sarah Klanderman is a Ph.D. student at Michigan State University studying algebraic topology, with a specific emphasis on connections between homotopy theory and algebraic K-theory. As an instructor, she has particularly enjoyed teaching Transitions to Proof and Math for Elementary Teachers, and she recently received Excellence-in-Teaching Citations from both the MSU College of Natural Science and the university. She is an avid Harry Potter fan and a Ravenclaw.

Having completed her Master's Degree in mathematics at the University of Wisconsin-Milwaukee, Alexa Schut is now an Assistant Professor of Mathematics at Culver-Stockton College in Canton, Missouri. With Bill and Dave, she coauthored a related article on Harry Potter and higher dimensions in the *Journal for Adolescent and Adult Literacy*. Outside of the classroom, she enjoys spending time hiking with her husband. She is a huge fan of Harry Potter and is a Slytherin.

Dave Klanderman has been a Professor of Mathematics at Trinity Christian College and begins as a Professor of Mathematics Education at Calvin College in the Fall of 2018. His research interests range from learning trajectories for length, area, and volume measurement to inquiry-based learning. He recently coauthored *A Pleasure to Measure*, a book of classroom activities for elementary school teachers. Although he has only read some of the Harry Potter books, his daughter assures him that he is a Ravenclaw.



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Abstract

As one of the most beloved series in children's literature today, the Harry Potter books excite students of all ages with the adventures of living in a magical world. Magical objects (e.g., bottomless handbags, the Knight Bus, time turners, and moving portraits) can inspire generalizations to

[†] In this paper, instead of providing a rigorous definition for *dimension*, we refer to a more familiar and less technical use of the term. For example, polygons are normally thought of as two-dimensional, while polyhedra are three-dimensional, etc. Although precision in mathematical terms is necessary, we believe that students will be more motivated to learn these rigorous definitions after exploring less technical (and arguably more interesting) analogies.

mathematical concepts that would be relevant in an undergraduate geometry or topology course. Intuitive explanations for some of the magical objects connect to abstract mathematical ideas. We offer a typology with a total of five categories, including Three Dimensions in Two Dimensions, Higher Dimensions in Three Dimensions, Two and Three Dimensional Movement, Higher Dimensional Movement, and Higher Dimensional Traces. These categories attempt to explain supernatural events from the wizarding world using mathematical reasoning in order to increase engagement in topics from topology to differential geometry. Our pedagogical approach is to pique student interest by linking these abstract concepts to familiar examples from the world of Harry Potter. Put on your Ravenclaw robe or Gryffindor scarf and join us!

1 Introduction and Overview of Research Project

The Harry Potter books by J.K. Rowling are not only incredibly popular, but they also provide wonderful analogies for higher dimensional thinking. Many of the experiences of a typical witch or wizard include magical objects, such as handbags enlarged via engorgement charms, time turners (which come in handy for taking two or three classes that meet at the same time), or moving portraits whose subjects can interact with people outside the frameall of these can in fact inspire generalizations to mathematical concepts that would be fitting in an undergraduate geometry or topology course. Many of the magical properties demonstrated in these books have a reasonable explanation if one considers the notion of higher dimensions or other abstract mathematical concepts. Our goal in this paper is to use analogies to motivate interest among students and to explain complicated and precise ideas in a more intuitive and less formal manner. That is, these vignettes from the novels serve as an anticipatory set, i.e. an introductory activity for the lesson that creates enthusiasm among students for later rigorous definitions, theorems, and further mathematical examples.

As part of a senior undergraduate research project, Alexa Schut explored the links between higher dimensions and young adult fantasy literature. Dave Klanderman and Bill Boerman-Cornell served as faculty mentors for Alexa and all three collaborated to find numerous examples of higher dimensional concepts in a wide variety of young adult fantasy literature. These include not only the Harry Potter novels of J.K. Rowling, but also the *Chronicles of Narnia* and the *Space Trilogy* by C.S. Lewis, *Flatland* by Edwin Abbott, and *A Wrinkle in Time* by Madeleine L'Engle. Based upon this research (which continued as Alexa pursued graduate degrees), five categories emerged that describe different types of dimensional concepts. Although these categories apply to all of the aforementioned novels, and although we included references to them in the typology chart below, for the sake of clarity, we will restrict our discussion in this paper to the Harry Potter series.

This first phase of the research led to an article published in the *Journal of Adolescent & Adult Literacy* that describes ways for middle school and high school mathematics and English teachers to entice strong mathematics students to explore higher dimensional concepts in the wizarding world of Harry Potter. It also provides the motivation for avid Harry Potter readers to desire a deeper understanding of these higher dimensional concepts linked to recommended topics in the middle school and high school mathematics curriculum.[1]

The next phase of research began with an observation by Sarah Klanderman, currently a Ph.D. student

[1]BoermanCornell, W., Klanderman, D., & Schut, A. (2017). Using Harry Potter to Bridge Higher Dimensionality in Mathematics and HighInterest Literature. Journal of Adolescent & Adult Literacy, 60(4), 425-432. Available at http://onlinelibrary.wiley.com/doi/10.1002/jaal.597/full

3D in 2D	Higher D in 3D – Pocket Dimensions	3D/2D movement	Higher D movement	Higher D Traces
~ Portraits (HP) ~ Newspaper (HP) ~ Chocolate Frog Cards (HP) ~ 2D world (WIT) ~ Books (EA) ~ The first conversation between Mr. Square and his grandson about how a square moving parallel to itself must create a geometric shape. (Flatland) ~ When Mr. Square puts squares on top of each other to create a solid while in the 3D world. (Flatland) ~ Tom's Journal (HP) ~ Room of Requirements (HP) ~ The Veil (HP -5th) ~ Doors in the Department of Mysteries entrance (HP) ~ Pensieve (HP) ~ Portrait in Hogs Head (HP) ~ Scar (HP)	~ Hermione's Bag (HP) ~ Tent (HP) ~ Train Station in the pillar (HP) ~ Wands (HP) ~ Eldila (Space T) ~ Oyarsa (Space T) ~ Diagon Alley (HP) ~ Sword in the Sorting hat (HP) ~ The fourth dimension as seen only by the inside of a 3D being as Mr. Square proposed to the Sphere. (Flatland) ~ Carriage (HP-4th) ~ Moody's Trunk (HP) ~ Room of Requirements (HP) ~ The Veil (HP-5th) ~ Doors in the Department of Mysteries entrance (HP) ~ Pensieve (HP) ~ Horcruxes (HP) ~ Scar (HP) ~ Weasley's Car (HP) ~ Ministry of Magic Underground (HP) ~ Vanished Objects (HP-7th) ~Woods between Worlds (Narnia)	~ Grim Place (HP) ~ Marauder's Map (HP) ~ Knight Bus (HP) ~ Car (EA) ~ Mirror of Erised/Sorcerer's Stone (HP) ~ Portrait in Hog's Head (HP) ~ St. Mungo's Mirror Entrance (HP)	~ Tessering (WIT) ~ Time Travel (WYRM) ~ Times linearity or lack of (WYRM) ~ Chromogaurd (EA) ~ The idea that if a four dimensional being is moving then we would see at as progression in time from a 3D view. (Flatland) ~ Train (HP) ~ Dobby's popping (HP) ~ Floo Powder (HP) ~ Time Turner (HP) ~ Portkey (HP) ~ Ship (HP-4th) ~ Triwizard trophy (HP) ~ Vanishing Cabinet (HP) ~ Gamp's 5 laws of magic (HP) ~ Vanishing Objects (HP-7th) ~ Apparating(HP) ~ Rings (Narnia) ~Food from a drop of liquid (Narnia)	~ Magic (HP) ~ Ghost (HP) ~ Resurrection Stone (HP) ~ Eldila (Space T) ~ Oyarsa (Space T) ~ Hypercube (video) ~ Leaky Cauldron (HP) ~ Stare of the Basilisk (HP) ~ Thestrals (HP) ~ Money in Gringotts that duplicates (HP) ~ Limbo state (HP) ~ The fact that when the Sphere enters Mr. Square's house all Mr. Square sees is the movement of a circle not the whole sphere. (Flatland). ~ "Ghosts" that appeared when Harry and Voldemort's wands meet (HP-4th)

Figure 1: *Typology Chart of Categories of Links between Young Adult Fantasy Literature and Higher Dimensions. Note:* ***HP*** *= J.K. Rowling's Harry Potter series (book number where noted),* ***WIT*** *= Madeleine L'Engle's A Wrinkle in Time,* ***Narnia*** *= C.S. Lewis's Chronicles of Narnia series,* ***SPACE T*** *= C.S. Lewis's Space Trilogy,* ***WYRM*** *= Rebecca Stead's When You Reach Me,* ***EA*** *= Jasper Fforde's Eyre Affair,* ***Flatland*** *= Edwin A. Abbott's Flatland,* ***Video*** *= Hypercube YouTube video*

in algebraic topology at Michigan State University. She noted that many of these five categories of dimensional concepts would lend themselves well to in-class examples in an undergraduate topology course. Sarah joined our research team at this juncture, and we plan to discuss three of these topological applications in detail in this paper.

The following sections of the paper will introduce each category, provide a comprehensive description for one specific example within the category along with a list of other examples, and, where appropriate, a discussion of mathematical concepts from topology that provide natural links from the Harry Potter novels to the undergraduate mathematics classroom. Here, the categories attempt to explain supernatural happenings in the wizarding world using mathematical reasoning in order to motivate topics in courses in topology or differential geometry. The descriptions in this article are designed to be only an introduction of the mathematics in order to clarify the literary connection to Harry Potter but can easily be supplemented with textbooks such as those referenced in this article. The goal is to pique students' interest by introducing upper- or graduate-level topics within the context of popular novels in order to motivate student understanding.

2 Category 1: 3 Dimension in 2 Dimensions

The first category of *Three Dimensions in Two Dimensions* can be briefly explained as three-dimensional objects that are in fact bounded in two-dimensional space. In *Harry Potter and the Sorcerer's Stone*, we see examples of this category as manifested in newspaper photos, wanted signs, Tom Riddle's journal, and the portraits of Hogwarts: Harry's legs were like lead again, but only because he was so tired and full of food. He was too sleepy even to be surprised that the people in the portraits along the corridors whispered and pointed as they passed.[2] These moving portraits, not unlike our own flat screen televisions, could be used to introduce topics such as stereographic projections, the real projective plane, or diffeomorphisms as defined using charts and atlases. Although the painting or photo is inherently contained in a two- (lower-) dimensional canvas like the projection or local neighborhood, the depicted scene (original space) is inherently three- (higher-) dimensional, which nevertheless allows them to move even beyond the bounds of their own painting into neighboring ones.

3 Category 2: Higher Dimensions in 3 Dimensions/Pocket Dimensions

The second category is *Higher Dimensions within Three Dimensions*, often more easily understood as pocket dimensions. Similar to the first category, this topic covers higher dimensional objects that move, operate, and exist within the confines of three dimensions. A well-suited example from the Harry Potter books is Hermione's bag, which particularly comes in handy during their travels in the seventh book, *Harry Potter and the Deathly Hallows*.[3] Because the handbag has an engorgement charm on it, it has the capacity to store many times more than it naturally should. In this way, the bag is a pocket dimension of sorts, containing more in its interior than is suggested by its small outer lining.

One way that we may consider this idea of pocket dimensions is as an outer reality (i.e. X = the outer appearance of Hermione's bag), which is homotopy equivalent[4] to that which is contained in the pocket dimension (i.e. Y = the inner contents of Hermione's bag). According to Hatcher's *Algebraic Topology*, "a map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $f \circ g \simeq 1$ and $g \circ f \simeq 1$. The spaces X and Y are said to be homotopy equivalent or to have the same homotopy type."[5] So although it isn't clear how one would define the continuous map from the inside of Hermione's bag to the outer beaded appearance in this situation (hence, why the idea is to intrigue students so that they are encouraged to bring their own imagination and investigate more), it's still clear that we have two spaces between which we cannot seem to visually understand their equivalence.

Other examples from the books of this homotopy equivalence of a space and what could be considered its pocket dimension would be Platform $9\frac{3}{4}$ (the train station seems to be contained within a single support pillar in King's Cross) and the small tents that Harry and Hermione stay in along with the majority of the Weasley family during the Quidditch World Cup.[6] Each tent has been charmed to be bigger on the inside than the outside perimeter would suggest,[7] able to hold a small kitchen, bunks, and, in one case,

[2] J. K. Rowling (1997) Harry Potter and the Sorcerer's Stone New York: Scholastic Inc. p. 128

[3] J. K. Rowling (2007) Harry Potter and the Deathly Hallows New York: Scholastic Inc.

[4] We recognize that a variety of notions of equivalence could instead be introduced here, but opt for homotopy equivalence because it doesn't require us to define metrics in a fictional universe, and it is also a topologically specific concept that often seems obscure to students upon first encounter.

[5] Allen Hatcher (2001) Algebraic Topology https://www.math.cornell.edu/ hatcher/AT/AT.pdf

[6] J. K. Rowling (2000) Harry Potter and the Goblet of Fire New York: Scholastic Inc.

[7] Perhaps using technology similar to the dimensional transcendentalism of the TARDIS (from the BBC television show

eight inhabitants (Harry, Ron, Bill, Charlie, Percy, Fred, George, and their father). Similarly to how we think of Hermione's purse, we may view one of these four-person tents to be a space X while their vast interiors form a space Y. However, because they occupy the same space via a pocket dimension, the two spaces are in this sense homotopy equivalent.[8]

4 Category 3: 3D/2D Movement

The third category includes situations in which a three-dimensional object must transform into a two-dimensional object, or vice versa. The Knight Bus provides transportation for wizards who are looking for a different pace than apparition or Floo powder. This vehicle is able to cover large distances quickly, even in heavy traffic conditions. The secret lies in the ability of the Knight Bus to flatten itself to squeeze between large vehicles traveling on the same road in both directions. In this sense, it temporarily transforms into a two-dimensional object before popping back out into its more typical three-dimensional form. It might be helpful to think of origami constructions that can be three-dimensional after multiple folds. However, the same object can be flattened back into a two-dimensional object by reversing these folds.

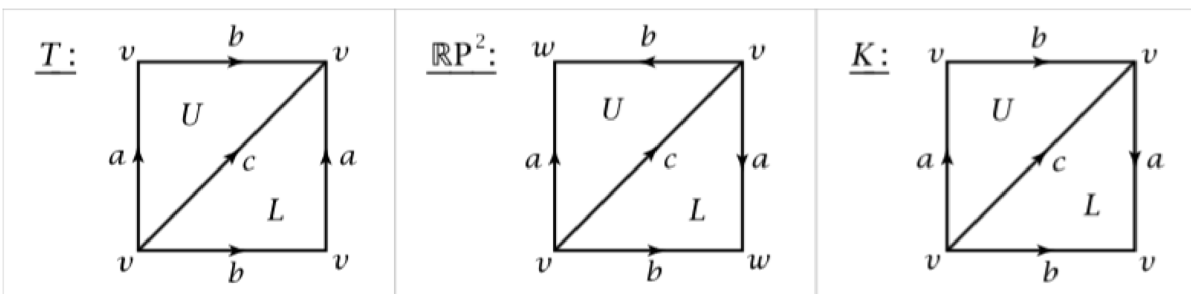

Figure 2: *Identification spaces of the torus, real projective place, and Klein bottle*[9]

From a more mathematical standpoint, this category links to the topological concept of an identification space. Identification spaces allow mathematicians to view higher dimensional objects through the analysis of two-dimensional figures that encode the higher dimensional information via gluings. For instance, a torus can be constructed from its identification space by gluing the sides labeled a to form a cylinder, and then gluing b to form its familiar donut shape. Unlike the torus or the other orientable surfaces shown in Figure 2, the real projective plane and Klein bottle live in four dimensions[10]; therefore we must use identification spaces in order to visualize them to the best of our abilities.

Other examples of this category would be the Marauder's Map, which depicts the movements of the

"Doctor Who"), which is accomplished via trans-dimensional engineering.

[8]As an example, the authors suggest implementing this motivating example for homotopy equivalence as follows. Near the beginning of an undergraduate topology course, an opening question could be, "When do we have two spaces that are equivalent?" Students may have a variety of answers, including ideas ranging from trivial labeling equivalences (e.g. "my bedroom" vs. "the room in which I sleep") to homeomorphisms/diffeomorphisms, etc. However, for a nontrivial example of distinct but co-existing spaces, the instructor could show a clip of Hermione pulling a large painting from her small handbag (thereby indicating its larger interior). The instructor could then move toward a more formal definition of homotopy equivalence, perhaps by using more standard mathematical examples (e.g. Möbius band and circle).

[9]Allen Hatcher (2001) Algebraic Topology https://www.math.cornell.edu/hatcher/AT/AT.pdf

[10]Technically, the fourth dimension is merely the smallest Euclidean dimension into which the Klein bottle embeds. However, due to our loose usage of the term *dimension*, one would first expect to see the Klein bottle "living in" four dimensions.

inhabitants of Hogwarts on a two-dimensional map, and 12 Grimmauld Place, the location of the headquarters of the Order of the Phoenix, which is a large apartment flat that exists in the space between two apartments which adjoin each other. This 3D/2D movement could also be extended to include the mirror of Erised and the appearance of the Sorcerer's Stone and the St. Mungo's Mirror Entrance.

5 Category 4: Higher Dimensional Movement

The fourth category also describes movement across different dimensions. However, these movements require access to a fourth or another higher dimension. Higher dimensional movement involves the idea of stretching and distorting to travel while still remaining equivalent to your original form, whereas higher dimensional traveling is the idea of entering higher dimensions in order to travel at greater speeds. Both of these ideas can be found throughout the Harry Potter books, including traveling via the Floo Powder network, taking portkeys or apparating, Dobby's popping/house elf apparition, vanishing objects, the train that carries wizards to Hogwarts, and Hermione's time turner.

As with any over-achieving student, Hermione decides to take all available classes in her third year, despite their overlapping schedules, and is able to do so with the aid of a time turner, which allows her to travel back in time in order to attend three courses during the same time period. Although the notion of time travel is common throughout science fiction, we emphasize it here in connection to higher dimensional movement in order to introduce the concept of invertibility. In order to rationalize the necessity of a higher dimension, it may be helpful to imagine an ant at one end of a long rope. If restricted to two dimensions, it would be very time consuming for the ant to move from one end of the rope to the other. However, if the two ends of the rope are picked up into a higher (third in this case) dimension, then the ends can be brought next to each other, the ant can travel between the ends of the rope in essentially no time at all.[11]

From a more mathematical standpoint, this category links to the concept of invertible functions. According to Wolfram MathWorld, an object is invertible if it admits an inverse. [12] In the case of time turners, the inverse requires the traveler to live back through time to the original time of departure. Via this pseudo-mathematical explanation, the invertible map has a specified return trip due to a time reversal charm that can have serious consequences if the travelers attempt to somehow interact with themselves or change their own experience of the past (as seen in the shenanigans that occur in *Harry Potter and the Cursed Child*). These bounds on how the traveler can relive the past are similar to how, for practical reasons, we require a function to be bijective in order to define its inverse. In this sense, such an inverse is limited and, extending back to Harry Potter, the invertibility of time travel is restricted to reliving the past in the Harry Potter canon (unlike in the *Back to the Future* films for instance).

6 Category 5: Higher Dimensional Traces

As in Abbott's beloved novel *Flatland*, the Harry Potter books explore the concept of higher dimensional traces through ghosts, the resurrection stone, duplicating money within Gringotts vaults, and the "shadows" that appear from Harry and Voldemort's wands when they meet in *Harry Potter and the*

[11] As described in *A Wrinkle in Time*

[12] Margherita Barile From MathWorld A Wolfram Web Resource, created by Eric W. Weisstein http://mathworld.wolfram.com/Invertible.html

Goblet of Fire.[13] Similarly, in *Harry Potter and the Deathly Hallows*, as Harry confronts Voldemort for a final time, he uses the resurrection stone that Dumbledore bestowed upon him and surrounds himself with the shades of his parents, Sirius, and Lupin: "They were neither ghost nor truly flesh, he could see that. They resembled most closely the Riddle that had escaped from the diary so long ago, and he had been memory made nearly solid. Less substantial than living bodies, but much more than ghosts, they moved toward him, and on each face, there was the same loving smile."[14] These traces that seem ethereal in our dimension are perhaps but shadows of beings that exist on another plane, as if we are only seeing a reduced version based on what is available in three dimensions.

7 Conclusion and Next Phase of Research

As the above discussion has documented, there are numerous examples of higher dimensional concepts in the wizarding world of Harry Potter. Some of these examples would serve as motivating examples for concepts in an undergraduate topology course. Although there are limits to these links, a topology professor would be well served to include such examples during an introduction of these more abstract mathematical ideas. As we mentioned earlier, our analysis of young adult fantasy literature extends beyond the Harry Potter novels. Our next phase of research will be a more thorough analysis of the works of C.S. Lewis, Madeleine L'Engle, and Edwin Abbott, among others. This research will build on the prior work of a fellow ACMS member.[15]

[13] J. K. Rowling (2000) Harry Potter and the Goblet of Fire New York: Scholastic Inc.
[14] J. K. Rowling (2007) Harry Potter and the Deathly Hallows New York: Scholastic Inc.
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Using Real-World Team Projects: A Pedagogical Framework

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Mike Leih holds a Ph.D. in information systems and technology from Claremont Graduate University and an M.S. in computer science from California State University Fullerton. His industry experience includes over 25 years in IT, holding positions in systems administration, application development, business analysis, project management and director of information technology. His research interests include IT governance, systems development practices, and technology course development.

Abstract

The use of team projects in a program capstone course for computer science or information systems majors has been a popular method for reinforcing and assessing program learning objectives for students in their final semester. Using real-world group projects as a learning activity is an excellent pedagogical approach in helping students develop critical thinking, team work, real-world problem solving, and communication skills. However, real-world group projects also provide many challenges to both the instructor and students alike. Instructors or students must find real-world projects appropriate for the learning objectives in the course. Instructors must determine how to provide teams with appropriate learning activities and provide effective feedback to reinforce learning objectives while fairly assessing project deliverables to individual team members. Students must find a common time to work together and learn to appropriately delegate project activities so each student fairly participates in the project. Finally, real-world projects have the real risk of failing due to circumstances outside the control of the instructor and students.

Papers have been presented in the past describing methods to address these challenges and successfully use real-world team projects. This paper gives a summary of these methods and presents a successful and practical approach that has been used for the past seven years in an Information Technology program capstone course. This framework is based on traditional project management methodologies which allow students the opportunity to successfully meet learning objectives even if the project success factors are not met.

1 Introduction

There are many excellent published articles demonstrating the effectiveness and benefit of using real-world team based projects as a major assignment within a course. These real-world team based projects can take the form of a service learning project, student consulting project, pilot project, or one of many other names to describe the engagement of students to complete a project whose origin is initiated by someone other than the student or instructor. Kolenko et al. describe their success of using service learning projects in a management course as a way to enhance the student's intellectual and moral aptitude. [4] Hatteche and Clayton describe how they successfully utilized service learning projects to have students develop websites and establish community partnerships with the university. [2] Heriot et al. write about their case study explaining how they used student consulting projects in a production management course to achieve high student engagement and reinforce student learning outcomes in their academic program. [3] Maloni et al. document how they used cross functional teams from multiple courses to provide students with a lifelike replication of what they can expect as systems analysts in the professional world. [5] Finally, Dixon provides another example of using service learning projects to provide students an experiential learning opportunity to enhance team skills and project management understanding. [1] These articles are only a few of the documented cases where real-world team based projects have proven to be a successful approach to improve student learning outcomes.

Although these articles demonstrate the advantages real-world team projects can bring to the classroom, many courses avoid using them due to the many challenges that come with utilizing real-world projects as a course assignment. Some of the challenges associated with these types of assignments are:

1. Finding appropriate real-world projects for the course and matching the student's skill level.
2. What to do when projects fail to meet initial objectives due to circumstances outside of the student's or instructor's control.
3. Projects that have an inappropriate scope for a given course or that experience significant scope creep during the course.
4. Dealing with team conflict among students.
5. Unbalanced team member effort where one or two team members complete most of the project work while other students have little or no project involvement.
6. Assessing and grading individual students fairly on the results of a team project.

This paper is intended to provide instructors a pedagogical framework to address these challenges in an Information Systems or Computer Science course. The framework is based on my experience using real-world team projects in an Information Technology capstone course for seven years and the published results of other instructors who have successfully utilized real-world team projects in their courses. The framework mimics the approach used in the traditional project management paradigm, which provides distinct project phases and deliverables, a team based development methodology, and regular progress check points. The framework also provides another opportunity to reinforce project management skills while providing students the challenge and experience of working on a real-world team project.

2 Pedagogical Framework Overview

The pedagogical framework consists of a set of strategies to help avoid the risks associated with real-world team projects and to allow students to meet the assignment learning objectives even if the main project deliverable fails. The strategies associated with this framework are as follows:

1. Find projects using prayer, professional networks, and local non-profits.
2. Evaluate and approve project scope before project start.
3. Use short time lines for project phases.
4. Utilize many-small process deliverables throughout the project.
5. Utilize small teams.

6. Have weekly project status update meetings.
7. Require weekly team activity reports.
8. Leverage traditional waterfall project management methodologies and deliverables.

3 Finding Projects and Building Teams

Finding appropriate real-world team projects is the first challenge in using them in a course. Depending on the experience of the students in the class, length of the course, and the types of skills students have, the instructor may choose to find and assign projects to project teams, or have student teams find their own projects. I recommend project assignments last between 5 to 7 weeks (the details regarding this recommendation are discussed later in this paper) so if the class is a traditional 15-week semester and the experience and skill level of the students is advanced, the instructor can have the student teams find their own projects. However, if the course is shorter in duration or the overall experience of students in the course is limited, then I recommend the instructor find the projects prior to the start of the course and assign them to the project teams based on the team's skill set.

Whether students find their own projects or the instructor finds the project for them, I recommend leveraging the following strategies. First, pray. Ask God to bless the class and the students as the instructor prepares for the course. Ask God to provide opportunities for the students to be a blessing to someone outside of the course as they learn through the project activities. I have often been amazed at the opportunities and impact the real-world projects have had on both the students and the organization being served. Second, the instructor should use their professional network. I have found many good projects by posting a request on LinkedIn or by sending out an e-mail to colleagues. I ask companies to provide me their ”number 11” on their top 10 list of technology projects. Something they would like to get done, but don't have the resources to initiate it. Finally, send an e-mail to local non-profit organizations asking if they have any technology needs the students can provide to them. As a final recommendation, the instructor should avoid using their own personal projects. This limits the learning experience of the students as the instructor tends to be more available to students and makes project requirements more clear than typical outside clients.

In general, as the instructor, I have had more success finding the projects for the students rather than requiring the students to find their own projects. This approach provides me the opportunity to initially evaluate the project scope to ensure the requirements will be challenging enough for the students to gain the learning objectives in the course, while keeping the project scope within the student's ability and the time constraints of the course. This also allows me to have an initial conversation with the project sponsor (the client) to ensure they will have the time needed each week to engage with the students to provide and clarify project requirements and to evaluate and test project deliverables.

Project teams can be established either before or after the projects are identified and can either be self-selected by students or assigned by the instructor. I have the most success assigning the project teams as the instructor after the projects have been identified. I have asked students for resumes highlighting their technology skills and experiences and have used those resumes to create project teams that best meet the needs of the project. Another approach in creating project teams is to build them based on GPA where the first three students with the highest GPA are placed on the first team and next three students are placed on the second team, continuing until all students are placed. This approach helps reduce the likelihood of one student doing the majority of the work, while the remaining team members do minor tasks.

4 Project Properties

When considering real-world team project, it is recommended the projects have the following properties:

1. The project has a clear set of deliverables.
2. The project sponsor (the client) has the time to commit to the project.
3. Major risk factors can be identified and mitigated.
4. The project has no more than one major risk factor.
5. The project duration is between 5 and 7 weeks.
6. Team size is between 3 and 4 students.
7. Total project effort is between 75 to 150 hours.

Clear Set of Deliverables: Having a clear set of project deliverables and project scope helps the project sponsor and the project team to carefully consider what the final project solution should be. Many sponsors have a general idea of what they would like, but requiring a detailed list of project deliverables establishes a clear understanding of project scope and provides the foundation to estimate project effort and successful project completion. Having a clear set of project deliverables is also required to leverage the traditional ”waterfall” project management approach. Although agile project management has gained wide acceptance in industry, the traditional ”waterfall” approach is more appropriate for students working on their first real-world team project. It provides greater predictability and control and is more easily understood when compared to an agile project management approach. [6]

Project Sponsor has Time: Ensuring the project sponsor has the time available to work with students is important to establishing the communication expectations of the project. Whether the instructor finds the projects for the students or the students find their own projects, it is important to be very clear with the project sponsor at the beginning of the project that they will need be available each week to engage with the students. In addition, the instructor will need to teach and train students to respect the time they do have with their project sponsor. They must be taught to act with professional comportment when communicating or meeting with their project sponsor. Students should have questions prepared beforehand, to be early to meetings, to dress appropriately, and to respect time constraints. I have used a ”letter of understanding” which highlights these expectations and have both the project sponsor and the project team sign it to emphasize this principle.

Major Risk Factors: The instructor will need to help the project sponsor and the project team to identify any major project risks. These risks could include; lack of commitment or availability from the project sponsor to work on the project during the project execution, the implementation of new technology, high estimated project cost, high potential of student distraction (such as athletics or extracurricular activities) or any other high risk element. As project failure tends to grow exponentially with the number of high risk elements, it is recommended teams avoid any project that has more than one identified major risk factor.

Project Length and Team Size: Finally, I recommend the instructor limit project time frames from 5 to 7 weeks and project team sizes to 3 or 4 students. The reason for the short project time frame is to help limit project scope, to keep the project sponsor engaged in the project, and to allow students to better experience the full project life cycle over a shorter time period. When real-world team projects are longer, the risk increases that the project sponsor will need to disengage due to urgent business demands or the team will lose a high level perspective as they get distracted with other school activities. Having a small team size also helps control project scope and reduces the risk of one or two students ”free-riding” on the project as others do the majority of the work.

5 Project Management Class Activities

Each week, student teams should submit a written project status update document, project deliverables and present their project status update to the class. The written project status update document and the team project status update presentation should contain the following elements.

1. A team activity report.
2. A dashboard or high level project status document containing the following:
   (a) The name of the project, the project team and the report date.
   (b) The overall status of the project (red, yellow or green).
   (c) The status of the project scope (red, yellow or green).
   (d) The status of the project schedule (red, yellow or green).
   (e) A list of risks, issues and concerns and their respective mitigations.
3. A review of the project task list and timeline.
   (a) Completed tasks.
   (b) Late tasks.
   (c) Upcoming tasks.
   (d) Who is assigned to each task.
   (e) Estimated task duration.
4. A review of any project deliverables due for the given week.
5. Team meeting minutes.
6. Questions and answers with the entire class.

Team Activity Report: The weekly team activity report is a simple report listing all the team member names, what project related activities they worked on during the past week and the amount of time they spent on each activity. Although this is a self-reporting team report and students misrepresent their actual effort, it does help to hold each team member accountable for working on the project each week. If a team member reports limited or no activity for a given week, it provides an opportunity for the instructor to question the team regarding the disproportionate effort on the project and adjust team points for a given individual if the team member is not fairly participating in the project.

Project Status Dashboard: The project dashboard document is a single page report showing the overall status of the project and any risks, issues or concerns pertaining to the project. This document helps teach the students to be concise and clear about the overall status of the project and provide the major risks relating to the projects. Teams should be encouraged to display this document during the project status presentation and respond to questions from the rest of class regarding the status of their project. It is good to challenge students about optimistic status indicators and to bring up project risks the students might be overlooking due to lack of experience.

Project Task List: The task list can be represented as a simple list of tasks in a document or spreadsheet, or be as formal as work breakdown structure in a Gantt chart. Students should provide as much detail as possible in the task list. This helps the team to distribute the workload of the project among the team members and keep track of all details that need to be completed as part of the project. It can be effective for the entire class to review each team's task list and to recommend adding tasks are missing and to see what other teams are doing to gain ideas on task items might be missing from their own project plan.

Project Deliverables: The project team should present any project deliverables due for a given week and respond to questions and comments from the rest of the class. It is recommended the instructor require one or two questions from the class during project status update presentations. This encourages students to stay engaged while other teams are presenting. It also helps teach the team presenting how to consider questions and respond to them. The presentation of project deliverables also helps hold teams accountable to move forward with their project each week and gives other teams an opportunity to learn from any quality work being presented.

Team Meetings: Project teams are encouraged to meet outside of class and required to provide team meeting minutes as part of their weekly status report documents. Meeting minutes should include when and where the team met, who was in attendance, what was discussed and what decisions were made. It should also include the time and location of the next team meeting. This encourages teams to formalize team meetings and to develop planning skills. If time permits, class time can also be used for short team meetings. This allows the instructor an opportunity to meet with teams and answer any questions the teams might have.

6 Project Schedule and Deliverables

The following is a suggested schedule for a six-week project time line and their respective deliverables for a typical technology project. This schedule assumes the projects have already been identified, evaluated and selected and project teams have already been established.

Week 0: The week before the project starts

1. Establish teams and assign team projects

Week 1

1. Project requirements document draft
2. Weekly status update reports
3. Project schedule and task list

Week 2

1. Final project requirements document
2. Project design documents draft
3. Updated project schedule and task list
4. Weekly status update reports

Week 3

1. Final project design documents
2. Project solution prototype
3. Updated project schedule and task list
4. Weekly status update reports

Week 4

1. Project testing plan draft
2. Project implementation plan draft
3. Project solution prototype update
4. Updated project schedule and task list
5. Weekly status update reports

Week 5

1. Final project testing plan
2. Final implementation plan
3. Project documentation and training material draft
4. Project solution prototype update
5. Updated project schedule and task list
6. Weekly status update reports

Week 6

1. Final product solution demonstration
2. Final project documentation and training material
3. Customer acceptance and sign-off
4. Weekly status update reports

The following provides a short description of each of the project deliverables referenced above. Providing examples or templates is the most effective way to teach students about what should be expected. Having students submit a draft provides the instructor and the class the ability to provide feedback on the draft document before the final document is submitted. Many of these documents can be fully developed regardless of how effective or complete the final project solution is when submitted or if the project sponsor chooses not to implement the project solution. This allows students to meet the learning objectives of the assignment and earn the majority of the points available even if the final project solution is not completed or implemented due to circumstances outside the control of the student or instructor.

Project Requirements Document: This document might contain many project initiation elements, such as the project charter, project scope, project assumptions, and requirement details. The level of detail and the amount of information here should be adjusted to meet the needs of the course and scope of the project.

Project Design Documents: These documents can be a collection of user interface designs, detailed descriptions of project solution elements, database design diagrams, network design diagrams, or any number of design elements. This collection of documents will vary greatly depending on the type of project solution required. The instructor should work with each team to establish expectations and provide directions to students.

Project Solution: The solution can be presented in numerous ways depending on the type of project being worked on. However, prototypes of the solution can often be submitted as a document with screen captures, source code, pictures and other graphics along with descriptions of the solution. The final solution submission can also be a document, in the case of a network implementation or computer server installation, or be the final product, such as a source code, in the case of a software project or website development project. The instructor should give each team specific instructions on what is expected for their given project.

Project Testing Plan: A detailed plan on how the solution will be tested to ensure it meets all project requirements. This can be a narrative of the testing processes needed, along with a check list of testing items or testing scripts. The completed test plan, showing all tests completed and passed, should be submitted with the final project documentation.

Project Implementation Plan: This document should describe the processes and procedures needed to move the project solution from the development and testing phase into a production phase. This document is often shorter than other project deliverables, but it does help the students think through what needs to occur to deliver and implement a project solution.

Project Documentation and Training Material: These documents should contain design and development notes, any necessary account or password information, any upgrade and maintenance items to support the project solution and training material on how to use the project solution. These documents will vary depending on the type of project solution being developed, so the instructor should provide specific direction to each team on what is expected.

The point value and level of detail for each of these project deliverables should be established based on the type of course being offered and emphasis the instructor wants to give to various learning objectives. Additional documents can be added or suggested documents can be removed to further emphasis learning objectives.

7 Conclusion

The framework and approach described in the paper is based primarily upon my own trial and error experience as I used real-world team projects in my courses over a seven-year period and upon the published experiences of other teachers using similar assignments. The overall framework should also be adjusted to meet the needs of the course. Even when using the framework, utilizing real-world team projects remains a challenging venture. However, with careful planning, a willingness to make adjustments throughout the project life cycle, and working closely with project teams, these challenges can be managed and students will experience a rich learning activity.

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Ten Mathematicians Who Recognized God's Hand in Their Work (Part 2)

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Abstract

Leading mathematicians of the past commonly affirmed that God created and sovereignly rules the universe and that He providentially sustains and nurtures His creatures. History teaches us that faith often informs rational inquiry and vice versa. In many cases Christian commitment stimulated intellectual activity; sometimes mathematical understanding led to spiritual insight.

In my former paper, ten of history's most influential mathematicians expressed the role that faith in God and religious conviction played in their work in their own words. This paper explores the same for mathematicians numbered eleven through twenty.

1 Introduction

"For since the creation of the world His invisible attributes are clearly seen, being understood by the things that are made, even His eternal power and Godhead, so that they are without excuse." (Rom. 1:20)

Enlightenment philosopher Denis Diderot (1713 – 1784) once observed that "Mankind have banned the Divinity from their presence; they have relegated him to a sanctuary; the walls of the temple restrict his view; he does not exist outside of it." Evidently his lesson was lost on Euler, the Bernoullis, Gauss, Riemann, and other notable mathematicians who believed God was actively working *outside the box* in commissioning and equipping them to glorify Him in their pursuit of truth through mathematics. Isaac Newton once declared, "All my discoveries have been made in answer to prayer."

My earlier paper [1] featured (in chronological order) the following mathematicians who clearly articulated their assurance of God's unmistakable presence in their lives and work:

1) Nicholas of Cusa (1401 – 1464)

2) Johannes Kepler (1571 – 1630)

3) Blaise Pascal (1623 – 1662)

4) Gottfried Wilhelm Leibniz (1646 – 1716)

5) Johann Bernoulli (1667 – 1748)

6) Colin Maclaurin (1698 – 1746)

7) Leonhard Euler (1707 – 1783)

8) Maria Agnesi (1718 – 1799)

9) Augustin-Louis Cauchy (1789 – 1857)

10) Georg Cantor (1845 – 1918)

This paper highlights ten more mathematicians who endeavored to do the same as those listed above. The essay for each mathematician includes an overview of his accomplishments, a brief description of his faith, and one or more quotations. We again proceed in chronological order.

2 Mathematicians Eleven through Twenty

11) Nicolaus Copernicus (1473 – 1543)

A native of the German-speaking territory of Prussia in the Kingdom of Poland, Nicolaus Copernicus launched one of the greatest revolutions in the history of science with the publication of his treatise *On the Revolutions of the Heavenly Spheres* in 1543. Eminent as a mathematician, physician, jurist, governor, military leader, classical scholar, Catholic cleric, economist, translator, and artist, Copernicus pursued astronomy only as an avocation [2]. Yet his heliocentric model of the universe, which opposed Ptolemy's time-honored geocentric view and was unacceptable to most scientists and churchmen of his era, was championed by Kepler and Galileo in the next century and was ultimately brought to widespread acceptance by Newton [29].

Near the beginning of Book One of *On the Revolutions*, Copernicus extols the virtues of astronomy as the pinnacle of the liberal arts in that it elevates the mind to the contemplation of the orderliness and goodness of the universe and its *Divine Architect*, as is captured in the following quotes.

> *"For who would not, while constantly studying the universe so clearly arranged in the most beautiful order and directed by divine wisdom – who would not, through the constant contemplation of this, I might almost say through intercourse with it, be induced to do everything good and to admire the Architect who created all this, in whom there is the highest bliss and in whom all good reaches its summit? The inspired psalmist would not sing that he is enraptured by God's creation and that he rejoices in the work of His hands, if he, like all men, were not moved by the sight of this Creation to the contemplation of the Supreme Good [25].*"
>
> *"So vast, without any question, is this divine handiwork of the Almighty Creator [19].*"
>
> *"For my part I believe that gravity is nothing but a certain natural desire, which the divine providence of the Creator of all things has implanted in parts, to gather as a unity and a whole by combining in the form of a globe [9].*"
>
> *"But we should rather follow the wisdom of nature, which, as it takes very great care not to have produced anything superfluous or useless, often prefers to endow one thing with many effects. And though all these things are difficult, almost inconceivable, and quite contrary to the opinion of the multitude, nevertheless in what follows we will with God's help make them clearer than day – at least for those who are not ignorant of the art of mathematics [8].*"

Regarding an exact determination of the trajectories of the planets, Copernicus observes:

> *"This more divine than human science, which inquires into the highest things, is not lacking in difficulties... In the case of the other planets (besides earth) I shall try – with the help of God, without Whom we can do nothing – to make a more detailed inquiry concerning them. [9]."*

12) Tycho Brahe (1546 – 1601)

Tycho Brahe, a Danish astronomer, mathematician, and alchemist, is credited with compiling the world's most accurate and comprehensive astronomical data prior to the discovery of the telescope. He made nightly observations of the stars and planets from Uraniborg, the celebrated observatory he built on the island of Hven off the coast of Denmark. After a disagreement with Denmark's king in 1597, Tycho became imperial mathematician and astronomer for Rudolph II, the Holy Roman Emperor, in Prague, Bohemia. Here he was assisted in his work by Johannes Kepler until the former's death in 1601. Kepler later used Tycho's observations to formulate his laws of planetary motion and in 1627 had them published as the Rudolphine Tables [5].

Tycho lost most of his nose as a result of a duel with a fellow Dane in 1566, and wore a realistic replacement made of brass for the rest of his life. Reputedly, the dispute was over the relative mathematical skills of the combatants! Also, Tycho kept a pet moose at his home near Hven that met an untimely end when it fell down stairs after it drank too much beer [14]!

The publishing of Tycho's geoheliocentric view in 1588 stimulated a lively discussion between him and a pair of German mathematician/astronomers, namely Cristoph Rothmann and Caspar Peucer. Rothmann, a supporter of Copernican heliocentrism, maintained that the Bible should not be taken literally in passages relating to science, as it was not written for that purpose but for matters relating to salvation. Below is a portion of Tycho's response.

> *"Much less do the things you affirmed deserve a place because you excuse those things that Holy Scripture asserts to the contrary. The reverence and authority due to the sacred writings is and ought to be greater than that of dragging them into common discussion. For although they adjusted themselves to the common method of understanding in physics and some other matters, yet let it be far from us to think of them as speaking in such a common manner that we do not believe them to be speaking the truth. Thus Moses, even if he does not refer to the deep things of astronomy when treating the creation of the world in the first chapter of Genesis, because he is writing for the common people, nevertheless he does introduce that which our astronomers can concede [24]."*

Peucer favored Ptolemaic geocentrism and held to an even more literal view of Scripture than did Tycho. In a letter to Peucer Tycho wrote:

> *"Nevertheless, the substantial absurdity of the earth's ordinary and continual revolution presented quite an obstacle to me, to say nothing of its being contrary to the unquestionable authority of the Sacred Scriptures [24]."*

Tycho's views of God as Creator and Ruler of the universe and the authority of the Holy Scripture are expressed in the following passages.

> *"So the heavenly bodies necessarily teach their meaning by the power placed in them by God and so one can infer that there are causes which signify. Nor does this in any way detract from divine omnipotence or liberty which are tied to secondary causes. Although God is a perfect and free agent, unrestrained by any natural law, yet he did not want to pervert the order of nature that he set up. And although God could have done everything without intermediaries, yet he was pleased by his inscrutable wisdom that all these things that normally happen in the world come from Him through means [4]."* *"For this reason I judge that, after the true and appropriate knowledge of God, revealed in the Word which he has given us, nothing is more suitable to the nature of man, and more agreeable to the purpose for which he has been created and placed on Earth, the centre of the universe, than that, while beholding as from a central place the things which shine forth in the whole structure of the world, but especially in that heavenly and brilliant court of so many everlasting stars, he should agreeably spend his life in this pleasant and studious contemplation, and that, while acknowledging God as Creator in these his wise and varied works, he should worship him with due veneration and praise [4]."*

Inscribed at the entrance to Tycho's observatory in Hven were the following words:

> *"Consecrated to the all-good, great God and Posterity. Tycho Brahe, Son of Otto, who realized that Astronomy, the oldest and most distinguished of all sciences, had indeed been studied for a long time and to a great extent, but still had not obtained sufficient firmness or had been purified of errors, in order to reform it and raise it to perfection, invented and with incredible labour, industry, and expenditure constructed various exact instruments suitable for all kinds of observations of the celestial bodies, and placed them partly in the neighbouring castle of Uraniborg, which was built for the same purpose, partly in these subterranean rooms for a more constant and useful application, and recommending, hallowing, and consecrating this very rare and costly treasure to you, you glorious Posterity, who will live for ever and ever, he who has both begun and finished everything on this island, after erecting this monument, beseeches and adjures you that in honour of the eternal God, creator of the wonderful clockwork of the heavens, and for the propagation of the divine science and for the celebrity of the fatherland, you will constantly preserve it and not let it decay with old age or any other injury or be removed to any other place or in any way be molested, if for no other reason, at any rate out of reverence to the creator's eye, which watches over the universe. Greetings to you who read this and act accordingly. Farewell [37]!"*

13) John Napier (1550-1617)

A Scottish mathematician, physicist, astrologer, and theologian, John Napier is credited with the discovery of logarithms, which revolutionized mathematical calculations and facilitated advances in many of the sciences, particularly astronomy. He also brought the decimal point into widespread use in mathematics and created a computational device known as Napier's Bones, which greatly simplified such operations as multiplication, division, and extraction of roots [13].

A student of the Bible, Napier wrote a treatise on the book of Revelation entitled *A Plaine Discovery of* *the Whole Revelation of St. John*, important in Scotland as the first biblical commentary authored by a Scot and highly regarded in France, Germany, and the Netherlands as well [45]. In *A Plaine Discovery* he consistently demonstrates his desire to glorify God and to edify His church and readily acknowledges man's impotence apart from God. Below are some excerpts from this treatise.

> *"And, surely, this that I have, how small soever it be, till God enlarge me more, I offer it gladly unto the glory of God, and education of his true Church. To God, therefore, the disposer of this and all other godly works and meditations, who liveth and reigneth eternally in Trinity and Unity, be glory, praise, laud and thanks for ever and ever. Amen [28]."*

> *"In vain are all earthly convictions, unless we be heirs together and of one body, and fellow partakers of the promises of God in Christ, by the Evangel [7]."*

> *"There be three equal persons of the Deity; Father, Son, and Holy Ghost. So be there here of this Jerusalem three equal dimensions of longitude, latitude, and altitude. None of the three persons of the Deity is separable from the other; so none of these three dimensions of a city, or of any solid body, can be separable one from another, for then should it become a superfice and no solid body. The three persons of the Deity and their functions cannot be confounded; so are not these three dimensions confounded, for the length is not the breadth, nor the breadth the height [27]."*

Napier wrote the following as a dedication of his work to King James:

> *"For shall any Prince be able to be one of the destroyers of that great seat, and a purger of the world from Antichristianisme, who purgeth not his own country? Shall he purge his whole country, who purgeth not his own house? Or shall he purge his house, who is not purged himself by private meditations with his God? I say, therefore, as God hath mercifully begun the first degree of that great work in your inward mind by purging the same from all apparent spot of Antichristianisme, as that fruitful meditation upon the 7, 8, 9, and 10 verses of the 20 Chapter of the Revelation, which your highness hath both godly and learnedly set forth, doth bear plain testimony to your Majesty's high praise and honour, so also we beseech your Majesty (having consideration of the treasonable practices in these present days, attempted both against God's truth, your authority, and the common wealth of this country) to proceed to the other degrees of that reformation, even orderly from your Majesty's own persons to your highness' family, and from your family to your court; till at last, your Majesty's whole country stand reformed in the fear of God, ready waiting for that great day, in the which it shall please God to call your Majesty or yours after you, among other reformed princes, to that great and universal reformation... [7]."*

14) Galileo Galilei (1564-1642)

Hailed as one of the founders of modern science, Galileo Galilei was an Italian mathematician, physicist, astronomer, philosopher, and musician. Improvements he made to the telescope revolutionized observational astronomy and his support of Copernicus' heliocentric view of the universe helped to promote its eventual acceptance by the scientific community. The telescope enabled him to explore the moon's surface, discover several moons of Jupiter, confirm the phases of Venus, and observe sunspots. His work in experimental physics included numerous innovative applications of mathematics [39].

Well documented is Galileo's persecution by the scientific community and ultimately the Roman Catholic Church for his outspoken support of Copernicanism, then considered to be both unscientific and unbiblical. Admittedly, Galileo had a hand in drawing the Church's ire. The Roman Inquisition tried and convicted him of heresy for his 1632 work *Dialogue Concerning the Two Chief World Systems*, in which he caricaturized the Pope as foolish for holding to a geocentric position [23]. He was forced to recant and was confined to house arrest for the last ten years of his life [39].

Galileo's awe of the Creator and the esteem in which he held His Word and Works are clearly demonstrated in the following passages of *Dialogue*.

> *"Let us, then, exercise these activities permitted to us and ordained by God, that we may recognize and thereby so much more admire His greatness, however much less fit we may find ourselves to penetrate the profound depths of His infinite wisdom [16]."*
>
> *"Many times have I given reign to my fancies about these things, and my conclusion is that it is indeed possible to discover some things that do not and cannot exist on the moon, but none which I believe can be and are there, except very generally; that is, things occupying it, acting and moving in it, perhaps in a very different way from ours, seeing and admiring the grandeur and beauty of the universe and of its Maker and Director and continually singing encomiums in His praise, I mean, in a word, doing what is so frequently decreed in the Holy Scriptures; namely, a perpetual occupation of all creatures in praising God [16]."*

Galileo demonstrates in his correspondence with his eldest daughter Maria Celeste his zeal to credit God for the astronomical discoveries he made by telescope and his acknowledgement of God's sovereign wisdom in human suffering.

> *"I render infinite thanks to God for being so kind as to make me alone the first observer of marvels kept hidden in obscurity for all previous centuries [31]."*
>
> *"Whatever the course of our lives, we should receive [sufferings] as the highest gift from the hand of God, in which equally reposed the power to do nothing whatever for us. Indeed, we should accept misfortune not only in thanks, but in infinite gratitude to Providence, which by such means detaches us from an excessive love of Earthly things and elevates our minds to the celestial and divine [31]."*

Galileo observes that although the study of the heavenly bodies cannot bring complete knowledge, it does better equip us to honor God as Creator and to learn lessons imparted by Him in general.

> *"Finally, lifting us to the final purpose of our efforts, namely, the love of the Divine Architect, [efforts toward learning some of the properties of sunspots] can sustain our hope of learning all other truths from Him, source of light and truth [17]."*

Galileo responds to what he considers to be a misapplication of Scripture to support claims of a stationary earth in a letter to friend and fellow mathematician Benedetta Castelli.

"The Holy Scriptures can never lie or err, and its declarations are absolutely and inviolably true. I should have added only that, though Scripture cannot err, nevertheless some of its interpreters and expositors can sometimes err in various ways [17]."

In an effort to persuade Christiana of Lorraine, the Grand Duchess of Tuscany, that Copernicanism does not violate the teachings of Scripture or of the Church, Galileo writes:

"To prohibit the entire science would be no different than to reject hundreds of statements from the Holy Writ, which teach us how the glory and the greatness of the supreme God are marvelously seen in all of His works and by divine grace are read in the open book of the heavens [17]."

15) Rene Descartes (1596-1650)

Rene Descartes, a French philosopher, mathematician, physicist, and writer, has been hailed as the father of modern philosophy and of analytic geometry. His philosophical works, notably *Discourse on the Method, Meditations on First Philosophy, Principles of Philosophy, Passions of the Soul*, and *Rules for the Direction of the Mind*, have greatly influenced Western philosophical thought and *Meditations* continues to be used in modern university philosophy programs. In mathematics, he devised the Cartesian coordinate system and developed analytic geometry, which provided a framework in which Newton and Leibniz would formulate the infinitesimal calculus. As a physicist, he made valuable contributions to mechanics and optics [38].

In his correspondence with French polymath Le Pere Mersenne, Descartes expresses his views of God as the omnipotent and omniscient source of all truth (including mathematics) and man as His limited and dependent subject.

"As for the eternal truths, I repeat that they are true or possible only because God knows them as true or possible; and he doesn't have this knowledge in a way that implies that they are true independently of him... In God, willing and knowing are a single thing in such a way that by the very fact of willing something he knows it and it is only for this reason that such a thing is true. So we mustn't say that even if God didn't exist these truths would be true; for the existence of God is the first and the most eternal of all truths that exist and the only one from which proceeds all others. What makes it easy for this to be misunderstood is that most people don't regard God as a being who is infinite and beyond our grasp, the sole author on whom everything depends; they get no further than the syllables of his name... Those whose thoughts go no higher than that can easily become atheists; and because they perfectly grasp mathematical truths and don't perfectly grasp the truth of God's existence, it's no wonder they don't think the former depend on the latter. But they should rather take the opposite view that because God is a cause whose power goes beyond the limits of human understanding and the necessity of these other truths doesn't put them out of our reach, these truths are less than, and subject to, the incomprehensible power of God [11]."

In Part I of his *Principles*, Descartes affirms God's infinitude and omnipotence in an effort to reconcile His sovereignty with man's free will.

"... that we shall have not the slightest trouble in ridding ourselves of the difficulty (which one may have in harmonizing the freedom of our will with the order of the eternal providence of God) if we observe that our thought is finite, and that the knowledge and the Omnipotence of God, whereby he has not only known from all eternity all that which is or which can be, but also has willed it, is infinite. We have therefore quite enough intelligence to recognize clearly and distinctly that this knowledge and this power are in God [26]."

"That there is freedom of our will and that we are able to assent or not assent is so evident that it should be counted among the first and most common notions that are innate in us. At the same time, God is all-powerful, and everything is preordained by Him. [12]."

"Although the light of reason, however clear and evident it is, may seem to suggest something different to us, we should put our faith exclusively in divine authority rather than in our own judgment [12]."

Meditations, a systematic exposition of Descartes' philosophical thought, explores the nature of truth and of material things, the distinction between mind and body, and the existence of God. *Meditation V*, entitled *Of the essence of material things, and, again, of God, that He exists*, includes Descartes' assertion that God's truth is foundational to all other truth.

"And thus I very clearly see that the certitude and truth of all science depends on the knowledge alone of the true God, insomuch that, before I knew him, I could have no perfect knowledge of any other thing. And now that I know him, I possess the means of acquiring a perfect knowledge respecting innumerable matters, as well relative to God himself and other intellectual objects as to corporeal nature, in so far as it is the object of pure mathematics [which do not consider whether it exists or not] [10]."

16) Jacob Bernoulli (1654 – 1705)

The eldest of eight prominent mathematicians belonging to the Swiss family Bernoulli, Jacob Bernoulli made outstanding contributions to the development of probability theory and Leibniz's infinitesimal calculus. In his treatise *The Art of Conjecture*, he systematized earlier work in probability theory and included his own pioneering work in permutations and combinations, the form of the binomial distribution, Bernoulli trials, Bernoulli numbers, and the law of large numbers. In collaboration with his younger brother Johann, whom he had tutored in mathematics, he demonstrated the usefulness of calculus by applying it to a number of areas of mathematics, including transcendental curves and isoperimetric inequalities. He also made important contributions to the subjects of geometry, mechanics, algebra, differential equations, infinite series, the calculus of variations, and catenary curves [34], [35].

The Bernoulli brothers' relationship devolved into one of jealousy and bitter rivalry, a sad footnote to the account of such otherwise illustrious men. Brotherly love was replaced by boasting, verbal assaults, priority disputes, accusations of plagiarism, and estrangement [36]. Nevertheless, each had an unwavering faith in God as Creator and Sovereign Ruler of the universe.

In *The Art of Conjecture*, Bernoulli uses a theological argument to help explain an important aspect of his law of large numbers. G. W. Leibniz had argued for an observable distinction between deterministic and probabalistic events and Bernoulli responds that from God's perspective all events are predetermined. He writes:

"They object first that the ratio of tokens is different from the ratio of diseases or changes in the air: the former have a determinate number, the latter an indeterminate and varying one. I reply to this that both are posited to be equally uncertain and indeterminate with respect our knowledge. On the other hand, that either is indeterminate in itself and with respect its nature can no more be conceived by us than it can be conceived that the same thing at the same time is both created and not created by the Author of nature; for whatever God has done, God has, by that very deed, also determined at the same time [3]."

Elsewhere in his great work Bernoulli, while observing that man's knowledge of the certainty of events is incomplete and subjective, elaborates upon God's complete and objective knowledge of and control over all such events:

"In themselves and objectively, all things under the sun, which are, were, or will be, always have the highest certainty. This is evident concerning past and present things, since, by the very fact that they are or were, these things cannot not exist or not have existed. Nor should there be any doubt about future things, which in like manner, even if not by the necessity of some inevitable fate, [all] nevertheless by divine foreknowledge and predetermination, cannot not be in the future. Unless, indeed, whatever will be will occur with certainty, it is not apparent how the praise of the highest Creator's omniscience and omnipotence can prevail [3]."

17) Carl Friedrich Gauss (1777 – 1855)

Generally included in the trio of greatest mathematicians of all time (along with Archimedes and Newton), German mathematician and physicist Carl Friedrich Gauss made major contributions to number theory, algebra, analysis, statistics, geometry, geodesy, astronomy, magnetism, and optics. His treatise *Disquisitiones Arithmeticae* (Number Research), which combined results of other mathematicians with new discoveries of his own, ushered in modern number theory. Statistics has benefited greatly from his derivations of the method of least squares and the normal distribution, and algebra has from his proof of its fundamental theorem and his method of Gaussian elimination. His work in non-Euclidean geometry predates that of Bolyai and Lobachevski, but priority of discovery goes to the latter pair because he didn't publish in the field. His Theory of Celestial Movement revolutionized computation in astronomy, and his collaboration with scientist Wilhelm Weber produced important results in electricity and magnetism [41], [40].

In defending the science of astronomy in his inaugural lecture on the subject at the University of Gottingen in 1808, Gauss recognizes God as the all-wise Creator of the magnificent structure of the cosmos.

"This sublime enjoyment which this study of astronomy guarantees, the peculiar satisfaction which occupation with serious sciences gives, and which cannot be described but only felt if one has the taste for it, the beneficial withdrawal from the frequently unpleasant material world by means of quiet contemplation arousing no passion, finally the magnitude and sublimity of the objects themselves – which extend our cosmic view, and so much that what we consider great and important in the hostile activity on our planet, appears petty, and why would we not confess finding again the vestiges of an Eternal Wisdom in the wonderful arrangement of the cosmic structure, the peaceful quiet which our nearsightedness

> *probably often believes lost to view in that hostile activity: According to my opinion these are worthy answers to the question, of what use is the study of astronomy [18]*?"

In his delightful collection of stories and anecdotes, *Mathematical Circles Squared*, Howard Eves relates Gauss' acknowledgment of God as his source of inspiration to complete a difficult mathematical proof.

> *"Finally, two days ago, I succeeded not on account of my hard efforts, but by the grace of the Lord. Like a sudden flash of lightning, the riddle was solved. I am unable to say what was the conducting thread that connected what I previously knew with what made my success possible [15]*."

In expressing his appreciation for philosophy as well as science, Gauss ranks the question of man's relation to God as one of highest importance.

> *"There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science [33]*."

Living with his eldest daughter after being widowed, Gauss conveys his hope that neither of them would have to live without the other:

> *"The best and greatest that God could grant us would be this one favor, that we two on the same day might die together [6]*."

18) William Rowan Hamilton (1805 – 1865)

Irish mathematician and physicist Sir William Rowan Hamilton made fundamental discoveries in mechanics, optics, algebra, and graph theory. His reformulation of Newtonian mechanics was foundational to the development of quantum mechanics and Einstein's theory of relativity a century later. His theory of optics gave new insights for the wave theory of light and conical refraction in biaxal crystals [22]. And Hamiltonian paths and circuits are central to the graph theory applications of business efficiency, optimal routing, and the traveling salesman problem [53].

The quaternions, for which Hamilton is (perhaps) most well known, generalized complex numbers to higher dimensions and yielded fruitful applications in algebra, computer graphics, control theory, signal processing, and orbital mechanics. The rules for quaternion multiplication dawned upon him as he and his wife were crossing the Broom Bridge in Dublin, Ireland, and in his excitement, he proceeded to carve these rules in the bridge with his pocket knife [42]!

Robert P. Graves' voluminous biography of Hamilton sheds much light on his Christian faith and his desire to glorify God in his life and work. In a speech given to honor the foreign attendees of a *British Association for the Advancement of Science* meeting at Liverpool in 1854, Hamilton acknowledges the accomplishment of French physicist Leon Focault, architect of the Focault pendulum, before looking heavenward to the source of all such intellectual triumphs.

"Essentially and throughout submissive to the laws which the great Creator of all things has been pleased to impress upon matter, I admired anew the activity of the French intellect; but I looked up with even greater reverence than before to the Supreme Giver of all intellectual and of all higher treasures [21]."

Admitting in correspondence with a friend his struggles with discouragement, Hamilton maintains that his faith, though wavering in intensity, has remained intact throughout his trials.

"My struggles and alternations in the spiritual life have not been (as that former expression of mine may for a moment have led you to fear) between belief and doubt; but between warmth and coldness. My intellect has never ceased to embrace Christianity with satisfactory and complete conviction [20]."

Upon learning that his fiance Helen Maria Bayly is suffering melancholy he encourages her to trust in God's attentive care.

"Our God indeed chasteneth those whom he loveth, but not because he grudges them prosperity. Let us commit ourselves to His hands without fear that He will visit us with affliction for its own sake, or because we are happy now [20]."

In a letter to his friend and correspondent the *Marquess of Northampton* he reflects upon hearing of learned men who had rejected Christianity.

"I could not resist the impulse to repeat that (declaration) of Him to whom was given the Spirit without measure: 'I thank thee, O Father, Lord of Heaven and Earth, that thou hast hid these things from the wise and prudent, and has revealed them unto babes [20]!' "

In the same letter letter to the *Marquess* he demonstrates his understanding of Christ's sacrifice on his behalf and the eternal life that is thereby attained.

"We (he and Mrs. Hamilton) went together to the Church of Castleknock, a place known to the readers of (Jonathan) Swift; and there enjoyed what I fear Swift never knew, as it ought to be and may be known, the pleasure of joining, with thankful hearts and minds, in the commemoration of that Last Supper upon Earth of Him who gave for us His body and His blood, and who has appointed to us a way whereby we may feed on them forever [20]."

19) George Stokes (1819 – 1903)

One of the most influential scientists of the nineteenth century, Irish mathematician and physicist Sir George Gabriel Stokes made major contributions to fluid dynamics, mathematical physics, and optics, and provided invaluable leadership to scientists at Cambridge University and the Royal Society of London for half a century. His work in fluid dynamics, e.g., motion, viscosity, Stokes' Law, the Navier-Stokes Equation, etc., enhanced the foundations and applicability of that science. The field of optics greatly benefited from his discoveries in wave theory and aberration of light, polarization, fluorescence, and spectroscopy, and mathematical physics did, in general, from the rigorous methods he employed. As Lucasian professor of mathematics at Cambridge and both president and secretary of the Royal Society, he found ample opportunity to encourage and mentor fellow scientists (most notably Lord Kelvin and James Clerk Maxwell) and to suggest possible areas of investigation [51], [52].

The design argument was for Stokes, an evangelical Protestant, overwhelming evidence of the existence of an omnipotent and benevolent God, and exploration of His Creation was a means of drawing closer to Him. That Scripture was divinely inspired and therefore authoritative was also a deeply held conviction of his. The following excerpts illustrate these views.

> *"Trials of some kind or other are almost sure to come, and death we know must come at last. But even death itself is only an incident in the bright course if we live as God grant we may. We must bear and forbear, love, comfort, and forgive, until by God's mercy we reach, as I trust we may, that City, into which must enter nothing that defileth, where there shall be no more death, neither sorrow nor crying, neither shall there be any more pain, for the former things are passed away; where, while faith shall have no more exercise, and human knowledge shall have vanished away, love shall abide forever [30]."*
>
> *"God be with you. Pray to Him to guide you aright, and if you see it to be right to go on, may He be with us in our journey through life, keeping us united in the ties of the deepest mutual affection and in the ways of His commandments, until it pleases Him to call one of us away; and at last when this transitory state of our probation is over may we dwell for ever before Him [30]."*

Stokes responds as follows to an inquiry from Mr. Arthur H. Tabrum, an official of the London Post Office, regarding the relationship between science and theology.

> *"As to the statement that recent scientific research has shown the Bible and religion to be untrue,' the answer I should give is simply that the statement is altogether untrue. I know of NO sound conclusions of science that are opposed to the Christian religion [30]."*

In another letter to Mr. Tabrum in which he points out the limitations of science in explaining the origins of life, Stokes argues for the existence of a Creator.

> *"I quite think that the existence of life is one of the strongest arguments for the existence of a Living Being who is the Author of life. I quite think with you that the great gaps which we find in the series of animated things, both plants and animals, weaken the theory that man came in an unbroken chain from some lowly form of life [30]."*

In the second of his three lectures on *light* delivered at Aberdeen University in the 1880s, Stokes reflects upon God's attentive care for man despite the vastness of the universe.

> *"In face of the views that thus open out to us, the feeling of the littleness of man comes upon us with almost overwhelming force, so much has modern research emphasized those*

words uttered of old, 'What is man that thou art mindful of him?' But when from the contemplation of such immeasurable distances we turn to an individual living being, when for example we consider the structure of our own bodies, and the wonderful adaptation of the various organs to their purpose, we see that the vastness of the universe has not caused the Creator to be unmindful of the least of his creatures. It is rather on the vastness of the scale and yet unity of plan of the universe that this year's course has led us to ponder [32]."

"Our inability to explain by mere natural causes the vast variety of forms of life, and their changes in geological time; difficulties of the Darwinian theory if regarded as a solution of the problem; evidences of design afforded by an examination of the structure of living things; marvelous adaptation of the eye to its uses; self-existence, beyond which we cannot go, not attributable to the visible universe; the evidence of design leads to the contemplation of a designing mind, of whom self-existence has been affirmed; further evidence derived from the study of the mind; inadequacy of the human mind to take in together the ideas of personality and exemption from limitation of time and space; the character of God revealed to us through the Son [32]."

Stokes also expressed in this lecture his view that the overwhelming evidence of design points unmistakeably to an omniscient and self-existence Creator.

"When I say we contemplate all this, it seems difficult to understand how we can fail to be impressed with the evidence of Design thus imparted to us. But that we want nothing more to account for the existence of structures so exquisite, so admirably adapted to their functions, is to my mind incredible. I cannot help regarding them as evidences of design operating in some far more direct manner, I know not what; but design is altogether unmeaning without a designing mind. The study then of the phenomena of nature leads us to the contemplation of a Being from whom proceeded the orderly arrangement of natural things that we behold. Thus we are led to place in a Being this attribute of self-existence which we failed to find in the races of living creatures, or even in the majestic march of the planetary bodies. And in the present connection it is noteworthy that it is precisely this attribute of self-existence that God himself chose for his own designation, when Moses was commissioned to go to his countrymen the Israelites and announce their coming deliverance from Egyptian bondage, and enquired by what name he was to call Him who sent him, the reply is, 'Thus shalt thou say unto the children of Israel, 'I AM hath sent me unto you [32].' ' "

Stokes further observes in his lecture that the reality of Jesus Christ as God's Son is evidence of a personal Creator God who loves and cares for His children.

"Are we then left to lose ourselves in an ocean of immensity, and driven to the conclusion that God is unknowable? Nay, as Christians we believe that the character of God has been revealed to us as it never had been before through that Diving Being who took our nature upon him and dwelt among us full of grace and truth. The greatness of the universe displays to us something of the greatness of its Author; but when we study the character of the Son, who is the image of the invisible God, we learn as never had been learnt before the lesson that God is love [32]."

20) Bernhard Riemann (1826 – 1866)

German mathematician Georg Friedrich Bernard Riemann's genius for abstraction profoundly impacted the fields of geometry, complex and real analysis, analytic number theory, and mathematical physics. He developed Riemannian (elliptic) geometry, a type of non-Euclidean geometry which served as the framework for Einstein's general theory of relativity and is foundational to algebraic and differential topology [48]; Riemann surfaces, configurations of the complex plane in which holomorphic functions, in particular multiple-valued functions, can be represented; the Riemann integral, which brought a new standard of rigor to integral calculus [49]; and the Riemann-zeta function, centrally important to analytic number theory. The Riemann hypothesis, which concerns roots of the aforementioned function, remains one of the most famous unsolved problems in all of mathematics [47]. His notion of hyperspace paved the way for James Clerk Maxwell's unified field theory for electricity and magnetism [43].

The son of a Lutheran minister, Riemann adopted his father's faith in God at an early age and it shaped the rest of his life. The following statements made by fellow mathematician and friend Richard Dedekind in his brief biography of Riemann provide a glimpse of the latter's faith and character.

> *The son of a Lutheran pastor, Riemann was sustained by a deep religious faith – not faith of the proud or proselytizing kind, but one which saw the main duty of the devout life as: "Daily self-checking before the face of God (Tgliche Selbstprfung vor dem Angesichte Gottes) [46]."*
>
> *Through all his troubles, he maintained a steadfast faith and conducted daily spiritual examination. As he was succumbing to tuberculosis, the Lord's prayer comprised the last words on his lips. His tombstone bears the inscription of Romans 8:28, "All things work together for good to them that love God [44]."*
>
> *"The gentle mind which had been implanted in him by his father's house remained with him all his life and he served his God faithfully as his father had, but in a different way [50]."*

3 Conclusion

Many of history's greatest mathematicians and scientists deny the Enlightenment/contemporary view that their respective disciplines have no need of religion or of God. As the quotes in this paper illustrate, many unashamedly proclaimed the greatness of God and his works and recognized that his enabling power is essential to discovery and advancement in mathematics and the sciences. Rather than substantiating Diderot's claim, they testified to an omnipresent, all-wise, and benevolent Creator who delights in enlightening the human mind and spirit in scholarship and in understanding of the universe around them. May today's God-fearing mathematicians and laypersons alike not follow the world's lead in applauding man's accomplishments apart from God, but instead learn from our illustrious forebears that He alone enables our endeavors to succeed and He alone is worthy of praise.

> *"Unless the LORD builds the house, they labor in vain who build it." (Ps. 127:1)*

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Variations on the Calculus Sequence

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Abstract

Many institutions have embraced a standard format for the Calculus sequence, comprising three four-credit courses covering a fairly consistent set of topics. While there is much to recommend this approach, it still leaves some fantastic concepts rushed or untouched, and it can be argued that it demands too much of students with weaker backgrounds. As such, some schools have experimented with variations on the standard format. This paper will present a model currently used at Eastern University, exploring the strengths and weaknesses of this particular approach. It will also suggest ideas, developed in conversation with other ACMS members, for how different approaches might be explored in a comparative study.

1 The Standard Calculus Sequence

Anyone who has spent time in a collegiate mathematics context is familiar with the standard calculus sequence. Nearly every college and university requires their mathematics students to take at least three semesters of calculus, and a wide array of other majors require some portion of the calculus sequence. These courses cover a common list of topics, though there may be some differences with regard to the number of credits awarded, the inclusion of technology, or the instructional techniques.

Calculus I:	Calculus II:	Calculus III:
Limits	Integration Techniques	Vectors
Derivatives	Applications	Partial Derivatives
Applications	Differential Equations	Multiple Integrals
Integrals	Sequences and Series	Vector Calculus

Historically, the standard sequence traces back to the 1960's and 1970's. In an engaging retrospective written for the Mathematical Association of America (MAA), Alan Tucker traces out the development of the mathematics major in America over the past century or so, to include the development of the modern calculus sequence [10]. In the early 20th century, calculus often served as more of a pinnacle of undergraduate mathematics, rather than the entry way that it is today. The significant contributions that mathematicians made to the Allied war effort helped to increase demand for advanced mathematics after World War II. In the 1950's, physicists began to regularly use calculus in introductory engineering courses, prompting a widespread adoption of a year-long calculus sequence for engineering and science freshmen. Throughout the next two decades, the MAA-sponsored Committee on the Undergraduate Program in Mathematics (CUPM) developed a variety of curricular recommendations (see, for example, [3], resulting in a common mathematics major - and calculus sequence - by the mid 1970's.

Despite its entrenched status, however, the standard calculus sequence is not free of recognized flaws. As some portion of the sequence has come to be a prerequisite to most of the sciences, calculus too often serves to filter out students who would otherwise be interested in science. Further, many professors can attest to the fact that even passing students too often scrape by with a superficial and formulaic understanding of the subject. By the late 1980's, a variety of reformation efforts were beginning to take shape with the goal of making calculus "a pump, not a filter" (see, for example, [2] or [9]). The MAA, along with many others, has continued this work, producing a wealth of research and resources (see, most recently, [1]). Most of these efforts, however, have focused on the first year of calculus.

That said, there have been some efforts that have looked at the sequence as a whole. Indeed, a number of schools have questioned whether the standard order of topics serves students effectively, particularly students in the sciences. In particular, students in a number of other disciplines do need to have some familiarity with the basics of multivariable calculus, but few students outside of the mathematics major have need of sequences and series. Therefore, a number of schools have been exploring a re-sequencing which moves the basics of vectors and partial differentiation into the second semester, while reserving sequences and series for the third semester (see [7], [4], or get in touch with colleagues at Gordon College, who have been running this sequence for some time). A recent project, Resequencing Calculus (www.resequencingcalculus.com), received significant NSF funding for exploring this approach, and the project will include the publication of a new textbook reflecting this resequencing.

In the 2015 CUPM curriculum guide ([8], see the section on the Calculus Sequence), another weakness of the standard sequence is identified. Far too often, the third semester of the sequence attempts to shoe-horn a wealth of Vector Calculus into the last few weeks of the semester. This leads to an abbreviated and unsatisfactory treatment of such beautiful topics as Green's, Gauss', and Stokes' theorems. The CUPM guide notes that some colleges have sought to address this problem by adding another course to the calculus sequence, but articles exploring this approach are not easily found. We shall now turn our attention to such an exploration.

2 Overview of Eastern's Approach

Eastern University is a small liberal arts school in the suburbs of Philadelphia. Our mathematics department has three full-time faculty, with an average of about 25 majors (about 6-7 graduating per year). The mathematics department has a partnership with Villanova University, whereby students can take engineering courses during their undergraduate career, allowing them to earn a BA in Mathematics from Eastern and an MS in Engineering from Villanova in a total of approximately 5 years.

For the past 15 years or so, Eastern has run a modified calculus sequence, consisting of four three-credit courses. Having only three credit hours per semester has limited the scope of the first two semesters somewhat. The first semester does not cover some common topics such as the Mean Value Theorem or the Intermediate Value Theorem, while other topics are covered only minimally and from a computational perspective; the second semester omits any treatment of differential equations or parametric equations, and the applications of integration are limited to the computation of area, volume, and arc length. The third semester then covers vectors, parametric equations, partial differentiation, and multiple integrals (to include an introduction to polar coordinates).

Calculus I:	Calculus II:	Calculus III:	Advanced Calculus:
Limits	Integration Techniques	Vectors	Generalized Derivative
Derivatives	Applications	Parametric Equations	Change of Variables
Applications	Sequences and Series	Partial Derivatives	Vector Calculus
Integrals		Multiple Integrals	

Where Eastern's approach most significantly deviates from the standard sequence, however, is the addition of a fourth semester, which we call Advanced Calculus. Most students take the course in their fourth semester, having by then completed the rest of the calculus sequence, Linear Algebra, and Discrete Math (which serves as a Transitions style class, introducing students to proofs). Thus, Advanced Calculus is partly a capstone on the first two years of collegiate mathematics. In this capacity, we intentionally revisit material from the previous courses, working to deepen students' conceptual understanding. As such, the course also serves as an additional bridge to upper level courses.

The full Calculus sequence is required of all math majors (to include students in the pre-engineering track), and students from a variety of majors regularly take between one and three semesters. The Advanced Calculus course is rarely taken by non-mathematics majors. It is offered every spring semester, and students regularly take it immediately following Calculus III. As such, the population is generally freshmen and sophomores (determined by how much credit students transfer in from the AP exam or elsewhere).

With regard to texts, Eastern uses Anton, Bivens, and Davis' *Calculus* book for the first three semesters. The Advanced Calculus course has experimented with a variety of texts, but has settled on using Marsden and Trombda's *Vector Calculus*. While no text is ever perfect (in particular, students have found a number of errors, and there does not seem to be a collection of errata for the sixth edition), we have not found any other resource that gives a sufficiently nuanced treatment of the material while remaining accessible to students.

Finally, with the limited number of credit hours in the first year of the sequence, we generally do not attempt to integrate technology beyond graphing calculators. That said, we do have a limited number of Mathematica licenses available to students, and we are glad to direct students to options such as Wolfram Alpha, Sage, or Geogebra (the later two of which are free and open source).

3 Eastern's Advanced Calculus Course

Before turning to an analysis of the strengths and weaknesses of this model, we will explore in more detail the content covered in Eastern's Advanced Calculus course. While there are some minor variations depending on who is teaching, it is more or less possible to break the course into five units, each of which review some material from previous courses and then generalize the material to new contexts. As we will see in the following exploration, a provocative question or example is used to introduce and motivate each unit.

3.1 Limits and Continuity

Due to the time constraints in Calculus I, there is little attempt made to develop more than an intuitive and computational understanding of limits and continuity. In the first unit, we recall that understanding, and an example such as the characteristic function on the rationals is used to demonstrate the need for a more nuanced definition. Students are introduced to basic topological concepts for Euclidean space (open and closed sets and boundary points), and rigorous definitions are given for limits and continuity. Computationally, students learn to show that limits do not exist by using different paths, and we consider various strategies for proving that functions are continuous (to include using polar coordinates and an opportunity to generalize power series in some of the more advanced homework problems).

3.2 Differentiation

In talking about differentiation, we review the definitions of derivatives and partial derivatives (which students rarely seem to remember), recalling that differentiability implies continuity. We then use the definitions to compute the partial derivatives of a function such as the following:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

Using material from the first unit, we observe that the function is not continuous at the origin, which calls for a more nuanced understanding of differentiability. Rearranging and generalizing the definition of the derivative, we are able to define a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at $\vec{x}_0 \in \mathbb{R}^n$ if

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\|f(\vec{x}) - \big(f(\vec{x}_0) + Df(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)\big)\|}{\|\vec{x} - \vec{x}_0\|} = 0 .$$

This approach emphasizes that differentiability is about the existence of a good linear approximation, and leads us to a generalized derivative. As we go through the process of generalizing the Calculus I derivative, we deduce that the derivative of our arbitrary function $f : \mathbb{R}^n \to \mathbb{R}^m$ must be the $m \times n$ matrix of partial derivatives:

$$Df(\vec{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}_0) & \frac{\partial f_1}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}_0) & \frac{\partial f_2}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial f_2}{\partial x_n}(\vec{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}_0) & \frac{\partial f_m}{\partial x_2}(\vec{x}_0) & \dots & \frac{\partial f_m}{\partial x_n}(\vec{x}_0) \end{pmatrix} .$$

The generalized derivative thus calls us to review material from Linear Algebra regarding matrix operations and linear transformations. By the end of the unit, we see that the generalized derivative allows us to simply generalize the chain rule without the need for the tree diagrams used in many of the standard calculus texts: $D(f \circ g)(\vec{x}_0) = Df(g(\vec{x}_0)) \cdot Dg(\vec{x}_0)$ (where the operation is matrix multiplication).

3.3 Implicit Function Theorem

Our third unit generalizes implicit differentiation, another topic that students either do not remember or understand only computationally, to the context of arbitrary functions on Euclidean space. As part of the generalization, we introduce the language of Jacobians and review the computation of the determinant. However, what really distinguishes this unit from the previous is an application of the mathematics. A 1998 article in *Siam Review* [6] provides a very readable description of the mathematics behind global positioning systems, showing how the generalized implicit function theorem determines the precision needed in the clocks used by the system. Thus, the question of how GPS works motivates the unit, and students are ultimately able to read the article for themselves. By the end of the unit, students also look at the Inverse Function Theorem as a specialization of the Implicit Function Theorem, leading them to review the computation of inverse matrices from Linear Algebra.

3.4 Change of Variables

Having generalized limits and derivatives to a broader context, the remaining units turn to integration. As noted previously, the first three courses in our calculus sequence do not touch much on other coordinate systems (polar coordinates are introduced at the end of Calculus III to aid in computing multiple integrals). Thus, after a brief review of polar coordinates, we ask students to consider how the change of coordinates relates to the technique of substitution learned in Calculus I. This leads us into studying change of variables in general, as well as cylindrical and spherical coordinates in particular.

3.5 Vector Calculus

The final unit of the course, and by far the most substantial, deals with Vector Calculus. Comprising 7 weeks of the 16 week semester, one goal of the unit is to avoid cramming the beautiful theorems of vector calculus (Green's, Gauss', Stokes') into the final week or two of class. We begin with computing line and surface integrals (taking pains to explain the concepts behind these computations and potential applications). Before moving on to the theorems, however, we introduce students to the language of differential forms (helpfully, the Marsden and Tromba text includes a section that introduces forms and basic operations on forms). We are then able to state the generalized Fundamental Theorem of Calculus, and introduce each of the theorems of Vector Calculus as a specification of this central theorem.

4 Strengths and Weaknesses

Having explored the content of Eastern's Calculus sequence, we now turn to a comparative analysis. Perhaps the most obvious advantage, especially in light of the overview of the Advanced Calculus course, is the opportunity to cover a number of interesting topics that are not touched by the standard sequence. In particular, informal feedback from conference talks seems to indicate that many undergraduate students never encounter the derivative as a matrix, the generalized implicit function theorem, or differential forms. However, Eastern students generally respond well to these topics, particularly appreciating the application of the implicit function theorem to GPS systems and the beauty of the Fundamental Theorem as it is interpreted in different contexts (under the standard sequence, it seems to be much more difficult for students to grasp the relationship between these theorems). Additionally, students participating in Eastern's engineering partnership with Villanova tend to benefit from the additional time spent covering line integrals, surface integrals, and the Vector Calculus theorems.

Moving beyond the Calculus sequence itself, Eastern's Advanced Calculus course is also able to connect in meaningful ways to core concepts from other courses. The treatment of the derivative makes use of matrices and matrix operations, as well as linear transformations. Likewise, the unit on change of variables deals with transformations, recalling the language of bijections from Discrete Math (again, this serves as a Transitions style course at Eastern). Throughout the course, an emphasis on precise definition and proof also connects to the work done in Discrete Math. In addition to reviewing old material, the course also looks ahead, explicitly preparing students for material they will see again in courses such as Real Analysis and Topology. Thus, this approach provides valuable opportunities to emphasize the unity of mathematics across the artificial boundaries of courses.

A final strength of this model is that it allows some tinkering with the standard sequence without causing too many extra complications. A more involved resequencing, such as that described in section 1, will necessarily complicate the awarding of credit to students who have scored well on the BC version of the AP Calculus exam (this exam covers all of the topics from the first two semesters in the standard sequence). Such a dramatic change also complicates the situation for students transferring in or out of the institution. In Eastern's model, we are able to award credit for the first two semesters to students scoring well on the AP exam. Meanwhile, transfer students who have completed the full standard sequence can be awarded credit for Calculus I-III; they still are required to take Advanced Calculus, but it covers enough new and unusual material to make it worthwhile. Thus, Eastern's model is a robust, credit-neutral way of exploring alternatives to the standard sequence.

While there is much to recommend Eastern's model, some challenges and weaknesses should also be apparent. The restructuring into four three-credit courses leaves less time in Calculus I and II, while still requiring students from other majors to sit through several weeks dealing with sequences and series. However, while a more radical restructuring may better serve non-mathematics majors, Eastern's approach does provide them with a more computational focus (again, leaving the deeper conceptual work until later in the sequence). The shortened time-frame in the first year also precludes the integration of a technological component to the course (and while we do provide both Mathematica licenses and open source options to students, few take advantage of the opportunity). As discussed in Section 2, the Advanced Calculus course also requires an additional text, which increases the financial burden on students. Finally, with regard to the content of the Advanced Calculus class, the use of differential forms in connecting the theorems of Vector Calculus leaves little or no time for covering proofs of these theorems. Instead, students are asked to trust that this is a valid generalization of the Fundamental Theorem. All in all, we have found that these challenges are easily surmountable, but they could prove more problematic for others.

5 Future Study

As noted in Section 1, there is a good deal of literature dealing with dramatic restructurings (those that include multivariable calculus in the second semester while saving sequences and series until the third semester). However, explorations of other variations are harder to find, as are comparative studies. In closing, we will explore what such a study might look like. (Note: these ideas were developed in conversation with Dave Klanderman and Amanda Harsy at the 2015 ACMS conference.) Should others be interested in a study, we propose measuring overall student outcomes by administering some shared assessments throughout the Calculus sequence. Further, by tracking the assessment data by student throughout the sequence, we can investigate whether one approach or another better fosters student improvement. Finally, we propose also measuring student attitudes toward the material by using an attitudinal assessment at several points throughout the sequence. Thus, we close with an invitation for further conversation and collaboration. If your institution has explored variations on the standard sequence, or if you are interested in working to develop a comparative study, please contact the author.

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Reading Journals: Preview Assignments that Promote Student Engagement, Productive Struggle, and Ultimate Success in Undergraduate Mathematics Courses

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Abstract

We spend a lot of time searching for the *best* textbook for students. We want our students to have a reliable and useful resource to reference, as needed. We even ask them to read over certain material before classes. Often, however, we fail to guide our students in how to read that textbook productively.

Having students journal about reading their mathematics textbooks allows us to simultaneously help students struggle and persevere when encountering new problems, help students develop strategies for reading mathematical text productively, and help capitalize on what the students already have to offer. In this paper, we will look at how *Reading Journals* motivate students in a variety of mathematics courses across the undergraduate curriculum. We will further share how to develop different types of prompts for journal entries and important lessons learned.

1 Introducing *Reading Journals*

In the Fall of 2016, I taught two sections of Calculus I. Throughout that first semester, I sought to engage my students and instill in them a deeper understanding of the concepts. Thus, I wholeheartedly implemented various active learning teaching strategies. Since I was working with a new population, I thought that students would read through the material prior to coming to class. Very quickly, I suspected that very few students were ever reading the textbook and even fewer were reading before class. In the second week of classes, I recognized that I should explicitly ask who had read over the material before we worked on it as a class. When I did so, I realized that only one student from either class was reading ahead sometimes. I encouraged my students to make this a priority. I even wrote up specific *Reading* prompts in the third and fourth weeks to help guide them in their reading endeavors. Unfortunately, since these readings had no direct impact on their grades., they did not prioritize them.

In planning for the Spring semester, I still wanted active learning to be the major component of class time. Moreover, I wanted to devote the majority of class time to those parts that cause the particular students trouble rather than wasting their time going over the parts that they already knew. I believed (and still do believe) that we would be able to engage in more active learning and focus class time together more effectively if students read ahead. As a consequence, I wanted to find a more effective way to convince my students to read ahead.

During the Project NExT session *Alternative Assessment Techniques for the Active Classroom* at the Joint Mathematics Meetings in Atlanta (January 2017), Dr. David Bressoud from Macalester College shared how he had incorporated reading ahead into one of his courses. In his talk on *Assessment Practices*, Bressoud shared that his students had been required to post *Reading Reflections* one hour prior to class. The students answered various types of questions, from recognizing patterns to stating the most important point. They were always asked to describe anything that had confused them. He typically awarded credit based on completion and did not accept late submissions. When analyzing his students' final performance, Bressoud found a higher correlation between Reading Reflection scores and final grades than between mid term grades and final grades. This high correlation caught my attention and inspired me to adopt my own version. I was so encouraged by his positive experience, I decided to incorporate something similar into three of my courses that Spring. In particular, I required my students to complete *Reading Journal* assignments in Calculus II, Discrete Mathematics, and Excursions in Mathematics.

1.1 Commonalities

At Lenoir-Rhyne, Calculus II, Discrete Mathematics, and Excursions in Mathematics are all 100-level courses. As such, one of my main goals was for students to **struggle, make mistakes, and persevere**. This student outcome was a major driving force behind the motivation for, and the design of, the *Reading Journal* assignments. In each syllabus, I included the following description of *Reading Journals*:

> This allows you to take on a more active role in learning about the new material. Further, your solutions provide us with the invaluable opportunity to note where your understanding is strong already and where your understanding can grow further. You will have a number of opportunities to share your work from your reading journals during class.

I wanted my students to read over specific material before we covered it in class. I also wanted my students to be resourceful in using their textbooks as a reference. Therefore, I required my students to read through upcoming material. Like Bressoud, I also asked my students to respond to specific questions from the material that they had read. Further, I graded the *Reading Journal* assignments based on completion. Unlike Bressoud, I was not available to read my students' responses the hour before class. Therefore, my *Reading Journal* assignments were due at the start of class. My students were expected to share their solutions at this time. While they were engaging in initial discussion of their solutions with partners, I was able to verify whether each student had completed their assignment. To encourage students to take these assignments more seriously, the *Reading Journal* assignments accounted for 10% of their final grades. In summary, these assignments followed the following basic format in all three courses.

- Students read through the material in the upcoming section or chapter. While reading, students were expected to:
  - compile notes from reading; and
  - attempt the assigned problems.
- Any time the student was unsure of how to proceed in a particular problem, the student was expected to:

  - read over the examples and detailed solutions provided in the text;
  - mark any conflicts or differences; and
  - write down questions as they arise.

- Students were expected to share their solutions and/or responses at the start of class.
- As a class, we went over the shared solutions. We added any missing details and addressed all of the content questions students posed with the full group.

1.2 Course Specific Information

As mentioned above, the underlying motivation and structure of *Reading Journal* assignments were the same for Calculus II, Discrete Mathematics, and Excursions in Mathematics. Yet the content and, subsequently, textbooks were so different in these three undergraduate courses that the *Reading Journal* assignments for each class had its own style. As such, we will elaborate on the remaining student learning outcomes and textbook for each course. Then we will detail the particularities of the assignments and how they were incorporated into the corresponding class.

Calculus II

In this second course of calculus, our students study integration, sequences and series, applications of the integral, and inverse functions. In addition to struggling productively, students who successfully completed my course this past Spring should have been able to:

- Understand, and apply, integration techniques to determine areas of regions, volumes of solids, and lengths of arcs;
- Understand, and be able to apply, various techniques of integration to determine the antiderivatives of a wide variety of functions in various situations;
- Understand infinite sequences and series, discuss the convergence or divergence of certain series, and use Taylor polynomials and Taylor series to approximate functions and functional values;
- *Understand and successfully apply various problem solving strategies, when appropriate;
- *Communicate about, and analyze, various problem solving strategies;
- *Analyze and clearly present multiple solutions to a given problem, both orally and in written form; and
- *Understand, and analyze, multiple ways to learn.

The *Reading Journal* assignments were designed to encourage students to meet all these goals, especially the ones highlighted by an * not specifically about content. We used *Calculus Early Transcendentals*, (Sixth Edition), by James Stewart. Each section of this book provides conceptual explanations along with several problems and their solutions. The student learning outcomes together with the structure of our textbook led me to use this additional structure for Calculus II *Reading Journal* prompts:

- I assigned 2 to 3 problems, *whose solutions were provided*, from the upcoming material at least one day before our class meeting in our online learning management system; and
- In class, students were expected to write up their solutions on the board.

Discrete Math I

In our first semester of Discrete Mathematics, students gain exposure to, and an appreciation for, various mathematical applications to computer science. Some of the topics include the nature of mathematics, basic logic, lists and sets, relations and partitions, functions and their properties, graph theory, and discrete probability theory. Beyond developing productive struggle strategies, students who successfully completed my course this past Spring should have been able to:

- Apply basic counting and probability techniques to solve specific problems;
- Apply proofing techniques, such as induction, to prove simple mathematical statements;
- Construct truth tables and apply deductive reasoning;
- Calculate basic sums and/or products;
- Verify certain properties of a given relation, such as reflexivity, transitivity, *etc.*;
- Apply certain basic techniques from Graph Theory and Trees to solve specific problems; and
- *Be familiar with, and communicate about, some skills and knowledge that can be applied in Computer Science.

In addition to mastering the content specific student outcomes, my students were challenged to develop their endurance in problem solving and their ability to communicate effectively. The *Reading Journal* assignments were designed not only to lay a stronger foundation for learning new content but also to encourage students to pursue these latter two goals. The *Reading Journal* questions were to be completed after reading through appropriate sections in our text, *Mathematics: A Discrete Introduction*, (Third Edition), by Edward R. Scheinerman. This book provides definitions followed by several examples and a few highlighted notes. The student learning outcomes together with the structure of our textbook led me to use the following structure specifically for Discrete Mathematics *Reading Journal* prompts:

- I posted a few questions, *whose solutions were not given*, spanning the upcoming section at least one day before our class meeting in our online learning management system;
- In class, students were expected to share their responses with one or two people sitting next to them. During this sharing, students had an opportunity to get feedback from peers and myself;
- Then a couple of students shared their responses with the entire class. I wrote up the information as they shared; and
- We discussed the rest of the material through the lens of these student responses.

Excursions in Mathematics

Excursions in Mathematics is a three hour course for students who need a mathematics course but do not have to take Calculus. To simultaneously expose my students to various mathematical concepts and encourage them to develop an appreciation for mathematics, I focused on developing various problem solving strategies throughout the course of the semester. Students who successfully completed my course this past Spring should have been able to not only struggle productively but also:

- Understand and successfully apply various problem solving strategies, when appropriate;
- *Communicate about, and analyze, various problem solving strategies;
- *Analyze and clearly present multiple solutions to a given problem, both orally and in written form; and
- Argue about the significance of having access to various problem solving strategies.

In this course, the *Reading Journal* assignments were designed to help students tackle the above outcomes which are emphasized by an *. They were meant to be completed as students read a particular chapter of *Crossing the River with Dogs: Problem Solving for College Students*, (Second Edition), by Ken Johnson, Ted Herr, Judy Kysh. Each chapter of this book focuses on one problem solving strategy. As you read a given chapter, you encounter a few problems. As the reader, you are expected to attempt these problems before continuing to read. Following each problem, the authors share at least one actual student solution and sometimes even discuss it. To reinforce the goals of this text, I incorporated these problems into the course by having students tackle them for their Excursions in Mathematics *Reading Journals*:

- Every time the text introduces a new problem, students were expected to attempt to solve that problem using the current problem solving strategy;
- Any time the student was unsure of how to continue solving a particular problem, *which was followed by at least one solution*, the student was expected to keep reading; and
- Students were expected to present their own solutions, offering explanation and answering questions from their peers and myself.

In addition to the 10% of their grade coming from these regular *Reading Journal* write ups and presentations, students were asked to write a reflection of their personal development as a problem solver throughout the course; in particular, as they developed their skills through the process of attempting to answer the *Reading Journal* problems. This assignment counted an additional 5% towards their final grades.

2 Developing Prompts

2.1 Calculus II

There are a number of problems together with solutions in every section of this text. One could assign students to work through and understand the solutions to all of the problems provided in a given section. Since there are so many problems, though, I opted to carefully select about two or three problems that were a representative sample of all of the ones provided. The representative example problems often served as the question prompts, as the following six prompts indicate.

- Read over section 5.5 *The Substitution Rule*.

  Work out Example 3, Example 5, and Example 9 from the reading.

  *Yes, these examples are all worked out in your text. Try to work them out without looking and then compare! Some of you will present your solution to the class (with your own notes and without using your book).

- Exam I Review Reading Journal

  Recall that we have covered Chapter 5 *Integrals*, 6.1 *Areas Between Curves*, 6.2 *Volumes*, and 6.3 *Volumes by Cylindrical Shells*. As you are reviewing for your first exam, think about different types of problems.

  i. Create two of your own problems that address two different topics we have covered. This exercise is meant to help you review further.
  ii. Attempt to set up, and solve, each of your own problems.

- Read over section 7.3 *Trigonometric Substitution* pages 467 through 472. As you read, work through Example 3, Example 6, and Example 7 on your own.

  Make note of any questions that arise!

- Section 7.4 RJ Part I

  Read over section 7.4 *Integration of Rational Functions by Partial Fractions* pages 473 through 481. As you read, rewrite the integral as the integral of a sum of partial fractions in Example 2, Example 4, and Example 5 on your own. Do not evaluate the integral.

  Make note of any questions that arise!

- Reread section 7.5 *Strategy for Integration* pages 483 through 488. As you read, write out the method of attack for Example 1, Example 2, Example 3, Example 4 and Example 5 on your own.

  You do not have to fully work out each integral.

  Make note of any questions that arise! Are there any you would like to work out as a class?

- Section 11.8 *Power Series* RJ 2

  Complete problems 21 and 29 on page 727. You should at least write down the problems and attempt to solve them.

  Make note of any questions that arise!

  Please, bring homework questions.

2.2 Discrete Mathematics

My Discrete Mathematics *Reading Journal* prompts took the longest to create. Rather than borrowing problems from the actual reading, I developed my own prompts. Here are ten of my *Reading Journal* prompts presented in the order in which we discussed them as a class.

- A. Read *To the Student* on pages xvii-xviii.
  - i. In your own words, explain the difference between continuous mathematics and discrete mathematics.
  - ii. Include an example.

  B. Read Section 4 *Theorem*.
  - (a) Create your own A and B so that the statement "If A, then B." is true.
  - (b) Write out the following statements using your A and B. For each statement, determine whether it is true, sometimes true, or false.
    1. "If not A, then not B."
    2. "If B, then A."
    3. "If not B, then not A."

- A. Read Section 4 *Theorem*.
  - i. In your own words, explain the difference between nonsensical statements and vacuous statements.
  - ii. Write your own nonsensical statement.
  - iii. Write your own vacuous statement.

  B. Read Section 7 *Boolean Algebra*.
  - i. Write out a program to evaluate Boolean expressions. Your program should test if these expressions are tautologies, test if two expressions are logically equivalent, etc.

- A. Read Section 8 *Lists*.
  - i. The author discusses two different types of lists and how to count them.
    - (a) Create your own example of each type and
    - (b) highlight their differences.
  - ii. Attempt problem 8.16 on page 39 in your book. (You should at least write the problem down and draw boxes.)

- A. Read Section 10 *Sets I: Introduction, Subsets*.
  - i. In your own words, explain the difference between $\subseteq$ and ϵ.
  - ii. Create your own set.
    - a. Identify at least two different elements of your set and
    - b. at least two different subsets of your set.

    Notice that this restricts the possible sets you may consider.

- A. Read Section 14 *Relations*.

  Pay close attention to the definition of a relation and the unpacking of the different properties of relations. In the past, I have generated questions to encourage close reading and sufficient understanding of the text. Studies show that reading students develop a deeper understanding when they formulate their own questions over the reading. Thus, for this reading journal, you are going to take on the role of instructor.

i. What Would You Ask? - Develop two to three questions you would ask the class, as the instructor, to check their understanding of a particular relation. Your questions should check for a deeper understanding of the material.

Note: These are questions to examine the understanding of your peers. You may or may not know the answer yourself before class.

*You will be given a particular relation in twos or threes. Then each of you will ask your fellow group member(s) the questions you have created. As a group, your questions will be shared and discussed.

- Read Section 24 *Functions*. Notice that there are definitions for function and inverse function.
  1. Create your own function whose inverse relation is not a function.
  2. Create your own function whose inverse relation is a function.
  3. Write your own question over the material in this section.
- Carefully read Section 5 *Proof*.
  1. Prove that the sum of two odd integers is even. (Propositions 5.2 and 5.5 will be extremely helpful here).
  2. Describe the two proof templates presented in this section. Write down specific steps associated with each.
- Carefully read Sections 30 *Sample Space* and 31 *Events*.
  1. The outcomes of an experiment are the elements of the sample space. Suppose you are rolling two dice. List all the possible outcomes. (There are 36, not 11, altogether!)
  2. Create your own sample space. Determine what the associated outcomes and events are.
  3. In your own words, describe the difference between an outcome and an event.
- Carefully read Section 53 *Planar Graphs*.
  1. Sketch your own (a) simple curve and (b) simple closed curve.
  2. Consider the crossing-free drawing of a connected planar graph on the top of page 369.
     (a) Make 4 other crossing-free drawings of connected planar graphs.
     (b) For each of those 4 graphs, record how many vertices, edges, and faces the drawing has.
- Carefully read Sections 49 *Connection* and 50 *Trees*. Consider the graph

$$G = (\{a, b, c, d, e, f, g\}, \{ab, cd, de, df, dg, eg\}).$$

  1. Sketch G.
  2. Identify a walk in G that is not also a path.
  3. Identify the components of G.
  4. Identify at least one cut vertex of G.
  5. Identify at least one cut edge of G.

6. Identify a longest cycle in G.
7. Identify a spanning subgraph F of G that is a forest. Sketch F.
8. Determine how many leaves F has.

As these different prompts illustrate, I was intentional about asking different types of questions in their *Reading Journal* prompts. Students were challenged to answer specific prompts, explain a new phenomena in their own words, generate and study their own examples, and/or generate their own questions over the new material. Occasionally, I would even ask students to complete one or two of the problems at the end of the section. I was conscientious about writing prompts that spanned as much of the section as possible.

2.3 Excursions in Mathematics

The book *Crossing the River with Dogs: Problem Solving for College Students* is set up for interactive reading. The authors intend for students to read the material prior to coming class. Moreover, as it says in the introduction, the authors want students to struggle and make mistakes as they attempt the problems in the reading. Then the students are meant to check for mistakes and persevere as they continue reading the text and, subsequently, encounter alternate solutions. Thus, this text lends itself most naturally to *Reading Journal* assignments which foster productive struggle. I have included four specific *Reading Journal* prompts below.

- Read Chapter 1. Attempt the problems as they arise in your reading. There are four problems.
- (The class was evenly divided between Chapters 5, 6, 7, 8, and 9 for this *Reading Journal*.)
  1. Read your assigned chapter.
  2. Take notes of the main ideas.
  3. Write down questions that arise.
  4. Completely work out two of the problems discussed in the text of your chapter.
- Under announcements, you will find “Geometry Unit 4.4 Reading and Exercises”.

  You are responsible for reading the attached file *GeometryUnit4.4_ReadingJournal.pdf* before class tomorrow. Complete the exercises as you read. Bring your work and any questions to class.
- (1) Carefully read Chapter 17 Section 1. (469-484)
  (2) Attempt the problems as they arise in your reading.
  (3) Take note of how the problems are solved in the reading.
  (4) Create your own questions. - Develop 2 to 3 questions over the material from Chapter 17 Section 1.

As these four representative *Reading Journal* prompts indicate, the question prompts are the problems included in the chapters already and the assignment descriptions became more thorough as the semester progressed. As stated above, students were always challenged to use a particular problem solving strategy in their work. As in Discrete Mathematics, I sometimes required the students to generate their own questions.

3 Lessons Learned

The first time we implement a new teaching strategy in the classroom, teachers often learn so much about how to make that new strategy even better the next time they use it. Not surprisingly, I learned a lot about incorporating *Reading Journal* assignments into different classes. While reflecting on my experience and feedback from students (both informally and formally), I believe that there are several changes that can be made in future *Reading Journal* assignments to help students value them more and to make them more beneficial to students.

There were two common themes that were most prominent in all three courses. First, it is important to support students in developing resourcefulness. Second, it is crucial that students understand what is expected of them and why. Answering these questions once is not enough. Students need to hear the what and why repeatedly. Looking back, I wish we had spent more time emphasizing what the purpose of the *Reading Journals* was and why I was having students complete them. By not emphasizing these ideas enough, I invited the students to draw their own conclusions. While I was excited that students were frustrated and had hit a wall in their understanding, the students were upset and felt like they had been abandoned to figure everything out solo. By more openly discussing these feelings, I hope to help students focus on overcoming the roadblocks rather than the roadblocks themselves.

3.1 Calculus II

Almost all of the Calculus II *Reading Journal* prompts were exercises from the text. Therefore, students had access to a full solution right away. This immediate feedback allowed students to build up their confidence before volunteering to share their own solution with the entire class. I had hoped that students would spend a reasonable amount of time working on a problem before referring to the solution provided. In practice, though, a majority of students simply copied down the solutions provided in the book. Although copying does not tell us as much information as honestly trying a problem before comparing our work to others' solutions, I had hoped the process of writing the solutions down would force students to actually think about the material.

However, in an anonymous survey at the conclusion of the semester, several students shared that one of the most frustrating aspects of the *Reading Journal* assignments was that they were so easily able to copy down the work without giving the problems and ideas enough thought. Perhaps this frustration explains why students' final grades did not correlate uniformly with their *Reading Journal* scores. Most students recognized that *Reading Journal* assignments were 'easy' points, so they wrote them up late at night, often only copying. I remember students quickly wrote up solutions to the questions from *Reading Journal*s 5.5 and 7.4 above and asked little to no questions. Since they had not struggled enough with the material before class, they did not know which pieces they did not understand or where they were stuck. On days when students did not come to class with questions, the majority of students did not even recognize that we were going over the material. When asked to give advice, my students warned future students to start working on *Reading Journal*s early enough to think about them. They further advised students to actually write down questions as they worked. While many students did admit to copying without paying attention, they did recognize that a more focused attempt would be more beneficial. In addition to sharing pointers, I need to lead discussions about the importance of finding where we struggle most in problems.

Sometimes these assignments did help students recognize where their understanding was weaker and where they had lingering questions. Not surprisingly, our most memorable class discussions originated from students asking questions about particular steps and confusing transitions. For example, students wanted to understand why the authors had chosen a particular trigonometric substitution in the examples from 7.3. This allowed us to work together to carefully analyze the differences in those problems. To address questions raised about 7.5, we discussed why the authors made certain choices and how to set up the integral for a given technique. That day, students had thought about which problems they most wanted to go over together. In retrospect, I think having students articulate why they saw certain problems as more challenging than other problems would have led to an even more fruitful conversation about where different people get stuck and strategies for overcoming these hurdles. Students constantly expect immediate feedback and ask for step-by-step instructions while working, independently or otherwise, on problems. Since *Reading Journal*s seem to go against these perceived needs, it is important that we spend more time discussing productive struggle and its value with our students.

Unlike for most *Reading Journal*s, students could not copy solutions from the text for the Exam 1 Review and 11.8 Part 2. These assignments pushed students to use their understanding of material to write their own solutions. I still wanted students to identify key ideas and figure out where their understanding needed refinement. Students reported that they found it easier to write their own questions than to write their own solutions. I used their sticky points to make the review more meaningful. Once more, it would have been extremely helpful to explicitly bring this fact to their attention. These prompts were challenging students and causing them to struggle. In the future, I want to spend more time discussing how we persevere in those moments.

Although students often had access to full solutions, a couple of students did not regularly attempt to solve these exercises. They did not even copy down the solutions in the book. Rather than reaching out for help, they gave up. Some students admitted to feeling under prepared to present new material. Yet most students completed their *Reading Journal* assignments regularly. Sometimes students would explicitly express that they had experienced frustration with a particular concept while completing the problems and then felt much better about these concepts after discussing them in class. By having students practice some problems before class, we actually spent most of our time discussing the concepts. Rather than wasting time going over the parts that the class already understood, we spent the majority of our time working out the finer details together. As mentioned above, students had trouble recognizing what we were doing and why. By allowing students to struggle and make mistakes before class, we were able to persevere and develop a deeper understanding altogether. Since it led to a richer and more focused discussion, I viewed the student frustration as fruitful. Unfortunately, a number of students let their frustration bring them down. Two challenges are selling the frustration feature in the context of productive struggle and helping students understand what it looks like to spend a reasonable amount of time working on the material before class.

3.2 Discrete Mathematics

Recall that I created the *Reading Journal* prompts for Discrete Mathematics based on the assigned readings. It is important to stress that these questions were not taken directly from the text. However, they were still designed to check student understanding of the material before class discussions. Since students were not able to copy down solutions from their textbook, they were challenged to think more deeply about the material. The students who were more willing to engage with material before class tended to perform better overall. Thus, it makes sense that the correlation between *Reading Journal* scores and final course grades was more linear in Discrete Mathematics than in Calculus II. As in Calculus II, some students in Discrete Mathematics expressed feeling frustrated while completing their *Reading Journals*. Unlike in Calculus II, students did not have immediate feedback for the problems assigned. Subsequently, some students expressed feeling completely lost while reading the text. In general, though, students were more comfortable creating their own examples than analyzing them. Sometimes students did not feel like they had enough understanding to even attempt to create their own examples. In spite of our discussions about expectations for making an honest effort and first steps in making an attempt, some students still reported discomfort in working on material before discussing it in class. During the semester and in their survey, students were most consistently vocal about their dislike in creating their own questions. One important change is to better equip students with tools to persevere through their feelings of frustration and being lost. Another big challenge is helping students internalize that I am looking for completion, not accuracy.

When students felt better about their *Reading Journal* responses, more students actually spoke up sooner in our full class discussions. Perhaps as expected, students preferred creating their own examples over attempting to analyze them. During class, many students were willing to share their own examples while only a handful of them would share their analysis of these examples with the entire class. For instance, we developed a more thorough understanding of the two types of mathematics by sharing our individual ideas for part A of the first prompt included above. When discussing part B of that same prompt, a large number of students shared their initial statements for (a). Of these students, only a couple were willing to share their responses for (b). As a class, we analyzed these statements, verified whether they were correct or not, and determined whether they were true or false. Students had struggled when writing their own statements and determining whether they were true on their own. Then we were able to uncover the subtleties together. In the future, I want to emphasize how working together is one valuable way to overcome our frustrations more consistently.

Throughout the semester, students repeatedly expressed their frustrations from not immediately knowing whether they were correct or incorrect in their responses. During our second exploration of Section 4 as a class, a few students shared their statements and which type they had labeled them. When we analyzed them further, we were able to pinpoint misconceptions resulting from the seemingly subtle differences between the two types of statements. We encountered a similar situation with Section 10. We first discussed a couple of their more straightforward examples together. Then we struggled through a more interesting example all together. As a result of grappling with this harder example, students were better able to internalize the difference between elements and subsets. When comparing answers to Sections 30 and 31, I was overjoyed to see the students who only found eleven outcomes realize which outcomes were missing from their list. Once more, the students who make a mistake are more likely to learn the material when correcting that mistake. The different responses led to a more productive discussion about the importance of knowing whether we distinguish between objects or not. While I was excited that we so often debated ideas and overcame misconceptions together, a number of students were annoyed. A number of times, I verbally expressed that I was happy that they had struggled with these ideas. I also pointed out how working together allowed us to all have a better understanding of the ideas. By resolving disagreements, we are better able to remember what we have learned. Even after sharing these benefits with the class, some students were still not happy. A few students were so frustrated at making mistakes initially that they were not able to appreciate how much more they learn this way.

Usually students were happy to generate their own examples. However, there were times when students felt ill equipped to do so. In particular, a number of students struggled to create their own examples for Sections 24 and 53. During class, they even complained about not having enough support to actually create functions satisfying the given restrictions and sketches satisfying the given conditions, respectively. Luckily, in these situations, peers would often point out how helpful it was to consider the examples from the assigned reading. Sometimes it was even possible to choose one of the examples from the text. In the future, I would like to spend more time discussing how failing to create functions and/or sketches actually fuels a richer discussion. Moreover, overcoming mistakes allows us to develop a better understanding of those concepts and, subsequently, remember them.

More generally, a few students were uncomfortable working through problems before we discussed the material as a class. Sections 8 and 5 were particularly challenging for students. For Section 8, they completed one of the exercises immediately following the reading. In the past, some students had voiced concerns about not knowing how to start a response. We had discussed the value in actually writing down the problem. As a reminder, I gave them the first step in the prompt. Together we discussed how far everyone had come in solving the problem. We shared strategies for moving from the problem statement to actually solving the problem. For Section 5, I included a hint to focus on trying to use specific propositions in the hopes that students would at least write down these propositions. Since writing proofs is all about communicating in a logical way as effectively as possible, this concept proved to be extremely challenging. When students voiced their frustrations with this *Reading Journal*, I emphasized the importance of using the context clues and all the given information. We explicitly discussed the fact that I expected everyone to write down the statement to be proved and the two propositions from the hint. Then we would be ready to untangle all of the ideas together. While such conversations were helpful in the moment, some students had more trouble applying the ideas to new situations. Thus, it is important to have these conversations more often. In the future, I want to further discuss the importance of not giving up on ourselves when we cannot completely answer something.

Overall, students were most vocal about not wanting *Reading Journal* prompts which required them to generate their own questions, as with Section 14. This surprised me. To me, generating questions is akin to generating examples. However, a majority of students struggled with wanting to completely understand and answer correctly. While I wanted students to struggle and uncover where their understanding was weakest, students wanted to show that they were correct. As a result, students were uncomfortable writing down questions for which they did not already have valid solutions. Switching the focus from finding complete valid solutions to giving an honest effort was a constant struggle in this class. While we did address this struggle together, some students were still not able to break that habit. Like 14, Section 49 frustrated the students. They wanted to be able to answer everything correctly right away. Next time, I would lead a discussion on how to initiate the solution to a given problem without stressing out about whether we can completely do so.

The most memorable *Reading Journal* prompts required students to create and analyze their own examples of a particular concept. Students were often more willing to share their own examples with the class. Moreover, students seemed more engaged in the material when they created their own examples. Focusing on creating an example rather than answering a question correctly helped more students be okay making mistakes. Whenever students created their own examples, we were able to discuss the entire section through further analysis of their examples. The conversations centering on student-generated examples encouraged much more animated conversation from the class. In addition to being more interesting to our class, these examples encouraged deeper conversations as we sought to analyze them. Their examples did not necessarily have one correct answer. In fact, there were often more possibilities. Thus, we worked more on developing deeper reasoning skills. Therefore, I felt that these *Reading*

Journal prompts worked exceptionally well for our class.

3.3 Excursions in Mathematics

Due to the nature of the text, students were able to see alternate solutions immediately after working on a problem. Thus, they were able to receive instant feedback on their understanding of the question and its solution in the context of a particular problem solving strategy. This access to full solutions allowed them to feel better about the parts that they had done correctly. The text also allowed them to compare their own write up to a possible solution and to see where their write up differed. While I had hoped that everyone would share different approaches with the class, few students did so in practice. However, student feedback on the anonymous survey and in their final reflection on these assignments was very insightful. The Reflection assignment asked students to chose three to five problems from different reading journals to submit with a reflection on how their ability to solve problems throughout the semester evolved and how reading journals contributed to helping them learn to solve problems successfully. Overall, students in this course were more willing to take advantage of the opportunity to struggle productively in and out of class than students in the other two courses.

The students who completed more *Reading Journals* tended to perform much better in the course overall. While I did check for physical evidence, it was easy to note which students were spending more time and energy on completing the *Reading Journal* problems through the presentations and our full class follow-up discussions. The end of semester survey completed by half the class further revealed that students spending less time on these assignments were more negative about them. They did not feel that the *Reading Journal*s were beneficial and some even stopped doing them completely. On the other hand, students who spent more time on these assignments reported much more positive feelings about them. When asked for advice, they encouraged future students to take these assignments seriously. They shared how beneficial it was to really think through the problems and compare to the solutions provided.

Students who had spent less time with the material had to focus on the big picture during class and struggled to figure out the more subtle details on their own later. For *Reading Journal*s like for Chapter 1, they had little to work down for the different problems. They rarely wanted to share a solution with the class. They tended not to pose, nor answer, questions. When students were responsible for highlighting main points for an assigned Chapter, as in the assignment for Chapters 5, 6, 7, 8, and 9, these students contributed less to their small group discussions and were not as comfortable sharing the work with other students during the speed interview discussions. They also tended not to have any questions written down when coming to class. For example, they were not as likely to have questions for the Geometry text of Chapter 17. Their final reflections emphasized that they did not find it beneficial to see the solutions in the text. One student even admitted to not doing them, because there was not value in doing so. They did not see the value in struggling productively. As expected, they reported that they did not perceive any growth in the ability to solve problems.

The students who spent more time digesting the material before class were better equipped to deepen their understanding during class. They had a better understanding of what they knew and where they had lingering questions. Since they had spent time with the material already, they were able to focus more on the nuances. In their reflections, these students highlighted how valuable it was to really think about a problem and its solution. They recognized how helpful it was to read about a problem solving strategy and work through some problems before talking all together. They appreciated the opportunity to check their work with the step by step solutions provided. One student even stated that every math class should use *Reading Journal* assignments so that students can look over what is coming next and have the opportunity to ask questions that allow them to deepen their understanding even more. As the semester progressed, one student mentioned that (s)he started thinking about each problem in multiple ways and tried to choose the most efficient solution. They saw the value in struggling to tackle various problems using a new technique. Further, they were able to persevere in their journals. They continued to persevere by developing an even better understanding when we went over problems as a class.

As in the other two courses, the students who fell between the low and high achievers tended to want my help and immediate feedback while working on these assignments. They also felt like they would have benefited more from me presenting a correct solution than a peer presenting their own solution. While I did address that there is far greater value in learning from each other and mistakes, my message did not seep in with everyone. In the future, I plan to more regularly ask who got stuck while working and how they got unstuck or how we will get unstuck together. Unlike the other courses, there was a stronger correlation between students who completed *Reading Journals* and students who performed well overall. Therefore, one of the biggest challenges in Excursions for Mathematics is getting the students to spend time completing the *Reading Journals*.

4 Looking to the Future

4.1 Calculus II

If I expect students to read their math textbooks productively, I have a responsibility to model what this looks like. When we are reviewing some integration concepts from Calculus I, I would like to find a way to intentionally model how to read the textbook productively. I also want to be conscientious about asking the class how many people struggled with the reading and how many people had felt frustrated. I want to follow up by complimenting them for doing so and reminding them of the value of getting stuck and struggling productively.

4.2 Discrete Mathematics

As in Calculus II, I feel that it is crucial to better prepare students for reading mathematical texts productively. In addition to modeling how to read, I want to generate an ongoing list of tools and resources available to us when we feel stuck or lost. On a regular basis, I want to recognize that we should all feel lost at different points when reading. I want to regularly emphasize that where we get stuck is where we most need help. I want to be conscientious about reminding students that the places we get stuck on our own are the places we need to focus on together. So the purpose of completing *Reading Journals* is not to have everything correct. The real purpose of these *Reading Journals* is to guide us to what we need to spend the majority of our time together discussing. In this way, we work together to struggle even more productively.

4.3 Excursions in Mathematics

Although we would spend at least two days discussing a new chapter, I had students read an entire chapter and complete all of the problems in that chapter before our first meeting. My students would benefit from a more regular schedule and more focused *Reading Journal* assignments. Thus, I would like to split the *Reading Journal* assignments for each Chapter into two parts. I hope that this change would encourage students to spend a little more time reading each part. Since there would be fewer problems due each day, I also hope that they will spend a little more time tackling those problems. I think that this would alleviate pressure on the students and support a more focused conversation each day. Another consequence is encouraging us to more regularly discuss strategies for making mistakes and learning from those mistakes in a productive way.

5 Closing Remarks

As I mention above, one of my main goals in teaching is to help students struggle, make mistakes, and persevere. Many of my students tend to fear making mistakes and are reluctant to try tackling a problem before seeing an expert tackle a few problems of that same type. Rather than giving in and doing what my students are most comfortable with, I push my students to take more ownership of their learning. I am driven by my desire to equip my students to develop a stronger foundational understanding of mathematics.

Since we spend so much time and energy finding the best possible resources for our students, why would we not find productive ways to get them to use those resources? *Reading Journal* assignments not only encourage students to actually read their textbooks but also help direct the focus of our class time to those concepts that cause students to struggle the most. These are the places we most need tools for persevering and would most benefit from discussing together. They allow students to recall the information they already know and to figure out what new information makes sense to them. These assignments also show students where they are confused and frustrated. These latter parts are the ones we should be discussing in class. *Reading Journal* assignments help students and teachers find out where to spend the most time. Further, they allow us to practice struggling productively.

Finding Meaning in Calculus (and Life)

Doug Phillippy (Messiah College)

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Abstract

The 2015 publication, *Insights and Recommendations from the MAA National Study of College Calculus*, noted that "students taking college calculus exhibited a reduction in positive attitude toward mathematics, which can affect their career aspirations and desire to take more mathematics". The study concluded that students' confidence to do mathematics, enjoyment of mathematics, and desire to persist in their study of mathematics had all decreased by the end of their college calculus experience. This paper will summarize the findings of the MAA's study of college calculus and then suggest how a Christian foundation can prove to be beneficial in improving student attitude toward calculus and mathematics in general.

I have seen all the things that are done under the sun; all of them are meaningless, a chasing after the wind. Ecclesiastes 1:14 (NIV)

1 Introduction

I have been teaching college calculus for more than 30 years now. Not only do I enjoy teaching the subject, I enjoy the material itself. Moreover, I believe there is value in studying calculus. I believe that those who master the subject will be better equipped to think about and understand God's creation. As a teacher, one of my greatest desires is to instill that same appreciation and understanding of calculus into my students. Unfortunately, many of my students do not share my affinity for the subject. Some struggle with course content from the onset of the course while others see little or no connection to their future careers. Apparently, my experience is not unique. The Mathematical Association of America's (MAA) recent study of college calculus students concluded that "students taking college calculus exhibited a reduction in positive attitude toward mathematics, which can affect their career aspirations and desire to take more mathematics" [3].

What leads to a reduction in positive attitude toward mathematics among students who take college calculus? The aforementioned study mentions one potential explanation for this reduction in attitude, namely increased rigor when compared with pre-college work. It is important to note that reduction in student attitude is a result of increased rigor and not just rigor alone. After years of training at the secondary level, it is natural for students to have developed some expectation of what their college mathematics experience will entail. Research shows that expectation and potential value derived are key factors when it comes to achieving learning objectives [1], so it should not be a surprise when students who begin their study of calculus with unrealistic expectations experience a reduction in positive attitude.

As mentioned above, one of my objectives as a teacher of calculus is to help students find meaning in their study of the subject matter. Because achieving this objective is closely related to my students' values and expectations, it is crucial to identify and understand these values and expectations. Therefore, my purpose in this article is two-fold. First, in order to better understand the values and expectations of college calculus students, I will summarize the context surrounding these students as they prepare to enroll in Calculus 1. Second, I will present some thoughts on helping students think about what they value and how their value systems may be perpetuating gaps in their understanding and also leading to a lack of meaning in their education. I will share my experience during the past thirty years of teaching college calculus, as well as some of the findings of the MAA's national study of college calculus. It is important to keep two things in mind as you read this article. First, my focus is on Calculus 1 as a college course. That is, I am not considering the calculus sequence as a whole. Second, the thoughts proposed in this article rely heavily on Christian values and as such any implementation of these ideas is likely limited to the Christian college classroom.

2 Setting the College Calculus 1 Stage

According to the MAA's study of college calculus, 500,000 students enroll in a college Calculus 1 course each year. About half of these students are seeing calculus for the first time. My goal in this section is to get a general sense of who these students are. What was their high school mathematics experience like? Are they prepared to take college calculus? What are their expectations? I make no claim to be exhaustive here. I rely heavily on the 2015 publication, *Insights and Recommendations from the MAA National Study of College Calculus* [3] for most of my answers and suggest it as a resource for the reader who is interested in more extensive answers to the questions posed above and other questions like them.

Much has changed during my 30 years of teaching college calculus. Perhaps the biggest change has been in the area of technology. Graphing calculators were unheard of when I studied calculus back in the early 1980's. I owned and used a calculator while learning calculus, but only for difficult numeric computations. In contrast, my students can do most of what I try to teach them just by punching a few buttons on their calculators. According to David Bressoud, author of the introductory chapter in the MAA's report on college calculus, "permission to use graphing calculators on exams is one of the sharp discontinuities between high school and college calculus" [3]. In the report, Bressoud notes that graphing calculators are almost universally used in high schools. This creates a potential problem for students who enroll in a college Calculus 1 course for which the use of a graphing calculator is either restricted or not allowed at all.

Some of the changes that have occurred during the past few decades are a direct consequence of students' struggles with the content and relevancy of the subject matter. Nearly 35 years ago, Robert White (President of the National Academy of Engineering) suggested that it was time to turn calculus from a filter to a pump, a reference to calculus as the reason some aspiring engineers were leaving the major. The calculus reform movement was born shortly after this statement was made (about the time I was beginning my teaching career). During my thirty-plus years of teaching, I have attended many talks about new and innovative ways to teach mathematics. My bookshelves are filled with textbooks emphasizing different approaches to learning calculus. Terms like student-centered learning and the flipped classroom, which were not part of my educational experience, are now very familiar to me, and I have experimented with some of these approaches in my classroom.

Have the changes mentioned above impacted student attitude toward calculus? According to the MAA's report, educational technology "was found to have no impact on attitude", while ambitious teaching "had a small negative impact on student attitudes". In contrast, the MAA's report noted that "instructors who employed generally accepted good teaching practices (e.g., clarity in presentation and answering questions, useful homework, fair exams, help outside of class) were found to have the most positive impact, particularly with students who began with weaker initial mathematics attitudes" [3]. Despite all the advances in technology and innovative approaches to teaching that have been made during the past three decades, Bressoud opened the MAA's report by admitting that calculus reform has made little progress, and calculus in many ways still remains a filter.

In order to understand why college calculus has remained a filter, it is not enough to only consider a student's experience in the college classroom. More than three-quarters of all students who study calculus have their first experience with calculus in high school, and as previously mentioned, approximately half of all students enrolled in a first-semester college calculus course have already studied calculus in high school. In discussing the impact that a high school calculus course can have on a college student, David Bressoud makes the following observation, "Mathematics is unique among all disciplines in having created a course, calculus, which is both the lodestar of the K-12 curriculum and the bedrock of post-secondary preparation for science and engineering. These distinct perspectives on this course create much of the discontinuity that students experience as they transition from high school to college" [3].

It has become virtually impossible to talk about the transition from high school to college mathematics without considering the impact of the AP exam. In 1989, 79,000 students took the AP exam [2], while in 2010 more than 400,000 United States students took the exam [3]. For some students, a good score on the Calculus AP exam exempts them from taking any mathematics classes in college, while for others it serves as a jump-start to their college education. According to Bressoud, the problem with jump-starting a college mathematics curriculum using the AP exam is that success in AP Calculus does not require the level of proficiency in mathematics that post-secondary faculty want their students to have when it comes to their mathematics, science, and engineering courses. He adds, "In some sense, the worst preparation a student heading toward a career in science or engineering could receive is one that rushes toward accumulation of problem-solving abilities in calculus while short-changing the broader preparation needed for success beyond calculus" [3].

The impact of the AP exam is especially felt in a college Calculus 1 class. That might seem like a strange statement to make given that the purpose of taking the AP exam is to bypass Calculus 1 in college, but according to the MAA's findings, nearly 75 percent of the students who take the AP AB exam and attend a four-year college score 3 or less (out of 5), and almost 60 percent of the students who take the BC exam score 2 or less [3]. Because the College Board calibrates the AP scores so that a score of 3 on the AB exam corresponds to C work (The AB exam is usually one point higher than the BC exam), this means that the majority of students who take the AP exam did, at best, C work in their high school calculus class. The strongest students are testing out of Calculus 1 and starting their college mathematics in a subsequent course, while those who scored low on the AP exam are retaking Calculus 1 in college.

Moreover, because Calculus 1 is foundational to many different majors, it is populated with first-year students at the beginning of their journey toward a future career. Some of these students are unsure of their major while others may even be questioning whether or not college is right for them. Additionally, many first-year students have not established good study habits. For these reasons, many institutions administer a placement exam to students who wish to enroll in a college calculus class. Students who are deemed not ready for Calculus 1 by a placement exam are often encouraged to enroll in some type of remedial program. While the MAA's report on college calculus identifies different types of placement strategies and even offers advice on what a successful placement strategy looks like, it does not provide any data on the percentage of students who are deemed not ready for calculus by some type of placement strategy.

During my tenure at Messiah College, the Mathematics Department has implemented several different types of placement strategies. Each strategy included a placement exam that was written in-house. This exam has remained virtually unchanged for the past 15 years, though we made the switch from pencil and paper to administering the exam on-line in 2013, and this past year we changed the name of the exam from placement exam to proficiency exam (for consistency, I will continue to refer to our exam as a placement exam). Using data that was collected since moving to the on-line exam, that is between 2013 and 2017, we found that almost 30 percent of the students (n = 342) in our Calculus 1 course scored less than 60 percent on the placement exam (this does not include students who withdrew from the course). Moreover, a statistical analysis of the data suggests that our placement exam has done well at identifying those students who are going to struggle in our Calculus 1 course.

Although our placement exam has done well at identifying those students who are likely to need supplemental instruction in order to succeed at calculus, our strategy for delivering that instruction has not been as effective. When I first arrived at Messiah College in 1993, there wasn't even a placement exam, much less a course in which to place students who were not ready for college calculus. Students were placed into Calculus 1, even if their algebra and trigonometry skills were not calculus-ready. Not surprisingly, these students struggled in Calculus 1. An internal assessment of our calculus sequence led to several changes in our Calculus 1 course, including the creation of a common final exam, a placement exam, and an option for students who were not ready for calculus. At first, students who were deemed not ready for calculus were placed in a traditional precalculus course. Our second attempt placed these students in a two-semester stretch-calculus course that included precalculus remediation delivered in conjunction with the standard material in a Calculus 1 course. Our current approach, which we have been using for six years, allows these students to remain in Calculus 1, but requires an additional one-credit course that emphasizes algebra and trigonometry in the context of the problems encountered in the calculus homework.

While some students have benefited from each of the strategies mentioned above, none of these strategies has been noticeably successful as a whole. Unfortunately, we have no data to substantiate this claim for the first two strategies, but the fact that each was abandoned to pursue a new strategy is evidence of at least the perceived ineffectiveness of both strategies. Although we (Messiah College) have no data to substantiate our claim about the ineffectiveness of our precalculus course, the MAA's report cites evidence that precalculus as a remedial college mathematics course has not been very effective in getting students on track to take Calculus 1. The report notes that there is ample documentation showing that "precalculus as currently taught in most post-secondary institutions in the United States does very little to improve student chances of success in Calculus 1 and can actually be detrimental" [3]. The report also mentions another problem with the traditional precalculus course, one that we found to be true at Messiah College as well: many students who intend to study calculus and who do well in precalculus, do not go on to take Calculus 1 after successfully completing their precalculus course. The report cites multiple studies with attrition rates as high as 65 percent for students who did well in precalculus. The strategy that is currently in place at Messiah College has not fared much better. Analysis of data that was collected at Messiah College between 2013 and 2017 showed no statistically significant evidence that our current approach increased the probability of success in our Calculus 1 course.

With this context in mind, I would now like to begin to answer the question of how students can find meaning in their study of calculus. I will argue that success (earning a good grade) is not enough to find meaning and will suggest a strategy that will help every student, even those who struggle with the material, find meaning in their study of calculus.

3 Finding Meaning in Calculus

Much good work has been done and is being done to help students succeed in Calculus 1. Personally, I have been involved in this work for years. I have written placement exams, researched and implemented three different remedial strategies, made use of technology to aid in student learning, and used some of the latest research to inform my teaching. My latest writing project is a calculus pretext that uses simple examples and simple language to expose students to course content before they see it in lecture. This pretext is not intended to replace the traditional calculus text but to supplement it. After using the pretext for a semester, I surveyed my students to see if they found the text useful. The results of the survey showed that some of my students found the pretext helpful, but others did not, and a good number of my students did not even make use of this resource. I have found this to be true in general, regardless of the resource. Whether it is the calculus pretext mentioned above, the slides I created to summarize my lectures and give students a visual representation of the problems presented in class, or the reference sheet I created to help students struggling with trigonometry, I put a lot of effort into creating the resource, and only some of my students use it and find it useful. Why is this? That is, why are some students motivated to use these resources and others are not?

The authors of *How Learning Works* note that “students’ motivation generates, directs, and sustains what they do to learn”. The authors add, “the importance of motivation, in the context of learning, cannot be overstated” [1], especially for college students who have a new found freedom to determine what, when, where, and how they study and learn. How can a lack of motivation be overcome in those students who are not inspired by the thought of studying calculus? According to the authors of *How Learning Works*, when a goal is set before a student, the subjective value that student associates with achieving that goal is a key factor in motivating that student to pursue that goal. The higher the value a student places on a particular goal, the more motivated that student will be to pursue that goal when confronted with multiple goals [1]. At this point in my career, I am not interested in creating another resource that is likely to help only some of my students. I am beginning to think a bit more holistically about some of the problems facing students taking a college calculus course. Rather than focusing on what I can do to fill a gap in student understanding by creating another resource, I want to focus on what the student can do by examining his or her value system in light of a Christian view of learning.

I am not alone in this endeavor. The importance of helping students examine what they value as part of their education was one of the main themes of the June 2015 issue of *Perspectives on Science and Christian Faith*. This particular issue of the journal was dedicated to mathematics, and Russell Howell was invited to write the lead article. Howell’s article, *The Matter of Mathematics*, made use of Arthur

Holmes' [6] four categories of faith integration [1] to stimulate conversation regarding mathematics and faith. James Peterson, editor of the journal, noted that Howell had much to say about the interaction of faith and mathematics, especially at the metalevel. I found this to be true, with many of the questions posed by Howell having a philosophical flavor to them. But Howell also posed several questions related to teaching, noting that "additional Christian perspectives are needed in evaluating the ever-increasing approaches to education" [7]. Despite Howell's emphasis on ideas with a philosophical theme, three of the four responses were related to teaching, with two of those responses offering a Christian perspective on pedagogy.

The authors of these two responses, Valorie Zonnefeld and Joshua Wilkerson, both address the idea of subjective value by considering in some way the age-old question students have asked about mathematics: "When will I ever have to use this?" In her response, Zonnefeld says that it is imperative that Christian educators can answer this question because "some students do not readily see the beauty in mathematics, but they may be drawn to its incredible utility" [11]. This statement implies that value can be found in some aspect of mathematics itself. That is, some students may find value (and hence motivation) in the beauty of mathematics while others find value (and hence motivation) in the utility of mathematics.

As an applied mathematician, I have always valued the utility of mathematics. I find great satisfaction in creating a mathematical model that does well at describing reality. My work building a mathematical model of the foot and shoe for Nike Inc. is among the most satisfying of my career. For this reason, I have often looked to the utility of mathematics to inspire my students. On the other hand, I have found the beauty of mathematics to be less inspiring, though as I have matured as a mathematician, this aspect of mathematics has becoming increasingly more attractive. My point is that value is not only a function of the individual but also of that individual's experience. In other words, what a person values may change as a person matures. While some students may find value in certain aspects of mathematics itself (i.e., its beauty or its utility), I do not believe that most of my Calculus 1 students are at a point in life where they assign value to their study of mathematics in this way.

Therefore, if I want to motivate my students, I must first identify and understand what they value. But it is not enough to understand what my students value if their values are misplaced. In this case, I must help them establish an appropriate value system. The authors of *How Learning Works* suggest six strategies to establish value [1].

1. Connect the material to students' interests.
2. Provide authentic, real-world tasks.
3. Show relevance to students' current academic lives.
4. Demonstrate the relevance of higher-level skills to students' future professional lives.
5. Identify and reward what you (the teacher) value.
6. Show your own passion and enthusiasm for the discipline.

I agree with every strategy in this list. I try to get students excited about the material they are learning by showing excitement. As stated earlier, I find enjoyment in constructing models of the real world

[1] The four categories proposed by Holmes are the attitudinal, the ethical, the foundational, and the worldview.

and seek to bring those models into the classroom. I understand the importance of making the material relevant to students' interests, current academic lives, and future careers. Each of these strategies can help students find value in their study of calculus. However, I think there is potential danger here as well. Care needs to be taken to ensure that students do not view their education with only self-serving eyes. In her response to Howell, Zonnefeld hints at this danger. She suggests that when too much emphasis is placed on the student in the education process, it can feed the distorted view of individualism that exists in western culture, a view that finds value only in things that are self-serving. She argues for a "subject-centered" approach to learning, where "God's truth takes center stage" [11].

Personally, I like the subject-centered approach described by Zonnefeld. However, I believe she is idealistic in its implementation. At one point she says, "Curiosity along with cognitive dissonance are harnessed to draw students in to learn more about topics in mathematics" [11]. She then gives an example, asking if it is possible to add four odd integers and obtain a sum of 19. Her description kindled my curiosity and drew me in, but I found myself wondering if it would do the same for my calculus students at 8:00 in the morning.

Wilkerson is not as idealistic in his response to Howell. He believes that most students who ask "When will I ever use this?" have already formulated the answer in their minds. That answer goes something like this: "I will never use this, so learning it is a waste of time". He then suggests that when students ask this question, what they really mean is, "Why should I value this?" [10] I tend to agree with Wilkerson. That is, I believe many of the students in my Calculus 1 class want to know why they should value calculus. They don't simply want to find value in calculus they want to know why they should value it in the first place.

Having said this, I believe that most of my students have already assigned some sort of value to calculus anyway. I believe this is true because they are there; that is, they were motivated to enroll in Calculus 1. To be sure, that value may be misplaced. For example, some students assign value to calculus because it helps them earn the degree they want to earn, while others value calculus because of the money that they will earn in a particular profession, and still others value calculus because it earns their parents' approval. In the past, I have looked to the utility and beauty of mathematics to help students find meaning in their study of calculus. Though this may still be a worthy goal, it will only motivate those students who value the utility and beauty of mathematics. This is not true for many of my Calculus 1 students. I now realize that before I can help these students find meaning in their study of calculus, I must first help them understand why they (as Christians) should value calculus in the first place.

Helping students understand why they should value calculus will enable them to better assign value to their efforts to learn calculus. This is crucial because misplaced value usually leads to lack of meaning. The writer of Ecclesiastes found this to be true of his work: "Yet when I surveyed all that my hands had done and what I had toiled to achieve, everything was meaningless, a chasing after the wind; nothing was gained under the sun" (Ecclesiastes 2:12). A peek back through chapter two of Ecclesiastes sheds light on what the writer valued and hence his state of mind. He valued pleasure (verse 1), achievement (verses 4-6), possessions and wealth (verses 7-8), and status (verse 9). Because these are some of the same values that motivate many students' pursuit of an education, the book of Ecclesiastes is a great place to start a conversation with students about values.

And that is what I plan to do. In other words, I believe that if we are to see progress toward making calculus a pump and not a filter, it has to begin by correcting the flawed value systems that motivate many of our students to study calculus in the first place. The answer is not another resource or innovative approach to teaching calculus (though these are good things), but Christian educators willing to help students learn the subject matter in the context of biblical values. In the *Abolition of Man*, C. S. Lewis writes, “education without values, as useful as it is, tends to make man a more clever devil” [8]. It is foolish to assume that students’ value systems are biblically intact when they are constantly confronted with the value systems of the world. In his book, *Faith and Learning on the Edge*, David Claerbaut argues that “the ability to apply Christian values and analysis to what one studies and learns” [4] is of utmost importance in every discipline, including mathematics. I know of no better tool than Scripture itself to help students (and me as well) see the flaws in what we value. Hebrews 4:12 says, “For the word of God is living and active. Sharper than any double-edged sword, it penetrates even to dividing soul and spirit, joints and marrow; it judges the thoughts and attitudes of the heart”.

In order to help students work through an examination of their value systems, I have written twelve reflection exercises consisting of questions, personal thoughts, and Scripture emphasizing learning from a Christian perspective. One such exercise uses the Ecclesiastes passage mentioned above to help students take an honest look at why they enrolled in a calculus course in the first place. A second exercise focuses on a proper foundation for an education. Proverbs 9:10 says that the fear of the Lord is the beginning of wisdom. In this exercise students are asked to consider the foundation of their education and what it means to make the fear of the Lord the beginning of their pursuit of wisdom. A third exercise focuses on distraction. Hebrews 12:1 tells us to fix our eyes on Jesus, the author and perfecter of our faith. In this exercise, students will be asked to consider those things that tend to distract them from pursuing wisdom, with an emphasis on technology (e.g., phones, video games, and social media).

4 Personal Observations

I conclude this article with an example of one of the reflection exercises that I intend to use with my Calculus 1 students and a brief summary of my experience to this point trying to implement these ideas in the classroom. The goal of the example exercise that follows is to have students examine their assumptions about learning mathematics and contemplate how those assumptions might impact their learning. Students are asked to complete the following:

1. Write down everything that you assume to be true about this class.
2. What does it take to be successful in a mathematics course?
3. In her book, *Mindset: The New Psychology of Success*, Carol Dweck distinguishes between a fixed mindset and a growth mindset [5]. What is your mindset with regard to mathematical ability? To answer this question, identify which of the following statement(s) best describe your understanding of mathematical ability:
    - It is something very basic about me that I cannot change very much.
    - I can always substantially change my ability to do mathematics.
    - Someone can learn new things, but they can’t really change their ability to do mathematics.
    - No matter how much mathematical ability someone has, they can always change it quite a bit.
4. Read the parable of the talents found in Matthew 25:14-30. What assumptions did the man who received one talent make? How did his assumptions impact his actions?

5. In his talk identifying characteristics of a successful education system, Andreas Schleicher summarizes two different responses (see below) that he received to the question: What makes you successful at mathematics [9]?

   - One response suggested that success is all about talent and that if someone was not a genius at mathematics, that person should study something else.
   - A second response suggested that success is a direct result of study and trust in the instructor.

   Schleicher attributes one of the responses to American students and the other to Asian students. Which response is the response of the American students? Why do you think the response of American students is different than the response of Asian students?

I have not yet had a chance to use this particular reflection exercise with my Calculus 1 students, but I have been able to begin implementing my plan to help students examine their value systems. This past semester, I piloted the ideas mentioned in this article in a small group consisting of junior and senior engineering students. We met weekly (outside of the classroom) throughout the semester. I distributed the exercises to the students prior to our meeting and during our weekly meetings we discussed responses to the questions posed in the exercises. Three of the five students involved regularly engaged in the discussion. Two students offered little input to the discussion, but all agreed that the exercises were helpful.

In addition, I focused my devotional time in Calculus 1 (the first five minutes of class) on the same questions found in the exercises mentioned above. However, unlike the small group mentioned above, I did not distribute the questions to the students beforehand, and I did not allow for in-class discussion. Despite these circumstances, I did see some interesting results. One of my students had this to say about the class: "I would like to formally thank you for the way in which you taught [Calculus 1] this semester. I truly appreciate your passion for mathematics and your desire for the success of each and every one of your students. There were multiple times when I wanted to give up because Calculus was no longer necessary for my new field of study, however you inspired me to keep working. Thank you for the short messages you began each class with. Although some people may have looked past these, I found great worth in each bit of wisdom you shared with us. Please continue to teach the way you are now. Many will benefit from the way in which you are using the talents with which God has gifted you." Another student asked if I would hold him accountable in his struggle with pornography, a first in all of my years of teaching.

To summarize, I am encouraged by the results I saw this past semester. I am not teaching Calculus 1 this semester, but I plan to use these exercises in a first-year seminar for mathematics majors. I am hoping the experience in a seminar setting will allow me to not only answer some of the questions I have about implementing my plan in a large Calculus 1 class, but it will also allow me to refine some of the questions in the exercises.

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Axioms: Mathematical and Spiritual: What Says the Parable?

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Abstract

Pastors such as A.W. Tozer have described their preaching as an attempt to extract from scripture axioms that are universal across circumstances. Similarly, academicians in various disciplines have tried to characterize the foundational principles of their field. For example, the social psychologist Gerard Hofstede proposed an onion model for comparing cultures in which core values of a nation help explain its rituals, feelings, and artifacts. On a personal level, many of us have likely tried to identify root causes of our beliefs and feelings.

These attempts to find bedrock are somewhat analogous to the axiomatization of mathematics attempted by Euclid, Hilbert, Frege, Russell and Whitehead, and others. As is well known, the success of these efforts ranges from that of Euclid (whose book arguably guided western mathematical reasoning for two millennia) to that of Russell and Whitehead (whose book was essentially defeated by Gödel's incompleteness theorem within two decades).

In this paper, I will propose insight that mathematical axiomatization (considered as a parable) may have for our efforts to understand God and explain our core spiritual beliefs. Is it really possible to identify our personal spiritual axioms? Is it possible to classify spiritual axioms as consistent and complete? Are there things that God does and does not want us to know? How should Christians respond when deductive reasoning fails to explain a situation?

1 Introduction

A *parable* is an example from a familiar context used to illustrate a point from a less familiar situation. Jesus frequently used examples from His first century agrarian society to illustrate concepts of the kingdom of heaven. An *axiomatic base* is a set of statements accepted without proof which is then used to derive other facts. In this paper, I will use ideas arising from the study of mathematical axiom bases to explore the nature of psychological and spiritual axioms. Much credit for the overall idea is due to Chapter 6 of [4].

I will first give three brief examples, from mathematics, cultural psychology, and theology, to illustrate the concept of axioms from each of these three areas. Next, I will return to each area in more detail. I will then comment on similarities and differences between mathematical and spiritual axioms, before finishing with brief overall conclusions.

1.1 A Mathematical Example

To avoid circular definitions, every field of mathematics must simply accept some terms as a starting point for further definitions. In this example taken from [8], the undefined terms are *spirit, substance,* and *possess.*

Two spirits are defined to be *distinct* if there is a substance which one possesses and the other does not. The following axiomatic base is then given:

1. There is at least one spirit.
2. Each spirit possesses at least one substance.
3. Given a substance, there is a spirit not possessing that substance.

These axioms then allow proof of theorems, such as the following, given here merely to illustrate the methodology of deductive mathematics.

Theorem: There are at least two distinct spirits.

Proof: By Axiom 1, there is a spirit s. By Axiom 2, s possesses a substance S. By Axiom 3, there is a spirit t not possessing S. Therefore, s and t are distinct spirits. QED

1.2 A Social Psychology Example

The Dutch social psychologist Geert Hofstede studies similarities and variations of culture between societies [6]. A modified version of his "onion model" is shown in Figure 1.

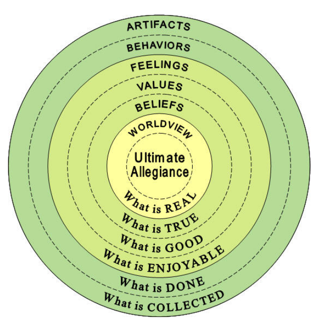

Figure 1: Hofstede Onion Model

The model illustrates that a visitor to this culture from the outside would first notice its artifacts and behaviors such as how weddings are performed, what recreational activities are common, and what literature is currently being produced. Increased familiarity with the culture would allow the visitor to learn the feelings, values, and beliefs behind these artifacts and behaviors such as the relative importance of time, the goals of the educational system, and the reasons the society's heroes were chosen. Finally, the visitor could, perhaps with great difficulty, learn people's ultimate foundations for their values and beliefs.

1.3 A Spiritual Example

The pastor and theologian A.W. Tozer once said at the beginning of a sermon [10], "My method in preaching is to extract from the scriptures certain basic spiritual principles and to turn those spiritual principles into axioms; they are valid everywhere, anywhere at all times, always." In his sermon, he then gave the example "There are different kinds of working, but the same God works all of them in all men" (I Corinthians 12:6, [1]). Restated another way, Tozer accepted as an axiom the biblical claim that all good works of humanity are inspired by the one God.

2 Mathematical Axioms

2.1 Definition of Axiom

In logic, the definition of an axiom has weakened over time. Classically, an axiom was understood to be a statement so evidently true that it should be accepted without question. Because of the philosophical difficulty of the meaning of "true," today an axiom is simply taken to be any statement accepted without justification as a starting point for reasoning.

In mathematics, there has also been a shift of meaning. In the time of Euclid, a distinction was made between logical and non-logical axioms. A *logical axiom* was a statement inherited from the general system of logic. For example, the transitive property of equality, $A = B$ and $B = C$ implies $A = C$, is commonly used without being stated as an explicit assumption. A *non-logical axiom* (also known as a *postulate*) on the other hand, is a statement consciously chosen for the specific situation. For example, deliberate inclusion or exclusion of a commutative axiom of multiplication $AB = BA$ characterizes the type of algebra being studied. Because most modern practicing mathematicians do not become involved with the foundations of formal logic, over time this distinction has somewhat faded. The term axiom is now often used interchangeably for both logical axioms and postulates, as will be the case in this paper.

2.2 Models

A *model* is an attachment of meaning to the undefined terms of an axiomatic system. The usage of the words "spirit," "substance," and "possession" in the example of Section 1.1 has likely already led the reader to try to form some sort of mental image of the situation. To illustrate how helpful our mental images can be, note that the exact same axiomatic system can be described by replacing these terms by "mug", "table," and "on" as follows.

1. There is at least one mug.
2. Each mug must be on a table.
3. Given a table, there is a mug not on that table.

Now the reader probably begins to get a more intuitive understanding of the axiomatic system through a mental image such as Figure 2.

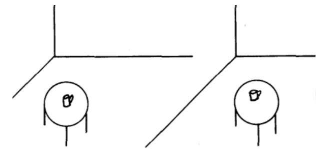

Figure 2: Mugs Model

This example with mugs and tables was prompted by statements of the famous mathematician David Hilbert pointing out that the process of formal deductive reasoning used in geometry does not depend on our intuition regarding the meaning of the undefined terms. In fact, this system is an example of a simple geometry described still more intuitively (at least to a mathematician) by the following axioms and pictured by the model in Figure 3

1. There is at least one point.
2. Each point must be on a line.
3. Given a line, there is a point not on that line.

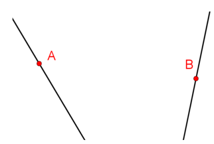

Figure 3: Points and Lines Model

Note that though different equally valid meanings (models) can be associated with a given axiom set, our understanding is greatly helped by finding familiar and intuitive models. We likely find (perhaps subconsciously) models for our spiritual axioms as well.

2.3 Desirable Features

When choosing a set of axioms, mathematicians strive for three desirable characteristics.

1. An axiom set is said to be *consistent* if no contradictions can be proven from the set. Consistency is a necessity for meaningful logic since if one contradiction can be proved, then any statement can be proved and the described system becomes meaningless. However, proving that a axiomatic base is consistent is usually quite difficult as will be illustrated shortly.
2. An axiom set is said to be *independent* if none of the axioms can be proven from the others. Independence is considered desirable since it avoids redundancy and extra complexity in the axiom set.
3. An axiom set is said to be *complete* if every well-defined statement can either be proved or disproved. One reason that completeness is desirable is that it avoids having to make the awkward distinction between "true" and "provable."

The concept of a model is a powerful means of determining the consistency, independence, and completeness of an axiomatic base. [9] Referring back to the points and lines axiom set of Section 2.2:

1. This axiom set is consistent since it has the model shown in Figure 3.

2. This axiom set is independent. For example, Axiom 1 is independent of Axioms 2 and 3 since empty space with no points or lines at all is a possible model for the latter two axioms but not of the full axiom set including Axiom 1.

3. This axiom set is not complete since, for example, the statement ”there exist three lines” can neither be proved nor disproved. Both models shown in Figures 3 and 4 are possible models for the axiom set, yet one has three lines and the other does not.

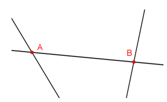

Figure 4: Alternative Points and Lines Model

We conclude this section with a summary of the status of “standard” mathematics in terms of consistency, independence, and completeness. There are elementary axiom sets, such as those for Presberger arithmetic (which is essentially the addition of whole numbers), that have been proven to be consistent, independent, and complete. Presberger arithmetic, however, is of mostly theoretical interest since it does not allow for standard arithmetic and thus for the fields of algebra and analysis.

While the body of mathematics as a whole has no official axiomatic base, since the early 20th century this role has been unofficially played by the Zermelo-Frankel axioms for set theory supplemented by the Axiom of Choice. In particular, the properties of standard arithmetic can be derived from the ZFC axioms. Set theorists originally hoped that these axioms would eventually be demonstrated to be consistent, independent, and complete. However, the two incompleteness theorems proved by Kurt Gödel in the 1930’s showed that:

1. Any consistent axiomatic system allowing the standard arithmetic of the integers must be incomplete.

2. Any axiomatic system allowing the standard arithmetic of the integers cannot prove its own consistency.

The Incompleteness Theorems showed that ZFC could not prove itself consistent, so today though essentially all mathematicians believe ZFC is consistent, it remains an open question. The Incompleteness Theorems also suggest (and this was later proven) that ZFC is incomplete. An example of a statement that ZFC can neither prove nor disprove is the Continuum Hypothesis, which asserts that the cardinality of the real numbers is the next larger infinity to that of the integers. It is interesting to note that the (logically troubling) incompleteness of the ZFC axioms has spurred a search for a new axiomatic base for set theory that has been ongoing for decades. Because of the competing demands from the mathematical academy on such a new axiom set, the selection process has actually in many ways resembled an ad hoc empirical approach as might be used in the natural sciences to test hypotheses. This empirical approach is itself controversial and no universally accepted resolution seems likely soon. In short, the foundations of modern mathematics are neither currently truly anchored to "bedrock," nor have they ever been.

3 Social Psychological Axioms

I now turn to constructs in sociology and psychology that resemble mathematical axioms. Hofstede's work as illustrated by the onion model in Section 1.2 is one such example. Another is a study on "Social Axioms" by Bond, Leung, et. al. [2]. Their main goal was to characterize certain types of beliefs, which they called social axioms, that are distinct from values and that might help predict human behavior. According to the social psychologist Leung, a *value* is an assertion that something, such as ending poverty, is good, desirable, or important. A *normative belief* is a prescriptive assertion such as "we should help poor people." Finally, a *social axiom* is a belief about "what is possible" and is often of the form of a correlation between two items such as "ending poverty would be possible via social programs."

In the early 2000's, Bond, Leung, and others conducted a survey of nearly 10000 individuals from over forty different culture groups. Subjects completed a Likert scale questionnaire indicating their level of agreement with statements such as "hard working people will achieve more in the end." It is important to note that allowing Likert scale responses means that derived social axioms will not be Boolean (either accepted or rejected) as are mathematical axioms, but will rather be held with a variable strength of belief that may be more analogous to the idea of subjective probability in mathematics.

As mentioned, one reason for attempting to identify social axioms is that social psychologists know that values alone are an imperfect predictor of behavior – humans do not always do what their claimed values would dictate. Among other results, the study confirmed that using social axioms and values together rather than values alone better predicted human behavior in coping style, conflict resolution style, and vocational choice. [3] Statistical principal component analysis of the responses showed two main factors of clustered social axioms described below, one of which is further broken down into four subcomponents.

1. Dynamic Externality
   (a) *Social complexity* is the belief that there are multiple ways of achieving an outcome and one must act according to specific circumstances.
   (b) *Reward for application* is the belief that planning and effort will lead to positive results.
   (c) *Religiosity* is the belief in a supreme being and that religious activities have positive results.
   (d) *Fate control* is the belief that people can influence the outcomes of life's events.
2. *Social cynicism* is a negative view of human nature, believing that people and institutions are exploitive and that life tends to unhappiness.

Different culture groups share social axioms, but with different strengths of belief. Like Hofstede's onion model, these differences in belief can help explain and predict differences in cultural practices in an analogous way to, for example, choices of mathematical axioms leading to different types of geometry.

It is extremely unlikely that an individual's set of social axioms will be entirely consistent. In fact, psychologists have long studied the *cognitive dissonance* resulting from an individual holding contradictory beliefs. For example, many of us value both physical fitness and comfort food and thus have the dilemma illustrated by Kevin Hosey in Figure 5. On a more serious note, in Romans 7 the Apostle Paul discusses essentially the same dilemma of his spiritual mind living in a body with a sin nature.

Figure 5: Cognitive Dissonance

4 Spiritual Axioms

I next address the possibility of identifying spiritual axioms (which are also related to the linguistics term *presuppositions*). I first explore the role of foundational statements in Christian apologetics, then consider the possibility of identifying one's own (perhaps implicitly held) axioms.

4.1 Axioms of Apologetics

Apologetics is the systematic defense of the truth of religious doctrines. Though most, if not all, religions have some culture of apologetics, the idea of a logical defense seems most highly developed in the Christian religion. Many Christians believe that the scripture "Always be prepared to give an answer to everyone who asks you to give the reason for the hope that you have" (I Peter 3:15) is a directive that all believers should be able and willing to engage in some level of apologetics.

Because a likely goal of an apologist is to persuade his audience to his own point of view, he should be aware of the presuppositions of that audience. For example, Peter's sermon in Jerusalem to the Jews in Acts 2 and Paul's sermon to the Athenians in Acts 17 begin from very different perspectives even though repentance and turning to Christ was the central message of each. One choice any modern Christian apologist must make is how much to use the Bible itself as an assumed foundation for Christian beliefs. In Christian theology, one accepts scripture (God's special revelation) and perhaps other religious experiences essentially as axiomatic and argues logically from that basis. On the other hand, in natural theology, one argues from observation of nature (God's general revelation). Both approaches have advantages and disadvantages.

If one accepts as an axiom that the Bible is inerrant, then many arguments are simple. For example, the first verse of the Bible "In the beginning, God created the heavens and the earth" (Genesis 1:1) already proves the existence of God as well as several things about His nature. However, there is still much room for disagreement. For example, Christians have historically struggled (and sometimes literally fought wars) over the scope of ecclesiastical authority. Some Christian leaders have used the text "I will give you the keys of the kingdom of heaven; whatever you bind on earth will be bound in heaven, and whatever you loose on earth will be loosed in heaven" (Matthew 16:19) as proof that they should be obeyed. It is interesting to note here that the controversy is partly due to differing definitions of "keys," "you," "bound," and "loosed." But perhaps the most serious weakness of reasoning solely from scripture is that it will likely be ineffective with an audience that is unfamiliar with or hostile to the Bible.

Arguments from general revelation trade the dependence on an axiom of scriptural inerrancy with dependence on various other presuppositions. Consider the "5th Way" argument of Thomas Aquinas for the existence of God, parts of which were actually used in other ways as far back as Aristotle.

Theorem: God exists.

Proof: Many non-intelligent objects behave predictably. This cannot be by chance, so their behavior must be set. It cannot be set by themselves, so must be set by an intelligent entity, understood to be God. QED

Aquinas' argument has been criticized both by people who agree with its conclusion (such as Immanuel Kant) and those who do not (such as Richard Dawkins). Both groups of critics agree that debatable presuppositions lie behind the statements in the argument. In linguistics, a *presupposition* is an implicit background assumption whose truth is taken for granted when a statement is made. A famous political example is the trap question "Have you stopped beating your spouse?" in which either a yes or no response confirms the implication that such beating has already occurred. Similarly, the statement "he forgot to mail the letter" presupposes that he intended to mail the letter or that it was his duty to mail the letter.

When listening to a logical argument, we all hear the words through the context of our own background. The Aquinas argument is repeated below with some of the implicit assumptions identified in italics as they are used.

[the universe can be rationally understood, so proof is possible]

Many non-intelligent objects *[such objects commonly exist]*

behave predictably. *[our senses and memory are accurate]*

This cannot be by chance, *[randomness cannot produce order]*

so their behavior must be set. *[every action is necessarily caused]*

It cannot be set by themselves, *[non-intelligent objects are passive]*

so must be set by an intelligent entity, *[law of excluded middle]*

understood to be God. *["God" is the only "prime mover" possible]*

The presuppositions that must be accepted in order to believe this argument weaken its persuasive force. This situation is somewhat analogous to the 18th century reluctance by mathematicians to accept non-Euclidean geometries because the required non-Euclidean parallel postulate is so "obviously wrong." Still, the concept of an apologetic is in essence the same construction as that of a mathematical proof.

4.2 Personal Presuppositions

An interesting exercise I have occasionally posed to students in our Math Senior Seminar class at Indiana Wesleyan University is to identify their set of personal "spiritual axioms." Recall that an axiom is a statement accepted without proof, and it is desirable that axiom sets be consistent, independent, and complete. The list below is a combination of student inputs and some of my own when I have tried to do the same thing.

1. "Truth" is a meaningful concept.
2. My senses are reliable.
3. I am capable of logical reasoning.
4. Logical reasoning can improve my life.
5. At least one "god" (a being overwhelmingly more powerful than humans) exists.
6. At least one omnipotent God exists.
7. *Exactly one omnipotent God exists.*
8. God is consistent.
9. God is "good" (holy and benevolent).
10. The original manuscripts of the Bible were inspired by God and contain truth beyond what can be discerned from any other source.
11. *The original manuscripts of the Bible are inerrant words from God.*
12. My copy of the Bible (e.g. NIV) is essentially the same in meaning as that of the original manuscripts.
13. *God is a Trinity – Father, Son, and Holy Spirit.*
14. *God created the physical world.*
15. *Events with no "natural" explanation can happen.*
16. *I have a nonphysical part of my being.*
17. *I have sufficient free will to significantly impact my future.*
18. *No man comes to the Father but by [Christ].*

This activity turns out to be surprisingly difficult for most people, including me. My personal opinion is that I must accept Axioms 1-3 in order to even be able to consider the task, and I must accept Axiom 4 in order to want to do so. Then comes the first real decision. Axioms 1-4 by themselves are highly incomplete and more must be assumed to reason further. Though I believe the nature of the physical universe and of human conscience are strong evidence for the existence of spiritual beings, I also believe naturalism is a partially intellectually defensible position. Therefore, there is still some measure of faith involved, so I choose to accept Axiom 5 as the best explanation for what I observe.

Next I must choose between monotheism and polytheism, both of which are believed by billions of intelligent people, so more faith is required. I accept Axiom 6 because to me a limited god really does not answer the "big questions" one expects from a deity. More than one omnipotent being seems self-contradictory to me, so Axiom 7 seems to be a theorem rather than an axiom to me (and therefore is written in italics). Axioms 1-6 have lead me to one Supreme God. I then immediately want to accept Axioms 8-9 for emotional reasons, as the idea of a petty or vindictive God is depressing.

Now there are several monotheistic religions to choose from. I believe God, being good, wants to communicate with me. I cannot personally see or hear Him, so I must rely on what I see in nature and hear or read from others. Monotheistic religions seem to have similar views on the order of the universe, so nature does not help me with this decision. I do not personally know anyone who claims to authoritatively speak for God (and probably would not trust such claims anyway), so I must rely on writings. Here I believe the Bible has the best case, both in terms of the consistency and appeal of its message as well as in the scholarship involved in its translations and analysis. So I am led to accept Axiom 10 and follow Christianity. I believe it would be deceptive, and thus neither good nor consistent of an omniscient God to inspire an only partially correct scripture, so Axiom 11 again seems to me to be a theorem following from Axioms 1-10. Again, the history of scholarship connected with the Bible gives me confidence in its modern translations, so I accept Axiom 12.

Once I believe the authority of scripture, I both want and am compelled to believe that God created the world and can continue to work miracles as He chooses, to accept my responsibility for choosing my soul's destiny, and to accept Christ as Lord and Savior. So Axioms 13-18 are theorems, as well as many similar statements, are theorems for me, and I have arrived at who I am today. My resulting set of axioms are the ones above in regular (non-italic) type. I believe they are independent but am sure they are not complete. Although I believe this axiom set is consistent, I know I have many other life axioms which taken in totality are inconsistent.

And now the confession. Did I really go through this methodical process when I became a Christian at age eighteen? I did not. As someone who grew up in a Christian home, I never remember not believing in the Christian God. I did not know enough about Islam, for example, to choose between Islam and Christianity. And for better or for worse, there is a good deal of peer pressure from Christian family and friends to fit in to society by "doing the right thing." In the years after I became a Christian, I (sometimes subconsciously) went back and filled in many of the missing links detailed above. Interestingly enough, this "after-the-fact" chronology is very similar to the historical development of some mathematical axioms as explained later in Section 5.3.

5 Application

I next present three types of observations regarding the use of mathematical axioms as a parable for spiritual axioms. First, I summarize what I believe are the essential similarities and differences between mathematical and spiritual axioms (I think the case with the social axioms discussed is similar to that of spiritual axioms). Next, I give my opinion regarding the consistency, independence, and completeness of spiritual axiom sets. Finally, I make some observations on the relative importance of this type of study. Some of these ideas are adapted from [4]

5.1 Mathematical and Spiritual Comparisons

Mathematical and spiritual axioms are similar in several ways. In both contexts, some statements (axioms) must be accepted "by faith," while other concepts (theorems) can be arrived at by logical reasoning. Consistency, independence, and completeness are valuable in both contexts. There is a need for undefined terms in both situations, which are then used to define further concepts. The deductive reasoning used in apologetics is essentially the same system of logic as used in mathematical proof, and the use of intuitive models is helpful in both areas. (Jesus' parables can be considered models of spiritual concepts as can the divine institutions of the family, communion, etc. Probably we all also form personal mental models of concepts such as heaven).

There are also some significant differences between mathematical and spiritual axioms. The most obvious has already been noted – mathematical axioms are Boolean (either accepted or rejected) by definition, while spiritual axioms often seem to be tentatively accepted with varying degrees of certainty. Another difference is the scope of applicability. Mathematical axioms are usually intended to apply only to a hypothetical sub-universe. Incompatible axiomatic systems, such as Euclidean and hyperbolic geometry, can therefore easily coexist in our minds as we understand that in a given situation we should use the system that is most useful at present. Spiritual axioms, on the other hand, often make universal statements about the only physical universe we know. When I claim that "murder is wrong," I do not just mean it is wrong for myself but rather that it is wrong for everyone. The resulting impact is that it is often hard for different religions to coexist without clashing over different foundational beliefs.

As an aside, it is interesting to speculate on the proper role of spiritual axioms in literary fantasy and science fiction where a different physical universe is hypothesized. Is it reasonable to expect Harry Potter or Frodo Baggins to share exactly the same values as someone in our own world? Because the author gets to choose his own physical universe in such fictional writing yet there is usually some attempt to connect with the reader on a believable human level, spiritual axioms in these stories may occupy a role intermediate between that of mathematics and religion.

5.2 Status of Spiritual Axioms

Regarding the consistency, independence, and completeness of spiritual axiom sets:

1. Achieving consistency in one's own spiritual axioms seems unlikely.

The concept of a spiritual axiom seems too complicated and ambiguous to allow for the elimination of self-contradictions. As we live life and learn more, no doubt our internal axioms are changing without our full conscious knowledge. Even at a corporate level, Christianity has learned to live with such ambiguities as that between "For by grace are ye saved through faith...not of works" (Ephesians 2:8-9) and "Ye see then how that by works a man is justified, and not by faith only" (James 2:24). I believe the human mind has been created to be able to handle such cognitive dissonance and that we will all necessarily deal with it throughout our lives.

2. Full independence is likely impossible, but at least reducing the number of axioms seems both useful and feasible.

   It is common practice for organizations to identify and publicize their "core values" (which is a similar concept to that of an axiom), perhaps as part of or associated with a mission statement. Such publications are more likely to be referenced and used if they are short, so minimizing the number of overall items by selecting an independent or near independent set is thus helpful. Similarly, in terms of apologetics and evangelism, it seems more persuasive to ask the audience, especially those familiar with formal deductive reasoning, to accept only a small number of items without justification.

3. Achieving completeness clearly seems permanently impossible.

   It would seem an act of great hubris to claim possession of a complete set of spiritual axioms which could then be used to determine the truth of all statements. The Christian perspective on such a claim could be summarized by " 'For my thoughts are not your thoughts, neither are your ways my ways,' declares the Lord. 'As the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts.')." (Isaiah 55:8-9). Most other religious faiths share the perspective that not all truth is knowable.

5.3 Importance of Spiritual Axioms

Finally, I briefly address the relative importance of this topic as a whole. On the one hand, one's choice of spiritual axioms is critically important. The extent to which I accept an axiom such as "logical reasoning can improve my life" may impact how I approach Bible study. More importantly, my acceptance or rejection of the axiom "writings outside scripture can be authoritative" may determine which denominations within the Christian faith are comfortable for me. And finally if I accept "Jesus is the Son of God" as an axiom then subsequently reject this axiom, nearly all Christians would say I have switched to a different religion entirely. The famous Christian saying "in necessary things unity; in uncertain things freedom; in everything compassion" (whose attribution seems uncertain) notes the differing levels of importance between different tenants of faith, but does not help in deciding which ones are necessary.

On the other hand, explicit identification of axioms may not be as important, either in mathematics or in religion, as we sometimes believe and claim. R.W. Hamming writes "The idea that theorems follow from the postulates does not correspond to simple observation. If the Pythagorean theorem were found to not follow from the postulates, we would again search for a way to alter the postulates until it was true. Euclid's postulates came from the Pythagorean theorem, not the other way." [5]) I agree with this statement as I believe most mathematicians would. Also, though Gödel's Incompleteness Theorems crushed the attempt to axiomatize all of mathematics, yet advances continue to be made in mathematics (in fact, made at unprecedented pace) since proof of these theorems. This idea also calls into question the extent to which we literally return to the foundations of our faith as we make moral choices. It also mirrors my own faith journey as explained in Section 4.2

Another reason to doubt the importance of explicit statements of axioms is related to an observation by Penelope Maddy. "Philosophers gave up the search for [epistemological certainty] arguments in natural science long ago; its retention in the philosophy of mathematics can only be traced to an outmoded vision of the nature of mathematical knowledge. No one would expect even the best scientific arguments to be absolutely justifying. Our epistemological inquiries in mathematics will be hampered if we set an unreasonably high standard." [7] Maddy is not advocating a relaxation in proofs of mathematical propositions, but rather in the methodology of mathematical epistemology. If it is unreasonable to expect certainty arguments in mathematical philosophy, the prospect certainly looks cloudy for systematic proof of broader spiritual propositions.

6 Conclusion: What Says the Parable?

So what can be concluded from using mathematical axioms as a parable for spiritual presuppositions? First, I believe the connection is strong enough that there is significant value in using the parable. Many of the same methodologies and difficulties occur in both fields. Compared to mathematical axioms, most spiritual axioms claim greater scope but have more ambiguity, which is exactly the situation in which a parable can effectively improve understanding.

It is also comforting from a faith perspective that mathematics has progressed despite the difficulties encountered in forming axiomatic bases. Most practicing mathematicians do not worry (indeed they may not even know) that the axiom set for the field in which they are working is not complete, for example. So as Christians we should not be discouraged or intimidated that we have trouble cleanly articulating our own spiritual axioms. Nor should we feel obligated to have an answer to every question posed by skeptics; the skeptics likely do not have answers either.

Finally, it is thought-provoking to consider the fact that in mathematics, the axiomatic foundation is usually constructed after much work in the field is already done. However, the development of axioms then often leads to additional insights. Axioms are important but in practice do not always play their stated role. Similarly, Christianity began and rapidly developed as a faith transmitted orally among people largely untrained in theology. But later development of formal doctrine has helped keep the religion relevant through cultural changes that impact methods of reasoning and communication.

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Cultivating Mathematical Affections through Engagement in Service-Learning

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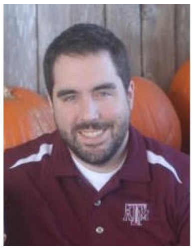

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Abstract

This research explores the impact of service-learning on the affective outcomes of secondary mathematics curricula. This was a qualitative case study of high school students who recently completed a service-learning project in their mathematics course. Data was gathered from student interviews, reflection journals, and field observations. The framework for the analysis follows the definition of "productive disposition" offered by the National Research Council (2001). The major themes that emerge from the data indicate that through service-learning students see math as sensible, useful, and worthwhile. This supports the potential of service-learning as a pedagogical tool that can be utilized to develop a productive disposition in students; addressing at a practical level how the affective objectives of national policy documents can be achieved.

"When am I ever going to use this?" is probably the most common statement uttered in a mathematics class. Please notice that I referred to this as a *statement* and not as a *question*. It is a statement of frustration. It is the culmination of confusion and stress and typically serves as an exclamation by the student of their withdrawal from the mental activity at hand. The real issue being raised by students is not one of application, but rather one of *values* . I have found that the best response to such a statement/question is to first translate it into what I believe the student truly meant to express, turning "When am I ever going to use this?" into "Why should I *value* this?" In his *Confessions*, St. Augustine states "You have made us for yourself, O Lord, and our heart is restless until it rests in you." As Christians we of all people should realize that statement does not stop being true when students walk into math class. Even (or perhaps especially) in math class, students desire to find value (through content/application) and to be valued (through pedagogy). While the utility of mathematical concepts is certainly important, we as mathematics educators, and especially as Christian educators, need to utilize the mathematics classroom to address the more fundamental issue of fostering a proper sense of values.

Affective/value language permeates national published standards on the teaching of mathematics as an ideal we should strive to inculcate into students but there is little discussion on *how* to go about doing this. The NCTM Standards for Teaching Mathematics (1991) states that "Being mathematically literate includes having an appreciation of the value and beauty of mathematics as well as being able and inclined to appraise and use quantitative information." Mathematical literacy involves a proper valuation of the discipline of mathematics. *Adding it Up: Helping Children Learn Mathematics*, a report published by the National Research Council (2001), the basis for the modern Common Core State Standards Initiative (2016), argues that mathematical proficiency has five strands, one of which is termed "productive disposition." Productive disposition is defined as "the habitual inclination to see mathematics as sensible, useful, and worthwhile" (p. 116). To be mathematically proficient the valuation of mathematics must lead to a habit of seeing mathematics as worthwhile. This definition of productive disposition is a clear example of an affective objective for students of mathematics, yet it contains no supporting information on how to practically reach the objective.

The purpose of this case study was to analyze the role of service-learning in the cultivation of a productive disposition for students in a statistics class as they participated in a service-learning project addressing chronic homelessness. Service-learning was examined as a potential pedagogical tool that can be utilized to develop a habitual inclination to see mathematics as worthwhile. The issue at hand is whether service-learning offers a vehicle for *how* to go about instilling the values that the math education desires to see in its students. The study sought to answer the following two research questions: (1) To what extent does service-learning impact the cultivation of a productive disposition among students? and (2) What is the alignment between the affective objectives of national policy documents on the aims of mathematics education and the affective outcomes on students participating in a service-learning project?

1 Framework

The theoretical framework of this study builds primarily off of the work of philosopher James K.A. Smith (2009). Smith describes education as not primarily a heady project concerned with providing information; rather, education is most fundamentally a matter of formation, a task of shaping and creating a certain kind of people (Smith 2009, pp. 26-27). Smith explains further that an education is a constellation of practices, rituals, and routines that inculcates a particular vision of what is good by inscribing or infusing that vision into the metaphorical heart by means of material, embodied practices. For Smith, there is no neutral, non-formative education.

The importance of affect and its positive development through habits, practices, and routines ties in directly to the development of positive disposition in classroom settings. Gresalfi and Cobb (2006) define learning as a process of developing dispositions - ways of being in the world that involve ideas about, perspectives on, and engagement with information that can be seen both in moments of interaction and in more enduring patterns over time. Thomas and Brown (2007) note that dispositions involve "attitudes or comportment toward the world" and are "generated through a set of practices" (p. 8). In mathematics education, specifically, it has been argued that the modification of student belief structures comes not through addressing content but through sufficiently rich educational practices (Goldin, 2002). McCloskey (2014) proposes ritual analysis as a lens for viewing the math classroom as a series of embodied practices that rise above a purely rational enterprise. The specific practice, or ritual, of guided reflective activities has been demonstrated to increase student appreciation of a given subject (Hulleman et al., 2010).

Hadlock (2005) stresses the importance of regular (habitual) practices of reflection throughout service-learning activities. Service-learning in its most effective and well-developed sense involves a multilayered reflection process that asks the learner to become more aware of what he/she brings to the learning process: values, assumptions, biases - many of which are unexamined and potentially problematic (Zlotowski 2005, p. ix). The reflective process is vital for students to gain the most from a service-learning experience (Webster & Vinsonhaler, 2005, p. 257). Combining Smith's view on rituals with the emphasis on the ability of reflective processes to impact student value systems in the classroom as proposed by Hulleman (2010), it becomes clear how service-learning might serve to best impact the affective learning of students in the mathematics classroom. Service-learning will be studied as a viable means to cultivate mathematical affections of students by providing a habitual practice of reflection in an educational context where the aim of the project is not primarily the increase of student cognition. As will be described below, all students in the course participated in a weekly reflection journal component of a service-learning project throughout the course of the entire year.

The conceptual framework of this study centered on the description of "productive disposition" offered by the National Research Council. The collected data was analyzed as to how it gives evidence of students seeing mathematics as sensible, recognizing the usefulness of mathematics, and understanding mathematics as a worthwhile task to be performed.

2 Methodology

This study was conducted as a qualitative case study, focusing on a select group of five students in a statistics class as they engaged in a service-learning project. As a course project, all students participated in a group which provided the following four service components: meeting with a non-profit agency and developing a survey instrument, conducting the survey, compiling data and performing statistical inference procedures, and presenting results. All students completed a shortened version of the Fennema-Sherman Math Attitudes Scale (FSMAS) (Fennema & Sherman, 1976; Mulhern & Rae, 1998) prior to the assignment of the service-learning project. From the responses to the FSMAS, a small group of five students was identified to be the members of the case study. The following variables were considered when selecting the group of five students: gender, grade level, section of course, achievement level in the course, FSMAS scores, and whether or not I had taught the student previously in a math course that involved a service-learning project. The intention was for students to be selected in a way that that makes the case study group representative of the classes as a whole.

The experience of these students was documented through observations, primarily of students as they interacted with one another in their group and the ways in which they interacted with the partner service organization. Student interviews and collected artifacts, such as weekly reflection journals employed throughout the project, were also documented. All students in the course participated in the reflection journal component of the project but only the responses of the students in the case study were analyzed thoroughly. The observational framework was based on the work of Schorr and Goldin (2008) in researching student affect in a math classroom - it focused on the visible student cues that could lead one to infer the affective engagement of the student. The interview protocol was shaped based upon prior student interviews over a service-learning project from a pilot study. Finally, the artifacts that were collected were designed around successful examples of reflection guides as presented by Hadlock (2005) and other appropriate research on service-learning in a mathematics context.

The partner organization (henceforth referenced as "PO") that students worked with on their service-learning project is a homeless outreach program in a central Texas city. PO operates under a philosophy that homelessness is more than house-less-ness, rather it is a severe break in community from others. PO purchased land just outside of the city on which they developed a community of affordable housing for the chronically homeless. This property also has amenities such as a gardening center, small livestock animals, a health clinic, a carpentry workshop, and a meeting space for continuing education and other such classes. Everything about the property is designed to foster a sense of complete community.

The founder and president of PO was interested in joining with the students in the statistics course to complete a study based largely on Bruce K. Alexander's "Rat Park" experiment (as referenced in Hari, 2015, p. 170ff). Seminal studies that had proven the addictiveness of drugs such as heroin had done so through administering the drug to rats in cages in isolation. Alexander set up a study in which the rats were allowed to operate in community and found that the amount of drugs consumed went down drastically, indicating that environment and community (or lack thereof) can play a significant role in drug use. PO was interested in having students survey the residents of their new community development on issues related to their life on the streets (physical, psychological, and spiritual) prior to moving to the new community and how those issues may have changed since moving to the community. With this basic premise, the students were tasked with developing the complete survey, methodology, and appropriate analysis as part of the service-learning project.

3 Results

Below, Table 1 summarizes the research questions of this study and the manner in which data was collected and analyzed to assess the outcomes of this project.

3.1 Quantitative Results

At the end of the year, all students (N = 39) in the course completed a community based service survey in which they responded to statements about the project on a Likert scale. The responses from the statements related to seeing math as sensible, useful, and worthwhile were combined to give every student a "productive disposition" score on a scale of 3 - 15, with 3 meaning the student responded 1 (strongly disagree) to all three topics and 15 meaning the student responded 5 (strongly agree) to all three topics. The results are shown in figure 1 below:

The lack of responses for 3-6 are left for emphasis to clearly show that no students responded near the bottom of the productive disposition scale. Because the first score is a 7 that means no student responded below 3 for all three areas of productive disposition, so then no student responded negatively to all three areas. Scores starting at 10 indicate that those students had to have included a 4 response at minimum for at least one area of productive disposition. A t-test was run against a null hypothesis of $\mu = 10$, the t value was 2.32 and the p-value is 0.01297, indicating significant evidence that the true mean disposition is greater than 10.

3.2 Qualitative Results on Productive Disposition

Research Question 1

Service-learning appeared to have a positive effect on students' valuation of mathematics. At an individual level it is worth noting that every student in the case study made an explicit reference to the service-learning project as indicating some change or impact that occurred for them at an affective level. Table 2 summarizes the levels of Krathwohl's affective domain (Krathwohl, 1964) and the terms/concepts that

Research Question	Variable	Indicators	Measurement
To what extent does service-learning impact the cultivation of a productive disposition among students?	Development of productive disposition tied directly to involvement in community experience	• Role of community experience in learning • Role of community experience in engagement • Changes in perspective on course content	Interviews Surveys Reflection Journals Observations
What is the alignment between the affective objectives of national policy documents on the aims of mathematics education and the affective outcomes on students participating in a service-learning project?	Understanding course content (Sensible)	• Role of community experience in understanding course content • Perceived relevance of community experience to course content	Interviews Surveys Reflection Journals Observed interactions with students and instructor
	Applying course content (Useful)	• Role of community experience in applying course content • Recognition of practical application of course content	Interviews Surveys Reflection Journals Observed interactions with community partner
	Valuing course content (Worthwhile)	• Role of community experience in producing a rewarding sense of work committed to course content • Recognition of community experience to sufficiently important to justify effort spent	Interviews Surveys Reflection Journals

Table 1: Questions and Data Collection

were used by students that were coded to correspond to each level.

One example of student responses and the corresponding association with advancement in Krathwohl's taxonomy comes from Tabitha (pseudonym). When Tabitha was asked if she would recommend doing the service-learning project for other classes:

> Tabitha: I would recommend that they do so because it's a really cool concept.... I think learning to practically apply what your learning in the classroom is important.

Elsewhere in her journal, when asked at the end of the year to reflect back on the experience of the project:

> Tabitha: As the semester has progressed I have slowly understood more and more about what we are doing in this project. Honestly, at the beginning I really didn't like it, but I think the group work and the articles helped get us invested in the project. It is also fun to

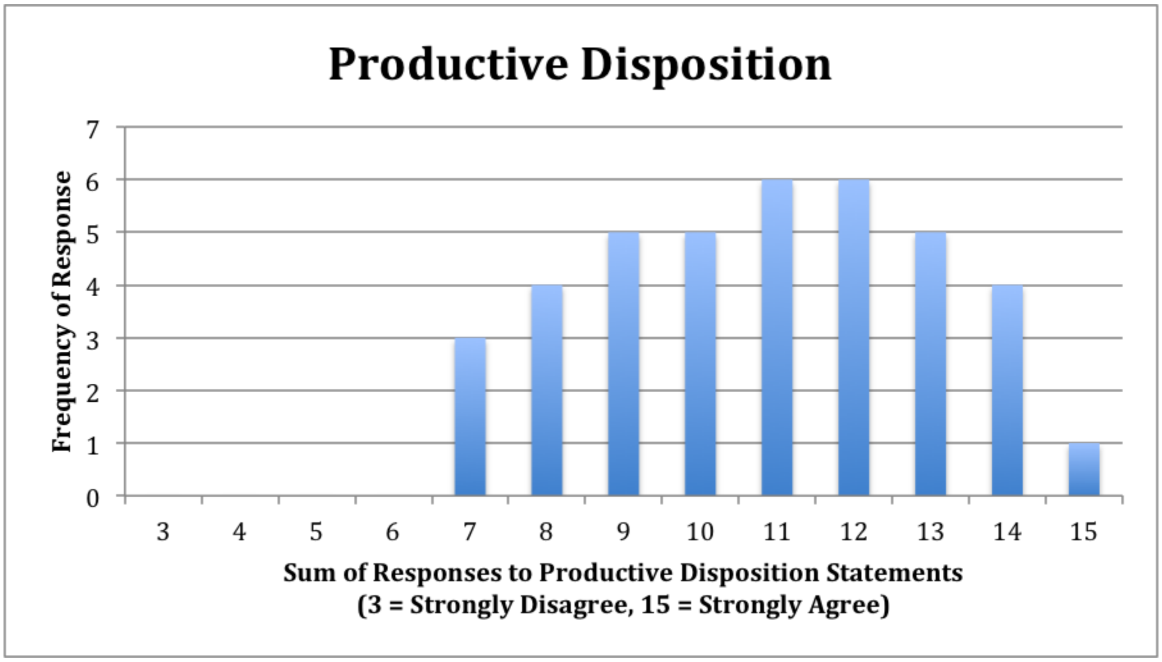

Figure 1: Community service survey responses related to disposition

> mix school work with community work. And even though I struggle with statistics I think it is helpful to see the practical use of math in real life situations, even if they are some what [sic] simulated. One of my favorite parts of the project has been reading the different articles specifically the news articles. It's exciting to see how what we are doing applies to current events.

Words Tabitha uses in reference to her experience on the project: cool, important, invested, fun, exciting. "Cool," from the teenage vernacular, in this context referencing a "cool concept," can probably be best understood as meaning " admirable." By definition, to be "admirable" is to inspire approval, reverence, or affection. "Cool," "fun," and "exciting" are all verbal indicators of Tabitha being at the *receiving* stage of Krathwohl's taxonomy, in that Tabitha is demonstrating a willingness to attend to a particular stimuli, in this case the service-learning project. By Tabitha indicating that she (and her classmates) are "invested" in the project she is indicating that she is at the *responding* stage, communicating an active participation on her part. And finally, Tabitha's reference to the project as "important" indicates the worth that Tabitha has attached to the service-learning project, thereby reaching the *valuing* stage.

Another student, Charlotte, indicated a significant change in her perspective of mathematics through the course of the project and she attributed that change directly to the service-learning project:

> Charlotte: I think that I do more math, like this year I've done more math, and my outlook has changed on that, just because the service-learning project has been more engaging and more exciting than sitting down and taking notes, and I really wanted to use what I've learned throughout the year and actually apply it, so I think it was more engaging and more fun.

When asked at the end of the year it she thought it was reasonable to say that her attitude towards

		Krathwohl's Affective Domain	Summary of domain category	Associated Verbs for Student Learning Objectives	Terms, concepts, descriptors used by students and coded:
Virtues Practiced by Students	Behaviors from simple to complex	Characterizing	individual has a value system that has controlled his or her behavior for a sufficiently long time for him or her to develop a characteristic "life-style" - thus the behavior is pervasive, consistent, and predictable	Revise, require, rate, avoid, resist, manage, resolve	Comments on change of perspective on mathematics in a service context
		Organizing	bringing together different values, resolving conflicts between them, and beginning the building of an internally consistent value system	Discuss, theorize, formulate, balance, prioritize	Comments on conflicts with prior negative experiences
Values Instilled in Students		Valuing	the worth or value a student attaches to a particular object, phenomenon, or behavior	Measure proficiency, subsidize, support, debate	Important, good, having meaning, fulfilling
		Responding	active participation on the part of the student	Comply, follow, commend, volunteer, acclaim, engage in	Invested, rewarding
		Receiving	student's willingness to attend to particular phenomena of stimuli	Differentiate, accept, listen for, respond to	Cool, fun, exciting, admirable

Table 2: Responses and Advancement

mathematics had become more positive, Charlotte noted:

> Charlotte: Yes, I do feel like my attitude in math has become more positive because ... I really think it's because of the service-learning project and because ... Math is easier to understand when it's used outside of the classroom, and it's more relatable to me when I'm using it in real life situations, so I think just this realization that I can use math in every day activities helps me see it more positively because then I realize it's more useful, and it actually does matter.

Charlotte references the project as "exciting" and "engaging" and as something that "actually does matter." This gives a clear indication of Charlotte reaching the *valuing* stage. In both comments above Charlotte notes the change that has occurred in her perspective from the beginning of the year until the end, indicating that she has also reached the *organizing* stage where she is resolving conflicts between values, and beginning the building of an internally consistent value system. Charlotte also recalls the lasting impact of her prior experience of being involved in service-learning during her freshmen year:

> Charlotte: Before, my answers are kind of the same, but thinking math is a waste of time, because before, once again, I just thought that math didn't really apply to anything besides

> math classes, because I never thought I'd be using sine and cosine in the real world or any of geometry in the real world, but now, through our service project and through the geometry project of creating a little tent thing for the RVs [students designed an awning for an RV park being used as affordable housing], I realized that math can be used in more ways than I thought. They can be used in the real world and not just in a classroom.

Charlotte is offering increased evidence of the long-lasting impact that the change of routines/experiences/liturgies of the mathematics classroom can have on a student. Charlotte is referencing changes in her perspective from three years ago indicating that perhaps she is entering the *characterizing* stage to some extent.

Overall, the five students in the case study seemed to regularly reach the *organizing* stage of Krathwohl's affective domain. The students also regularly attributed the method by which they reached the *organizing* stage to a change in the routine of the mathematics class - from the expected lecture-practice-assess cycle to breaking for periods of application and reflection. This study then seems to give clear evidence of how service-learning engages students at a deeply affective level and provides a venue for students to wrestle with their valuation of mathematics.

Research Question 2

For the five students in case study the interviews, field observations, and collected reflection journals were coded following the three major themes of a productive disposition: seeing math as sensible, useful, and worthwhile. These codes initially derived from the definition of a productive disposition offered by the National Research Council (2001), followed in the vein of Jansen (2012), and were confirmed as these themes emerged through an open coding of the pilot study interviews. It is important to note that the purpose of this study is not to argue for what entails a productive disposition and the best way to define and analyze it. Rather, the focus of this study is to examine if service-learning can cultivate a productive disposition as it is currently defined by the National Research Council (2001). That is why the concepts of sensible, useful, and worthwhile were used in seeking to determine if students were developing a productive disposition through the course of the service-learning project. While these terms have quite a bit of overlap in their usage (students tend to see one as encompassing or necessarily following from the other) for the purposes of coding and analyzing the student interviews and reflection journals an attempt was made to treat these terms as distinctly as possible.

Each student in the case study was able to articulate, in his or her own unique way, their understanding of mathematics as sensible, useful, and worthwhile. Through the project each student was able to articulate the sensibleness of the course material, how the math he/she was learning was useful in his/her immediate context, and the rewarding nature of the work and effort the student contributed.

The working definition of being sensible is "to be reasonable or comprehensible, rational." The main idea for this term is that it implies mathematics is understandable, that the service-learning project has in some way aided the student in making intellectual sense of the mathematics involved. In the case study group, each student came into the class and the project with very different views on how much sense mathematics made to them. While students might make assent to math being logical and rational, when that abstract statement was made more personal to their own experience they tended to express frustrations or confusions with the mathematics they see in school. However, through the project each student was able to articulate the sensibleness of the course material. The project served to solidify each students' understanding of the course and their ability to make sense of the mathematics involved.

The working definition of being useful is “being of practical use, serving some purpose.” The main idea for this term is that it implies mathematics has a purpose, that the service-learning project has in some way aided the student in seeing the practical applications of mathematics. While all students in the case study expressed their understandings differently, all of them indicated that the service-learning project improved their perspective on the usefulness of mathematics. A typical response at the beginning of the year was to offer an assent to math being useful, but for somebody else in some different job, in some distant future. After the project, every student was able to articulate how the math they were learning was useful in their immediate context. The usefulness of mathematics became a more personal experience.

The working definition of being worthwhile is “being rewarding, valuable to justify time or effort spent.” The main idea for this term is that it implies mathematics is worth putting time and energy into learning; mathematics offers something valuable and rewarding for everyone. In the context of this study this means that the students indicate that the mathematics involved in the service-learning project was an important task to undertake, has beneficial outcomes, and was worth the effort that was committed. In determining if the students in the case study found mathematics to be worthwhile, one of the hardest distinctions to make was if they were expressing the worthwhileness of the experience in terms of the mathematics involved or purely in terms of the service. In other words, could students find value and worth in the service but still not see the mathematics as worthwhile? Ideally the answer to this question should be ’no’ for any well-designed service-learning project. A well-designed service-learning project necessarily involves integrating the content of the course into the service being performed. So then, if a student says that they found the service valuable, that service involved performing mathematics. In this study, while some students were initially drawn to the service-learning project because they found the concept of service in general as worthwhile, by the end of the project each student was able to articulate that the mathematics involved in the project was worth the effort spent to learn and apply it.

Tabitha came into the project having a positive attitude towards mathematics but through the project she realized that the positive attitude was misplaced as she had an incorrect understanding of mathematics. Tabitha expressed discomfort in transitioning from seeing mathematics as formulaic classroom learning to creative, real-life application, but she also expressed a recognition that this change was for the best. So while Tabitha’s FSMAS scores dropped at the end of the year, in reality she developed a disposition towards mathematics that was more productive. Ava came into the year classifying herself as not a ” math person” but gave intellectual assent to the notion that mathematics is an important field to study because the value of being a well-rounded educated person had been instilled in her. By the end of the service-learning project she was able to articulate the worth and value of a math education in much more personal and immediate terms; mathematics was no longer something abstractly beneficial, but practically beneficial to her. Ava never fully left behind the notion of not being a “math person” but her disposition towards mathematics certainly became more productive over the course of the year. John came into the year with a high FSMAS scores and a high aptitude for mathematics, even expressing his intention to major in math in college. The engaging nature of the service-learning project pressed John to deepen his understanding of how mathematics can applied and brought him to see that mathematics is not just about applications in science and engineering, but also in service contexts. While John’s disposition could have already been summarized as productive coming into the project, that disposition was arguably strengthened through the project.

What follows is an extended analysis of one student, Charlotte, to indicate how student responses were coded and analyzed to see growth in productive disposition. Charlotte began the year with very low FSMAS scores and, like Ava, described herself as not a “math person.” Through the project Charlotte became one of the most vocal students in the case study on how the service-learning project influenced her change to a more positive view of mathematics. In her initial interview and initial journal entry, Charlotte indicated not seeing mathematics as a worthwhile endeavor. She indicated very negative feelings towards the subject with indications that mathematics was not worth the time and energy that she committed to it.

> Charlotte: [Math] makes me feel uncomfortable, restless, irritable, impatient because unlike literature or history you can just write out your thoughts or whatever. Math is like, certain numbers and certain things. There's a right or wrong answer. If you don't get the right answer then it's like, oh, you get everything wrong basically .... Math problems seem to be more confusing to me because there's intricate little steps that you have to do. It makes me feel restless because the math problems that we're doing are way more complicated than 2 plus 2. You have to go through all of the things and work a long time on the problems. It takes time. I feel like you have to have like, a mathematical brain to understand a lot of math things. Taking mathematics is a waste of time to a certain extent. Learning sine and cosine will probably never come in handy in my life.

For Charlotte it is important to note that these negative sentiments, seeing mathematics as not being worth the effort, were only expressed at the beginning of the year. As she engaged in the project her expressions became much more positive about mathematics. Despite feeling uncomfortable with mathematics Charlotte still expressed excitement (which is really just having a willingness to put forward energy towards something) when asked about the service-learning aspect of the project:

> Charlotte: I like how this project we're able to actually make a difference and serve someone instead of doing it for our own benefit. I was excited.

In her journal entries as the year progressed, Charlotte indicated confidence in finding the service-learning project rewarding.

> Charlotte: I did not expect this to be what we were doing for the project. For some reason I thought it was going to do with estimating numbers or something. I am glad that we are doing a survey now.... I am so excited to go out and give the survey.... I'm excited to go out there and give a real survey to the people. I know it will be rewarding when the whole project is done. I'm dreading writing all the papers but I know it will be useful in the long run.

Despite the mathematical work involved in the project, the type of work that Charlotte expressed made her feel uncomfortable and restless, by placing that mathematical work in the context of a service-learning project Charlotte is now able to see the work as being worthwhile. When asked at different points during the year why she should value her math education, Charlotte's responses can clearly be seen to be evolving; to seeing mathematics as valuable to her life currently (specifically in the context of service) and not just valuable at some undetermined future time:

> Charlotte: (Beginning of year) I should value by math education because it will be important to me later in life. I may not see a use for it now but I know the basic understanding of math will help me solve problems later on.

> (Later in the year) I think we should value everything about our math education. I think it can be extremely helpful in everyday uses. I think my opinion has changed because I've grown to see math in a new and different way. There are several different types of math which can be useful in different areas of our lives. Although I'm not super interested in math I think we should all value it to some degree.
>
> (End of year) I still think math is very valuable even more so after the project is over. I also think ministry is a way for us to serve the community with gifts and talents. Now, I know I can serve the community with mathematics! That is something I would have never thought of until this year.... I have had a more positive attitude on math thanks to our awesome survey project!

By the end of the year Charlotte was expressing her work in mathematics not in terms of it bringing discomfort and uneasiness but rather as enjoyable, valuable, and worthwhile.

> Charlotte: This is a lot of [hypothesis] tests to run and a lot of data to report. It has been cool seeing our project come all the way through. I have enjoyed each and every step.... Yes I would recommend doing it for other classes! It was neat to see how the things we learned in class played out in the real world. It was a lot of work, but it was all worth it in the end.

In her final interview, Charlotte is able to look back over the year and speak of all the work put into the project as "paying off" - that is, being worth the time and energy put into it.

> Charlotte: The most exciting part of it was seeing that all of our hard work had paid off and that all of the notes that we had taken in class and everything that we learned this year was able to be applied to something other than math, in a math class.

The final student, Mason, began the year with one of the lowest FSMAS scores that was recorded from the entire class. While harboring a very negative attitude towards mathematics, the prospect of being involved in a service project was extremely appealing to Mason as he greatly valued forming relationships with others. While only the relational side grabbed his interest at the beginning, by the end of the project Mason was articulating how the experience had begun to change his views of mathematics and he reported one of the largest increases in FSMAS scores by the end of the year. When asked to comment on if he feels the FSMAS survey was correctly relaying that his attitude towards mathematics had become more positive:

> Mason: Yeah, definitely, much more positive. It was hard, don't get me wrong and I'm not saying I'm no good at math thing didn't change, but I do think ... I should've changed my ... I am sure that I can learn it, because I am sure I can learn it. It just will take longer and when you don't feel so completely discouraged about it ... When you do feel that you do have shot to understand it and learn it, for me at least it really raises my attitude towards it. It doesn't feel like it's this hopeless thing that I just have to suffer through. It is kind-of just a hill you climb, right?

This statement by Mason embodies the ideal of a productive disposition: while the student doesn't expect to perform math perfectly or always enjoy it, math is no longer seen as hopeless and discouraging, but something the student is capable of doing and succeeding at.

4 Conclusions

The data gathered from this study clearly indicates that the practice of service-learning in a mathematics course led to engaging students at a deep affective level, with every student demonstrating wrestling through Krathwohl's advanced stage of organizing; bringing together different values, resolving conflicts between them, and beginning the building of an internally consistent value system. Not only were students engaged at a deeply affective level, they were engaged in a positive way that led to a more productive disposition; of seeing mathematics as sensible, useful, and worthwhile (National Research Council, 2001, p. 116). Students were clearly demonstrating the building of an internal value system in a positive way about mathematics, thereby cultivating mathematical affections.

To be clear, service-learning is not being advocated as a "fix-all" pedagogical approach. Students (particularly Tabitha, Ava, and Mason) still harbored negative feelings towards mathematics; feelings that were deep-seated and had been formed over the course of years during their schooling. While some students (particularly those who had previously been involved in service-learning in mathematics) indicated that they may be on the threshold of Krathwohl's most advanced domain of characterizing (individual has a value system that has controlled his or her behavior for a sufficiently long time for him or her to develop a characteristic "life-style" - thus the behavior is pervasive, consistent, and predictable), the reality is that this could not be measured over the course of a single school year. Much work would need to be done to unseat negative characterizations students have about mathematics.

While service-learning doesn't complete this task fully, this study has demonstrated that it does make substantial progress. Students deepened their understanding of the ways in which mathematics can be applied, seeing it as useful in their immediate context rather than as some potential skill in the future, and seeing it as useful for the service of others rather than for the student's own advancement. The regular liturgies of the classroom that were instituted in order to emphasize these points, such as interacting with the partner service organization, outside speakers, readings, and reflections, were mentioned by every student as having contributed to their growth in a productive disposition. Having success in mathematics is no longer hopeless. It will take work, but it is no longer hopeless. This is the essence of cultivating mathematical affections.

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Appendix 1: Conference Schedule and Abstracts

21st Biennial ACMS Conference

Charleston Southern University
Charleston, SC

May 31 - June 3, 2017

Tuesday, May 30

6:00-7:00 pm	Dinner for early arrivals

Wednesday, May 31

7:00-8:00 am	Breakfast
8:30 am - 12:00 pm	Pre-Conference Workshops I
12:00 - 1:00 pm	Lunch
1:00 - 4:30 pm	Pre-Conference Workshops II
5:30 - 6:30 pm	Dinner
7:00 - 7:15 pm	Welcome
7:15 - 8:15 pm	*Shaping a Digital World: Connecting Bytes and Beliefs* Derek Schuurman
8:30 - 9:30 pm	Reception

Thursday, June 1

7:15 - 7:45 am	Morning Prayer
7:45 - 8:45 am	Breakfast (board meeting in a side room)
9:00 - 9:30 am	Devotions and announcements
9:30 am - 2:45 pm	Excursion to Downtown Charleston (including box lunch)
3:00-4:00 pm	*The Mathematics of Faith: Euler's anonymous work on the limits of mathematics, science, and faith* Dominic Klyve
4:00-6:00 pm	Parallel Sessions I
6:15-7:15 pm	Dinner
7:30 - 8:30 pm	Parallel Sessions II
8:30 -9:15 pm	Discussion of the role of CS/IS in ACMS

*Note: All meals (except Friday's banquet) will be in the Cafeteria. The locations for parallel sessions can be found on the parallel session schedule. All other activities will be held in WCCL room 102-3.

Friday, June 2

7:15 - 7:45 am	Morning Prayer
7:45 - 8:45 am	Breakfast
9:00 - 9:30 am	Welcome and Devotional
9:30 - 10:30 am	*De Morgan's Budget of Paradoxes*
	Sloan Despeaux
10:30 - 11:00 am	Break
11:00 - 12:00 pm	*Responsible Automation: Faith and Work in an Age of Intelligent Machines*
	Derek Schuurman
11:45 - 12:45 pm	Lunch (board meeting in a side room)
1:00-3:00	Parallel Sessions III
3:00-3:30 pm	Break
3:30-6:00 pm	Parallel Sessions IV
6:15 - 7:30 pm	Banquet
7:30 - 8:30 pm	*Translation and Betrayal: "Euler's Letters to a German Princess"*
	Dominic Klyve
8:30 - 9:15 pm	Discussion of the Role of Statistics in ACMS

Saturday, June 3

7:45 - 8:45 am	Breakfast
9:00 - 10:00 am	*Fit to print? Referee's Reports of Mathematics in Nineteenth-Century London*
	Sloan Despeaux
10:30 - 11:00 am	ACMS business meeting
11:00 - 12:00 pm	Worship Service

Schedule of Parallel Sessions

Parallel Sessions I - Thursday, June 1

	WCCL 201	WCCL 115	WCCL 116
4:00-4:15	*Models, Values, and Disasters* Michael Veatch	*Some Silly Analogies and Mnemonic Devices for Upper-Level Math Topics* Aaron Allen	*Gaudi's Geometry of Nature* Donna Pierce
4:20-4:35	*P-values Considered Harmful* Michael Stob	*Warm-up Problems in Linear Algebra* Kristin A. Camenga	*Portuguese Mathematical History* Maria Zack
4:40-4:55	*The Daily Question: Building Student Trust and Interest in Undergraduate Intro to Probability and Statistics Courses* Matthew A. Hawks	*Inquiry-Oriented Instruction in an Abstract Algebra Course* Jill Jordan	*Algebra without Signed Numbers: Mr. Frend's Universal Arithmetic* Richard Stout
5:00-5:15	*Mentoring as a Statistical Educator Within the Context of a Christian College* L. Marlin Eby	*Reading Journals: Promoting Student Engagement and Success in Undergraduate Mathematics Courses* Sarah A. Nelson	*The Resolved and Unresolved Conjectures of R.D. Carmichael* Brian D. Beasley
5:20-5:35	*Future Medical Professionals and the Statistics Classroom* Paul Lewis	*Revitalizing Complex Analysis* Russell W. Howell	*Ten Mathematicians Who Recognized God in Their Work (part 2)* Dale McIntyre
5:40-5:55	*Conceptual Understanding of Inference via Simulation* Justin Grieves	*The Complex Moduli Project and Mathematica-Based Modules in Complex Analysis* Bill Kinney	*Mathematics and an Epistemology of Love* Chris Micklewright

Parallel Sessions II - Thursday, June 1

	WCCL 201	WCCL 115	WCCL 116
7:30-7:45	*Hybrid Courses Across the Curriculum: What Works and What Doesn't* Catherine Crockett	*Influence on Students' Dispositions Toward Mathematics* Patrick Eggleton	*The Topology of Harry Potter: Exploring Higher Dimensions in Young Adult Fantasy* Alexa Schut, Sarah Klanderman, Dave Klanderman, William Boerman-Cornell
7:50-8:05	*Teaching Math and Computer Science Using Graphic Novels* Eric Gossett	*Poster Projects in Math for Liberal Arts* Brandon Bate	*Book Review: Redeeming Mathematics - A God-centered Approach by Vern Poythress* Kevin Vander Meulen
8:10-8:25	*Including a Writing Project in a Service Learning Course for Mathematics and Computer Science Students is a Win for Both Students and Professor* Lori Carter	*Finding Meaning in Calculus (and Life)* Doug Phillippy	*God: One, Part Two* Daniel Kiteck

Parallel Sessions III - Friday, June 2

	WCCL 201	WCCL 115	WCCL 116
1:00-1:15	*Using Machine Learning and Data Mining to Analyze Retention Rates at Bethel University* Deborah Thomas	*Precalculus and Calculus Students' Understanding of the Concept of Function* Lauren Sager	*An Overview of Specifications Grading* Mike Janssen
1:20-1:35	*An Analysis of SNU Chapel Attendance Data* Nicholas Zoller	*A Pre-Calculus Controversy: Infinitesimals and why they really matter* Karl-Dieter Crisman	*A "Big Ideas" Reflection Assignment Inspires Students to Make Valuable Connections* Jeremy Case and Mark Colgan
1:40-1:55	*Mathematics and Statistics Service Learning: Beyond the Project* Alana Unfried	*Using Inertial Navigation to Demonstrate Basic Calculus Concepts* Ron DeLap	*Mathematics/STEM Study Abroad across Europe* Nicholas J. Willis
2:00-2:15	*The Heart of Mathematics Through the Eyes of Faith* Matt D. Lunsford	*Why You Should Move Your Infinitesimals to the Top Drawer* Troy Riggs	*Axioms: Mathematical and Spiritual* Melvin Royer
2:20-2:35	*Classical Mathematics: An Attempt to Integrate Mathematics and Christian Worldview in a General Education Mathematics Course* Jamie K. Fugitt	*Variations on the Calculus Sequence* Chris Micklewright	*Integrating Faith and Discipline - Beyond the Classroom* Kevin Hopkins
2:40-2:55	*Teaching by Not Teaching: The Power of Collaboration* Audrey DeVries	*A Pre-Lab Style Approach to Calculus III* Michael Martinez	*Living at Work: My First Year as a Faculty-in-Residence* Rachel Grotheer

Parallel Sessions IV - Friday, June 2

	WCCL 201	WCCL 115	WCCL 116
3:30-3:45	*Logical Axioms and Computational Complexity: A Correspondence* Danny Rorabaugh	*Creating a Natural Progression Through a College Algebra Course* Denise K. Dawson	*The Role of Informal Learning in Supporting Mathematics Teacher Education* Alice E. Petillo
3:50-4:05	*Techniques for Integrating Faith Into the Computer Science Classroom* CSU computer science department	*The Corset Theorem* Owen Byer	*Math Teacher Circles: How and Why to Start Yours* Amanda Harsy, Tom Clark, Dave Klanderman
4:10-4:25	*Using Real-World Team Projects: A Pedagogical Framework* Michael Leih	*The Set of Zero Divisiors of Factor Rings* Jesús Jiménez	*Math Teacher Circle Problems: This Year's Brightest and Best* Mike Janssen, Mandi Maxwell, Sharon Robbert
4:30-4:45	*ScratchFoot: A Tool for Transitioning Students From Scratch to Greenfoot* Victor Norman	*Bicycle Routes and Euler Double-paths* C. Ray Rosentrater	*A Framework for Integrating Faith and Learning in the K-12 Mathematics Classroom* Ryan and Valorie Zonnefeld
4:50-5:05	*Teaching Introductory Computer Programming with "Processing"* Jeff Nyhoff	*Tightened Relaxations of the Traveling Salesman Problem* Audrey DeVries	*Cultivating Mathematical Affections through Engagement in Service-Learning* Joshua Wilkerson
5:10-5:25	*A Practical Mechanism to Perform Secure Computation* Benjamin Mood	*Sculpting Seifert Surfaces: Shepherding a Student Through a Mathematical Art Project* Lisa Hernández	*Revamping the Path Through High School Mathematics - The Impossible Dream?* Clayton R. Hall II
5:30-5:45	*Brokering Trust in Knowledge Management with the use of Data Visualizations* Kari Sandouka	*Manifold Methods for Averaging Subspaces* Justin Marks	*On Beyond Calculus: A Day for Community Outreach* Rebekah Yates

Abstracts for Invited Talks

De Morgan's *Budget of Paradoxes*

Sloan Despeaux

De Morgan's often-quoted and highly entertaining work, *A Budget of Paradoxes*, was a natural outgrowth of his anonymous book reviews for the weekly London-based literary magazine, *The Athenaeum*. This talk will give a sample some of the Budget's most enjoyable excerpts, discuss De Morgan's motivations for writing them, and consider why the work enjoyed such wide-ranging appeal. It will also consider De Morgan's treatment of mathematical "cranks," that is, people who tried to convince the general public that they done mathematically impossible feats such as doubling the cube, squaring the circle, or trisecting the angle. Exposing these "paradoxers" was part of De Morgan's program of a responsible and truthful presentation of mathematics to the public at large.

Fit To Print? Referee's Reports of Mathematics in Nineteenth-century London

Sloan Despeaux

The Royal Society represents one of the first British scientific societies to establish a peer review process for papers submitted to its journals. While peer review procedures were initially at best informal, by the 1830s, they became a formal, required gateway for all Royal Society submissions. This talk focuses on the refereeing of mathematical papers submitted to the Society from 1832 to 1902, the years covered in the first fifteen volumes of referee's reports archived at the Royal Society Library. Besides judging the mathematical content of papers, these reports give us sometimes tantalizing glimpses into the politics, relationships, and personalities of 19th-century mathematicians.

The Mathematics of Faith: Euler's anonymous work on the limits of mathematics, science, and faith

Dominic Klyve

In 1747, Euler wrote (and anonymously published) his Rettung der gottlichen Offenbahrung (E92), defending the validity of divine revelation as a valid source of knowledge, while considering perceived inconsistencies in mathematics and science. Drawing on a new translation of the Rettung, we shall attempt to elucidate some of Euler's religious views, and draw connections between this work and the work he was doing at the time in astronomy and mathematics. A close familiarity with Euler's mathematical works of the 1740's is necessary to understand Euler's religious document, and we thus demonstrate some of the mathematical theorems and problems besetting Euler at this time.

Translation and Betrayal: "Euler's Letters to a German Princess"

Dominic Klyve

Perhaps no work of popular science was more widely read, and more widely mistranslated, than Leonhard Euler's Lettres à une Princesse d'Allemagne. In this talk, we shall examine Euler's book, trying to understand why it was so popular and influential in its time. We shall then look at its dissemination and translation into eight languages over the first century following its publication. We shall discover that much of Euler's book, including almost all of his writing on God and faith, was systematically eliminated or edited by Condorcet in an attempt to make the book "more appropriate" for the modern reader. We will conclude by using this one example as a meditation on the problem of translation in general.

Shaping a Digital World: Connecting Bytes and Beliefs

Derek Schuurman

What does the Bible have to say about technology? What do bytes have to do with Christian beliefs? This talk will present some ideas from the book "Shaping a Digital World" (InterVarsity Press, 2013) in which a Christian perspective of technology will be sketched based on the Biblical themes of creation, fall, redemption, and restoration. It will be argued that computer technology is not neutral, but value-laden with substantial legal, ethical, social, cultural and faith implications. A discussion of norms for computing will be presented as guidelines for working towards more responsible technology. The talk will conclude by sketching common technological visions of the future and contrasting them with what is described in the biblical story.

Responsible Automation: Faith and Work in an Age of Intelligent Machines

Derek Schuurman

Ever since the Luddites revolted in opposition to the industrial revolution in the early 19th century, concerns over automation have persisted. Throughout the Industrial Revolution, the introduction of new technology did displace certain jobs, but it simultaneously created new jobs. And so it went for much of the 20th century. However, the digital revolution and the increasing pace of automation are now rapidly transforming the economy and employment. With recent developments in robotics and artificial intelligence, many have begun to warn of the "end of work" in what has been called "the second machine age." If pursuing absolute efficiency through automation leads to undesirable consequences, what normative guidelines can help us responsibly harness the possibilities of new technologies while simultaneously ensuring flourishing for humans and the rest of the creation? A set of norms informed by a Christian perspective for responsible technology will be presented to provide a helpful framework as we face the second machine age.

Abstracts for Parallel Session Presentations

Some Silly Analogies and Mnemonic Devices for Upper-Level Math Topics

Aaron Allen

There exist a number of topics, particularly in upper-level math courses, that are difficult to explain. Moreover, even when the professor explains an abstract topic clearly, there's the question of whether or not the student is able to grasp what the professor is communicating. The aim of this talk is to simply provide some examples of analogies and memory devices that help with student understanding of certain difficult topics. Examples include cooking (or baking) as an analogy for explaining vector spaces and archery as an analogy for explaining functions. Also included are memory devices such as pirate integrals and a spherical coordinate dance.

Poster Projects in Math for Liberal Arts

Brandon Bate

Over the past few years, I have been assigning poster projects to my Math for Liberal Arts class. For these projects, students identify an area of mathematics that they would like to learn independently and then present what they've learned to their peers in a poster session. In this talk, I will share about my experience using this assignment, including ideas for how to help students find appropriate topics, advice for how to encourage independent learning, and some thoughts on what I feel students gain through completing this assignment.

The Resolved and Unresolved Conjectures of R. D. Carmichael

Brian D. Beasley

Even before heading to Princeton University to work on his doctoral degree, Robert Daniel Carmichael started influencing the path of number theory in the 20th century. From his study of Euler's totient function to his discovery of the first absolute pseudoprime, he set the stage for years of productive research. This talk will present a brief overview of Carmichael's life, including his breadth of mathematical interests and his service on behalf of the Mathematical Association of America. It will focus mainly on his two most famous conjectures - which one has been settled, and which one remains open to this day?

The Corset Theorem

Owen Byer

In this presentation, we explore how to interpret the base and altitude of a triangle relative to a coordinate system in order to generalize the standard formula for the area of a triangle. This generalization will lead to an elementary proof of a formula for the area of a general polygon in terms of the coordinates of its vertices. A surprising corollary is that if every other vertex of a polygon is translated by a fixed vector, the area of the polygon is unchanged.

Warm-up problems in Linear Algebra

Kristin A. Camenga

Inspired by Ryan Higginbottom's article in the February/March 2016 MAA Focus, I used warm-up problems as the only homework in my linear algebra class of 31 students in Spring 2017. After every class I assigned several homework problems and at the beginning of the next class, I (mostly) randomly selected students to present some of these problems to their peers. Goals included increasing student responsibility, focusing attention on the communication of mathematics, giving students detailed and quick feedback, and limiting my time commitment to grading homework. I will share specifics of how I used warm-up problems, challenges faced, and how these goals were achieved (or not).

Including a Writing Project in a Service Learning Course for Mathematics and Computer Science Students is a Win for Both Students and Professor

Lori Carter

At Point Loma Nazarene University, we faced 2 dilemmas with our Mathematics and Computer Science graduating seniors. One issue became apparent in the capstone course taken the last semester before graduation. A major component of the course is a research paper. Each year we saw many seniors who still could not write well about concepts from their disciplines. The second issue was with the Service Learning course taken by students in our department. In this class, students use the skills they have acquired from their coursework and work in groups to complete a project as a service for a non-profit client. There are times when progress is not made on a project either because the client has not had time to get the data or resources required to move forward, or the students just lack motivation for this credit/no credit course. The addition of a project-based research paper, written with instruction and over the course of the semester-long Service Learning course has appeared to improve the experience in both Service Learning and Capstone for the students and professors alike. The presentation will focus on the instructive methods used to help students progressively write the paper in a way that motivated them to make progress on their projects and increased their confidence in writing without huge additional burden on the professor.

A “Big Ideas” Reflection Assignment Inspires Students to Make Valuable Connections

Jeremy Case and Mark Colgan

While participating in a Faculty Learning Community, we explored the “big questions” we wanted our students to take away from our mathematics courses. We called these questions the Big Ideas of the course and developed a Big Ideas Reflection Assignment, which we continue to assign at the end of each of our courses. Students are able to demonstrate understanding and application of their learning as well as their values and appreciation of mathematics. The assignment encourages students to move beyond a focus on technique and symbolic manipulations towards a broader and more holistic approach, including making connections between their learning and the Christian faith. We also noticed that the assignment changed the way we approach our teaching and designed our courses, so that our students are beginning to see our courses not as a collection of neatly packaged isolated chunks of material but as a unified collection of important mathematical ideas.

A Pre-Calculus Controversy: Infinitesimals and why they really matter

Karl-Dieter Crisman

In teaching calculus, it is not uncommon to mention the controversy over the role of infinitesimals with Newton’s and Leibniz’ calculus, including Berkeley’s objections. In a history of mathematics course, it is a required topic! But rancor over infinitesimals and their role in mathematics predates calculus - so much so that a popular new book is dedicated to this topic.

In this talk, I will discuss not just the relevant controversies between Cavalieri and the Jesuits, between Thomas Hobbes and John Wallis, but also why they matter to us today, both as mathematicians and as Christians. Perhaps most importantly, I will present the modern life of the controversy as an example of overreach in the history of science, along with suggestions for how to bring the whole story into the mathematics classroom.

Hybrid courses across the curriculum: what works and what doesn’t

Ryan Botts, Lori Carter and Catherine Crockett (presenter)

Recent hype around online and blended courses touts the benefits of immediate student feedback, flexible pace, adaptive learning, and better utility of classroom space. Here we aim to summarize the results of a 3-year pilot study using blended courses across the quantitative science curriculum (Mathematics, Statistics and Computer Science), in both upper and lower division, major and GE courses. We present findings on student attitudes towards this format, most helpful course components, time on task, progress on learning outcomes and faculty perspectives. This summary can be used to inform best practices in hybrid design, implementation and faculty expectations in the quantitative sciences.

Creating a Natural Progression Through a College Algebra Course

Denise K. Dawson

By approaching College Algebra as a study of functions and their graphs we are able to present the material in a way where each new topic builds on previous topics and naturally adds a level of complexity. The hope is that students will be able to see the connections and common themes running through the course. This presentation will compare this graphical approach to a more classical presentation of the material and will discuss a general map through the course.

Using Inertial Navigation to Demonstrate Basic Calculus Concepts

Ron DeLap

Students often absorb concepts more fully when they see concrete demonstrations of how the concepts can be applied. One such example is the relationships between acceleration, velocity and position. A lesson on inertial navigation provides a creative and interesting way to demonstrate these basic physics and calculus concepts in the classroom. “Smart cars” or autonomous driving vehicles are just one example of a known technology that uses inertial navigation. In this talk I will discuss several ways to illustrate the concepts of acceleration, velocity and position via theory and demonstrations of how inertial navigation works.

Teaching by Not Teaching: The Power of Collaboration

Audrey DeVries

The first ten minutes of class can either be the most wasted time or the most valuable time. A “Problem of the Day”, completed in groups and given at the start of each class, can serve to review the prior day’s material, encourage discussion and interactive learning, assess the students’ understanding, provide the opportunity for instant feedback, and altogether result in learning by means other than teaching - by means that extend beyond the classroom walls - by means of collaboration. We all learn a great deal from what others know, and every student brings a unique perspective to the table. Collaboration allows students to see and understand concepts in a new light. Furthermore, a daily warm-up exercise frees up the teacher to answer questions, interact with students individually, observe and assess student understanding, critique student work, and build rapport with the class. It’s well worth the sacrifice of ten fewer minutes of lecturing! Moreover, promoting collaboration within the classroom paves the way for collaboration amongst students outside the classroom. All in all, collaboration is a means of teaching by not teaching and leads to continued learning far beyond the scope of the teacher.

Tightened Relaxations of the Traveling Salesman Problem

Audrey DeVries

This paper derives new mixed 0-1 linear representations of the asymmetric quadratic traveling salesman problem (AQTSP). The AQTSP is a variant of the classical ATSP which allows for "quadratic" objective coefficients to record costs associated with travel along pairs of consecutive arcs. Emphasis is given to exploiting the problem structure so as to obtain representations with tight linear programming relaxations. In the process, different subtour elimination constraints are tightened, including the Miller-Tucker-Zemlin restrictions, and the restrictions invoking precedence-constrained variables. A consequence is improved valid inequalities for the (linear) ATSP. Computational results are provided to compare the relative merits of the different approaches.

Mentoring as a Statistical Educator Within the Context of a Christian College

L. Marlin Eby

In this paper, I present principles based on more than thirty years of intentional mentoring as a statistical educator in a Christian college. I believe this mentoring has been enhanced due to the setting - a Christian college, and the discipline - statistics. I discuss distinctives of the Christian college setting that positively impact mentoring in any discipline with respect to the mentor, the mentee, and the pervading campus atmosphere. I focus on mentoring as a statistical educator by specifically considering the following: attracting students to the discipline of statistics, preparing students for careers using statistics, and preparing students for graduate study in a statistics related field. For each, I consider principles of successful mentoring in statistics at the undergraduate level regardless of the type of institution and how these principles can be expanded within the context of a Christian college.

Influences on Students' Dispositions toward Mathematics

Patrick Eggleton

This session is intended for presenting the findings from a Spring 2017 research study conducted at Taylor University regarding influences that contribute to a student's disposition toward mathematics. In the foundation level mathematics course taught for non-majors at Taylor, students are asked to share a reflection on their past mathematical experiences. Analysis of these reflections shows general themes regarding the influences, both good and bad, that have contributed to how these students approach mathematics. We would like to use this information as well as related studies to help instructors of mathematics develop positive dispositions toward mathematics in their students.

Classical Mathematics: An Attempt to Integrate Mathematics and Christian Worldview in a General Education Mathematics Course

Jamie K. Fugitt

As part of the faculty review process, Classical Mathematics, a freshman-level, general education mathematics course designed for students who wish to take a broad look at the history and structure of mathematics was developed. In this course students explore mathematical topics related to numeration, geometry, number theory, symbolic logic, and statistics through processes such as reasoning, proof, the axiomatic method, and application. Throughout the course, related philosophical and theological ideas are explored. At this point the course has not been taught and is still in the developmental stage.

In this session the journey that led to the development of this course curriculum and the major components of this curriculum will be presented. Participants will be encouraged to share similar efforts so that participants can learn from each other.

Teaching Math and Computer Science Using Graphic Novels

Eric Gossett

Find out how graphic novels can supplement textbooks in a math or computer science class. The presentation will also demonstrate how to produce your own graphic novel supplement at a reasonable cost. Here are a few advantages that graphic novel supplements can provide:

- They are fun to read.
- They provide easy to read overviews of many standard topics. Students can read the appropriate section of the graphic novel, then read, in more depth, the material in the textbook.
- They can be a useful part of an exam review. Repeatedly rereading a textbook does not aid learning. The graphic novel provides a quick reminder of the topics. The student can then look at some key sections of the main textbook to deepen that review.
- A graphic novel provides more opportunity to explore some common errors in student thinking.
- The story can model the kinds of questions that a successful student would ask.
- Many helpful research-based study tips can be included throughout the novel.

Conceptual Understanding of Inference via Simulation

Justin Grieves

In this talk we will look at an example in which hypothesis testing about a proportion is motivated and discovered. A story about a woman claiming to smell Parkinson's Disease motivates this lesson in which a simulation based randomization test is conducted. The lesson has been presented to 7th grade students, AP Statistics students, and college level introduction to statistics students and emphasizes the concept of testing and p-values.

Living at Work: My First Year as a Faculty-in-Residence

Rachel Grotheer

Research has shown that increased interaction between faculty and students outside of the classroom improves student achievement and the students perception of their educational experience. In an effort to increase this interaction and bridge the gap between a student's academic and residential life, Goucher College has adopted a Faculty-in-Residence program in its new first-year dorm. In this talk I will share the challenges and successes of developing this program and determining just how to create that bridge. I will also share challenging issues such as the blurred boundaries between life and work, maintaining a welcoming but still personal space, deciding how to bring mathematics into a dorm, and the unique perspective faith brings to it all.

Revamping the Path Through High School Mathematics - The Impossible Dream?

Clayton R. Hall II

In high school, most college-bound students follow the same general path of courses in mathematics and take Algebra 1, Geometry, Algebra 2, Trigonometry, Pre-Calculus, and Calculus. However, a significant number of students fall off this path at some point, ending their mathematical experience with negative feelings and possibly failure. For those students who survive, and even for those that have good mathematical ability, it is often the case that one of the two semesters of Calculus is for them a terminal course. The purpose of this talk is to look at the narrow view of mathematics held by most students and to ask if it is possible to broaden their perspective. Or is this really the impossible dream?

Math Teacher Circles: How and Why to Start Yours

Amanda Harsy, Tom Clark, Dave Klanderman

Many K-12 math teachers are not ready to teach from a conceptual and inquiry-oriented perspective because they have an algorithmic understanding of mathematics. One solution is to create a math teacher circle (MTC), which provides conceptual and inquiry-based learning activities and builds professionalism among the teachers. With the support of grants from the American Institute of Mathematics (AIM), over 100 math teacher circles (MTCs) have been formed over the last decade. Each circle is a partnership among faculty members at colleges and universities along with local elementary, middle, and high school math teachers. In this session, we describe the origins of two such MTCs, highlighting the process of identifying leadership team members, submitting the grant proposal for seed money, and hosting launch events, intensive summer workshops, and monthly meetings during the academic year. We will also share opportunities for professional development for college and university faculty, including research linked to shifts in in-service teacher attitudes and impact on student learning. Join us for a clear path to starting your own circle!

The Daily Question: Building Student Trust and Interest in Undergraduate Introductory Probability and Statistics Courses

Matthew A. Hawks

Introducing probability or statistics to disinterested undergraduate students is challenging. Adding faith in such a classroom at a secular institution only increases the complexity. We share an unobtrusive way to build trust with students, creating a medium to both naturally share your faith and have your students look forward to attending each class. The context is the United States Naval Academy, a four-year undergraduate institution with an emphasis on leader development. In addition to a calculus sequence, Humanities majors enroll in Probability with Naval Applications or Introductory Statistics. These sophomores or juniors are split between those who have no intention of enrolling in subsequent statistics courses (English, History, Language majors), and those who will take a follow-on course as part of their major (Economics, Political Science). Based on a technique of daily questions suggested by Penn State University Lecturer Dr. Heather Hollerman (2015), we integrate daily questions with the course content. The daily question offers opportunities for expressions of faith and invitations to further conversations. Anonymous midterm assessments and end of term student opinion forms demonstrate initial success of this method.

Sculpting Seifert Surfaces: Shepherding a Student Through a Mathematical Art Project

Elliott Best and Lisa Hernández (presenter)

A Seifert surface is a surface whose boundary is a particular knot or link. This talk will give a brief introduction to knot theory and Seifert surfaces followed by an exposition on guiding a student through a mathematical art project involving making physical models of Seifert surfaces.

Integrating Faith and Discipline - Beyond the Classroom

Kevin Hopkins

For those teaching at a Christian College, the idea of integrating faith and discipline is not a new one and is probably an expectation of the job. We hopefully have ideas on how to do that in the classroom. In the past couple of years I have been challenged to think of ways to integrate my faith and discipline beyond the classroom. This talk will share how I have attempted to integrate faith and discipline in some opportunities God has blessed me with outside the classroom. I hope the ideas shared will encourage others in their integration of faith and discipline both inside and beyond the classroom.

Revitalizing Complex Analysis

Russell W. Howell

Complex Analysis, despite its beauty and power, seems to have lost some of the prominence it once enjoyed in undergraduate STEM fields. Thanks to a grant from NSF a team of scholars convened in the summer of 2014 at Westmont College to discuss what might be done to rectify that situation. As a result, there have been contributed paper sessions for the past three years at the winter joint mathematics meetings, and just recently a theme issue of *PRIMUS* was dedicated to the revitalization effort. This talk will focus on some of the interesting pedagogical and research ideas that have grown out of this movement, and discuss a tip or two for applying for an NSF grant.

An Overview of Specifications Grading

Mike Janssen

There has been much recent interest in grading and assessment structures which do not use the traditional, points-based partial credit model. I will discuss my experiences using one such system, known as specifications grading, in courses throughout the math major. Several examples will be given and student feedback discussed.

Math Teacher Circle Problems: This Year's Brightest and Best

Mike Janssen, Mandi Maxwell, Sharon Robbert

The key to a good Math Teacher Circle (MTC) is have a great collection of problems that are both deep and engaging but also approachable. We present several of the year's best activities that were used at our MTC meetings. We will describe activities such as fair division, extensions and generalizations of numerical and algebraic patterns, and applications in cryptography. In each case, feedback from the MTC meeting will include reactions from teachers at all grade levels, potential implications for classroom teaching, and possible extensions for future meetings. Come to learn more about these important partnerships among math teachers at elementary, middle, and high school levels as well as college and university mathematics and mathematics education faculty members.

The set of zero divisors of factor rings

Jesús Jiménez

Let A be a ring and $\mathfrak{a}$ an ideal of A. In this paper we show how to construct factor rings $A/\mathfrak{a}$ and a finite set of ideals $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k$, of $A/\mathfrak{a}$, such that: each ideal $\mathfrak{a}_j$ is contained in the set of zero divisors of $A/\mathfrak{a}$, the factor ring $A/\mathfrak{a}$ is a direct sum of these ideals, and each ideal $\mathfrak{a}_j$ is a ring with unity when endowed with addition and multiplication modulo $\mathfrak{a}$. Explicit examples are given when A is the ring of integers, Gaussian integers or the ring of polynomials over a field.

Inquiry-Oriented Instruction in an Abstract Algebra Course

Jill Jordan

I will briefly discuss why some mathematics instructors are choosing to move away from lecture and toward classes that emphasize inquiry-based learning, and explain what specifically is meant by the phrase "inquiry-oriented instruction." The focus of the talk will be my own experience in teaching Abstract Algebra I using an inquiry-oriented approach. I will discuss the primary curriculum I have used, give examples of specific activities from the curriculum, and share suggestions for adapting the curriculum as necessary to fit individual situations. Comments from end-of-course evaluations will be shared to give insight into students' perspectives on the course.

The Complex Moduli Project and Mathematica-Based Modules in Complex Analysis

Bill Kinney

The Complex Moduli Project (semi-pun intended) is a website under construction whose purpose is to house materials for educational modules in complex analysis. Educational modules can take a variety of forms, from thirty-minute learning activities to semester-long projects. In any form, it is important to relate educational modules to the rest of the course content, and to include background information and goals. Personally, I am creating Mathematica-based modules to help students learn how to use Mathematica as a tool for exploration in complex analysis. I will show the website and some of these modules in my talk. I welcome collaborators on the website.

God: One, Part Two

Daniel Kiteck

At the last ACMS Biennial conference in 2015, I gave a talk titled: God: One, where I explored the relationship between the ontology of the number one and God. I have continued to be fascinated with this theme, and, thus, I have continued looking into this. This talk is my continued explorations.

Near the end of my last talk, God: One, I asked "What if God had chosen not to create anything? Is [the concept of the number one] necessary then?" I wasn't sure where to go with the question, as it had just occurred to me the week I gave the talk. After further reflection, I have more to say on that now. This lead me to explore the concept of the Simplicity of God, a traditional aspect of God that is not as prominent in modern theology compared to ancient theology, including Christian greats such as Basil of Caesarea and Gregory of Nyssa who were influential in the final version of the Nicene Creed, a universally accepted creed for Christianity. I read Basil of Caesarea, Gregory of Nyssa, and the Transformation of Divine Simplicity by Andrew Radde-Gallwitz. This has helped shape my concept of God and one. If God is Simple in a sense that nothing else could be, perhaps anything meaningful to us when we say "one" would not necessarily have any truly equivalent concept to if there was God alone and no creation.

Using Real-World Team Projects: A Pedagogical Framework

Michael Leih

The use of team projects in a program capstone course for computer science or information systems majors has been a popular method for reinforcing and assessing program learning objectives for students in their final semester. Using real-world group projects as a learning activity is an excellent pedagogical approach in helping students develop critical thinking, team work, real-world problem solving, and communication skills. However, real-world group projects also provide many challenges to both the instructor and students alike. Instructors or students must find real-world projects appropriate for the learning objectives in the course. Instructors must determine how to provide teams with appropriate learning activities and provide effective feedback to reinforce learning objectives while fairly assessing project deliverables to individual team members. Students must find a common time to work together and learn to appropriately delegate project activities so each student fairly participates in the project. Finally, real-world projects have the real risk of failing due to circumstances outside the control of the instructor and students.

There have been papers presented in the past describing methods to address these challenges and successfully use real-world team projects. This presentation gives a summary of these methods and presents a successful and practical approach that has been used for the past seven years in an information technology program capstone course. This framework is based on traditional project management methodologies which allow students the opportunity to successfully meet learning objectives even if the project success factors are not met.

Future Medical Professionals and the Statistics Classroom

Paul Lewis

Students seeking admission to Texas medical and dental schools are required to take a course in statistics. In fall of 2014, I began teaching a statistics course offered specifically to meet the needs of pre-health students. In this talk, I will present a few medical applications that could be explored in most probability and statistics courses. I will then share some of the benefits of offering a course intended primarily for pre-health students and how my interactions with students through this course have shaped my perspective on teaching statistics.

The Heart of Mathematics Through the Eyes of Faith

Matt D. Lunsford

Over the past several years, I have developed and taught a freshman-level mathematics course for non-math/science majors, which fulfills the general core requirement in mathematics. The primary text for the course is *The Heart of Mathematics* by Burger & Starbird. The senior capstone course in mathematics, which lies at the opposite end of the undergraduate experience, requires mathematics majors to read and respond to *Mathematics Through the Eyes of Faith* by Bradley & Howell. Upon noticing the many similarities in mathematical content presented in the two texts, I have worked to synthesize portions of these two texts into a coherent framework that could be used in the lower-level course. The ultimate goal of this endeavor is to demonstrate a mutually beneficial relationship between the discipline of mathematics and the Christian faith. This talk will focus on how the course and the integration project have evolved.

Manifold Methods for Averaging Subspaces

Justin Marks

Applications of geometric data analysis often involve producing collections of subspaces, such as illumination spaces for digital imagery. For a given collection of subspaces, a natural task is to find the mean of the collection. A robust suite of algorithms has been developed to generate mean representatives for a collection of subspaces of fixed dimension, or equivalently, a collection of points on a particular Grassmann manifold. These representatives include the flag mean, the normal mean, the projection mean, and the Karcher mean. In this talk, we catalogue the types of means and present comparative heuristics for the suite of mean representatives. We respond to, and at times, challenge the conclusions of a recent paper outlining various means built via tangent-bundle maps on the Grassmann manifold.

A Pre-Lab Style Approach to Calculus III

Michael Martinez

The Calculus sequence plays a foundational role for math and science majors of all stripes and is usually taken either concurrently with or immediately preceding the core upper-division math courses. At the same time, the traditional third course in the calculus sequence can be characterized as revisiting the concepts of the first two in higher dimensions and so much of the material is familiar to the students. As such, Calculus III is positioned wonderfully to more intently teach students many of the "soft skills" of mathematics. In this talk I will present some lessons learned after a year of restructuring the Calculus III course and discuss ideas for further implementation in the context of a liberal arts university

Ten Mathematicians Who Recognized God's Hand in their Work (part 2)

Dale McIntyre

Scottish philosopher David Hume (1711-1776) once observed that

> "Whoever is moved by faith to assent to [the Christian religion], is conscious of a continued miracle in his own person, which subverts all the principles of his understanding, and gives him a determination to believe what is most contrary to custom and experience."

Evidently Hume's cynical pronouncement did not apply to Descartes, Newton, Riemann, and other profound thinkers who believed God had commissioned and equipped them to glorify Him in their pursuit of truth through mathematics - And based on their extraordinary achievements the principles of their understanding do not appear to have been subverted too badly!

Leading mathematicians of the past commonly affirmed that God created and sovereignly rules the universe and that He providentially sustains and nurtures His creatures. Despite Hume's assertion, history teaches us that faith often informs rational inquiry and vice versa. In many cases Christian commitment stimulated intellectual activity; sometimes mathematical understanding led to spiritual insight.

In my former paper, ten of history's most influential mathematicians expressed the role faith in God and religious conviction played in their work in their own words. This paper explores the same for mathematicians numbered eleven through twenty.

Mathematics and an Epistemology of Love

Chris Micklewright

N.T. Wright often describes the resurrection as ushering in a new paradigm for understanding and inhabiting the world, calling on scholars to explore how an "epistemology of love" transforms their discipline. This talk will engage in such an exploration, beginning with our understanding of mathematical objects. In particular, we will consider how both platonist and formalist views of mathematics have the potential to draw our imagination and our love toward abstract ideas and certainty, but away from God's good creation – and, perhaps, away from the Creator as well. One response, articulated by Boyer and Huddell in a recent Christian Scholar's Review article, is to follow Augustine in locating all of mathematics in the very nature of God; such an approach accounts well for the value and beauty of mathematics, but still has the potential to leave the mathematician adrift from creation. It can also be argued that the Augustinian approach improperly elevates mathematics above other disciplines, privileging an epistemology that falls short of Wright's ideal. As such, we will consider a Christian Humanist perspective on mathematics, recognizing mathematics as a part of human culture that seeks to "make something of the world" (to borrow from Andy Crouch). Such a perspective has the potential to fulfill much of what Wright calls for, and has significant implications for the ways in which we practice and teach mathematics.

Variations on the Calculus Sequence

Chris Micklewright

Many institutions have embraced a standard format for the Calculus sequence, comprising three four-credit courses covering a fairly consistent set of topics. While there is much to recommend this approach, it still leaves some fantastic concepts rushed or untouched, and it can be argued that it demands too much of students with weaker backgrounds. As such, some schools have experimented with variations on the standard format. In this talk, I will present the model that my institution currently uses, exploring the strengths and weaknesses of our particular approach. I will also suggest ideas, developed in conversation with other ACMS members at the 2015 meeting, for how different approaches might be explored in a comparative study. It is my hope that this talk will open productive conversation on the most effective ways of structuring the Calculus sequence, and that it will help to guide and inform the development of the comparative study.

A Practical Mechanism to Perform Secure Computation

Benjamin Mood

Secure computation allows multiple parties to compute on secret input and receive secret output without revealing sensitive information to any other party. For instance, two or more parties are able to find out which contacts they share on their mobile phones without revealing any information about the contacts they don't share by using a secure computation. Two health care companies could perform statistical queries on their data to find out new correlations between diseases without revealing the highly confidential data to each other. Unfortunately, the computational cost to perform secure computation by using cryptography is orders of magnitude slower than performing that same computation without any security. This work will discuss a new set of instructions for Intel processors, the software guard extensions, and how they can be used to perform secure computation efficiently.

Reading Journals: Promoting Student Engagement and Success in Undergraduate Mathematics Courses

Sarah A. Nelson

We spend lots of time searching for the *best* textbook for students. We want our students to have a reliable and useful resource to reference, as needed. We even ask them to read over certain material before classes. Often, however, we fail to guide our students in how to read the text productively.

Incorporating reading journals into your classes is an excellent way to simultaneously develop your students' ability to read mathematical text and capitalize on what the students already have to offer. In this presentation, we will look at how reading journals motivate students in a variety of mathematics courses across the undergraduate curriculum. As time permits, we will share important lessons learned and how to develop different types of prompts for journals.

ScratchFoot: a tool for transitioning students from Scratch to Greenfoot

Victor Norman

Students in middle school and high school are often introduced to computer programming with the online blocks-based programming environment, Scratch. Scratch is an ideal tool because students can learn computational thinking, while working in an environment in which little keyboard typing is required and it is difficult to make syntax errors. However, students do become frustrated with some of the limitations of Scratch. They also learn that "real" computer programming uses text-based languages, like Java, Python, or C++.

Another Computer Science educational tool is Greenfoot. Like Scratch, Greenfoot has a large user base and many online resources, including curriculum, lesson plans, tests, etc. However, Greenfoot helps students learn introductory programming using Java, a text-based language.

For years, educators have been seeking ways to gently help students make the transition from blocks-based program to text-based programming. In fact, a recent conference was convened to discuss, among other things, exactly this problem.

Over the last 3 summers students and I have been developing a tool to transition Scratch users to Greenfoot. The similarities between Scratch and Greenfoot are notable: 1) both offer a 2-dimensional canvas on which programmers place images (or *sprites*); 2) both offer an API to allow the programmer to move the image, to changes the direction it is facing, to detect collisions between images, to detect collisions with the edges, to change their images, and to react to user events - keyboard presses, mouse movement, etc.; and 3) both allow students to post their creations online easily, to be shared with others.

The tool, ScratchFoot, consists of 2 parts. First, the tool includes a Greenfoot library that provides a Scratch-like API, so that students who are familiar with Scratch "calls" can easily recognize the ScratchFoot equivalents. Second, the tool can automatically convert a Scratch project into a Greenfoot scenario.

My presentation will demonstrate the tool and discuss the challenges in making Greenfoot's execution model act like Scratch's. It will also discuss my first round of testing with middle-school and high-school students during the summer of 2017. The presentation will include a live demo of the tool.

Teaching Introductory Computer Programming with "Processing"

Jeff Nyhoff

"Processing" is a free, open source, and multiplatform computer programming language that was originally created 16 years ago at MIT Media Lab to make it easier for digital artists to create works involving graphics, animations, and interactivity. Since then, Processing has evolved into a stable, powerful, and versatile programming environment with a worldwide popularity. However, Processing remains relatively unknown in academic computer science. This presentation will describe and demonstrate how, at Trinity, Processing has continued to prove to be an enjoyable and highly effective way for college students from a variety of majors to learn the basics of computer programming and acquire a solid basis for further learning. Because Processing is a considerably simplified version of Java, using Processing in Trinity's "CS1" course has been found to prepare students well for the "CS2" course that uses Java. Processing can be used to write graphical programs as well as numerical programs. Processing has also proven to be suitable for instruction of high school students: this past fall, 8 students from Chicago Christian High School enrolled in the introductory computer programming course taught at Trinity using Processing and continued on to the second programming course in Java, performing strongly in both courses. Trinity students have also used Processing in more advanced courses to create Android smartphone apps and interactive body-tracking games that incorporate the Microsoft Xbox Kinect camera.

The Role of Informal Learning in Supporting Mathematics Teacher Education

Alice E. Petillo

Informal learning in mathematics opens a door for pre-service teachers to experience mathematics in a constructive and interconnected way. This research addresses a gap in research and practice in that many future teachers are not currently exposed to informal learning opportunities. Data from undergraduate students was collected as part of IRB-approved studies over the last three years at the National Math Festival 2015 and 2017, and the USA Science & Engineering Festival 2016 in Washington DC.

The purposeful implementation of these community-based informal learning experiences supports pre-service teachers' mathematical development while still undergraduate students. These experiences may promote a positive shift in beliefs and feelings towards mathematics as a subject they will one day teach. Implications include recommendations for teacher educators to facilitate pre-service experiences in informal and service-based learning. Since many informal learning events are free and open to the public, these events provide opportunities for students from all types of backgrounds and have the potential to support educational equity.

Professional education associations like the American Educational Research Association (AERA), National Council of Teachers of Mathematics (NCTM) and National Science Teachers Association (NSTA) recognize that informal education should be a part of a core STEM strategy and be supported by professional development of in-service teachers and pre-service teachers. Educational research describing how informal learning environments in mathematics can create interest and connections with mathematics is needed. In addition, mathematics teacher educators can benefit from support in designing these kinds of experiences for their pre-service teachers.

Finding Meaning in Calculus (and Life)

Doug Phillippy

A 2015 publication of the Mathematical Association of America (*Insights and Recommendations from the MAA National Study of College Calculus*) noted that "students taking college calculus exhibited a reduction in positive attitude toward mathematics, which can affect their career aspirations and desire to take more mathematics". The study pointed out that students' confidence to do mathematics, enjoyment of mathematics, and desire to persist in their study of mathematics had all dropped by the end of their college calculus experience. The study also suggested possible reasons for the negative drop in student attitude and made several recommendations to mathematics departments on how to address this problem. The focus of this talk will be on a uniquely Christian approach to addressing this problem, one that the presenter believes is the only long-term solution to find meaning in the calculus classroom.

Gaudi's Geometry of Nature

Donna Pierce

Antoni Gaudi (1852-1926) was a Catalan architect whose distinctive style was characterized by its range of forms, textures, polychromy and sense of unity with the natural world. His architecture, inspired by the Christian message and nature, broke from the established order both in form as for the constructive solutions employed in his buildings. Gaudi took his inspiration from the Christian message and nature. He saw creation as being the supreme work of the Creator and the model from which he should take inspiration for his buildings. Employing complex geometries such as ruled surfaces, hyperboloids, paraboloids, double twisted columns, caternary arches, his architectural structures give the appearance of being a natural object in complete conformity with nature's laws. In this talk we will look at his most famous building, La Sagrada Familia in Barcelona, and learn how he employed his geometry of nature to create a place of worship resembling a wood that invites prayer and is fitting for celebrating the Eucharist. We will see how Gaudi took his observations of nature and passed them through the sieve of his enormous personal creativity and ingenious methodologies to create the structures and forms which give his buildings their organic unity of nature, faith and many arts.

Why you should move your infinitesimals to the top drawer

Troy Riggs

Anyone who has taught applications of calculus has used infinitesimal arguments, even if only heuristically. I will argue that based on A. Robinson's foundations, B. Dawson's approximation relation and a very positive response from students; we should all formalize this process and use it as an alternative to limits from the very beginning of calculus.

Logical Axioms and Computational Complexity: A Correspondence

Danny Rorabaugh

Relational structure $\mathbb{A}$ is compact provided for any structure $\mathbb{B}$ of the same signature, if every finite substructure of $\mathbb{B}$ has a homomorphism to $\mathbb{A}$ then so does $\mathbb{B}$. The Constraint Satisfaction Problem (CSP) for $\mathbb{A}$ is the computational problem of determining whether finite structures have homomorphisms into $\mathbb{A}$. We explore a connection between the hierarchy of logical axioms and the complexity hierarchy of CSPs: It appears that the complexity of CSP for $\mathbb{A}$ corresponds to the strength of the axiom "$\mathbb{A}$ is compact". At the top, the statement "K_3 is compacts" is logically equivalent to the compactness theorem. Thus the compactness of K_3 implies the compactness of all finite relational structures. Moreover, the CSP for K_3 is NP-complete. At the bottom are width-one structures; these are provably complete from ZF and their corresponding CPSs are polynomial-time solvable.
This is joint work with Claude Tardif and David Wehlau, arXiv:1609.05221 [math.LO].

Bicycle Routes and Euler Double-paths

C. Ray Rosentrater

What configurations of roads are amenable to creating a bicycle route that traverses each segment exactly once in each direction? As stated, every configuration of roads can be so traversed. However, the question becomes much more interesting when U-turns are not allowed. This talk investigates the conditions under which a connected graph has a non-reversing, Euler double-path.

Axioms: Mathematical and Spiritual

Melvin Royer

Pastors such as A.W. Tozer have described their preaching as an attempt to extract from scripture axioms that are universal across circumstances. Similarly, academicians in various disciplines have tried to characterize the foundational principles of their field. For example, the social psychologist Gerard Hofstede proposed an onion model for comparing cultures in which core values of a nation help explain its rituals, feelings, and artifacts. On a personal level, many of us have likely tried to identify root causes of our beliefs and feelings.

These attempts to "find bedrock" are somewhat analogous to the axiomatization of mathematics attempted by Euclid, Hilbert, Frege, Russell and Whitehead, and others. As is well known, the success of these efforts ranges from that of Euclid (whose book arguably guided western mathematical reasoning for two millennia) to that of Russell and Whitehead (whose book was essentially defeated by Gödel's Incompleteness Theorem within two decades).

In this talk, we will propose insight that mathematical axiomatization (considered as a parable) may have for our efforts to understand God and explain our core spiritual beliefs. Is it really possible to identify our personal spiritual axioms? Is it possible to classify spiritual axioms as consistent and complete? Are there things that God does and does not want us to know? How should Christians respond when deductive reasoning fails to explain a situation?

Precalculus and Calculus Students' Understanding of the Concept of Function

Lauren Sager

Function is a central topic to the precalculus course, and many researchers have suggested the importance of a strong conception of function to students' mathematical development. Following a framework of understanding suggested by Anna Sfard (1991), I developed a questionnaire on functions for precalculus and calculus students. This talk will focus on interesting trends found in the solutions, and the differences between precalculus and calculus students' solutions.

Brokering Trust in Knowledge Management with the use of Data Visualizations

Kari Sandouka

Discussion of a research proposal that I developed to measure and broker trust within knowledge management using data visualizations. A conceptualized view is provided to show how the four modalities of knowledge creation are intertwined with creating and maintaining trust in an organization with data visualizations. In addition to the conceptual model, a measurement scheme to identify how effective the data visualizations are in building trust between the hierarchical levels of employees. Overall, I am proposing a research model to identify if, and if so how effectively, data visualizations are brokering trust by providing transparency into the data used for decisions at the administrative and/or executive levels of an organization.

The Topology of Harry Potter: Exploring Higher Dimensions in Young Adult Fantasy

Alexa Schut, Sarah Klanderman, Dave Klanderman, William Boerman-Cornell

As one of the most beloved series in children's literature today, the Harry Potter books excite students of all ages with the adventures of living in a magical world. Magical objects (e.g., bottomless handbags, the Knight Bus, time turners, and moving portraits) can inspire generalizations to mathematical concepts that would be relevant in an undergraduate geometry or topology course. Intuitive explanations for some of the magical objects connect to abstract mathematical ideas. We offer a typology with a total of five categories, including Three Dimensions in Two Dimensions, Higher Dimensions in Three Dimensions, Two and Three Dimensional Movement, Higher Dimensional Movement, and Higher Dimensional Traces. These categories attempt to explain supernatural events from the wizarding world using mathematical reasoning in order to increase engagement in topics from topology to differential geometry. Our pedagogical goal is to pique student interest by linking these abstract concepts to familiar examples from the world of Harry Potter. Put on your Ravenclaw robe or Gryffindor scarf and join us!

Techniques for integrating Faith into the Computer Science Classroom

Valerie Sessions, Yu-Ju Lin, Paul West, Sean Hayes, Fred Worthy, Jim Roberts

The computer science department at Charleston Southern University uses a variety of methods to integrate faith into the classroom. The department strives to integrate faith in the general education course, Freshman Seminar, all the way up to our masters level courses in computer architecture, user interface, and distributed data mining. Mapping Bloom's taxonomy to a series of courses, we show a few of our methods and how they map with the type of skills we wish to see in our students. Over the course of the majors' time at CSU we strive to grow the student from knowledge of the Christian faith, to how our faith can illuminate the type of work we do in computer science, to how CS (particularly Artificial Intelligence) could inform our very faith. We will also give some specific examples of class discussion topics, ethics papers and out of classroom club activities. This will be a very practical look at faith integration by the entire department.

P-values considered harmful

Michael Stob

In 2016, the American Statistical Association adopted a statement on null hypothesis statistical testing. This unprecedented statement was motivated by increasing criticism of both the contemporary practice and theoretical foundations of this method which is taught in every introductory statistics class as an important method for gaining scientific knowledge from data. In this talk, we look at just one strand of that criticism, the replicability crisis, to see what all the fuss is about.

Algebra without Signed Numbers: Mr. Frend's Universal Arithmetic

Richard Stout

Toward the end of eighteenth century negative numbers had gained wide acceptance yet, there were still pockets of resistance, especially in England. One particularly vocal and irritating opponent of the use of negative numbers was the British mathematician William Frend (1757-1841). Frend adamantly maintained that signed quantities are meaningless, that they have no place in rigorous mathematics, and that their use was essentially why students struggled with algebra, which at the time was thought of as a universal arithmetic.

Frend was educated at Cambridge and held a position as a tutor at Jesus College until he became disenchanted with the Church of England and was forced to leave. Nevertheless, he continued to be active in London intellectual circles and maintained a connection with the British mathematical community. Some historians believe that his campaign against the negatives was a factor in Peacock's developing a formal system of algebra.

Convinced that the use of negative quantities was a major cause of confusion among algebra students, Frend wrote a text devoid of signed quantities, *The Principles of Algebra*, published in 1796. Because he eschews the use of negative quantities, Frend must resort to clever arguments to obtain basic algebraic identities. In this talk we will introduce Frend and his text, considering several illustrative examples of his reasoning. Our goal is to appreciate the methods he uses and critique his claim that deleting negative quantities simplifies the subject and facilitates understanding.

Using Machine Learning and Data mining to analyze retention rates at Bethel University

Deborah Thomas

Academic institutions, such as ours, are always looking for ways to increase retention rates by increasing our percentage of successful students. Two indications of success at Bethel are the percentage of returning students between freshman and sophomore year as well as the graduation rate for a particular class after four years. Since we have acquired information about our students over the past many years, we can use it to describe students who are successful and those who aren't. Often, the underlying patterns may not be obvious with simple analysis techniques, making it a challenging problem. Once we identify them, it will help us better assist our students so that we can increase our success rates. This is where data mining can be used to analyze the data and look at relationships, across student types and across years. The presentation will include a discussion of such techniques as well as our results so far, including summative descriptions of students who tend to succeed at Bethel and return the following year. We will also do similar analysis on the graduation rates of Bethel students.

Mathematics and Statistics Service Learning: Beyond the Project

Alana Unfried

"What is service?" "Is mathematical consulting a form of service?" "How do I consult effectively?" "Why does it even matter?" A Service Learning course or project is a powerful tool for helping mathematics and statistics students realize the impact that their field of study has on the world, as well as the role that they themselves might play in bettering society. This type of course or project can often be a student's first exposure to using their mathematical and statistical skills to work with a real client. This talk will discuss not only the implementation of consulting projects, but also the logistics of creating a reflective, engaging learning environment for our students at secular and religious universities alike. I will share resources and ideas for reading assignments, presentations, journals, in-class activities, and more. Service learning is more than just a project; it is a way for students to realize the deep need for their skill sets in society today and to explore how these skills can be used in service to others.

Book Review: Redeeming Mathematics - A God-centered Approach by Vern Poythress

Kevin Vander Meulen

The title of the book by Vern Poythress should peak the interest of members of the ACMS. I will review what the book has to offer, and give a critique of the approach given in the book.

Models, Values, and Disasters

Michael Veatch

Response to a major disaster involves rapid movement of people and supplies in a challenging environment. Mathematical planning models have been proposed, analogous to those used in commercial logistics. Part of the model building challenge is to capture the decision-maker's objective, which is multifaceted and not simply monetary. How is equity balanced with efficiency? How is cost and donor interest considered? How should uncertainty be modeled? The objective functions of several models are described. Like any decision tool, the models have values embedded in them. For the decisions to reflect the right values, the model must align with the organization's values. I will report on a study of how the values and priorities of Christian relief organizations differ in ways that can - or should - be reflected in their logistics procedures and these models. The study used interviews with World Vision and secular organizations and a survey of logisticians. I will also describe plans to assess these decision-maker's values.

Cultivating Mathematical Affections through Engagement in Service-Learning

Joshua Wilkerson

Why should students value mathematics? While extensive research exists on developing the cognitive ability of students, very little research has examined how to cultivate the affections of students for mathematics. The phrase "mathematical affections" is a play on the affective domain of learning as well as on the general notion of care towards something. Mathematical affections are more than a respect for the utility of the subject; the term is much broader and includes aesthetic features as well as habits of mind and attitude.

This paper will analyze the findings from a research project exploring the impact of service-learning on the cultivation of mathematical affections in students. This was a qualitative case study of high school students who recently completed a service-learning project in their mathematics course. Data was gathered from student interviews, reflection journals, and field observations. The framework for the analysis follows the definition of "productive disposition" offered by the National Research Council (2001) as well as the concept of formative "cultural liturgies" offered by the philosopher James K.A. Smith (2009).

The major themes that emerge from the data indicate that through service-learning students see math as sensible, useful, and worthwhile. This supports the potential of service-learning as a pedagogical tool that can be utilized to develop a productive disposition in students; addressing at a practical level how the affective objectives of national policy documents can be achieved.

Mathematics/STEM Study Abroad across Europe

Nicholas J. Willis

Are you interested in taking your Math majors/STEM students on a trip to Europe? Nick Willis has lead 21-day study abroad trips to Europe four times and is ready to share his experiences with you. In this talk Nick will give ideas about places that you might take your Math students in Europe and some of the lessons he has learned about leading STEM trips. Countries Nick has lead trips to include Spain, France, England, Switzerland, Germany, Austria and Italy.

On Beyond Calculus: A Day for Community Outreach

Rebekah Yates

Each February, around 100 high school students descend on Houghton's small campus in western New York to take the American Mathematics Competition and participate in several sessions designed to inspire them in math or science. Our department's goal is always to show them that there is more to math than the march to calculus. We have offered sessions on a variety of topics from voting to game theory to spherical geometry. In this talk, I will share some of the materials I've used to introduce high school students to binary, non-standard metrics, and infinity.

Portuguese Mathematical History

Maria Zack

Portuguese mathematical history is quite different from that in the rest of western Europe. In the 1500's Portugal was a world power with a large shipping fleet that navigated the globe and took trade and settlers to new colonies. The Portuguese also built significant fortresses in their far-flung colonies. However by the 1700's, the mathematics taught at the universities in Portugal was significantly less advanced than what was being taught in neighboring countries. This talk will discuss Portuguese history and trace the unique role that the military academies played in modernizing Portuguese mathematics in the late 1700's.

An Analysis of SNU Chapel Attendance Data

Nicholas Zoller

Southern Nazarene University (SNU) requires its traditional undergraduate students to attend chapel services as an expression of its commitment to faithful living in Christian community. Chapel services are held regularly on Tuesday and Thursday mornings for the entire SNU community. Other chapel opportunities are available throughout the week. Traditional undergraduate students must attend at least 27 chapels each semester in order to pass the chapel requirement. Failure to meet the requirement results in a fine or suspension from SNU.

We present a descriptive analysis of SNU chapel attendance data for the Fall 2014, Spring 2015, and Fall 2015 semesters. We find that female students have better chapel attendance completion rates than male students, and student athletes have better chapel attendance completion rates than non student athletes. We also examine chapel attendance trends over the course of each semester by day of the week (e.g. Tuesday or Thursday) and by type of chapel (e.g. regular chapel or alternative chapel service).

This work was completed during the summer of 2016 by an SNU math major. We con- clude by noting some of the lessons learned from using institutional data for an undergraduate statistics research project.

A Framework for Integrating Faith and Learning in the K-12 Mathematics Classroom

Ryan Zonnefeld and Valorie Zonnefeld

As middle school and high school mathematics teachers, we were challenged by the desire to integrate our faith in the mathematics classroom. Over the past year, in consultation with kindergarten through college-level mathematics educators, we have co-authored a document to give K-12 teachers ideas for teaching mathematics faithfully organized by grade level at the elementary level and domain at the secondary level. The document is intended to be a springboard for teachers and is organized around the Teaching For Transformation Through lines while using the structure of the Common Core State Standards for mathematics. As educators continue to speak into the document it expands and morphs. This talk will briefly share the development process, give an overview of the design, and present the current version of the framework for you to use. There will also be time for questions and feedback allowing you to "speak into" the document.

Appendix 2: Conference Attendees

First	Last	Institution	email
Aaron	Allen	Milligan College	aaallen@milligan.edu
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