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Indiana Wesleyan University Conference Attendees, May 29 - June 1, 2019

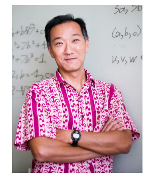
Table of Contents

Introduction

The twenty-second biennial conference of the Association of Christians in the Mathematical Sciences was held at Indiana Wesleyan University from May 29 until June 1, 2019. Thanks go to Melvin Royer and his colleagues at IWU for all their efforts that went into hosting it.

Many thanks also to the three invited speakers:

- Ken Ono (University of Virginia)



  - *Why does Ramanujan, "The Man Who Knew Infinity," Matter?*
  - *The Jensen-Pólya Program for the Riemann Hypothesis*

  Ken Ono (B.A., University of Chicago; Ph.D., UCLA) is the Thomas Jefferson Professor of Arts and Sciences at the University of Virginia. He has been awarded an NSF CAREER grant, a David and Lucile Packard Fellowship, a Presidential Early Career Award, and a Guggenheim Fellowship. He is currently a Vice President of the American Mathematical Society. He is a renowned expositor with recognition as the MAA George Pólya Distinguished Lecturer, and recent winner of the Prose Award for Best Scholarly Book in Mathematics. His passion for developing and sharing the ideas of Ramanujan is illustrated by his recent role as associate producer of the film *The Man Who Knew Infinity.*

- Joan Richards (Brown University)

  - *From Theology to the Negative Numbers in the World of William Frend*
  - *The Foundations of Calculus in the World of Augustus De Morgan*

  Joan Richards (B.A., Radcliffe College; Ph.D., Harvard University) is the Director of the Program of Science and Technology Studies and Professor of History at Brown University. She received a Guggenheim Fellowship, and served as fellow at the Max Planck Institute. Her book *Mathematical Visions: Non-Euclidean Geometry in Victorian England* focuses on the reception of a geometrical theory in the wider culture of nineteenth century England. Her book *Angles of Reflection* is at once a memoir and an exploration of the logical work and family life of Augustus De Morgan. Her projects are linked by an abiding interest in the ways that mathematics has served as a model of human thinking.

- Michael Alford (FBI, Cyber Division)

  - *Threats and Successes of Cybersecurity*

  Michael Alford has been a Special Agent with the FBI in Indianapolis for over ten years. Special Agent Alford has a Master's of Science in Digital Forensics from the University of Central Florida, and has over 15 years of experience in computer security and computer forensics. Special Agent Alford routinely investigates complex computer intrusions, including national security intrusions and intrusions into industrial control systems.

Two workshops were offered on May 29. Ryan Botts, Judith Canner, and Alana Unfried organized a seminar on using *R Studio*, and in a parallel session Greg Crow, Lori Carter, Catherine Crockett, Matt DeLong, Derek Schurrman, and Amanda Harsy Ramsey discussed early career issues for new faculty and graduate students. Many thanks to them, not only for organizing the sessions, but also for the helpful information they provided.

The conference schedule is presented in Appendix 1, Appendix 2 gives the abstracts for the parallel sessions, and Appendix 3 lists information for the individual participants.

There were a total of 79 papers presented by the 138 conference attendees. Not every paper was submitted to these *Proceedings*, but the following pages contain the ones that made their way through the single-blind review process, each having been scrutinized by a minimum of two reviewers. Thanks go to the authors for their good work. Too numerous to mention are all the referees that were involved, but heartfelt thanks go to them for their diligence.

The twenty-third biennial conference for ACMS, set to be hosted by Azusa Pacific University, is slated for June 23–26, 2021. Plans are to hold it in conjunction with the Christian Engineering Society. Details can be found at acmsonline.org, which is the official ACMS website.

Russell W. Howell (Westmont College)
ACMS *Proceedings* Editor

From Perfect Shuffles to Landau's Function

Brian D. Beasley (Presbyterian College)

Brian Beasley (Ph.D., University of South Carolina) has taught at Presbyterian College since 1988. His professional interests range from number theory and history of mathematics to problems sections in various journals. Recently he has helped recruit mentors for ECM and Project NExT fellows on behalf of the MAA Southeastern Section. Brian and his wife Kathy are now in the empty nest stage, thankful that both sons are gainfully employed college graduates.

Abstract

If we view a given shuffle of a deck of cards as a permutation, then repeatedly applying this same shuffle will eventually return the deck to its original order. In general, how many steps will that take? What happens in the case of so-called perfect shuffles? What type of shuffle will require the greatest number of applications before restoring the original deck? This paper will address those questions and provide a brief history of the work of Edmund Landau on the maximal order of a permutation in the symmetric group on n objects. It will also note some recent progress in refining his results.

1 Shuffling Cards

1.1 Perfect Shuffles as Permutations

Given a deck with an even number of cards, we define a *perfect shuffle* as one which splits the deck into two halves and then interlaces them perfectly, as noted in [5] and [18]. Our goal is to view such a shuffle as a permutation, so we start by providing an example with $n = 6$ cards.

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline 6\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline 5\\\hline 6\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 4\\\hline 2\\\hline 5\\\hline 3\\\hline 6\\\hline\end{array}$$

Permutation: $p = (1)(2\ 3\ 5\ 4)(6)$

This type of perfect shuffle, known as an *out-shuffle*, leaves the top card on top and the bottom card on bottom. A perfect shuffle which moves the top card to the second position in the deck is an *in-shuffle*. We compare the in-shuffle on $n = 6$ cards with the previous corresponding out-shuffle.

$$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline 5\\ \hline 6\\ \hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\quad \begin{array}{|c|}\hline 4\\ \hline 5\\ \hline 6\\ \hline\end{array} \longrightarrow \begin{array}{|c|}\hline 4\\ \hline 1\\ \hline 5\\ \hline 2\\ \hline 6\\ \hline 3\\ \hline\end{array}$$

Permutation: $q = (1\ 2\ 4)(3\ 6\ 5)$

We have written these permutations as products of disjoint cycles, in order to view them as elements of the symmetric group $\mathcal{S}_n$ on n objects (where the group operation is composition). In particular, we wish to compute their orders in the group, so we apply the fact that the order is the least common multiple of the cycle lengths. When $n = 6$, the order of the out-shuffle in $\mathcal{S}_6$ is 4, which means that four repeated out-shuffles will return the deck to its original order. Similarly, the order of the in-shuffle is 3, so three repeated in-shuffles will return the deck to its original order. In general, we would like to know the order of an out-shuffle and an in-shuffle in $\mathcal{S}_n$ when n is even.

Even more generally, we also hope to calculate the maximal order of an element in this group and to determine how many elements achieve the maximal order. Equivalently, we seek the type of shuffle of a deck of cards that would take the most repeated applications to restore the original deck order. In the next section, we examine the order structure of $\mathcal{S}_n$ for small values of n and give results for the standard 52-card deck.

1.2 Partitions and Orders for Out-Shuffles and In-Shuffles

In abstract algebra classes, the symmetric group on n objects provides students with a useful early example of a non-abelian group for $n \geq 3$. Since $\mathcal{S}_3$ is the smallest such group, we start by determining its order structure and counting the number of elements which have maximal order. Listing the elements of this group as

$$\mathcal{S}_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\},$$

we may express its order structure in the form 1-1, 3-2, 2-3 (one element of order 1, three elements of order 2, and two elements of order 3). So the maximal order in $\mathcal{S}_3$ is 3, achieved by two elements. Of course, the fact that every element in $\mathcal{S}_3$ is a cycle simplifies these calculations considerably, but $\mathcal{S}_n$ does not have this property for $n \geq 4$.

Accordingly, in moving to an analysis of $\mathcal{S}_4$ and $\mathcal{S}_5$, we note that the orders of elements correspond to partitions into disjoint cycle lengths, so we must apply the least common multiple property. For example, in $\mathcal{S}_5$ the order of any 5-cycle, such as $(2\ 5\ 1\ 4\ 3)$, is 5; similarly, the order of the product of a 3-cycle with a disjoint 2-cycle, such as $(2\ 5\ 1)(4\ 3)$, is 6. In particular, in order to count the number of elements of each possible order in $\mathcal{S}_n$, we must know all the different types of partitions of n. Examining $\mathcal{S}_4$ in detail yields:

Partition of 4	Order in $\mathcal{S}_4$	Number with Order
4	4	$4!/4 = 6$
3+1	3	$4!/3 = 8$
2+2	2	$4!/(2^2 \cdot 2!) = 3$
2+1+1	2	$4!/(2 \cdot 2!) = 6$
1+1+1+1	1	$4!/4! = 1$

Hence the order structure of $\mathcal{S}_4$ is 1-1, 9-2, 8-3, 6-4; in particular, six elements achieve the maximal order of 4.

Similarly, analyzing $\mathcal{S}_5$ produces:

Partition of 5	Order in $\mathcal{S}_5$	Number with Order
5	5	$5!/5 = 24$
4+1	4	$5!/4 = 30$
3+2	6	$5!/(3 \cdot 2) = 20$
3+1+1	3	$5!/(3 \cdot 2!) = 20$
2+2+1	2	$5!/(2^2 \cdot 2!) = 15$
2+1+1+1	2	$5!/(2 \cdot 3!) = 10$
1+1+1+1+1	1	$5!/5! = 1$

Hence the order structure of $\mathcal{S}_5$ is 1-1, 25-2, 20-3, 30-4, 24-5, 20-6; in particular, twenty elements achieve the maximal order of 6.

But this approach of listing all possible partitions is certainly not practical for a standard 52-card deck, since 52 may be partitioned in 281,589 ways [20]. Even so, we are able to determine which of these partitions corresponds to an out-shuffle and which corresponds to an in-shuffle, allowing us to calculate the order of both types of perfect shuffles in the corresponding group. Later, we will also find the partitions which correspond to the maximal order as well as calculating the number of elements in $\mathcal{S}_{52}$ with that order.

We start with the out-shuffle on a standard deck of 52 cards:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline .\\\hline .\\\hline .\\\hline 49\\\hline 50\\\hline 51\\\hline 52\\\hline\end{array}
\;\longrightarrow\;
\begin{array}{|c|}\hline 1\\\hline 2\\\hline .\\\hline .\\\hline .\\\hline 25\\\hline 26\\\hline\end{array}
\;
\begin{array}{|c|}\hline 27\\\hline 28\\\hline .\\\hline .\\\hline .\\\hline 51\\\hline 52\\\hline\end{array}
\;\longrightarrow\;
\begin{array}{|c|}\hline 1\\\hline 27\\\hline 2\\\hline 28\\\hline .\\\hline .\\\hline .\\\hline 25\\\hline 51\\\hline 26\\\hline 52\\\hline\end{array}$$

Permutation: $p = \ ?$

We represent this out-shuffle as the following permutation on $\{1, 2, \ldots, 52\}$:

$$p(x) = \begin{cases} 2x-1 & \text{if } x \leq 26 \\ 2(x-26) & \text{if } x \geq 27 \end{cases}$$

We also note that $p(x) \equiv 2x - 1 \pmod{51}$. Then $p = (1)(52)(18\ 35)uvwxyz$, where

$$u = (2\ 3\ 5\ 9\ 17\ 33\ 14\ 27),$$

$$v = (4\ 7\ 13\ 25\ 49\ 46\ 40\ 28),$$

$$w = (6\ 11\ 21\ 41\ 30\ 8\ 15\ 29),$$

$$x = (10\ 19\ 37\ 22\ 43\ 34\ 16\ 31),$$

$$y = (12\ 23\ 45\ 38\ 24\ 47\ 42\ 32),$$

$$z = (20\ 39\ 26\ 51\ 50\ 48\ 44\ 36).$$

Thus the order of p in $\mathcal{S}_{52}$ is 8. This means that a skilled card shark who is able to perform an out-shuffle each time without any errors will restore the deck to its original order after only eight of these shuffles.

Next, we compare the in-shuffle on 52 cards and compute its order:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline .\\\hline .\\\hline .\\\hline 49\\\hline 50\\\hline 51\\\hline 52\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline .\\\hline .\\\hline .\\\hline 25\\\hline 26\\\hline\end{array}\quad\begin{array}{|c|}\hline 27\\\hline 28\\\hline .\\\hline .\\\hline .\\\hline 51\\\hline 52\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 27\\\hline 1\\\hline 28\\\hline 2\\\hline .\\\hline .\\\hline .\\\hline 51\\\hline 25\\\hline 52\\\hline 26\\\hline\end{array}$$

Permutation: $q = \ ?$

We may represent this in-shuffle by $q(x) \equiv 2x \pmod{53}$, or by

$$q(x) = \begin{cases} 2x & \text{if } x \leq 26 \\ 2(x-26)-1 & \text{if } x \geq 27 \end{cases}.$$

Then $q = (1\ 2\ 4\ 8\ 16\ 32\ 11\ 22\ 44\ 35\ 17\ 34\ 15\ 30\ 7\ 14\ 28\ 3\ 6\ 12\ 24\ 48\ 43\ 33\ 13\ 26\ 52\ 51\ 49\ 45\ 37\ 21\ 42\ 31\ 9\ 18\ 36\ 19\ 38\ 23\ 46\ 39\ 25\ 50\ 47\ 41\ 29\ 5\ 10\ 20\ 40\ 27)$.

Thus the order of q in $\mathcal{S}_{52}$ is 52. Our card shark would find it more challenging to make 52 consecutive in-shuffles to restore the deck.

In general, Diaconis, Graham, and Kantor have noted and proved the following "well known" results [5].

Theorem 1. Given a deck of $n = 2m$ cards:

(i) The order of an out-shuffle equals the order of 2 modulo $2m - 1$.

(ii) The order of an in-shuffle equals the order of 2 modulo $2m + 1$.

(iii) The order of an in-shuffle in $\mathcal{S}_n$ is the same as the order of an out-shuffle in $\mathcal{S}_{n+2}$.

We use this theorem to verify the results in our previous examples. For $n = 6$, we note that the order of 2 modulo 5 is 4, since the powers of 2 modulo 5 cycle as 2, 4, 3, 1, ... ; similarly, the order of 2 modulo 7 is 3, as the powers of 2 modulo 7 cycle as 2, 4, 1, ... For $n = 52$, we leave it as an exercise for the reader to confirm that the order of 2 modulo 51 is 8, while the order of 2 modulo 53 is 52.

In an attempt to generalize the notion of a perfect shuffle, we offer the following question: Given a deck with $n = 3m$ cards, how would we define a *perfect 3-shuffle*? The natural approach for such a shuffle would be to divide the deck into three equal piles and then interlace them perfectly. Here is one example with $n = 6$ cards:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline 6\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\quad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\quad \begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline 2\\\hline 4\\\hline 6\\\hline\end{array}$$

Permutation: $r = (1)(2\ 4\ 5\ 3)(6)$

We note that for each deck with $n = 3m$ cards, there are $3! = 6$ perfect 3-shuffles, as opposed to just two perfect 2-shuffles. How do the orders of these six permutations compare? Given an element σ in $\mathcal{S}_3$, we define a σ-shuffle to be the perfect 3-shuffle in which the order of interlacing the three piles of cards follows the order of the elements as listed in σ. We calculate those orders for several small values of n.

n	(1 2 3)	(1 3 2)	(2 1 3)	(2 3 1)	(3 1 2)	(3 2 1)
3	1	2	2	3	3	2
6	4	4	4	6	6	6
9	2	4	4	6	6	4
12	5	21	21	11	11	3
15	6	30	30	15	15	4
18	16	90	90	52	52	18
21	4	24	24	38	38	5
24	11	48	48	51	51	20
27	3	6	6	9	9	6
30	28	28	28	360	360	30

We offer the following conjectures concerning perfect 3-shuffles:

1. The order of a (1 2 3)-shuffle equals the order of 3 modulo $3m - 1$.
2. The order of a (3 2 1)-shuffle equals the order of 3 modulo $3m + 1$.
3. The order of a (1 3 2)-shuffle equals the order of a (2 1 3)-shuffle.
4. The order of a (2 3 1)-shuffle equals the order of a (3 1 2)-shuffle.

1.3 Maximal Order in $\mathcal{S}_{52}$

If our goal instead is to determine the type of shuffle that would take the most repeated applications to restore a standard deck, then we must find the partition of 52 that maximizes the least common multiple of its part lengths. For example, dividing the deck into two piles, one with 25 cards and the other with 27, and simply cycling the cards in each pile separately yields a permutation with order $\operatorname{lcm}(25, 27) = 675$ in $\mathcal{S}_{52}$. Experimenting further leads to additional partitions in which the parts are prime powers that are pairwise relatively prime, such as:

$$\begin{aligned} 52 = 32 + 9 + 11: &\quad \text{order} = \operatorname{lcm}(32,9,11) = 3168 \\ 52 = 3 + 13 + 17 + 19: &\quad \text{order} = \operatorname{lcm}(3,13,17,19) = 12{,}597 \\ 52 = 1 + 3 + 7 + 11 + 13 + 17: &\quad \text{order} = \operatorname{lcm}(1,3,7,11,13,17) = 51{,}051 \end{aligned}$$

As noted in [11], the maximum order of an element in $\mathcal{S}_{52}$ is 180,180. The partition $52 = 1 + 1 + 1 + 4 + 9 + 5 + 7 + 11 + 13$ corresponds to such an element, as

$$\operatorname{lcm}(1, 1, 1, 4, 9, 5, 7, 11, 13) = 180, 180.$$

We observe that the partitions $52 = 1+2+4+9+5+7+11+13$ and $52 = 3+4+9+5+7+11+13$ also correspond to elements in $\mathcal{S}_{52}$ of order 180,180; in fact, this represents the maximal order in $\mathcal{S}_{49}$ as well, since any partition of the three "extra" cards does not contribute to an increase in the least common multiple.

We conclude this section by calculating the number of elements in $\mathcal{S}_{52}$ with maximal order 180,180. The number of permutations in $\mathcal{S}_{52}$ corresponding to the partition $52 = 1+1+1+4+9+5+7+11+13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} \approx 7.460889 \times 10^{61}.$$

Next, the number corresponding to the partition $52 = 1 + 2 + 4 + 9 + 5 + 7 + 11 + 13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 2} \approx 2.238267 \times 10^{62}.$$

Finally, the number corresponding to the partition $52 = 3 + 4 + 9 + 5 + 7 + 11 + 13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3} \approx 1.492178 \times 10^{62}.$$

Hence the number of permutations in $\mathcal{S}_{52}$ having maximal order is

$$\frac{6 \cdot 52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} \approx 4.476533 \times 10^{62}.$$

We also note that these permutations are rare in $\mathcal{S}_{52}$, as the probability of choosing such a permutation at random is

$$\frac{6}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} = \frac{1}{180,180}.$$

These results in $\mathcal{S}_{52}$ provide a transition into the next section, in which we address the same questions for $\mathcal{S}_n$ in general. Our focus will be to summarize the contributions of Edmund Landau and others in the study of what is now known as Landau's function.

2 The Life and Legacy of Landau

2.1 Background

Following the summary found in [17], we offer just a brief outline of the biographical highlights of Edmund Landau. Born in Berlin in 1877, Landau remained in the same city to pursue his entire mathematical education, culminating in his doctorate in 1899 under Georg Frobenius. He taught at the University of Berlin for the next ten years, during which he gave a simpler proof of the Prime Number Theorem. He also established an asymptotic result for the maximal order of an element in $\mathcal{S}_n$, which will be the focus of our next section. In 1909, Landau succeeded Hermann Minkowski at the University of Göttingen. Three years later, he challenged the Fifth Congress of Mathematicians to prove or disprove the following four conjectures concerning prime numbers [19]:

1. Goldbach: Every even integer $n \geq 4$ may be written as the sum of two primes.
2. Twin Prime: There are infinitely many primes p such that $p + 2$ is also prime.
3. Legendre: For every integer n, there is a prime p with $n^2 < p < (n + 1)^2$.
4. Euler: There are infinitely many primes of the form $n^2 + 1$.

All four remain unresolved to this day.

After World War I, both Landau and the University of Göttingen flourished, and many promising young mathematicians received inspiration and influence from Landau as well as his colleagues Felix Klein and David Hilbert [8]. Landau's research focused primarily upon analytic number theory, covering a broad range of topics from the distribution of prime ideals to the Riemann zeta function [12]. But in 1933, due to the increasing pressure of anti-Semitism in Germany, Landau was forced to retire from Göttingen; in fact, one of his own students, Oswald Teichmüller, played a major role in Landau's ouster [2]. After this tragic turn of events, Landau was able to lecture only outside Germany. Reflecting upon one of his visits to England, Hardy and Heilbronn noted in [8], "His enforced retirement must have been a terrible blow to him; it was quite pathetic to see his delight when he found himself again in front of a blackboard in Cambridge, and his sorrow when his opportunity came to an end." Ultimately Landau returned to Germany, and he died of a heart attack in Berlin in 1938 [17].

As previously noted, one of Landau's research interests was studying the maximal order of an element in the symmetric group $\mathcal{S}_n$. In his honor, the function $g(n)$ which gives this maximal order for any positive integer n is now known as Landau's function. Equivalently, as we have seen, $g(n)$ is the maximum least common multiple of the parts of a partition of n; the observation that the parts may be taken to be prime powers was first proved by Landau as well.

Before giving Landau's main result in the next section, we examine $g(n)$ for some small values of n and offer several conjectures.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$g(n)$	1	2	3	4	6	6	12	15	20	30	30	60	60	84	105	140	210	210

By its definition, g is a non-decreasing function, but we note that it is not strictly increasing. For instance, from our earlier card examples, we have $g(49) = g(50) = g(51) = g(52) = 180,180$. Also, g never seems to increase by more than a factor of two for consecutive values of n. As we will observe later in the paper, of the following five conjectures about Landau's function, three are true, one is false, and one remains unresolved.

1. For $n > 15$, $g(n)$ is even.
2. The function g is constant on arbitrarily long intervals.
3. For every n, $g(n+1) \leq 2g(n)$.
4. For infinitely many n, $g(n+1) = 2g(n)$.
5. The function g is strictly increasing on arbitrarily long intervals.

2.2 Landau's Theorem

As noted in [4] and [11], in 1903 Landau proved his famous result concerning $g(n)$, publishing his theorem in [9]. He first established the connection between $g(n)$ and the partitions of n, noting that

$$g(n) = \max\{\mathrm{lcm}(n_1, n_2, \ldots, n_r)\},$$

where this maximum is taken over all partitions $n = n_1 + n_2 + \cdots + n_r$. Landau also showed that without loss of generality, the maximum may be taken over all partitions with prime powers as parts; more precisely, he proved that if $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for distinct primes p_i with $e_i > 0$, and if $\ell(m) = p_1^{e_1} + p_2^{e_2} + \cdots + p_k^{e_k}$, then

$$g(n) = \max\{m : \ell(m) \leq n\}.$$

Using those observations, Landau was able to describe the asymptotic behavior of $g(n)$ in the following result.

Theorem 2. Using log to denote the natural logarithm, we have $\log g(n) \sim \sqrt{n \log n}$ as n goes to infinity; equivalently,

$$\lim_{n\to\infty} \frac{\log g(n)}{\sqrt{n \log n}} = 1.$$

In [11], Miller provided an alternative proof of Landau's result by applying the following technique. Given a positive integer n, find k such that the sum of the first k primes $2, 3, 5, \ldots, p_k$ is less than or equal to n but would exceed n if p_{k+1} were included. We note that this corresponds to seeking a partition of n using the first k primes, with 1's included as needed. Then let $f(n)$ equal the product of these k primes. For example, $f(18) = 2 \cdot 3 \cdot 5 \cdot 7 = 210 = g(18)$. On the other hand, as Miller observed,

$$f(52) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30,030 < 180,180 = g(52),$$

so at first glance, we would not expect $f(n)$ to keep up with $g(n)$ as n increases. Nevertheless, Miller was able to prove that $\log f(n) \sim \log g(n)$. He then established Landau's result by showing that $\log f(n) \sim \sqrt{n \log n}$.

As we noted earlier in the case $n = 52$, it is interesting to compare the size of $g(n)$ to $n!$, the order of $\mathcal{S}_n$. We also examine the question of how many elements in $\mathcal{S}_n$ can be expected to have maximal order $g(n)$. First of all, using Landau's theorem, we observe that $g(n)$ is very small compared to the order of $\mathcal{S}_n$, since Stirling's formula implies

$$\log n! \sim n \log n - n.$$

Next, we let $h(n)$ be the number of elements in $\mathcal{S}_n$ with order $g(n)$. The following table compares $g(n)$ and $h(n)$ for small values of n.

n	1	2	3	4	5	6	7	8	9
$g(n)$	1	2	3	4	6	6	12	15	20
$h(n)$	1	1	2	6	20	240	420	2688	18144
$n!$	1	2	6	24	120	720	5040	40320	362880

We observe that in some, but not all, cases, $g(n)h(n) = n!$. In fact, not many elements in $\mathcal{S}_n$ have order $g(n)$. More precisely, Erdős and Turán showed in [6] that almost all permutations in $\mathcal{S}_n$ have order k with

$$\log k \sim \frac{1}{2} \log^2 n.$$

2.3 Conjectures for Landau's Function

In this section, we return to the five conjectures previously mentioned, and we note several other results and open problems involving Landau's function. The first three of those five conjectures are true, having been proved by Nicolas [15, 16]:

1. For $n > 15$, $g(n)$ is even.
2. The function g is constant on arbitrarily long intervals.
3. For every n, $g(n+1) \leq 2g(n)$.

However, the fourth conjecture is false—there are not infinitely many values of n for which $g(n+1) = 2g(n)$. On the contrary, Nicolas showed in [14] that

$$\lim_{n\to\infty} \frac{g(n+1)}{g(n)} = 1.$$

As for the fifth conjecture, it remains open. In [13], Nicolas conjectured that for each $k \geq 2$, there are infinitely many n with

$$g(n-1) = g(n) < g(n+1) < \cdots < g(n+k-1) = g(n+k).$$

In [4], Deléglise, Nicolas, and Zimmermann noted that there are only nine values of $n \leq 10^6$ with $k \geq 7$, and the current "record" value of k is $k = 20$, starting at $n = 35,464$.

Another active area of research for Landau's function involves the prime factorization of values of $g(n)$. In order to examine some of the patterns in these factors, we offer the following table, taken from [16].

n	$g(n)$	prime factorization
25	1260	$4\cdot 9\cdot 5\cdot 7$
27	1540	$4\cdot 5\cdot 7\cdot 11$
28	2310	$2\cdot 3\cdot 5\cdot 7\cdot 11$
29	2520	$8\cdot 9\cdot 5\cdot 7$
30	4620	$4\cdot 3\cdot 5\cdot 7\cdot 11$
32	5460	$4\cdot 3\cdot 5\cdot 7\cdot 13$
34	9240	$8\cdot 3\cdot 5\cdot 7\cdot 11$
36	$13,860$	$4\cdot 9\cdot 5\cdot 7\cdot 11$
38	$16,380$	$4\cdot 9\cdot 5\cdot 7\cdot 13$
40	$27,720$	$8\cdot 9\cdot 5\cdot 7\cdot 11$
41	$30,030$	$2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$
42	$32,760$	$8\cdot 9\cdot 5\cdot 7\cdot 13$
43	$60,060$	$4\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$
47	$120,120$	$8\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$
49	$180,180$	$4\cdot 9\cdot 5\cdot 7\cdot 11\cdot 13$
53	$360,360$	$8\cdot 9\cdot 5\cdot 7\cdot 11\cdot 13$
57	$471,240$	$8\cdot 9\cdot 5\cdot 7\cdot 11\cdot 17$

Given a prime p and a positive integer m, we let $\nu_p(m)$ denote the largest integer k such that p^k divides m. In addition, for each n, we denote by P_n the largest prime that divides $g(n)$. Nicolas showed in [15] that: for primes $p < q$ with $\alpha = \nu_p(g(n))$ and $\beta = \nu_q(g(n))$, $\beta \leq \alpha+1$; $\nu_{P_n}(g(n)) = 1$ unless $n = 4$; and

$$P_n \sim \log g(n) \sim \sqrt{n \log n}.$$

In [13], Nicolas also established that for any prime p, there is a positive integer n such that the largest prime factor of $g(n)$ is p. Thus for each prime p, we may define n_p to be the smallest input for which p divides $g(n_p)$. Based on the following table for small values of p, it is tempting to conjecture that n_p is increasing as a function of p.

p	2	3	5	7	11	13	17	$\ldots$
n_p	2	3	8	14	27	32	57	$\ldots$

However, n_p is not increasing as a function of p. In fact, as seen in [10], the smallest counterexample occurs when $n_{67} > n_{71}$:

p	...	61	67	71	73	...
n_p	...	429	519	510	586	...

2.4 Recent Progress

Research continues on refining the asymptotic result in Landau's main theorem. For example, as noted in [10], Massias obtained the following bounds in 1984:

- For $n \geq 2$, $\log g(n) \leq \sqrt{n \log n}\left(1 + \dfrac{\log \log n}{2 \log n}\right)$.
- For $n \geq 906$, $\quad \log g(n) \geq \sqrt{n \log n}$.

He also showed that $\max\limits_{n>1} \left\{ \dfrac{\log g(n)}{\sqrt{n \log n}} \right\} = 1.05313\ldots$, with the maximum attained at $n = 1,319,166$. In 1989, Massias, Nicolas, and Robin established improved bounds [10]:

- For $n \geq 3$, $\log g(n) \leq \sqrt{n \log n}\left(1 + \dfrac{\log \log n - 0.975}{2 \log n}\right)$.
- For $n \geq 810$, $\log g(n) \geq \sqrt{n \log n}\left(1 + \dfrac{\log \log n - 2}{2 \log n}\right)$.

They also conjectured that for $n \geq 4$,

$$\frac{P_n}{\sqrt{n \log n}} \leq 1.265\ldots \ ,$$

with the maximum value occurring when $n = 215$. In 2012, Deléglise and Nicolas proved this conjecture [3].

For more research involving the largest prime P_n that divides $g(n)$, we note the 1995 result of Grantham [7]:

- For $n \geq 5$, $P_n \leq 1.328\sqrt{n \log n}$.

In 2012, Deléglise and Nicolas also established the following results [3]:

- For $n \geq 1755$, $\quad P_n \geq \sqrt{n \log n}$.
- There are infinitely many n with $P_n > \log g(n)$ and infinitely many n with $P_n < \log g(n)$.

On the computational side, in 2008, Deléglise, Nicolas, and Zimmermann proved the following result [4]:

Let $N = 2^{23}3^{15}5^{10}7^{8}11^{7}13^{6}17^{6}[19-31]^{5}[37-79]^{4}[83-389]^{3}[397-9623]^{2}[9629-192678817]$, where $[p-q]$ denotes the product of all primes between p and q, inclusive. Then the value of $g(10^{15})$ is:

$$\left(\frac{192678823 \cdot 192678853 \cdot 192678883 \cdot 192678917}{389 \cdot 9539 \cdot 9587 \cdot 9601 \cdot 9619 \cdot 9623 \cdot 192665881}\right) N.$$

Remaining computational challenges for Landau's function include finding the value of $g(10^n)$ for $n \geq 16$ and finding a value of $k > 20$ with

$$g(n) < g(n+1) < \cdots < g(n+k-1).$$

2.5 Final Word

Over one hundred years after Landau discovered his main result, his function continues to hold a certain fascination for mathematicians. With research still underway in both the theoretical and the computational realms, one may wonder whether this interest might wane at some point. On the other hand, as related in David Burton's *Elementary Number Theory* [1], Peter Barlow predicted in his own 1811 number theory text that the eighth perfect number $2^{30}(2^{31}-1)$ would be "the greatest that ever will be discovered; for as they are merely curious, without being useful, it is not likely that any person will ever attempt to find one beyond it." Perhaps Burton's response applies to the ongoing work on Landau's function as well:

"Barlow underestimated obstinate human curiosity."

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References

[1] D. Burton, *Elementary Number Theory*,7th edition, McGraw-Hill, 2010.

[2] M. Chowdhury, Landau and Teichmüller, *The Mathematical Intelligencer*, 17(4), 1995, 12-14.

[3] M. Deléglise, J.-L. Nicolas, Le plus grand facteur premier de la fonction de Landau, *The Ramanujan Journal*, 27, 2012, 109-145.

[4] M. Deléglise, J.-L. Nicolas, P. Zimmermann, Landau's function for one million billions, *Journal de Théorie des Nombres de Bordeaux*, 20, 2008, 625-671; updated October 25, 2018 on arXiv.

[5] P. Diaconis, R. Graham, W. Kantor, The mathematics of perfect shuffles, *Advances in Applied Mathematics*, 4, 1983, 175-196.

[6] P. Erdős, P. Turán, On some problems of statistical group theory, *Zeitschr. fŭr Wahrscheinlichkeitstheorie und verw. Gebiete*, 4, 1965, 175-186.

[7] J. Grantham, The largest prime dividing the maximal order of an element of $\mathcal{S}_n$, *Mathematics of Computation*, 64(209), 1995, 407-410.

[8] G. Hardy, H. Heilbronn, Edmund Landau, *Journal of the London Mathematical Society*, 13, 1938, 302-310.

[9] E. Landau, Über die Maximalordnung der Permutationen gegebenen Grades, *Archiv der Math. und Phys.*, Ser. 3(5), 1903, 92-103.

[10] J.-P. Massias, J.-L. Nicolas, G. Robin, Effective bounds for the maximal order of an element in the symmetric group, *Mathematics of Computation*, 53(188), 1989, 665-678.

[11] W. Miller, The maximum order of an element of a finite symmetric group, *The American Mathematical Monthly*, 94(6), 1987, 497-506.

[12] L. Mirsky, In memory of Edmund Landau: Glimpses from the panorama of number theory and analysis, *The Mathematical Scientist*, 2, 1977, 1-26.

[13] J.-L. Nicolas, On Landau's function $g(n)$, in *The Mathematics of Paul Erdős I*, editors R. Graham and J. Nešetřil, Algorithms and Combinatorics, Springer Verlag, vol. 13, 1996, 228-240.

[14] J.-L. Nicolas, Ordre maximal d'un élément d'un groupe de permutations, *C.R. Acad. Sci. Paris*, 270, 1970, 1473-1476.

[15] J.-L. Nicolas, Ordre maximum d'un élément du groupe de permutations et highly composite numbers, *Bull. Soc. Math. France*, 97, 1969, 129-191.

[16] J.-L. Nicolas, Sur l'ordre maximum d'un élément dans le groupe S_n des permutations, *Acta Arithmetica*, 14, 1968, 315-332.

[17] J. O'Connor, E. Robertson, "Edmund Landau." MacTutor History of Mathematics. Available at http://www-history.mcs.st-and.ac.uk/Biographies/Landau.html

[18] Su, Francis E., et al. "Perfect Shuffles." Math Fun Facts. Available at https://math.hmc.edu/funfacts/perfect-shuffles/

[19] E. Weisstein, "Landau's Problems." From MathWorld–A Wolfram Web Resource. Available at http://mathworld.wolfram.com/LandausProblems.html

[20] D. Wilson, "10001 terms of OEIS sequence A000041." Available at https://oeis.org/A000041/b000041.txt

The Applicability of Mathematics and The Naturalist Die

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Abstract

Philosopher and Christian apologist William Lane Craig has proposed a valid deductive argument for God's existence that is rooted in the applicability of mathematics to the physical universe. This argument was presented by Craig during a debate with philosopher and atheist Alex Rosenberg. During the debate, Rosenberg presented a rebuttal to the soundness of this argument by appealing to chance as an explanation to the applicability of mathematics to the physical universe. Rosenberg argues that some mathematics will apply to the universe when considering all the alternative mathematics that don't apply to the universe. In this paper, we will contend that the naturalist position is unable to produce chance application of mathematics when assuming a mathematical fictionalist position. We shall then defend the soundness of the argument in light of the ontology and effectiveness of mathematics. Pertinently, the problem of God and abstract objects will be addressed. A new modified version of the argument will be proposed to emphasize the unreasonable effectiveness of mathematics. The paper concludes with a biblical motivation for the study of the argument.

1 Introduction

William Lane Craig is a prominent Christian apologist that has debated atheists and scholars over various decades. His most well-known book and website are both titled *Reasonable Faith* (see [6] and [14]). On February of 2013, William Lane Craig had a debate with philosopher and prominent atheist Alex Rosenberg at Purdue University, West Lafayette, Indiana (see [11] and [12]). The debate, titled *Is Faith in God Reasonable?*, consisted of various arguments for or against the reasonability of God's existence. In this paper, we shall scrutinize Craig's deductive argument for God's existence via the applicability of mathematics as presented at the debate. We shall label this argument **AM**. The argument consists of the following premises:

AM:

1. If God does not exist, then the applicability of mathematics would be a happy coincidence.
2. The applicability of mathematics is not a happy coincidence.
3. Therefore, God exists.

This deductive argument is valid by virtue of its Modus Tollens form. The term "happy coincidence" and its contextual usage must be clarified. "Happy coincidence" is a term borrowed from philosopher of mathematics Mary Leng to seemingly indicate an unlikely (small probability) correspondence between mathematical objects and the physical world (see [13]). To be clear, mathematical theory indeed yields predictable results in all of the natural sciences. Put another way, premise 2 is accepted by everyone based on the empirical consistency of applied mathematics. The debate consequently lies in premise 1. Essentially, with premise 1, Craig attempts to point out the inability of naturalism to address the astonishing effectiveness of mathematics. The reason for this, as Craig points out during the debate, is a default anti-platonist (or mathematical fictionalist) position. That is, Craig and Rosenberg both believe mathematical objects are fictions; mathematical fictions do not exist in some metaphysical sense other than as concepts. In this paper we will see that if mathematical objects are just fictions, and there is no God, the applicability of mathematics is unlikely.

Given that the second premise of **AM** is universally accepted, Rosenberg's rebuttal to the **AM** argument consists of attacking the truthfulness of the first premise in order to avoid the guaranteed conclusive third statement. Videlicet, since mathematical fictionalists must avoid coincidences, Rosenberg must show that in a godless universe the applicability of mathematics is not by happy coincidence. To do this, Rosenberg invokes the infinitely many mathematical objects and geometries that do not apply to the real world in order to establish math applications by stating that "any one of an indefinitely large number could perfectly well apply in the universe" (see [12]). To establish an analogy, Rosenberg is saying that when we roll a traditional die, one of the six sides will land. The difference of course is that Rosenberg's naturalist die contains infinitely many sides, each with a different geometry or set of mathematical rules and objects. Craig seems to deny Rosenberg's rebuttal by essentially saying that even if Rosenberg's naturalist die allows for some math to apply, the link between this die and reality is still unexplained.

In this paper, we will take a different approach from Craig to denying Rosenberg's rebuttal by examining the naturalist math die itself. In passing, we will explore the ontology of mathematics and its relationship to naturalism and theism. From this examination on the ontology of mathematics the Christian shall adopt the anti-platonist position due to a concern with the problem of God and abstract objects (see [9]). Particularly, our anti-platonism will be established via mathematical fictionalism. In other words we shall specifically deny the existence of mathematical objects, not the abstract nature of mathematical objects. While the naturalist is not concerned with the problem of God and abstract objects, we shall also see that the naturalist should take a fictionalist position for a different reason. We shall then present a somewhat enhanced version of Craig's argument inspired by his quoting of physicist Eugene Wigner. The paper will conclude with a biblical motivation of the persuasiveness and personal benefit of studying the unreasonable effectiveness of applied mathematics.

2 Platonism at odds with both theism and naturalism

In order to better understand the **AM** argument, we must study the ontology of mathematics as it seems that both Craig and Rosenberg are anti-platonists in this debate. As we will now argue, both theism and naturalism should adopt an anti-platonist position, albeit each for different reasons. To begin, let's explore the Christian perspective on the ontology of mathematics.

Mathematical platonism certainly entices the Christian mathematician. After all, mathematics seems to fit very nicely as eternal ideas in the mind of God. This has surely been the *Christian Platonist* position of Newton and others in the past (see [4]). However, to view mathematics as concepts or ideas in a mind, albeit divine, does not constitute pure platonism. Rather, said position is a pseudo-platonist position at best since pure or traditional platonism means something ontologically stronger than a concept. Platonism is summarized by the following two statements (see [9]):

1. Abstract objects exist. [Platonism]
2. If Abstract objects exist, then they are independent of God. [Platonist assumption]

Once again, we remind ourselves here that platonist existence does not refer to a mere conceptual existence. If a person imagines a unicorn, the concept of the unicorn exists, but the unicorn itself does not exist. Platonism is stronger than this since platonic existence goes beyond a concept or idea. Thus, assuming pure platonism means assuming that there are objects that exist independently from God. A Christian must consequently be careful to classify his or herself as a mathematical platonist. The care is warranted by the aseity-sovereignty doctrine (see [9]):

(i) God does not depend on anything distinct from Himself for his existing, and (ii) everything distinct from God depends on God's creative activity for its existing.

It is clear that Christians traditionally subscribe to the aseity-sovereignty doctrine. If we are to assume that both traditional platonism and the aseity-sovereignty doctrine are true, we inevitably arrive at the so-called problem of God and abstract objects (see [9]). This problem consists of an inconsistent triad that emanates from subscribing to both traditional Christianity and to traditional mathematical platonism:

Problem of God and Abstract Objects

1. Mathematical objects exist. [Mathematical Platonism]
2. If mathematical objects exist, then they are dependent on God. [Aseity-Sovereignty doctrine]
3. If mathematical objects exist, then they are independent of God. [Platonist assumption]

The first and third premises of this inconsistent triad are due to traditional platonism. Particularly, the third premise of the triad states that mathematical objects are independent of God for their existence. On the other hand, the aseity-sovereignty doctrine of Christianity requires that all existing things depend on God for their existence. Hence, one of the three premises in this inconsistent triad must be eliminated. Christian mathematicians must therefore consider their position carefully. While we shall suppress a proper and philosophical exegesis of the following, these passages are of utmost importance for Christians to consider in light of the inconsistent triad:

> *For from him and through him and to him are all things. To him be the glory forever. Amen.* -Romans 11:36

> *Worthy are you, our Lord and God, to receive glory and honor and power, for you created all things, and by your will they existed and were created.* -Revelation 4:11

In consideration of these passages for support of a strict aseity-sovereignty doctrine interpretation, we realize that the first and third statements of the inconsistent triad, traditional platonism, are where we shall fix the inconsistency. Specifically, we shall sacrifice the very existence of mathematical objects (the third premise of the inconsistent triad). To clarify, we are not denying the existence of abstract concepts. Rather, we are denying an ontologically strong existence of abstract objects in the platonist sense. To reiterate our unicorn analogy, we are not denying the existence of the concept of a unicorn, rather we are denying the existence of a unicorn. It is worth noting that the independence of abstract objects does not threaten the aseity-sovereignty doctrine since it claims "everything distinct from God depends on God's creative activity for its **existing.**" Since we are assuming mathematical objects do not exist, the independence of mathematical objects from God is ill-posed. It is therefore advantageous for the Christian to agree with Craig and assume an anti-platonist position via mathematical fictionalism.

It is true that a naturalist has no reason to even consider the aseity-sovereignty doctrine. Therefore, we make no such claim. All we have asserted is that the Christian is forced to adopt mathematical fictionalism because of the aseity-sovereignty doctrine. We however do claim that the naturalist position also leads to fictionalism, but from a different assumption. Traditional platonism is not a possibility for the naturalist worldview since traditional platonism requires a metaphysical narrative to account for the non-physical and abstract existence of mathematical objects. That is to say, the naturalist would have to speak of and accept a platonic realm despite lacking explanations or observable evidence of such a realm, other than the applicability of mathematics. Nonetheless, some naturalists have used an indispensability argument (see for example [13]) to establish the existence of mathematical objects within naturalism. The indispensability argument is made of three premises and a conclusion as seen below:

1. Naturalism: We should look to science, and in particular to the statements that are considered best confirmed according to our ordinary scientific standards, to discover what we ought to believe.
2. Confirmational Holism: The confirmation our theories receive extends to all their statements equally.
3. Indispensability: Statements whose truth would require the existence of mathematical objects are indispensable in formulating our best confirmed theories.
4. Therefore, Mathematical Realism: We ought to believe that there are mathematical objects.

However, Mary Leng rightly argues that confirmational holism is in conflict with naturalism (see [13]). As Leng explains

> It appears, then, that one might reasonably make successful use of a theory while holding back from belief in some of its component parts, either because they are known to be idealizations, and thus to be contributing to theoretical success for reasons other than their truth, or because it has not yet been established that their contribution to the success of our theory will be best accounted for by assuming the existence of their objects, so that it is reasonable to remain agnostic about the objects posited. The fact that we can recognize cases where the practical success of a theoretical hypothesis is not a result of its truth means that we should should hold back from assuming, as Quine's

confirmational holism does, that a practical decision to adopt a hypothesis as part of our theoretical worldview should always be understood as providing us with a reason to believe that hypothesis.

In other words, there are at least two different reasons for one to not assume confirmational holism. For one, some assumptions in mathematical modeling are idealizations of reality. Secondly, we don't know if the success of a theory is best explained by assuming the existence of the mathematical objects involved in the theory. Whether by negating confirmational holism, or by realizing that naturalism would have to provide a naturalist framework for the platonic realm of mathematical objects, or by the lack of a naturalist link between the platonic real and physical reality (as briefly discussed in section 5 of this paper), we find that naturalism should not co-exist with platonism. Hence we find that both theism and naturalism are at odds with platonism. But this common ground that forces the fictionalist position upon both perspectives turns into an advantage to Craig's argument.

3 The Naturalist Die

In the previous section, I argued that both theism and naturalism have difficulties with a traditional platonist view of mathematical objects. For the theist, the problem of God and abstract objects arises because of the aseity-sovereignty doctrine. For a naturalist, confirmational holism stands in the way of traditional platonism. We now turn our attention to the **AM** argument and Rosenberg's rebuttal. As we have noted earlier, Alex Rosenberg's rebuttal says that "there are" indefinitely many mathematical objects and indefinitely many functions relating these mathematical objects for which only a small number of these have applications to the real world. Thus, since there are so many, it was inevitable that one or a few of these should apply to the real world. As Craig seems to counter, this still fails to account for why mathematical objects are applicable to reality. But there is a greater erroneous subtlety in Rosenberg's answer. To be fair, while not explicitly addressed by Craig, it is very likely Craig was or is aware of this error. The error lies in Rosenberg's words, "there are." Indeed, these words imply that there "exist" alternative mathematical objects or geometries. These abstract objects cannot exist within the naturalist worldview as we have previously mentioned. That is, naturalism does not support a platonist perspective. Consequently, Rosenberg's rebuttal, runs into a fundamental problem: the naturalist die that Alex Rosenberg wishes to roll, does not exist. In other words, the application of these "fictions" to the real world is in fact impossible under a naturalist perspective.

We should clarify why the naturalist die of mathematics does not exist. To do this, we shall further explain the analogy of the die and its associated terms. We shall characterize mathematics that apply to the real world as "linked" to reality. The die with its sides represents all of the possible outcomes of running an experiment (represented by rolling the die). Each side or outcome, represents a geometry that is to be linked or applied to reality. Consequently, rolling the die (running the experiment) is analogous to selecting and linking a geometry or mathematical objects to physical reality. In this analogy, Rosenberg's rebuttal to the **AM** argument is that any of the infinite mathematical sides could land with a roll. Here lies the issue. Simply put, what naturalistic process or experiment (die roll) links a random set of abstract fictions to physical processes? Naturalism, by its own default position cannot answer this question. For why would a naturalistic mindless process randomly select an arbitrary set of abstract fictions (mathematical objects and geometries)? Ergo, Rosenberg's fictionalism defeats his attempt to invoke chance.

It is worth noting that an atheist has a way to avoid this pitfall. The atheist would have to embrace platonism which would require abandoning naturalism. But to do so, the atheist will open his or herself to other arguments for God's existence, different from the one Craig has presented. In essence, the atheist runs into other dangers in retreating from naturalism. We shall address these in section 5 of this paper.

4 Unreasonable Effectiveness of Mathematics

The power of Craig's **AM** argument lies in the lack of a proper naturalistic explanation for the applicability of mathematics to the physical world. As we have noted in the previous section, Rosenberg's rebuttal is flawed in the sense that fictions or concepts cannot be linked to physical processes through any naturalistic mechanism. To clarify Craig's argument to Rosenberg's position, we shall propose a modification to Craig's argument in order to highlight the heart of the issue to the naturalist position. In this modified argument, we shall furthermore emphasize the "unreasonable effectiveness of mathematics." This unreasonable effectiveness, a phrase borrowed from Eugene Wigner's paper (see [8]), is something that Craig uses to defend his argument, though it could be emphasized within the stated argument itself to bolster its persuasive power.

The motivation for emphasizing the unreasonable effectiveness of mathematics is to distinguish unreasonable applications from certain straightforward applications that make much sense and are actually reasonable applications. That is, per Wigner, there are two types of mathematical applications: reasonable and unreasonable; expected and unexpected. Arguably, Craig's argument gains strength in the case of unreasonable applications. For it is the unreasonable applications that highlight the unlikely effectiveness. Therefore, here we shall require a definition that sets the unreasonable applications apart from the reasonable applications. Then we shall construct the modified argument with said definition.

Definition 1. *A physical phenomenon (phenomena) is said to be **inherently mathematical** if there is a mathematical model that is indispensable to the understanding of its properties and dynamics, indispensable to the intended and unintended accurate predictions of the physical phenomenon and other related phenomena, and does not lack a hidden variable. Furthermore, we assume a phenomenon is inherently mathematical if we assume a current model can be dispensed by a more complete unknown model that satisfies the same above criteria.*

The reader should note that the usage of "*indispensable*" in our definition is not equivalent to the usage of the same word in the previously discussed indispensability argument that is used to establish mathematical realism (platonism). For we have made a case for Christians and naturalists alike to adopt mathematical fictionalism. Thus, in our definition, we use "*indispensable*" to mean absolutely necessary to the scientific theory. In our definition of **inherently mathematical**, it is worth highlighting the usage of *unintended predictions* in the definition. This is is where the mystery of the effectiveness of mathematics lies, according to Wigner. In his paper (see [8]), Wigner notes:

> ... it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language. Let us consider...elementary quantum

> mechanics. This originated when Max Born noticed that some rules of computation, given by Heisenberg, were formally identical with the rules of computation with matrices, established a long time before by mathematicians. Born, Jordan, and Heisenberg then proposed to replace by matrices the position and momentum variables of the equations of classical mechanics. They applied the rules of matrix mechanics to a few highly idealized problems and the results were quite satisfactory. However, there was, at that time, no rational evidence that their matrix mechanics would prove correct under more realistic conditions. Indeed, they say "if the mechanics as here proposed should already be correct in its essential traits." As a matter of fact, the first application of their mechanics to a realistic problem, that of the hydrogen atom, was given several months later, by Pauli. This application gave results in agreement with experience. This was satisfactory but still understandable because Heisenberg's rules of calculation were abstracted from problems which included the old theory of the hydrogen atom. The miracle occurred only when matrix mechanics, or a mathematically equivalent theory, was applied to problems for which Heisenberg's calculating rules were meaningless. Heisenberg's rules presupposed that the classical equations of motion had solutions with certain periodicity properties; and the equations of motion of the two electrons of the helium atom, or of the even greater number of electrons of heavier atoms, simply do not have these properties, so that Heisenberg's rules cannot be applied to these cases. Nevertheless, the calculation of the lowest energy level of helium, as carried out a few months ago by Kinoshita at Cornell and by Bazley at the Bureau of Standards, agrees with the experimental data within the accuracy of the observations, which is one part in ten million. Surely in this case we "got something out" of the equations that we did not put in.

Here we see the unintended and accurate prediction when Wigner explains that "*we 'got something out' of the equations that we did not put in.*" Another important aspect of our definition of **inherently mathematical** is that the model must not lack a hidden variable in order for it to be indispensable. In quantum mechanics, we are guaranteed that there are no local hidden variables via Bell's theorem (see [3]).

Thus, we can say that the phenomena of quantum mechanics are **inherently mathematical** per our definition. Using this example, we construct our modified applied mathematics argument (which we shall denote **MAM**) as follows:

MAM

1. If God does not exist and mathematical objects do not exist, there are no inherently mathematical phenomena.

2. Mathematical objects do not exist.

3. The physics of quantum mechanics are inherently mathematical.

4. Therefore, God exists.

Mary Leng (see [13]) and others have tried to provide naturalist explanations for the applicability of mathematics to the physical universe. Mary states:

> I argued that a fictionalist account of the role of mathematical hypotheses was possible, which viewed those hypotheses as literally false but useful means of representing how things are taken to be with non-mathematical objects . In particular, if we take the axioms of our favourite version of set theory with urelements to be generative of a make -believe according to which non-mathematical objects can be the members of sets (and therefore can stand in various set-theoretical relations to further sets), we can provide an account of how participating in such a game of make-believe can provide us with a useful means of representing hypotheses concerning non-mathematical objects and their relations.

If there were phenomena for which this explanation would suffice, it could be for so-called "reasonable" applications. But not for unreasonable applications. Applications that are reasonable are therefore the result of selecting axioms that emulate reality. Mary Leng would likely then argue that mathematics is dispensable by arguing that it is possible to reformulate scientific theories without relying on abstract mathematical objects. But quantum mathematics is a known exception to this. Leng (see [13]) has tried to argue otherwise by invoking Balaguer's nominalization attempt (see [1]) to essentially argue that the mathematics of quantum mechanics are dispensable. But the success of this nominalization is easily and substantially contested (see [5]). Thus, in the case of quantum mechanics, the *inherently mathematical* status of its physical phenomena is currently irremovable. Furthermore, there is no heuristic derivation of the mathematics of quantum mechanics. The wave function is not emulating anything in reality. Thus, the effectiveness of mathematics in quantum mechanics cannot be explained by making assumptions that emulate reality.

One might contest that the failure of the nominalization of mathematical applications is a failure for fictionalism since typically, realists (platonists) invoke said failure to promote their view of mathematical platonism. But we must understand that the failure only points to realism if there is no God. The failure to dispense of mathematics in quantum mechanics indeed convinces us, if anything, that quantum mechanics is per our definition, inherently mathematical.

5 Does Platonism allow an atheistic view?

Given that atheistic anti-platonism cannot provide an explanation for the unreasonable effectiveness of mathematical application, the atheist might consider platonism as a viable and godless explanation of the applicability of mathematics. That is, the atheist might consider negating our second premise in **MAM**. But as Leng, Craig (see [7]), and others have noted, if anything, both share in having difficulties with godless explanations for the applicability of mathematics. For while platonism could say that a realist view of mathematical objects is compatible with the truthful predictions one can mathematically make, it still fails to supply a reason for why there is a link between the abstract mathematical objects in the platonic realm and the non-mathematical objects in the physical universe. As Ballaguer notes (see [2]),

> The idea here is that in order to believe that the physical world has the nature that empirical science assigns to it, I have to believe that there are causally inert mathematical objects, existing outside of spacetime."

Furthermore, since naturalism cannot hold hands with platonism per Leng and our previous discussion, the atheist platonist might need to abandon naturalism all together. This of course will provide all sorts of new problems for the atheist. For one, all arguments for the historicity of the resurrection of Christ are typically dismissed simply on naturalistic grounds. For while the New Testament is far more superior to the ancient classical texts in terms of its historical reliability, the historicity of the New Testament is typically dismissed because naturalism does not accept miracles. But without naturalism, an atheist must intellectually struggle with the validity of the historical attestation of the resurrection of Christ.

6 The applicability of mathematics in the Bible

The argument for God's existence based on the applicability of mathematics has a familiar biblical tone. In particular, one could argue that the depth of Romans 1:20 is discovered as we learn more about the mathematical nature of the universe. In Romans 1:20, Paul writes

> For his invisible attributes, namely, his eternal power and divine nature, have been clearly perceived, ever since the creation of the world, in the things that have been made. So they are without excuse.

First we notice that Paul equates God's eternal power and divine nature with his invisible attributes. It is clear that to natural eyes these attributes are invisible. But Paul holds that while these attributes are not visible, they are clearly perceived in creation. I do not intend to say that the original intent of Paul in this passage was to refer specifically to the inherently mathematical phenomena of the universe. Rather, that the perception of God's attributes only becomes "clearer" as we study the intricacies of God's creation. So while a naturalist might argue that science has cast a shadow on this passage, the Christian understands that the perception of the invisible only becomes clearer as we learn more of the mathematical backbone of the universe. As Christian mathematicians, we must continue the legacy of clearly perceiving his invisible attributes by studying the applicability of mathematics to the "things that have been made," as we respond with joyous wonder.

7 Conclusion and Acknowledgement

The goal of this paper has been to persuade the reader of the futility of Rosenberg's rebuttal to Craig's argument (the paradox of the naturalist die), strengthen the argument of the applicability of mathematics by considering unreasonable applications within inherently mathematical phenomena, and motivate the continued study of mathematics applied to the things made in order to gain a "clearer" perception of God's attributes. So we find that our motivation for appreciating the unreasonable applicability of mathematics is twofold: to apologetically persuade those that are not believers and to glorify God as we stand in awe of his mathematically created universe.

I would like to thank God for his manifold grace in my life, my colleagues in the Department of Mathematical Sciences at California Baptist University, the reviewers of this paper for their valuable comments that helped me improve it, the Association of Christians in the Mathematical Sciences (ACMS), the ACMS biennial conference and its respective organizers, and Russell Howell for his thoughtful insight and grace. I would also like to thank one of my students, Jeremy C. Duket, who is partially responsible for coining the "Naturalist Die" and whose questions forced me to think deeply about Rosenberg's rebuttal.

References

[1] M. Balaguer (1996). Towards a nominalization of quantum mechanics, *Mind 105*(418), pp. 209-226.

[2] M. Balaguer (1998). *Platonism and Anti-Platonism in Mathematics*, Oxford: Oxford University Press.

[3] J.S. Bell (1964). *Physics* **1**, 195.

[4] J. Bradley, and R. Howell (2011). *Mathematics Through the Eyes of Faith*, New York, NY: HarperCollins, pp. 226.

[5] O. Bueno (2003). Is it possible to nominalize quantum mechanics? *Philosophy of Science*, Vol. 70, No. 5, Proceedings of the 2002 Biennial Meeting of The Philosophy of Science Association-Part I: Contributed PapersEdited by Sandra D. Mitchell (December 2003), pp. 1424-1436.

[6] W. L. Craig (2008). *Reasonable Faith: Christian Truth and Apologetics*, 3rd ed. Crossway Books.

[7] W.L. Craig (2013). God and the “unreasonable effectiveness of mathematics,” *Christian Research Journal 36*(6)

[8] E. Wigner (1960). The unreasonable effectiveness of mathematics in the natural sciences, *Communications in Pure and Applied Mathematics 13*(1), pp. 1-14.

[9] P. M. Gould (2014). Introduction to the Problem of God and Abstract Objects, In P.M. Gould, (Ed.), *Beyond the Control of God? Six Views on the Problem of God and Abstract Objects*, New York: Bloomsbury Academic, pp. 1-19.

[10] Harstad, K. and Bellan, J., “The Lewis number under supercritical conditions”, *Int. J. Heat Mass Transfer,* in print

[11] Is Faith in God Reasonable? (2013). Available at https://www.reasonablefaith.org/videos/debates/craig-vs-rosenberg-purdue-university

[12] Is Faith in God Reasonable? (2013). Available at https://www.reasonablefaith.org/media/debates/is-faith-in-god-reasonable/

[13] M. Leng (2010). *Mathematics and Reality.* Ney York: Oxford University Press.

[14] Reasonable Faith. Available at https://www.reasonablefaith.org

Marin Mersenne: Minim Monk and Messenger; Monotheism, Mathematics, and Music

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Karl-Dieter Crisman is Professor of Mathematics at Gordon College, but first heard of Mersenne in both math and music courses as an undergraduate at Northwestern University. His main scholarly work focuses on open software resources for mathematics, and the mathematics of voting. He is certain Mersenne would see as many connections between those topics and faith as he did with the math of music and physics.

Abstract

Marin Mersenne is one of many names in the history of mathematics known more by a couple of key connections than for their overall life and accomplishments. Anyone familiar with number theory has heard of 'Mersenne primes', which even occasionally appear in broader media when a new (and enormous) one is discovered. Instructors delving into the history of calculus a bit may know of him as the interlocutor who drew Fermat, Descartes, and others out to discuss their methods of tangents (and more). In most treatments, these bare facts are all one would learn about him.

But who was Mersenne, what did he actually do, and why? This paper gives a brief introduction to some important points about his overall body of work, using characteristic examples from his first major work to demonstrate them. We'll especially look into why a monk from an order devoted to being the least of all delved so deeply into (among other things) exploratory mathematics, practical acoustics, and defeating freethinkers, and why that might be of importance today.

1 Introduction

The seventeenth century was a time of ferment in the sciences in Western Europe. For instance, the ancient discipline of mathematics was being dramatically changed. At the beginning of this period coordinate geometry did not exist, and 'calculus' was still largely at the place where Archimedes' investigations had left it; by its end, their combination was ready to have a huge impact. Fields such as chemistry and biology were illuminated by experiments on air pressure and by the microscope.

Particularly stunning was the change in how mathematics was applied to what we today call physics. As an example, Copernicus' new model for the heavens was still of essentially the same type as Ptolemy's, with epicycles required to make it as accurate[1]. In the span of less than one hundred years, Brahe's observations, Kepler's empirical laws, Galileo's experiments and dialogues, and finally Newton's justifications completely changed this situation, allowing the *calculation* of the movements of the heavens.

Near the center of these developments (and many others) for many years was a Parisian monk, Marin Mersenne. Because of the influence of the regular scientific meetings in his simple cell, and of his voluminous correspondence, he has become known to Western intellectual history as

[1] See [20] for a discussion of Brahe's view that each system had its places of more accuracy; see also [19].

the preeminent 'intelligencer' of the time, a clearinghouse for all the latest scientific developments. Beyond this more generally known activity, Mersenne also produced much of his own work – his publications are famously massive works of erudition.

The aims of this article[2] are twofold. We will first give a brief introduction to his life and overall work, as many mathematicians (and others) who know the name know little more about him. Then we will give a more in-depth introduction to his approach and some of the topics he cared most deeply about, by examining representative excerpts from one of his earliest books, *Quaestiones celeberrimae in Genesim*.

As we will see in brief here, a large part of his legacy is his willingness to tackle any subject, or any correspondent, that might help understand even a small of God's Scriptural revelation in the physical world. This is a notable cross-current to the prevailing winds of specialization – and hence Mersenne's attitude may be worthy of more attention even today, especially from people of faith in the academy. We will conclude by returning to this point.

2 Life and Work

Mersenne's dates and a few basic facts can be found in most number theory and math history texts – but usually little else. This section gives a longer synopsis of his life and most important work, which is in any case essential context for our discussion of *Quaestiones* in the next section.

We should note here that most sources of the basics of his life – including online ones such as [17] and [22] – really largely go back to the (published) encomium [4] of his comrade in the order, Hilarion de Coste. The standard biography[3], exploring the role of 'mechanism' at length, is [10]. Of the more recent English-language book-length works, [3] places emphasis on his debt to Catholic scholastic (and other) trends, while [2] focuses more on his mentoring and encouragement of both peers and younger scholars.

2.1 Early Life

Born in 1588 in the province of Maine, Mersenne showed early aptitude for and interest in study. His parents (whose economic status is variously described, but was not close to the top stratum) were able to send him first to a grammar school, and then the newly-formed Jesuit school/'college' of La Flèche. By 1611 he had finished further study in theology in Paris and joined the order of the Minims, being ordained a priest shortly thereafter.

As the name indicates, the Minims[4] considered themselves to be the 'least of these', who tried to emulate the humble life of their founder, Saint Francis of Paola. Among other things, the order lived on the Lenten fast rules year-round. For a sincere religious character like Mersenne's ([10] goes on at some length about this), it seems to have been a good match.

His initial activities for the order included teaching philosophy in central France. Once summoned back to Paris in 1619, the order apparently saw Mersenne's greatest value in his writing and studies, including the commentary on Genesis we will discuss below. Throughout his life they granted him

[2]Mersenne is never short-winded; likewise, this article is an expansion of just part of a talk given at the Conference of the Association of Christians in the Mathematical Sciences in 2019. We are grateful to the referees for suggestions on exposition and content.

[3]Unfortunately, still only available in French.

[4]The Minims still exist in a number of countries; see their website at http://www.ordinedeiminimi.it/index.htm.

fairly permissive rules to this end – not just of using their library, having his own equipment in his cell, or of leaving the premises of the monastery, but even of hosting outside scholars there on various occasions.

Most of his first publications[5], such as *L'Impiété des déistes*, in the mid-1620's were clearly aimed at combating the relatively new perils of not just deism, but even outright atheism. Certainly this was a main goal of *Quaestiones*; more subtly in a similar vein was his defense of the rational basis of scientific investigation against those skeptical of all knowledge in *La Vérité des Sciences* (see [3] at length on this issue). We must recall that at this time any such 'freethinking' variant (including in France, Protestantism) was considered not just theologically dangerous, but also as having potential to upend a broader social order with nothing to replace it. It is not surprising that a sincere and devout Catholic of some talent would focus on this threat[6].

2.2 Mature Work

Commentators disagree on the continuity between his more polemic early work and his more 'scientific' mature opera. Nonetheless, the topics we will see he waxes eloquent on in *Quaestiones* come to full fruition in his best-known works, of which we here briefly mention the two best known.

Mersenne's longest labor (and generally most famous) book is *Harmonie Universelle* [15], the 'Universal Harmony'. Constantly revised, digested, and even translated[7] up through his death, it brings together nearly everything known at the time about all manner of musical instruments, compositions, and practice, and it is still an important source for musicologists. Acoustics take special pride of place in this as in others of his writings, and he is generally acknowledged as being the first person to write down a formula for a vibrating string – and connecting this to vibrating air columns in organ pipes and wind instruments. Experimental matters come to the fore as well; for example, Mersenne precisely determines the difference in pitch arising from making strings of different materials such as iron, silver, or gold.

Not that this is a music textbook. For instance, his theology is unavoidable throughout. Interestingly, in personal correspondence, the author of the paper [21] on Mersenne's 'monochord' (a set of several strings formed in such a way as to investigate different musical intervals with great precision) points out that its three strings should be taken as a figuring of the Trinity. Likewise, Mersenne conceives of the concept of harmony[8] very broadly, interweaving (desired) harmony in politics, mathematics arising from harmony, as well as the physical attributes of harmony. As he was still forming his views on free fall and Galileo's mechanics at the same time, there is discussion of this as well – motion of a string is still motion.

Mechanics show up in many other of his later works, such as *Cogitata Physicomathematica* [16]. This is emblematic of what [3] and others follow Mersenne in calling 'physico-mathematics', the earlier mentioned slow unification of algebraic mathematics with questions about motion and other physical notions. These works are full of references to his own, repeatable and repeated, experiments. For instance, Mersenne's discussion of hydraulics in [16] comes complete with many figures of apparatus in use, doubtless based on ones like those used in his own cell. It is developed in a long series of definitions, 54 propositions, and various scholia.

[5] See [3] or [10] for fairly detailed discussion of these early works.

[6] See [10] pp. 171ff.

[7] Into Latin, for wider readership, though typically for him with many changes and additions. Mersenne cared about writing in the vernacular but also for international audiences, even in a pastoral sense.

[8] See [3] for discussion of this point.

This is the important shift in focus from an Aristotelian worldview to a 'modern' 'mechanistic' one that Lenoble names his book [10] after; this emphasis seems to postdate *Quaestiones*. Again, this is not the exclusive content – the *Cogitata* has a section on music theory, another on weights and measures, alongside one on ballistics! As Lenoble remarks about several of Mersenne's works, one should think of the *Harmonie* or *Cogitata* as a sort of modern *Summa* – just not one of theology, but of music or physics.

In all of these works there are astonishingly long interludes regarding essentially unrelated pure mathematics. For example, [16] is best known for being the first source of the so-called Mersenne primes of the form 2^p+1 for p a prime, his primary modern mathematical claim to fame[9]. However, Mersenne included pure mathematics at any appropriate opportunity; for instance, the *Harmonie Universelle* has much discussion of the combinatorics of musical composition.

Mersenne's mature work was clearly valued by contemporaries, even if they may not all have read it completely through. Boria has an impressive list of scholars from Cavalieri to Cavendish who were influenced by the *Cogitata*, and the *Harmonie Universelle*'s impact was even greater. We have only touched on it here; the website [17] is another good place to start, and Lenoble's biography [10] is mandatory reading on its impact among nascent scientists, in ways that (according to him) compare favorably to Descartes and Pascal – notably his empiricism.

2.3 Messenger

It is impossible to discuss Mersenne without mentioning his correspondence and publishing activity. His early work explicitly (as opposed to implicitly) aimed at defending Catholic thought does not seem to have had a huge impact on that per se. Rather, it attracted attention more from letter-writers who wished to discuss matters further – and whom Mersenne pursued in the furtherance of his investigations. With hindsight, it is clear that Mersenne's largest contribution to intellectual history[10] as a whole is certainly his efforts at making new discoveries known by the means of his correspondence.

Three well-known, and typical, episodes will suffice to give the reader an indication of the overall contribution. First, Mersenne is one of those responsible for collating the various 'Objections' to Descartes' *Meditations*, as well as Descartes' replies. Mersenne had no objection to printing both that which he agreed with and that not, if in the spirit of open inquiry and publishing defenses of his own views. (His toleration of near-atheist Thomas Hobbes as a friend often raises similar questions, but is consistent with his later practice on this score.) There are long stretches of back-and-forth between them long before Descartes finally published his work.

As another example, in 1637-8 Mersenne was so excited to have Fermat and Descartes share their respective methods for finding maxima and minima (and hence tangents) that he brokered their correspondence. As [13] points out[11], Mersenne had probably already overshared a number of Descartes' geometrical and other works, and in this case made some more missteps over Descartes' easily-bruised personality, including sharing Descartes' letters with some of Mersenne's circle in Paris who were known skeptics of his philosophy and approach. In the event, through these exchanges both Fermat and Descartes were forced to divulge – and, crucially, *clarify* – more of their

[9] It is obligatory to mention here [7], an internet-enabled search for larger ones; mathematical readers will also be interested in different points of view in [5] and [2] regarding whether errors in his computations are typographical in nature, and that Mersenne got the idea from Frenicle de Bessy.

[10] Even if [8] probably overstates the case over his relatively weak scientific skills.

[11] See Section IV.IV of [13] for every single detail.

proto-calculus than they had intended.

Finally, despite some initial qualms, and despite always treating Galileo's cosmology as a working hypothesis rather than fact[12], Mersenne not only ensured Galileo's publication in Holland, but created a translation-cum-commentary of the *Dialogues* in French for the local market. As various commentators have pointed out, the story of Galileo's reception by the church is more complex than the received wisdom, and Mersenne's bona fides with both church and king in France were well-used.

Mersenne's full edited correspondence runs to seventeen volumes[13]. Because he was persistent in asking questions of every correspondent, its sum has become invaluable for understanding much of scientific philosophy of this era, hence his universal acclamation as the 'intelligencer' for his times.

2.4 The End

As we have seen, over time Mersenne's writing came to focus more and more on the scientific and musical topics that he encountered while refuting his enemies. He also continued to write hundreds of (surviving) letters to all and sundry regarding these same topics. However, he did not travel much outside Paris – one notable longer trip was to Italy, where he hoped to learn more about Torricelli's barometric experiments. In 1648, after nearly thirty years of constant academic activity he fell victim to complications of a lung abscess – missing the chance to confirm the existence of air pressure, which Pascal soon after completed.

Despite a wide-ranging *ouevre*, he never lost his overall focus (despite some commentators' claims to the contrary). As he expressed to one of his first correspondents near his death, "We are in a strange century for the different kinds of libertinism ... neither reasoning nor Scripture can make them yield to the truth ... When videbitur Deus Deorum in Sion[14], we will no longer have wicked people to persecute us, nor impious people to mock God and religion" (quoted in [3], page 203). There does not seem to be any indication that his eulogies were inaccurate as to that he was a pious and sincere man to the end of his days.

3 The Most Famous Questions

We return now to Mersenne's early period, and his monumental early volley against the freethinkers, *Quaestiones celeberrimae in Genesim* [14]. In discussing these 'Most famous/celebrated questions in Genesis,' we will see indications of many of the themes broached in our brief history, exemplified by excerpts we hope will be of interest to the mathematically inclined reader of the twenty-first century.

3.1 Overview

The goals of *Quaestiones*, including two thousand tight columns of text with various adjuncts[15], were not few. The full title page (see[16] Figure 1 on the following page) continues by commending its 'accurate explanation of the text', defeating of 'atheists and deists', 'vindicating the Vulgate from the calumnies of heretics', and 'restoring the music of the Greeks and Hebrews'.

[12] See [6] or [11] for commentary.

[13] See metadata for the entirety at the Bodleian's website, http://emlo-portal.bodleian.ox.ac.uk/collections/?catalogue=marin-mersenne.

[14] When the God of Gods will be seen in Zion.

[15] And then hundreds more columns of *Observationes* usually appended to the *Quaestiones*.

[16] All figures are courtesy of the very generous terms of nonprofit usage of the Gallica online repository, https://gallica.bnf.fr, of the French National Library.


F. MARINI
MERSENNI
ORDINIS MINIMORVM
S. FRANCISCI DE PAVLA
QVÆSTIONES CELEBERRIMÆ
IN GENESIM, CVM ACCVRATA
TEXTVS EXPLICATIONE.
Græcorum, & Hebræorum Musica instauratur.
FRANCISCI GEORGII VENETI
CABALISTICA DOGMATA FVSE REFELLVNTVR,
OPVS THEOLOGIS, PHILOSOPHIS, MEDICIS,
LVTETIÆ PARISIORVM.
M. DC. XXIII.

Figure 1: Title page of *Quaestiones*

A brief summary of the contents is hopeless; the index of the ‘questions’ alone, on everything from angels to architecture, runs to six pages. A modern reader might wonder whether the title page promises too *little*!

Nonetheless, there is a clear structure. *Quaestiones* is assembled as a commentary on the first six chapters of Genesis, and begins with a discussion of the Hebrew word for ‘In the beginning’ and its connection to ‘head’. Even in these opening lines, we see how he uses this to create a different kind of ‘Summa’. His first ‘problem’ is that of whose voice is the first we hear in Scripture, immediately claiming ‘without God nothing may be understood’. He swiftly moves to the question of whether God indeed exists, and how to prove this against the atheists – eventually including no fewer than thirty-five rationales for God to exist. (See Figure 2.)

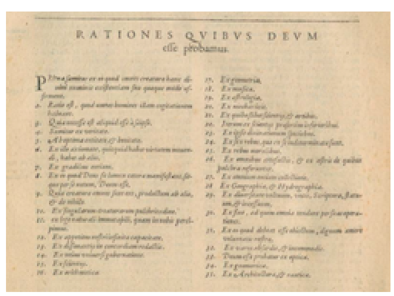

RATIONES QVIBVS DEVM esse probamus.

Figure 2: Thirty-five rationales for God’s existence

3.2 Examples and Structure

Mersenne’s real goal, then, is investigation of many, many questions (and other ‘articles’ and ‘problems’ of side commentary) raised by the contents of these verses, in any field of knowledge. All knowledge is part of God’s knowledge, to him, and so using what is nominally a commentary to share all sorts of information is a legitimate organizational principle. This is especially so if the information can bring people to a reasonable faith in the God of Scripture and counter poor reasoning as practiced by others (such as the long stretches arguing against witchcraft and deism).

A good example of this *not* connected to scientific investigation is provided by his commentary on Genesis 4:18-19, wherein Cain's descendant Lamech married two women, Adah and Zillah. In Figure 3 we have the (Latin) Vulgate, the Hebrew text, and the (Greek) Septuagint, followed by a commentary. Note the references on the margins, which in general quote quite extensively from Scripture, church Fathers, classical authors, and other primary sources.


VERSICVLVS XVIII. ET XIX.

Figure 3: Commentary on Genesis 4:18-19

This verse also demonstrates how a typical 'Question' arises. In Figure 4 we see Question 54, which answers whether Lamech sinned in having two wives, and under what law polygamy is prohibited. Mersenne quotes in his answer a ninth-century letter from the pope to a Holy Roman Emperor, other examples in Genesis, and Jesus' teaching about (and rejection of) divorce in Matthew 19 (*ab initio non fuit sic*, from the beginning it was not so).


QVÆSTIO LIV.

AN LAMECH PECCAVERIT DVAS VXORES

Figure 4: Question 54

But Mersenne intended *Quaestiones* to be much more. Consider his treatment of Genesis 1:14. The lights God creates in the sky give signs for days and years in this verse. But he does not just discuss God's providence in this passage, but leads into to a very long refutation of all manner of horoscopes—including detailed examples of their construction. Even by today's academic standards, Mersenne is usually punctilious in giving his opponents the best possible discussion of their own views before trying to defeat them.

Likewise, in discussing Genesis 4:20, wherein Lamech's son Jabal is the father of those who live in tents and raise livestock, Mersenne gives a comprehensive discussion of what we would today call the liberal and practical arts, including things like surgery, hydraulics, ethics, and rhetoric – each time with an at-times devotional connection to some episode in Jesus' life or the Apostle's creed[17].

The reader may now, with Lenoble, wonder whether we simply have a 'picturesque disorder' or even 'intellectual incontinence'[18]. Our view is closer to that of Boria; Mersenne may need an editor

[17]For instance, Caiaphas' comment on one man dying for all the people is connected to geography.

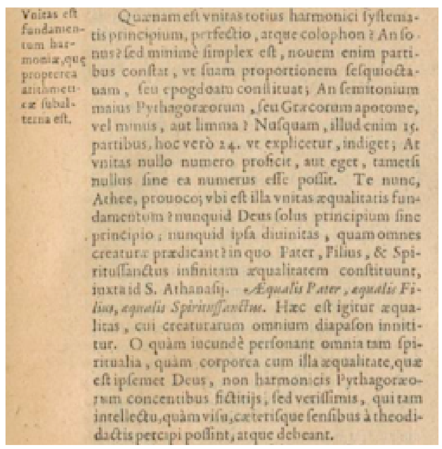
[18]See [10], pages 4 and 25; his language is often more picturesque than that of more recent scholarship.

and enjoys throwing in the kitchen sink, but the structure is not a rambling chaos. In Lenoble's own words, despite Mersenne's 'eclecticism', this is merely a new kind of 'Summa'.

3.3 Music in *Quaestiones*

As noted above, Mersenne's most original investigations were in music theory, particularly in what we would now call the physics of music. (Recall that of the traditional seven liberal arts, music was considered to be applied arithmetic.) Even at the time of *Quaestiones* he has plenty to say about this discipline, and it is appropriate to spend some time with this.

What might be surprising to the modern reader is how universally he considers music to be applicable. Already in the eighteenth rationale for God's existence (see Figure 5) Mersenne is quoting Augustine and Athanasius with respect to music and how it shows humans God.



Vnitas eſt fundamentum harmoniæ, quę propterea arithmeticæ ſubalterna eſt.

Quænam eſt vnitas totius harmonici ſyſtematis principium, perfectio, atque colophon? An ſonus? ſed minimè ſimplex eſt, nouem enim partibus conſtat, vt ſuam proportionem ſeſquioctauam, ſeu epogdoam conſtituat; An ſemitonium maius Pythagoræorum, ſeu Græcorum apotome, vel minus, aut limma? Nuſquam, illud enim 15. partibus, hoc verò 24. vt explicetur, indiget; At vnitas nullo numero proficit, aut eget, tametſi nullus ſine ea numerus eſſe poſſit. Te nunc, Athee, prouoco; vbi eſt illa vnitas æqualitatis fundamentum? nunquid Deus ſolus principium ſine principio; nunquid ipſa diuinitas, quam omnes creaturæ prædicant? in quo Pater, Filius, & Spiritusſanctus infinitam æqualitatem conſtituunt, iuxta id S. Athanaſij. *Æqualis Pater, æqualis Filius, æqualis Spiritusſanctus.* Hæc eſt igitur æqualitas, cui creaturarum omnium diapaſon innititur. O quàm iucundè perſonant omnia tam ſpiritualia, quàm corporea cum illa æqualitate, quæ eſt ipſemet Deus, non harmonicis Pythagoræorum concentibus fictitijs, ſed veriſſimis, qui tam intellectu, quàm viſu, cæteriſque ſenſibus à theodidactis percipi poſſint, atque debeant.

Figure 5: A bit of music and God

If this seems a little far from Anselm's ontological argument (which Mersenne approves of), you are right; as with his science, Mersenne is not looking for a full deductive proof of God's existence, but rather many reasonable arguments that, in total, cannot be confounded[19]. He grounds this reason in the harmony (recall our discussion of [15]) and unity that both God and music provide.

However, Mersenne leaves the majority of his musical thoughts in *Quaestiones* to his commentary on Genesis 4:21, where Cain's descendant Jubal is the father of musicians. Nearly five percent of the *Quaestiones* is a commentary on this verse!

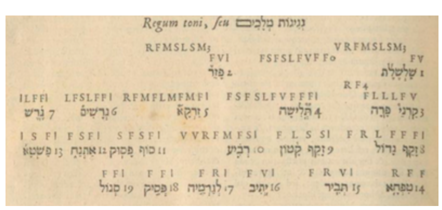


Figure 6: In the mode of the kings

From the title page promising a restoration of Hebrew and Greek music, we might have expected some reconstruction like in Figure 6. The Roman letters correspond to the solfege syllables like Re, Mi, Fa, where Ut, the predecessor to Do, becomes V. Mersenne waxes rhapsodic just before

[19]See [3] on this and its relation to his other writings.

this about how happy is the one who restores the wonderful music of the temple of Jerusalem, and happiest the one who delights in the harmony of the heavens, with which the saints praise God the all-powerful, as in Psalm 83. Perhaps this does belong in a commentary after all!

Yet just before this, Mersenne talks about the way music ought to be composed. Before *that*, there are long extracts describing the various diatonic intervals – perfect fourth, major second, and so forth – with tables. In various articles to the question for this verse he shows correct scansion in various languages, examples of proper four-part music in Latin and French, and a long discussion of diction and what various authors have to say about it. The lyre in Figure 7 is a prefiguring of his detailed accounts of instrument construction in the *Harmonie Universelle*.

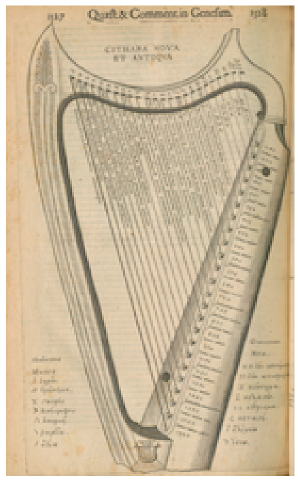


Figure 7: New and ancient lyre

Again, this is not simply the delight of sharing all he knows, though we would argue it is *also* that. The Parisian father is simply keenly interested in every aspect, practical and theoretical, of an earthly instantiation of God's harmony, and uses this verse to shine it through.

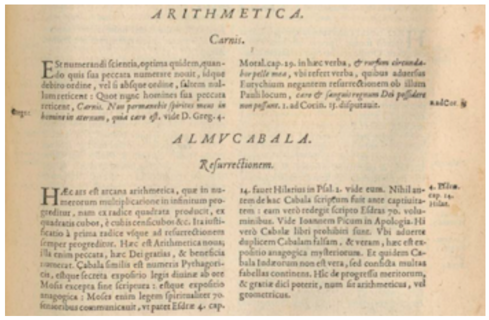
3.4 Basic Mathematics in *Quaestiones*

As is well known, it is a very short step from describing the different intervals in harmony to number theory. We should also not forget that Mersenne's Jesuit education at La Flèche would have included a fair amount of Euclidean geometry, courtesy of Christopher Clavius' innovations (see [1]). It is then small wonder that Mersenne was interested in mathematics as an adjunct (and more) to music and harmony, as well as anywhere geometry is important, and that it shows up many times in the *Quaestiones*. By examining the discussions of mathematics in this early work, we may get a sense of how he may have later become interested in deeper mathematical questions.

Many of the places where mathematical content appears in *Quaestiones*, in contradistinction to Mersenne's later work, focuses fairly directly on the theological questions at issue—and sometimes is not very deep. For instance, when attacking cabalistic numerology in the commentary on Genesis 3:20, he does some basic computations to refute it, including comparing the Latin and French versions of the names Maria and Stephen.

Somewhat more interesting is his mention of the number 11240025908271968000 of possible Hebrew 'words' (in a combinatorial sense), which apparently goes back to cryptographer Blaise de Vigenère in his treatise on ciphers of 1586. In *Quaestiones*, Mersenne simply mentions that the (unnamed) author was unable to explain it, and that rather than being practically infinite, isn't much bigger than 100 quintillion. The later Mersenne we would expect to try to recreate this number, even if to refute its usefulness.

Different, but still not (mathematically) deep, are his discussion of the sciences of mathematics in the earlier-mentioned commentary on Genesis 4:20. In Figure 8 we see arithmetic related to the flesh. Essentially[20], the best knowledge of counting is in knowing how to count one's own sins!


ARITHMETICA.

Carnis.

Est numerandi scientia, optima quidem, quando quis sua peccata numerare nouit, idque debito ordine, vel si absque ordine, saltem nullum reticent: Quot nunc homines sua peccata reticent, *Carnis. Non permanebit spiritus meus in homine in aeternum, quia caro est.* vide D. Greg. 4. Moral. cap. 29. in haec verba, *& rursum circundabor pelle mea*, vbi refert verba, quibus aduersus Eutychium negantem resurrectionem ob illum Pauli locum, *caro & sanguis regnum Dei possidere non possunt*. 1. ad Corin. 15. disputauit.

ALMVCABALA.

Resurrectionem.

Haec ars est arcana arithmetica, quae in numerorum multiplicatione in infinitum progreditur, nam ex radice quadrata producit, ex quadratis cubos, è cubis censicubos &c. Ita iustificatio à prima radice vsque ad resurrectionem semper progreditur. Haec est Arithmetica noua; illa enim peccata, haec Dei gratias, & beneficia numerat. Cabala similis est numeris Pythagoricis, estque secreta expositio legis diuinae ab ore Mosis excepta sine scriptura: estque expositio anagogica: Moses enim legem spiritualiter 70. senioribus communicauit, vt patet Esdrae 4. cap. 14. fauet Hilarius in Psal. 2. vide eum. Nihil autem de hac Cabala scriptum fuit ante captiuitatem: eam verò redegit scripto Esdras 70. voluminibus. Vide Ioannem Picum in Apologia. Hi verò Cabalae libri prohibiti sunt. Vbi aduerte duplicem Cabalam falsam, & veram, haec est expositio anagogica mysteriorum. Et quidem Cabala Iudaeorum non est vera, sed conficta multas fabellas continens. Hic de progressu meritorum, & gratiae dici poterit, num sit arithmeticus, vel geometricus.

Figure 8: Arithmetic and Al-Muqabalah

Perhaps even more surprisingly, the second paragraph in Figure 8 connects 'Al-muqabalah', or the growing area of algebra[21], to the resurrection. 'This art is the secret arithmetic, which in the multiplication of numbers proceeds to infinity . . . '

Though this is confusing to a modern reader at first, what Mersenne is referring to is the distinction between mere addition and subtraction and the raising to many powers, which is fantastically powerful. For someone educated in Euclidean geometry, discovering cubes, fourth powers, and many higher ones (part of the general sphere of algebra of the time, in addition to solving equations) was doubtless a fascinating new world, even if the vocabulary is as old as Diophantus. He says this is the 'new arithmetic, which is the grace of God, and counts its benefits'. (We would remark that this has parallels to Jesus' calling on us to forgive not just seven times, but seventy times seven, which indeed is a power in both senses.)

3.5 Geometry in *Quaestiones*

So far, we still see Mersenne essentially using rudimentary knowledge about mathematics to make theological points. However, when he turns to geometry, we see that Mersenne is quite capable of understanding the details as well, as proved true in his later work.

In *Quaestiones* this really comes to the fore in his knowledge of contemporary optics. In reason 33 for the existence of God, for thirty columns he uses the 'beauty of optics', bringing in astronomy, geometry, and arithmetic, to support his contention. This includes reiteration of quite detailed computations of various astronomical quantities (see Figure 9 for an example), including the ratio of the diameter of the sun to that of the earth. Mersenne doesn't even neglect to provide a table

[20] I am grateful to my colleague Graeme Bird for assisting and correcting my Latin, though all errors are solely mine.

[21] Al-Khwarizmi's treatise on simple algebra uses two words, of which today we typically only use 'al-jabr' or algebra, but at this time it was reasonable to use the other one to refer to algebra, as Mersenne does here.

of the multiples of sixty (with references to degrees, minutes, etc. of arc) to the *fourteenth* power in order to be sure everyone knows just how accurate[22] an astronomer could be if necessary!

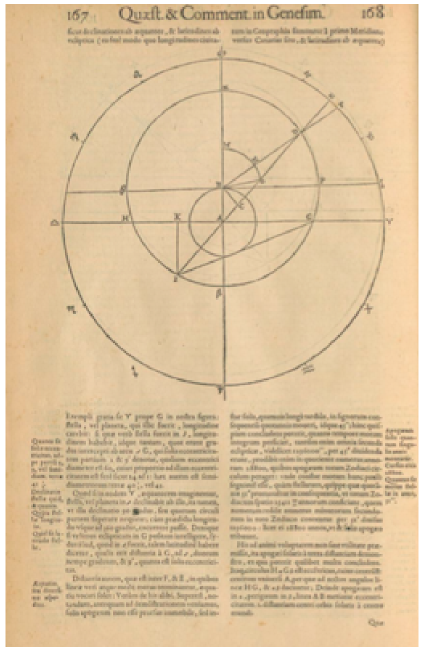


Figure 9: Optics and astronomical calculation

All the while he brings Euclidean propositions, Kepler's observations, ancient authors like Seneca, Plutarch, and Thucydides – and of course the usual orthodox sources – to bear in asserting the relevance and history of this one point. The preponderance of evidence from all respected sources is what will win the fight against the 'impious atheist' whom he addresses.

Along similar lines, Mersenne spends a number of pages refuting the possibility that angelic appearances could have been generated using sophisticated mirrors in some manner. This is an opportunity not lost for describing the full mathematical properties of various sorts of mirrors, including the 'most perfect' parabolic shape (see Figure 10). His goal of defending orthodoxy does not rule out bringing the reader to full scientific understanding as well.

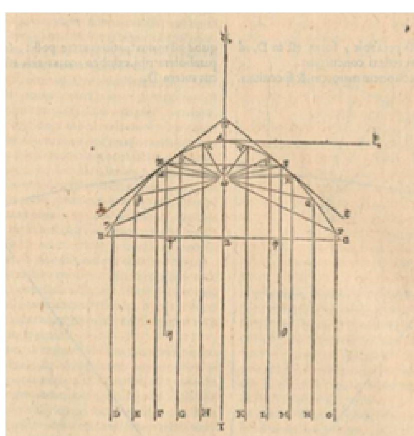


Figure 10: Parabolic reflection

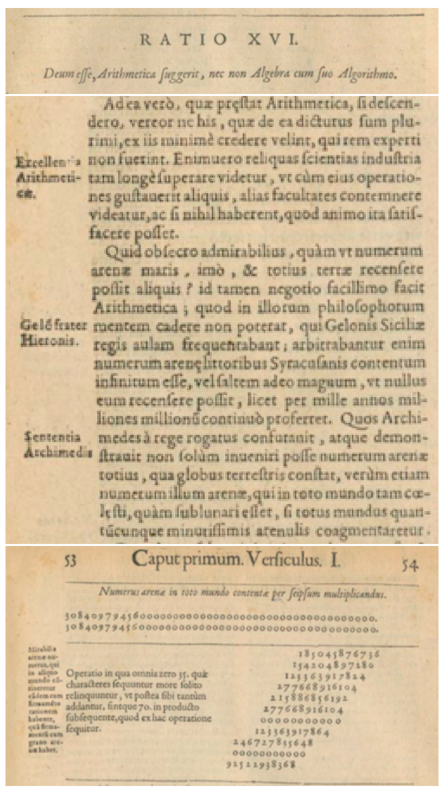
[22] In column 187 he even points out that a year has 525,960 minutes since it is slightly longer than 365 days.

3.6 Mathematics and God's Existence

We have just seen that geometry in the service of optics, to Mersenne, help in making God's existence reasonable. He is far more explicit about this in reasons 16 and 17 (see above Figure 2), where he enlists arithmetic and geometry directly to that cause.

It would be an entire study to go through these rationales in detail. Reason 16 may be roughly translated, 'Arithmetic suggests [that] God exists, as does algebra with its algorithms.' Mersenne starts immediately with the 'excellence of unity', from which the 'infinite storehouse' of arithmetic is generated, waxing quite eloquent about the many varieties of numbers and how God's unity is shown through this. He then continues to explain the excellence of arithmetic in describing the world by giving large parts of Archimedes' argument in *The Sandreckoner* for the number of grains of sand in the universe, particularly the various assumptions needed and then the computation of the cubes of various large numbers.

How does this relate? Because it is so easy for the human mind to compute all this, among other reasons. The explicit calculation of the square of a number of 46 digits (for all this, see Figure 11) needed to compute these numbers in decimal notation[23] helps in his argument: 'You, atheist, behold the prodigious intellect of man' which can calculate the grains of sand.



RATIO XVI.

Deum esse, Arithmetica suggerit, nec non Algebra cum suo Algorithmo.

Excellentia Arithmeticæ.

Ad ea verò, quæ præstat Arithmetica, si descendero, vereor ne his, quæ de ea dicturus sum plurimi, ex iis minimè credere velint, qui rem experti non fuerint. Enimuero reliquas scientias industria tam longè superare videtur, vt cùm eius operationes gustauerit aliquis, alias facultates contemnere videatur, ac si nihil haberent, quod animo ita satisfacere posset.

Gelō frater Hieronis.

Sententia Archimedis

Quid obsecro admirabilius, quàm vt numerum arenæ maris, imò, & totius terræ recensere possit aliquis? id tamen negotio facillimo facit Arithmetica; quod in illorum philosophorum mentem cadere non poterat, qui Gelonis Siciliæ regis aulam frequentabant; arbitrabantur enim numerum arenę littoribus Syracusanis contentum infinitum esse, vel saltem adeo magnum, vt nullus eum recensere possit, licet per mille annos milliones millionū continuò proferret. Quos Archimedes à rege rogatus confutauit, atque demonstrauit non solùm inueniri posse numerum arenæ totius, qua globus terrestris constat, verùm etiam numerum illum arenæ, qui in toto mundo tam cœlesti, quàm sublunari esset, si totus mundus quantūcunque minutissimis arenulis coagmentaretur.

53 Caput primum. Versiculus. I. 54

Numerus arenæ in toto mundo contentæ per seipsum multiplicandus.

30840979456000000000000000000000000000000000000.
30840979456000000000000000000000000000000000000.

Mirabilis arenæ numerus, qui in aliquo mundo cõtineretur eãdem cum firmamẽto rationem habente, quã firmamentũ cum grano arenæ habet.

Operatio in qua omnia zero 35. quæ characteres sequuntur more solito relinquuntur, vt postea sibi tantùm addantur, sintque 70. in producto subsequente, quod ex hac operatione sequitur.

185045876736
154204897280
123363917824
277668916104
215886856192
277668916104
00000000000
123363917864
246727835648
00000000000
92522938368

Figure 11: Reason 16

Similarly, the amazing power of algebra to solve problems involving high powers or related rates is lauded; for instance, Mersenne quotes and then solves an ancient Greek problem later appearing in many English-language puzzle books about a 'brazen lion'. He ends by calling the atheist to rethink the error of his ways with Plato's quote of God as geometrizing.

[23]Recall the Greeks did not use decimals in the Hindu-Arabic numeral sense.

Reason 17 continues with the same argument, now explicitly from geometry. As one of many mathematical examples, Euclid's Proposition XII.18 is provided, which shows that the volume of spheres are in proportion to the cubes of their diameters. Indeed, he looks at the circles' areas as well, as we see in Figure 12. Why the focus on circles? As he says at the beginning of the section, it is because God is the 'indivisible center, whose irradiation extends to the periphery of all things,' which perfectly understands all angles of creation. Later he says he wishes with all these arguments to recall from their error the detested atheists.

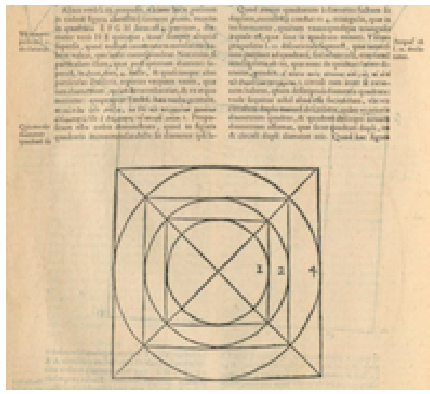


Figure 12: Euclid Proposition XII.18

Yet Mersenne is clearly going far beyond any relatively simplistic analogies, which at least for much of these sections is all he really needs to make his arguments against his targets. He addresses the angle between a circle and its tangent; he complains about some researchers' values for π; he examines some 'excellent' propositions from book 2 of Euclid, the Pythagorean Theorem, various aspects of the parallelogram, and the so-called Conchoid of Nicomedes.

To what purpose? We argue it is both his pure delight in these results (one which he wants the reader to share) *and* his desire to 'offer the atheists medicine'. Geometry offers such a treasury of interesting and useful results, but it also allowed (to Mersenne) God to build all things in number, measure, type, etc. We can see that Mersenne, in indulging his own 'youthful ardor' (so Lenoble of *Quaestiones*) looks forward to his later compilations while also meeting his immediate goal.

3.7 Reception?

In the discussion of his musical proof of God he says, "I provoke you now, Atheist," in asking where else such unity may be found but in the Trinity. This mission was accomplished, as a number of authors, notably Robert Fludd, proceeded to defend themselves and attack Mersenne. This in turn drew the interest of more orthodox commentators, who helped Mersenne obtain many of his contacts and led to his huge legacy of missives. However, it is interesting that none of the his major biographers discuss *Quaestiones*' influence on orthodox developments, and it is not clear there were many. Instead, along with his other early books, it served to introduce the topics he cared about most; at least some authors suggest his change in publication to more strictly scientific work was occasioned by his lack of success in convincing the heretics to change.

4 The Legacy

It took many years for Mersenne to be seen as a figure worthy of study in his own right. Perhaps partly this is because he "combine[d] sophistication and naivety as perhaps only a very kindly and cloistered savant could do"[24]. Certainly his understanding of what Fermat was doing with whole numbers, or Galileo's work, was not as insightful as that of some of his colleagues[25].

Crucially, however, he understood their *importance* – in the case of Fermat's number theory, he was one of the only ones to do so – and he was responsible for their ideas being promulgated in Northwestern Europe. In general his understanding of music, mathematics, and other fields was not simply that of a rank amateur. His was a quest to understand these ideas more fully, and add to the world's compendium of knowledge of how God organized the world. It is not just that Mersenne facilitated the transmission of knowledge, but that he actively sought out correspondents from the great organist Jehan Titelouze and the renowned philosopher Rene Descartes to many far more obscure figures in order to achieve this quest.

In doing so, Mersenne would tackle any subject, or any correspondent, that might help understand even a small part of creation – and bring it to others. A major thesis of [2] is that he was the only one who would deign to report much of this knowledge to many less famous, now-unknown intellectuals of the time (as well as future luminaries when they were unknown). As we mentioned earlier, his commitment to organizing and passing on these insights to a wide audience bore fruit in his later work being much valued for bringing together the known state of many fields (notably music).

Mersenne's true legacy is thus not any of his specific efforts, or his role in intellectual history or music theory, though these are well known to experts in those fields. He was also not precisely a polymathic 'Renaissance man,' though his interests and investigations did range very widely. Nor do his theological arguments from music or mathematics directly contribute to modern debates over the existence or nature of the Deity.

Instead, the example of Mersenne may be thought of as advocating a return to one's first love – both intellectually and spiritually. Above all, he wanted to defend the possibility of knowledge and revelation over against various adversaries (not just freethinkers). And as Dear points out at length in the last chapter of his book, all the various 'mixed mathematical sciences', including acoustics and hence music, were the best place to do this in the seventeenth century.

So Mersenne returned again and again to all of these topics in his work, while not necessarily seeing his experiments in air pressure or long combinatorial digressions[26] as being somehow unrelated to his main purpose. He investigated what he was interested in, then brought it back as best he could to how it showed God's authority, purpose, or handiwork. And he would talk to anyone who could help him in it, great or small, without worrying (too) much about fame or priority, or even whether he was bothering them with his long letters.

It is true that sometimes he also didn't worry too much about coherence for other readers; yet this, in the end, was also a positive for Mersenne. Consider Descartes, who took years to publish

[24]See [12], in the context of keyboard tuning, though it is more generally true.

[25]See e.g. [11] on French scientists preferring Galileo's Italian originals to Mersenne's French translations/condensations.

[26]Or infinitesimal angles of tangents to circles or showing how to properly set four-part music or . . .

his deductive superstructure, always worried about whether someone might scoop him (as the Fermat episode demonstrates). Their contemporary John Amos Comenius (Jan Komensky) never finished his 'Pansophic', fully organized compendium of human knowledge[27], despite many pleas from others. But Mersenne got people talking.

In this age of hyper specialization and showing only the polished final product of our work (at least in mathematics), Mersenne affords an alternate view of how to pursue and disseminate intellectual inquiry, even (or particularly) when motivated at root by theological concerns. Perhaps we would do well to consider emulating him a bit more, rather than being a Fermat or a Descartes, waiting until all is perfect to strike the iron.

Not that we should all follow Mersenne's lead, either. But he encourages us to step back a bit from our own legacy-seeking to consider the disciplines on their own merits, wherever they lead. For a father from the order of 'minimum' importance, that is a legacy of maximum value.

References

[1] Amir Alexander, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American Press, 2014.

[2] Vittorio Boria, *Marin Mersenne: Educator of Scientists*, American University diss., 1989.

[3] Peter Dear, *Mersenne and the Learning of the Schools*, Cornell University Press, 1988.

[4] Fr. Hilarion de Coste, *La vie du R P Marin Mersenne, theologien, philosophe et mathematicien, de l'Ordre der Pères Minim*, Sebastien and Gabriel Cramoisy, 1649.

[5] Stillman Drake, *The rule behind 'Mersenne's numbers'*, Physis: Rivista Internazionale di Storia della Scienza Vol. 13 No. 4, pp. 421-424.

[6] Daniel Garber, *On the frontlines of the scientific revolution: How Mersenne learned to love Galileo*, Perspectives on Science Vol. 12, No. 2, pp. 135-163.

[7] *Great Internet Mersenne Prime Search*. Available at https://www.mersenne.org (accessed Aug. 29, 2019).

[8] Justin Grosslight, *Small Skills, Big Network: Marin Mersenne as Mathematical Intelligencer*, History of Science Vol. 51, No. 3, pp. 337-374.

[9] Douglas M. Jesseph, *Squaring the Circle: The War Between Hobbes and Wallis*, University of Chicago Press, 1999.

[10] Robert Lenoble, *Mersenne ou la naissance du méchanisme*, Vrin, 1943.

[11] John Lewis, *Mersenne as Translator and Interpreter of the Works of Galileo*, MLN Vol. 27 No. 4, pp. 754-782.

[12] Mark Lindley, *Mersenne on Keyboard Tuning*, Journal of Music Theory Vol. 24 No. 2, pp. 166-203.

[27] See [3] on their correspondence and Mersenne's less-than-enthusiastic take on Comenius' project.

[13] Michael Sean Mahoney, *The Mathematical Career of Pierre de Fermat*, second ed., Princeton, 1994.

[14] Marin Mersenne, *Quaestiones celeberrimae in Genesim*, Sebastien Cramoisy, 1623.

[15] Marin Mersenne, *Harmonie Universelle*, Sebastien Cramoisy, 1636-7.

[16] Marin Mersenne, *Cogitata Physicomathematica*, Antonius Bertier, 1644.

[17] J.J. O'Connor and E.F. Robertson, *Marin Mersenne*. Available at http://www-groups.dcs.st-and.ac.uk/history/Biographies/Mersenne.html (accessed Aug. 15, 2019).

[18] Bernard Rochot, *La Correspondance Scientifique Du Père Mersenne*, Université de Paris, 1966.

[19] Mano Singham, *The Copernican Myths*, Physics Today 60, 12, 48 (2007).

[20] Frank Tipler and Wesley Bollinger, *Ptolemy versus Copernicus*. Available at https://inference-review.com/article/ptolemy-versus-copernicus (accessed Aug. 29, 2019).

[21] Carlos Calderón Urreiztieta, *The Monochord according to Marin Mersenne*, Perspectives on Science Vol. 18 No. 1, pp. 77-97.

[22] *Marin Mersenne*, https://en.wikipedia.org/wiki/Marin_Mersenne (accessed Aug. 15, 2019).

Developing Mathematicians: The Benefits of Weaving Spiritual and Disciplinary Discipleship

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Patrick Eggleton (B.S., M.Ed., University of South Florida; Ph.D., University of Georgia, all in area of mathematics education) serves as a Professor of Mathematics at Taylor University where he teaches mathematics and prepares future mathematics teachers. His interests focus on instructional strategies that emphasize active learning and reasoning, and in developing future mathematics teachers with a similar focus.

1 Introduction

Driving down the interstate in the early 2000s, I came upon a billboard advertising a Christian university with a single word – Discipleship. To be honest, I was quite impressed by that one-word explanation of this school's mission. Discipleship conjures an image of emerging adults engaging with mentors, to grow and learn through experiences, instruction, and reflection. While this mission is certainly commensurate with an emphasis on the faith development organized through chapel, campus life activities, and scriptural instruction, there is an implication that discipleship is equally at the heart of courses where students learn coding languages and integration techniques among the many other aspects of instruction ascribed to the university. Parks states, "Faith is often linked exclusively to belief, particularly religious belief. But faith goes far beyond religious belief, narrowly understood. Faith is more adequately recognized as *the activity of seeking and discovering meaning in the most comprehensive dimensions of our experience*." [12, p. 10] The mission of discipleship is a mission of faith development – a mission of meaning-making – and it is intricately weaved through every aspect of an emerging adult's development. James K. Smith suggests that the development of an individual's faith in God occurs primarily through the routines, rituals, and culture of the person's experience that seeps into their heart and transforms their identity. In this paper I explore how a similar process of routines, rituals, and culture are used to develop each student's disciplinary "faith" and how these processes complement each other.

2 Spiritual Discipleship

Most students choose a Christian university because they believe that somehow they will develop more of the faith that God calls them to exhibit into all aspects of their lives, including their vocations. It is not clear how this faith development will occur. For most, we believe that the combination of the campus culture, the interactions within the community, and the instruction in the classroom will somehow bring about faith development – and it will – to some degree or another. Our faith does not remain stagnant and our interactions with our environment affect our faith, for good or for bad. A university committed to biblical faith development seeks for students to develop the faith that is already manifest in their lives to grow closer and closer to the ideal faith encouraged by Jesus. This type of faith development requires intentional discipleship – much like the discipleship demonstrated by Jesus within the gospels.

"One of the most crucial things to appreciate about Christian formation is that it happens over time. It is not fostered by events or experience; real formation cannot be effected by actions that are merely episodic. There must be a rhythm and a regularity to formative practices in order for them to sink in—in order for them to seep into our *kardia* and begin to be effectively inscribed in who we are, directing our passion to the kingdom of God and thus disposing us to action that reflects such a desire." [14, p. 226] Jesus modeled this rhythm and regularity of formative practices in his teaching of the first disciples. First, Jesus served as a role model of faith and directed the attention of the twelve to others who modeled faith well, such as the Roman officer described in Luke 7 whose faith was praised by Jesus because he knew that just a word from Jesus would bring healing to his servant. The faith formation that Jesus provided also included a regularity of events and experiences that directed the passion of the twelve toward the kingdom of God. They witnessed miracles attributed to faith. They shared truths heard from Jesus and performed miracles. They also had experiences of faith, like Peter's walking on water, and challenges to their faith in the demand to take up a cross and follow. Jesus practiced a regularity of Sabbath study of the scriptures in the synagogue and participation in sacred observances, while also providing faith events and experiences that were a part of their daily existence.

Discipleship is a focus at many Christian universities. Where I teach, (Taylor University) spiritual discipleship has a regularity of formative practices. Some of these are more liturgical and systematic, like chapel services provided three times a week and the intentionality of most of the dorms sitting together over meal times. The chapel services expose students to role models in the faith, often sharing how faith has shaped the speaker's vocational life as well as their personal life. The "intentional community" demonstrated in one way through the dorm wing dining practices allows these emerging adults a venue for experiencing the messiness of relationships while haphazardly exploring practical outcomes of a faith-focused life. Other campus liturgies occur as opportunities are available, much like many of the lessons for Jesus's twelve. The campus emphasis on servanthood, while symbolically modeled at the beginning (the welcoming of freshmen to the campus) and end (graduation) of each school year with the Taylor towel, permeates the atmosphere in and out of the classroom through discussions and actions that continue to point the community toward the ideal of faith desired. Another less formal discipleship emphasis is in the development of leaders marked with passion. Much like Jesus sharing His passion for the Father and for His creation as He walked alongside His disciples, leaders at Taylor, both from the staff and emerging within the student body, share a passion for ministering love and truth within this community and beyond. These practices, among others, characterize a discipleship at Taylor that is much like the discipleship that Jesus offered His twelve. As summed up by the 2026 strategic direction, "the essence of the Taylor discipleship experience is . . . something woven into the fabric of our culture – the right kind of students having the right kinds of interactions with the right kind of faculty." [15, p. 10]

3 Disciplinary Discipleship

I would contend that an emphasis on spiritual discipleship at a Christian university can pervade the instructional practices of the Christian university faculty member. Those faculty members who weave their personal faith development with their disciplinary development provide a type of disciplinary discipleship for students emerging in those fields. Much like James K. Smith shared regarding spiritual formation, development within the discipline requires an education of more than the facts and procedures related to the given specialty. Exemplary education provides a discipleship within the discipline that seeps into the *kardia*, providing individuals with an opportunity to express their God-given unique gifts with a broader, life-encompassing emphasis. While what I call disciplinary discipleship is not necessarily unique to faith-based schools, the discipleship methods for developing the heart growth of faith are similar to the methods needed to develop the *kardia* connection within a discipline.

To better characterize this idea of disciplinary discipleship we can look at the journey of one of my former students through his preparation to become a mathematics teacher. Ken participated in a study designed to observe and analyze the development of emerging mathematics teachers [5]. These emerging teachers were enrolled in a program that sought to promote their ability to teach mathematics as described by the *Curriculum and Evaluation Standards for School Mathematics* that had been developed by the National Council of Teachers of Mathematics (NCTM) in 1989. The mathematics emphasized within this framework goes beyond the typical emphasis of mathematics computations and procedures to include what has been called "mathematical power" [9, p. 205], "mathematical proficiency" [11, p. 5], and more recently "mathematical practices" [10, p. 7].

> Strategic competence (i.e., the ability to formulate, represent, and solve mathematical problems) and adaptive reasoning (i.e., the capacity to think logically and to justify one's thinking) reflect the need for students to develop mathematical ways of thinking as a basis for solving mathematics problems that they may encounter in real life, as well as within mathematics and other disciplines [10, p. 7].

While most preparing mathematics teachers have excellent computational and procedural skills in mathematics, few have developed the more life-encompassing emphasis that is the goal of the more experienced community of mathematics teachers. Ken was observed in classes and student teaching, interviewed, and his course artifacts were analyzed over the last two years of his college experience, allowing a unique opportunity to note any affects from what I am calling disciplinary discipleship.

Ken was very gifted in mathematics, yet he struggled to see how mathematical ways of thinking provide a cognitive foundation for the strategic competencies and adaptive reasoning needed for problems encountered in life. The small advances Ken made over the 1.5 years of the program occurred due to what I would now call the discipleship aspects of the program. Although he was resistant to any changes in his formulaic view of mathematics, the regularity of experiencing, exploring, and reflecting on alternative views toward mathematics helped him open up to what he called a "deeper understanding of mathematics" [5, p. 98]. In an interview with Ken toward the end of his program, he shared about changes in his thinking and how those changes were connected to his experiences within the mathematics methods class. In the interview he shared how prior to the class his emphasis for students was to memorize procedures. At the end of his program he now saw daily, creative problem solving as a more ideal method of teaching mathematics.

The brief disciplinary discipleship he experienced within his mathematics teaching methods course allowed him to experience modeling of a broader emphasis in mathematics. He regularly met with mathematical experiences where he had to formulate, represent, and solve mathematical problems and where he had to think logically and justify his thinking. The "liturgies" advocated for future mathematicians and experienced by Ken in his methods courses are summed up by the "Mathematical Practices" in the Common Core State Standards: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-Construct viable arguments and critique the reasoning of others, 4-Model with mathematics, 5-Use appropriate tools strategically, 6-Attend to precision, 7-Look for and make use of structure, and 8-Look for and express regularity in repeated reasoning [4]. For example, he worked with others to develop regular patterns related to measurements taken with a viewing tube while adjusting distance from an object observed or the length of the tube. His group had to work together to provide firm justifications for the conclusions that they developed. His disciplinary discipleship also occurred through readings. In an interview related to his reading of mathematical power shared by the National Council of Teachers of Mathematics [9], Ken shared that this emphasis on mathematics provides a "deeper" understanding of mathematics. He shared that a "perfect world" would be characterized by students able to develop mathematical power [5, p. 98].

Having experiences where he had to formulate and solve problems and taking time to reflect on alternative views of mathematics provided a type of disciplinary discipleship for Ken. The discipleship-like influences that Ken experienced had started to work on his heart and passions, yet they were incomplete. Those serving as role models within this discipleship effort did not teach mathematics courses. They were researchers into methods of teaching. Classroom teachers contributing to Ken's university experience tended to strengthen his formulaic view of mathematics rather than providing a role model toward a broader view. The mathematics courses Ken had experienced, both at the university and before, also did not emphasize a more life-encompassing view of mathematics. Aspects of his disciplinary gifts were awakened, yet he lacked the rhythm and regularity in these formative practices in order to move him beyond the facts and procedures of his specialty.

In the Christian university we have an advantage when it comes to disciplinary discipleship. Since most of our students come to the university with a desire to grow in their faith, we can weave that path of spiritual growth with their path toward disciplinary growth. Both pathways require an element of humility, an acknowledgment that there is more for each of us to learn. Murray describes this essential disposition as follows: "This [disposition of humility] is the true self-denial to which our Savior calls us: the acknowledgment that self has nothing good in it except as an empty vessel which God must fill, and also that its claim to be or do anything may not for a moment be allowed." [8, p. 34] Humility was a rare disposition in Ken, who once referred to himself as a "math god" [5, p. 154]. Ball noted that one of the hindrances that developing mathematics teachers face in developing a broader understanding of mathematics is their success in a more formulaic approach to math. "Years of memorization, of focusing on answers, of inattention to meanings, have yielded reliably algorithmic ways of knowing and doing mathematics. Furthermore, the surrounding culture is even less oriented toward mathematical sense-making." [2, p. 15] Students usually pursue a specific discipline in their university studies due to success and gifting in that area. Pride in success and past achievements often serves as a hindrance to continued growth, but the Christian university's humble acknowledgment of a need for faith growth can contribute to a student's acknowledgment of a need for growth in their discipline.

4 Spiritual and Disciplinary Discipleship Intertwined

David Smith and Kevin Smith suggest that it is our practices or liturgies that form individuals. As such, if the pursuit of humility is not a regular practice within the Christian university, then this soil in which other virtues root [8, p. 17] will not be cultivated enough to produce the spiritual or disciplinary discipleship desired. Fortunately, many practices within Christian universities do promote humility. Taylor University's emphasis on servanthood through the Taylor Towel and its mission statement provides a foundation that encourages regular practical expressions of humility. Prayer in the classroom, while often discounted in efforts toward faith and learning integration, provides an opportunity for students and faculty alike to acknowledge our inadequacies and our dependence on God. Even regular chapel programs can communicate to those attending that our lives are on a continual journey of coming to know God more fully and learning to die to self daily. These practices serve to provide spiritual discipleship, yet they also provide a fertile soil for disciplinary discipleship.

Like spiritual discipleship, disciplinary discipleship requires formative routines that will allow the life-encompassing qualities of the discipline to seep into the *kardia* and broaden the abilities of the student. In mathematics, one of our liturgies comes in the form of problem solving. Liberal arts students explore the impact of compounded interest on loans they may pursue. Calculus students create models of a roller coaster and inquire into the rates of change associated with the falls and turns. Statistics students determine how carefully collected data can inform decisions that they make. This competence in problem solving is further developed as students articulate justifications for the solutions they develop. While students still experience formulaic exercises related to procedures and concepts in their coursework, the rhythm of problem solving explorations helps them move from the "memorizing" view of mathematics that Ken and many other students experience toward the life-encompassing mathematical practices promoted by experts.

Again, the Christian university has an advantage as we are able to weave the processes related to faith and disciplinary discipleship. One student wrote the following when reflecting on her mathematics class at a Christian university:

> Whenever you start with a problem, whether mathematical or personal, you have to ask yourself questions. For math, the questions are ones like, "What do I have to find? What information is given and what information do I already know without the information given in the problem?" You have to ask yourself the same kinds of questions when dealing with personal problem, whether dealing with earthly relationships or your heavenly one. When we want to find out more about God, we have to ask, "What exactly do we want to know? What do I already know relating to the subject and where can I find answers?" This course has taught me that there are many ways of finding these answers, whether the answers are from friends, family, or from God Himself. [7, p. 7]]

The spiritual discipleship of the student actually enhances the disciplinary discipleship and vice versa, allowing the student to accept more life-impacting understandings of the content.

Another liturgy common for our mathematics students is the need to work in groups. From a disciplinary aspect, the group work helps mathematics students formulate their problems and explore alternative perspectives to represent the problem and create solutions. Students develop communication skills that allow them to justify their thinking with one another and test their logic. These group attributes "disciple" them into a broader disciplinary mindset. The work in groups is also a focus in faith discipleship. Group work provides an opportunity for students to serve one another as they share their different backgrounds and gifts. They are challenged to be God's image bearers as they interact with one another, forgiving faults and building each other up. The disciplinary and spiritual discipleship work hand-in-hand as students seek and discover meaning both in their discipline and in their spiritual lives.

In the Taylor mathematics department, disciplinary discipleship is further encouraged through an emphasis on "big ideas". "According to Ken Bain's *What the Best College Teachers Do*, excellent teachers ask themselves what 'big questions' a course will help students answer, and how we can motivate students to care about these questions [1]. The best teachers stir up excitement and curiosity within students to explore the important issues and applications of the discipline." [3, p. 10] Case and Colgan reformatted the "big questions" into "big ideas" that are emphasized and analyzed by students within their mathematics courses. The "big ideas" help students move beyond the isolated skills and procedures learned within the course to gaining an appreciation for the beauty and relevance of the topics. "The Big Idea reflection assignment asks how students have learned to think more like a mathematician as a result of exploring the Big Ideas in a course." [3, p. 14] Much like the reflective assignments that helped Ken obtain a broader view of mathematics, reflecting on the "big ideas" requires students to commit in writing to a broader view of the mathematics that they have studied. They are being discipled to develop mathematical ways of thinking. Some of the "big ideas" emphasize a connection between faith and mathematics, weaving the spiritual discipleship with the disciplinary discipleship. Knowing students will reflect on these "big ideas" at the end of the course, instructors explicitly emphasize them more often throughout the course, further helping students move from their former ways of knowing math into a more developed mathematical proficiency. The emphasis on "big ideas" is moving the mathematics from the students' heads to their hearts.

5 Conclusion

Being a disciple of Jesus is not primarily a matter of getting the right ideas and doctrines and beliefs into your head in order to guarantee proper behavior; rather, it's a matter of being the kind of person who *loves* rightly – who loves God and neighbor and is oriented to the world by the primacy of that love. [14, p. 32]

The term "discipleship" beautifully encompasses the mission of the Christian university. Much as it was for Jesus's twelve, the rhythm and regularity in the interactions and experiences emerging adults sample in the university moves them beyond "right ideas and doctrines" and develops the heart. With the right liturgies, the student's faith develops significantly toward the ideal faith taught by Jesus. Woven with disciplinary discipleship, the gifting and calling of students develops, preparing them to share their talents with others as they continue on the journey God provides.

References

[1] Bain, Ken. *What the Best College Teachers Do*. Cambridge, Mass: Harvard University Press, 2004. Print.

[2] Ball, Deborah Loewenberg. "Breaking with Experience in Learning to Teach Mathematics: The Role of a Preservice Methods Course." *For the Learning of Mathematics*, vol. 10, no. 2, 1990, pp. 10–16.

[3] Case, Jeremy, and Colgan, Mark. "'Big Idea' reflection assignments for Learning and valuing mathematics." *Proceedings of the Association of Christians in the Mathematical Sciences Conference, May 31 – June 2, 2017,* edited by Russel Howell, Association of Christians in the Mathematical Sciences, 2018.

[4] Council of Chief State School Officers (CCSSO) *Standards for Mathematical Practice.* http://www.corestandards.org/Math/Practice; 2019.

[5] Eggleton, Patrick J. *The Evolving Mathematical Philosophy of a Preservice Mathematics Teacher*. 1995. Print.

[6] Eggleton, Patrick J. Faith Learning Application Paper. October 20, 1999. TS. Written for Huntington College New Faculty Orientation – Author's private collection.

[7] Eggleton, Patrick J. Mathematics: An Infrastructure for Inquiry. January 2, 2004. TS. Written for Huntington University Tenure – Author's private collection.

[8] Murray, Andrew. *Humility*. New Kensington, PA: Whitaker House, 2004. Print.

[9] *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: The Council, 1989. Print.

[10] *Principles to Actions: Ensuring Mathematical Success for All.* , 2014. Print.

[11] Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. *Adding It Up: Helping Children Learn Mathematics*. Washington: National Academies Press, 2001. Internet resource.

[12] Parks, Sharon D. Big Questions, *Worthy Dreams: Mentoring Emerging Adults in Their Search for Meaning, Purpose, and Faith*. San Francisco, CA: Jossey-Bass, 2011. Print.

[13] Smith, David, and James K. A. Smith. *Teaching and Christian Practices: Reshaping Faith and Learning*. Grand Rapids, MI: W.B. Eerdmans Pub. Co, 2011. Print.

[14] Smith, James K. A. *Desiring the Kingdom: Worship, Worldview, and Cultural Format*ion, 2009. Print.

[15] Taylor University. Strategic Directions 2026: Final Report. May 2015. TS. University archives.

Overcoming Stereotypes through a Liberal Arts Math Course

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Abstract

"I'm just not a math person." We've heard this comment countless times from our students. It is a mentality that both paralyzes and strangely comforts them. In this paper, I will describe how I use my liberal arts Joy of Mathematics course to help students address and overcome stereotypes. In particular, I will discuss a specific assignment as well as share some student comments and perspectives on how this course helped change their viewpoint on more than just math.

1 Introduction

When I tell people that I teach math I typically get one of two reactions: "Ew, I hate math!" or "Wow, you must be so smart!" What is even worse is that people do not realize how inappropriate both responses are. It has become widely accepted that math is hard and you have to be really "smart" to succeed at it. It is overwhelming to think about overcoming this mindset. How do we even begin tackling this monumental problem?

As I always tell my students, when you are faced with a problem that seems too difficult, try to solve a smaller related problem. I can not change everyone's view of mathematics. However, studies show that math anxiety comes from a variety of places, two of which are parents [6] and teachers [1]. I am a parent and a teacher and this is my smaller related problem to try and solve. Though the responsibility of a parent is greater, my job in promoting a growth mindset in mathematics is much easier, in a way, with my little boy than with my students. My little boy is a clean slate, so to speak, while my students come with a lot of mathematical baggage. So how do we, as educators, remove this baggage and start with a clean slate? I think one solution is through a liberal arts math course-a course unlike any other students have taken. Eugene Northrop developed one such innovative course for freshman at the University of Chicago in the 40s. He said that a liberal education is one which "liberates the student's mind" [4]. I think we can all agree that our students need to have their minds liberated, especially when it comes to mathematics.

I have created such a course at Winthrop University entitled Joy of Mathematics and have seen really positive results. I have seen student perspectives changed and minds liberated from the bondage of math stereotypes and it has been so encouraging. But then I asked myself some questions. If we can get students to overcome stereotypes about mathematics why stop there? Why not use their positive experience with mathematics to help them overcome other stereotypes, to see that getting to know someone (or something) can be eye opening and life changing? This is the motivation for my final assignment in my Joy of Mathematics course at Winthrop University. In this paper I will give you a small taste of this course, both the content and the population, describe a final reflection paper I give involving math and stereotypes, and discuss results I have seen over the past two years.

2 Setting

Winthrop University is a public liberal arts university in Rock Hill, South Carolina. As a believer it can be frustrating teaching at a public school. Many of my colleagues have vastly different views than I do. I can not read scripture or pray with students in the classroom. However, as Paul describes in 2 Corinthians, my goal is to be a "living letter" shining God's light and love to all I encounter. One way I share God's love is by showing my students that I care about them as people. I want them to succeed inside and outside the classroom. This desire for student success was one of the catalysts for designing Joy of Mathematics.

For years at Winthrop University, there has been only one general education math class other than college algebra/pre-calculus and statistics-Introduction to Discrete Mathematics. This notorious class has had one of the highest D/F/W rates of the entire university. Just the name of the course strikes fear and dread into our students. There are several reasons this course has such a high failure rate, one of which is that there are too many students who take it. The class is only required for our early childhood education majors, so many students taking the class do not need this specific course. They typically sign up for this course for one or both of these two reasons: they think it will be easy as it is one of the lowest numbered courses and it is not algebra, which many of them have had negative experiences with in the past. Students take Introduction to Discrete Mathematics to simply check a box in their degree checklist and nothing more. This problem became abundantly clear to our department. So we set out to solve it. A colleague of mine designed a 100-level course called Every Day Mathematics and I created Joy of Mathematics. Though the two courses are extremely different in content they have a common goal-to help students develop an appreciation for mathematics.

How does one develop an appreciation for something? The answer may be different depending on the person. However, I think we can all agree on things that will *not* develop an appreciation. These include rote memorization of formulas, methodical procedures for solving problems, and learning other mathematical topics seen in a vacuum. Morris Kline, when writing about liberal arts mathematics, said "to separate mathematics from other human subjects and endeavors is to lead to a hollow shell and...a perversion of the subject" [3]. In both Everyday Mathematics and Joy of Mathematics we show connections between mathematics and the world.

2.1 The Course

As mentioned above, Joy of Mathematics is a 100-level liberal arts math course. I use the *Heart of Mathematics* book by Burger and Starbird, and we cover a variety of topics. I always start the semester with fun problem solving. Day one is spent discussing the importance of a growth mindset and questions such as "what does it mean to be smart?" and "how do you know you've learned something?" After we discuss these topics together we start a fun problem-the problem of the zombies and hats. I include it here for your pleasure as well!

The Problem: Suppose you are out walking with some friends and are suddenly confronted by zombies. These are zombies who love problem solving so they decide to give you a challenge. They are going to put you all in a line and put either a black hat or a white hat on your head. You can see the color of everyone's hat in front of you but not your own and not anyone behind you. They will start in the back of the line (or wherever you want them to) and ask the color of the hat on your head. You can say only one word-black or white. However, you can discuss a strategy beforehand with your friends. If you say the right color you live. If not, your days of fun mathematics are over. The question is: how many people can you save?

Of course this is an overwhelming question to start with and they always try to find loopholes like using accents, coughing, stomping your feet, etc. none of which are allowed. I encourage them to start by trying to save just one person. We easily do that. Then I challenge them to try and save half, then 2/3. Then I ask can you save everyone? If not, how many can you save and how? This is a challenging question, but the students really enjoy it! I bring in two different types of party hats and we act it out so they can be a part of the problem. This forces everyone to actually think and understand the solution to the problem. This sets the tone for the entire semester. Students immediately see two crucial things: the content of this class will be different from ones they have had in the past and they will be expected to participate.

The next couple of days are spent solving other fun problems, many of which come from chapter one of the Burger and Starbird book. Other topics I cover include Fibonacci numbers, the golden ratio, modular arithmetic and cryptography, voting theory, exploding dots (which is based on activities created by James Tanton for the Global Math Project), and infinity. My last unit has varied. I have tried math and art, rubber geometry, graph theory, and this semester I might try math games. There is no set curriculum for the course but there are set goals.

My overall goal in this course, which I tell students on day one, is to change their perspective on mathematics. Most students (even beginning math majors) do not really know what math is. I want to give them a glimpse into the beauty and creativity of mathematics and help them develop an appreciation for mathematics. Overall, I have been successful in this endeavor. Below are two student quotes from my first time teaching this course in the spring of 2017.

"Even if I never use the methods that were taught to me in Math 112 I still left with an understanding that math is something that grows every single day, and even though I may think that I won't use math outside of class it's all around me no matter what."

"The Joy of Mathematics course is a course that truly allows for students to embrace mathematics to its full extent and have enough knowledge to spread its appreciation."

2.2 The Students

The students that typically take Joy of Math come into the course with feelings that are the opposite of the title. They enter with years of built up fear and anxiety towards math. They all classify themselves as "not math people", as if there was such a thing. In fact, just as we have axioms that govern our mathematics I believe we, and our students, should share the following axiom for our lives.

Axiom 1. $\{x \mid x \text{ is not math person}\} = \emptyset$

I often wonder to myself why this term is so widely accepted. No one would classify themselves as “not a reading person” or “not a writing person” . To believe that there are some people who are programmed for mathematics is precisely having a fixed mindset about mathematical ability. In Mindsets and Math/Science Achievement, Carol Dweck discusses this very issue. She indicates that the source of such a mentality comes from parents and teachers using this label as a source of comfort for students [2]. In fact, some of my own students have mentioned this phenomena. They use the label “not a math person” as a safety net for any struggles or failures they may face in mathematics. Below are a handful of student quotes regarding this issue.

“I’ve found myself saying ‘I’m just not good at math’ a lot because of how hard I’ve actually tried to understand it but could never understand.”

“It’s just easier to say ‘I’m not a math person’ than to explain that, try as I might, I can’t understand math. Or it’s easier to state so that people don’t think down on me because of how bad I am at math. It’s a blanket statement to hide behind I guess.”

“I absolutely hate failing so I often find comfort in telling my teachers and friends that I am not a math person.”

While I respect the honesty and introspection from these students it saddens me that they feel so hopeless in their mathematical abilities that they label themselves in this way. I try very hard to be sensitive to my students’ misconceptions of themselves and mathematics. I work all semester to stay positive, encourage a growth mindset, and show students that getting the right answer is not what is most important. I know I have not had complete success in this endeavor, but I have seen so many students change their perspective on math by the end of the semester. After seeing such positive results the first time I taught this course I decided to dream bigger. I wanted students to realize that if they can overcome their stereotype of mathematics, then they can overcome other stereotypes as well. This led to a final reflection assignment which I will now describe.

2.3 The Assignment

When you think about the problems with stereotypes you probably do not think about mathematics but there are many stereotypes associated with the subject. Our students stereotype math as hard, boring, and formulaic. They stereotype themselves as not math people and certain ‘types’ of people as good at math. I decided to ask students to seriously reflect on these issues, along with other stereotypes, in a final reflection paper. For this paper, I ask students the following questions.

- What stereotypes did you have about mathematics before this course? How have these stereotypes affected your view and/or performance in mathematics courses?
- Now that you’ve seen a different side of math do you think that stereotype has changed? If so, how?
- Now think about any stereotypes you may currently have towards other people, whether it be based on race, gender, religion, etc. Why do you think you have these stereotypes? How do these stereotypes affect the way you interact with this group of people?
- Do you believe that getting to know this group of people better would change the stereotype?

- Lastly, do you think you stereotype yourself in any way? (For example, you may have said that you just “aren’t a math person”. That’s a stereotype.) Think long and hard about this. How do these stereotypes limit who you can be and what you can achieve?
- Thinking about this course, your experience and what you’ve learned, how can you work to overcome stereotypes?

As you can imagine, not every student takes this assignment seriously. There have been many students who missed the opportunity to grow because they just wanted to complete the assignment and be done for the semester. However, as I will discuss in the next section, overall I have been pleasantly surprised with the students’ reactions to this paper and to the course as a whole.

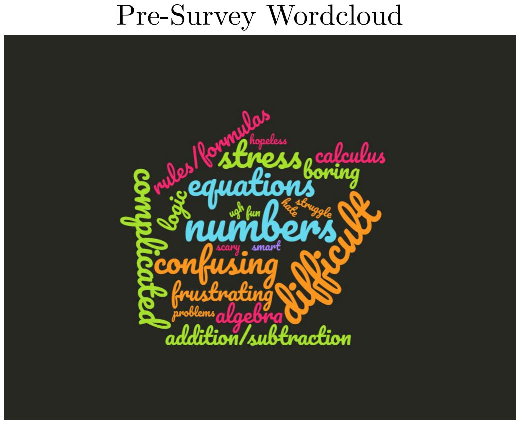
3 Results from Course

As mentioned, I have seen some great results from this course. I have broken the results into two sections. First I will discuss results from a pre and post survey that I give my students. Then I will discuss some student quotes from the end of the semester reflection assignment.

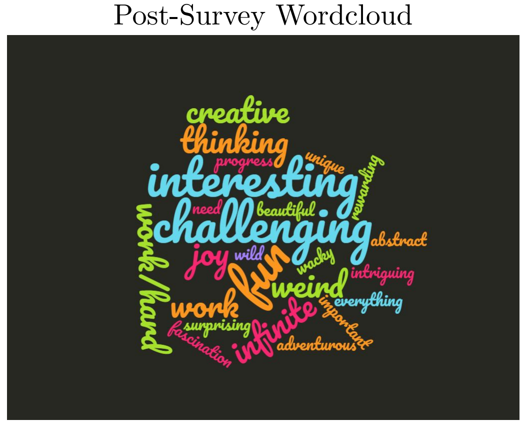
3.1 Pre- and Post-Surveys

On the first day of class before I introduce myself or the course I give students a survey. I do the same survey on the last day of the course. I ask them a number of things some of which I have included here. I start with a simple word association. I ask students to write down the first three words that come to their mind when they hear the word math. Below are the word clouds from student responses over the past two semesters of the course. The larger a word is the more often it was listed.

Pre-Survey Wordcloud



Post-Survey Wordcloud



As you can see, the pre-survey wordcloud is what you would expect-difficult, numbers, confusing, equations, etc. The post-survey includes words like fun, interesting, and challenging. I find it interesting, but not all surprising, that students still found math difficult/challenging. While most students who enter this course assume that it will be easy (because it is numbered 112 which is lower than 150 so must be easier) they quickly learn that the content is not easy, just different.

Another thing I ask students is to give a score from 1 to 5 to four questions, with 5 being the most extreme (i.e. 5 would mean very hard or very beautiful). The results are given below with $N = 48$.

Question	Pre-Survey Average	Post-Survey Average	t-score	p value
How hard is math?	3.64	3.5	.96	.18
How creative is math?	3.63	3.92	1.36	.1
How scary is math?	3.58	2.92	3.77	.000
How beautiful is math?	2.79	3.35	2.83	.004

I conducted a one-tailed t-test for a single population with two means. The null hypothesis was that the difference between the pre and post survey ratings would be zero. For the question "how hard is math" I had a t-score of .96 giving $p = .18$. This is not statistically significant and it was the question with the smallest change. However, as mentioned I am satisfied with this result because I do not want to convince students that math is not hard. I want them to believe that they can succeed despite how hard it is. Next the t-score for "how creative is math" was 1.36 giving $p = .1$. The last two questions had the most statistically significant results. The question "how scary is math" gave $t = 3.77$ and $p = .000$ and "how beautiful is math" gave $t = 2.83$ and $p = .004$. These results are exactly what I wanted. My main goal in this course is to change student perspective on mathematics, to help them overcome their fixed mindsets and believe that they can succeed in math. I want students to see the beauty of mathematics and develop a newfound appreciation for the subject and these results show that is exactly what is happening.

3.2 Student Quotes

Next, I will include some of my favorite quotes from the end of the semester reflection assignment on overcoming stereotypes.

"Math 112 taught me numbers are actually a small part of math . . . The reason why my view changed is because I was forced to open my mind to new things, and to critically think in order to find a solution. Outside of math, I've found myself having other stereotypic views. I believe the reason why I have those views is because I have not yet opened my mind into trying to understand new things."

"I had no idea that this math course would so heavily influence my opinions about a topic I had so intensely despised."

"I feel like [math 112] helped me in life also. I walk into situations with wider eyes and an open mind. I used to look at math and say "Math, That's hard. I can't do it.". . You can't go into every situation saying that you can't. Past experience and stereotypes are hard to overcome but all you have to do is try."

"I never thought that a math course would teach me about stereotypes and how to overcome them, but I am glad for it. Sometimes learning lessons in unexpected places can be more impactful than learning them where you would expect."

"Most of the fear in my life is because of the little amount of knowledge I possess about it. My previous fear of math worked the same way."

"Much like my stereotypes in math I think that my stereotypes towards others come from a misunderstanding of who they are. I believe that if I learn more about the people I stereotype I will end up connecting with them. I could make new connections and become a more open-minded person much like how I have become more open to learning about mathematics through learning and understanding more about the subject."

These results and student quotes go to show that perspectives can be changed, labels can be removed, and minds can be liberated.

4 A Call to Action

As I close this paper I have a challenge for all of us. The problem of stereotypes goes beyond our students and mathematics. We need to ask ourselves some serious questions and take time to genuinely consider them. How are we stereotyping our students? Do we go into our 100-level courses with a bad attitude already believing that our students are going to fail? I know it can be difficult teaching these lower level courses where we do not usually get to know the students very well. The sections are large and the students are typically uninterested. It is hard to stay excited when you care more than the students do, especially when you have been teaching for years, but I challenge you to enter the classroom with joy and excitement that is contagious. Infect your students with your love for mathematics!

Are we letting a bad past experience with a specific student influence how we view and interact with that student in a different course? I have personally been guilty of this before and have learned so much. Sometimes students, just like faculty, have a bad semester. We can not let past performance define who they are. We must show grace and believe that they can and will improve. God continually looks past our failures and sees us for who we are in Him. Let us all strive to do the same for our students.

Lastly, are you stereotyping yourself? Whether we would like to admit or not, almost everyone struggles with some sort of imposter syndrome. We believe we are not good enough as research mathematicians or as teachers and this influences how hard we try and how willing we are to fail. We need to humble ourselves and believe in the abilities God has given us. We can not do anything in our own strength but with Him we can do amazing things, in all areas. I close with one of my favorite verses. Let this verse inspire us to be the best we can in all classes, with all students, and in our lives.

"Whatever you do, work at it with all your heart as if for the Lord and not for man."
(Colossians 3:23)

References

[1] Beilock, S. L., Gunderson, E. A., Ramirez, G., and Levine, S. C. (2010). Female teachers' math anxiety affects girls' math achievement. *Proceedings of the National Academy of Sciences of the U.S.A.* 107, 1860-1863.

[2] Dweck, C. (2008). Mindsets and math/science achievement. *New York: Carnegie Corporation of New York, Institute for Advanced Study, Commission on Mathematics and Science Education.*

[3] Kline, M. (1962). Mathematics—A Cultural Approach, *Addison-Wesley Series in Mathematics.*

[4] Northrop, E. P. (1945). Mathematics in a liberal education. *The American Mathematical Monthly*, 54: 132-137.

[5] Steele, C. M. (1997). A threat in the air: How stereotypes shape intellectual identity and performance. *American Psychologist*, 52(6), 613-629.

[6] Wigfield, A., and Eccles, J. S. (2000). Expectancy-value theory of achievement motivation. *Contemorary Educational Psychology* 25, 68-81.

Analyzing the Impact of Active Learning in General Education Mathematics Courses

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Abstract

In this paper, we share the preliminary results of a study that explores the overall perceptions and attitudes of students in general education mathematics courses. Our work includes an analysis of survey data collected from two different general education mathematics courses on three occasions throughout a semester: pre, mid, and post. We compare students' responses in one course taught using primarily active learning-based methods such as group work, projects, and discovery learning to the responses of those in a different general education course taught using a more traditional, lecture-based method. The surveys explore students' disposition, mindset, mathematical confidence, mathematics anxiety, and perceptions of pedagogical methods. In both courses, our analysis showed that students indicated a growth-mindset view of learning mathematics. While our analysis did not indicate any significant difference in students' math anxiety level, the students in the active learning-based course experienced lower confidence levels compared to students in the lecture-based course. By comparing pre-survey and post-survey responses, our analysis also explored how these perceptions and dispositions evolved over the duration of each course. For example, our analysis indicated that student enjoyment of mathematics in general was increased throughout the semester in both courses. As we continue

to collect data, we predict the differences of student attitudes towards effective learning styles, math confidence, and math anxiety will become more pronounced over time.

1 Introduction

Undergraduate general education requirements account for, on average, approximately 30% of a student's curriculum [7]. Because of the large impact general education has on a student's academic career, its reform in higher education has remained an ongoing topic of discussion [2, 7, 13]. Many majors, including STEM, business, and the social sciences, have courses within their own curriculum that will satisfy the general education requirement for mathematics, analytical reasoning, or quantitative reasoning. For those whose discipline curriculum does not already have a required course that will fulfill their mathematics general education requirements, universities often create special mathematics courses.

In this paper, we share our preliminary analysis of the introduction of a nonstandard mathematics general education course taught using primarily active learning-based methods, such as group work, projects, and discovery learning. Specifically, we examine the overall perceptions of students in this new general education mathematics course and compare those to the attitudes of students in a more traditional mathematics general education course taught using a lecture-based method. Students in both courses filled out pre- and post-surveys that assessed their views of the impact of participation in a mathematics course. In particular, these questions explored the students' dispositions and mindsets toward learning in general and the learning of mathematics, their confidence in doing mathematics, their mathematics anxiety, and their perceptions of pedagogical methods used in mathematics courses. By comparing pre-survey and post-survey responses, we were able to analyze how these perceptions and dispositions evolved, or remained unchanged, over the duration of each course.

This paper first discusses relevant past research of active learning and mathematics anxiety in Sections 2 and 3, respectively. We then describe our study design in Section 4. Sections 5 and 6 share and analyze the preliminary results of our study. Finally, we conclude by describing our plans for future work in Section 7.

2 Exploration of Active Learning Techniques in Higher Education

Higher education aims to help students cultivate the ability to communicate effectively, think critically, and solve problems [23]. In [19], Halpern defines critical thinking as the "purposeful, reasoned, and goal-directed" use of cognitive skills that requires students to be actively engaged in applying, analyzing, synthesizing, evaluating, and communicating information [33]. In order to achieve this, a broad consensus is that it is important for professors to provide meaningful learning opportunities in which students can engage in open problems and tasks [18, 23]. Some teaching methods used to create these learning opportunities include active learning techniques such as cooperative learning, project-based learning, and discovery learning, Many studies have been completed in order to determine the impact of these different active learning techniques in general education and STEM courses [10, 23–25, 38].

Collaborative learning and small group learning is believed to foster critical thinking. For example, Scardamalia and Bereiter [31] and Vygotsky [37] found that social interactions between students often help them tackle problems they may not have been able to solve individually. In [38], Ward compared a group study approach to learning to a lecture-based approach in a general education science course. He found that lecture worked better for lower achieving students while a group-based method resulted in better content retention for higher achieving students. Furthermore, the study in [18] found that the use of open problems and tasks in small cooperative groups was effective for enhancing students' critical thinking skills in science courses. Kim et al. [23] implemented group-based learning modules in which students were asked to solve real-world natural disaster problems. They found that although students did improve in critical thinking, they did not "master" critical thinking. Instead, students were only able to reach the mid-level subcategory of "developing" critical thinking skills. Their conclusion was that their active small-group learning environment helped students engage cognitively and enhanced student engagement [23].

Discovery-based and project-based learning may also help student engagement with course material. We follow Bruner's definition of discovery-learning [8, 9], which states that in discovery learning, the instructor's primary goal is to assist students in discovering the concepts and ideas of the course and to facilitate students in developing knowledge through exploration and experimentation [25]. Kyriazis et al. [25] explored discovery learning techniques using Mathematica electronic worksheets, which allowed students to conduct computational experiments in mathematics and science courses. Although they found that students' beliefs about physics and mathematics did not change, they reported higher percentages both in passing grades and overall grades for the students who had the discovery-based methods included in their class [25]. Additionally, Havenga [20] found that project-based learning in programming courses contributed to the development of a variety of important critical thinking skills including solving complex problems, working within a team, and establishing self-directedness.

Cherney [10] also explored the impact active learning had on free recall in undergraduate courses. She found that active learning helped students have better recall of material across introductory level and upper level courses taught by the same instructor. Furthermore, connecting the course material to real-life, concrete examples and experiences enhanced student understanding in introductory psychology and statistics courses [10].

Studies have shown active learning techniques positively impact student learning; however, student perception of active learning does not necessarily reflect these positive impacts. For example, Vadav et al. [39] found that although students' learning gains from problem-based learning were twice their gains from traditional lecture, students thought they learned more from traditional lecture. Similarly, Lake [26] reported that students in the active learning sections of a course perceived that they had learned less than students in the lecture section of the same course. Additionally, students' perceptions of course and instructor effectiveness were lower in the active learning sections than in the lecture section [26]. Smith and Cardaciotto [35] found that although students participating in active learning activities reported greater retention of and greater engagement with course material, students participating in content review activities showed a greater enjoyment of the class and a more positive overall evaluation of the course. Not all active learning techniques are perceived equally among students. In [28], Machemer and Crawford compared eight teaching techniques utilized in a single class, each classified as either cooperative, independently active, or lecture. They found that students valued lectures and being individually active equally well and that students valued cooperative activities significantly less. Their conclusion was that active learning is valued from the students' perspective, but working with others significantly diminishes the value [28].

3 Mathematics Anxiety

A mathematics or quantitative reasoning component is typically required as part of a major's general education curriculum, and many students who are taking these quantitative reasoning general education courses have lower confidence in their mathematical abilities and/or have high anxiety levels when performing various mathematical tasks [27]. Both factors may contribute to student disengagement and ultimately failed learning outcomes. The first factor, known in the literature as "mathematics self-efficacy," is summarized as a student's conviction that he or she can successfully solve a math problem or complete a mathematical task. Students with low levels of mathematics self-efficacy are at a high risk of underperforming in mathematics despite their actual abilities [32]. The Programme for International Student Assessment (PISA) provides the most extensive set of data on mathematics self beliefs by collecting responses from students among the 34 Organisation for Economic Co-operation and Development (OECD) countries. In its 2012 assessment, 43% of students reported that they agree or strongly agree that they are not good at mathematics, whereas 38% reported to have always believed that mathematics is one of their best subjects [14]. The assessment showed mathematics self-efficacy is strongly associated with mathematics performance. In particular, students with low mathematics self-efficacy perform worse in mathematics than students who are confident about their ability to handle mathematical tasks [14]. On the other hand, the assessment showed mathematics self-efficacy tended to increase among countries that show reduced levels of math anxiety, another factor that can plague a student's experience in a mathematics general education course.

Mathematics anxiety is defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" [30]. A considerable number of children and adults have mathematics anxiety. In fact, according to Luttenberger et al. [27], 93% of adult US-Americans report experiencing at least some math anxiety and 17% report high levels of anxiety. This anxiety can lead to avoidance of mathematical activities as well as overloading working memory during mathematical tasks, both of which disrupt mathematical learning and performance in the classroom [11]. On average across OECD countries, 30% of students reported that they feel helpless when doing mathematics problems; 59% of the 15- to 16-year-old students reported that they often worry math classes will be difficult for them; 33% reported that they get very tense when they have to complete math homework; and another 31% stated they get very nervous doing math problems [14]. These self beliefs are observed to have substantial negative impact on mathematical performance. The assessment reveals that greater mathematics anxiety in OECD countries is associated with a 34-point lower score in mathematics, the equivalent of almost one year of school [14]. While math anxiety is universal, studies show that students who mostly enjoy humanities or social sciences subjects have higher mathematics anxiety compared to those who mostly enjoy mathematics-related subjects [16, 36]. This suggests mathematics anxiety is a particularly influential factor in the mathematics general education classroom where most students are pursuing non-mathematical majors.

Researchers continue to find that that having high levels of math anxiety has a negative impact on math performance, math confidence, and math self-efficacy [1, 5, 17, 22]. On the other hand, math confidence, self-efficacy, and growth-mindset are shown to positively influence math performance [1,5,6]. Therefore, identifying causes of math anxiety and working to alleviate this anxiety and math avoidance has become a special effort of researchers, universities, and societies like the Mathematical Association of America and the National Institute of Education [3, 21, 34]. One way that professors are attempting to ease math anxiety is through researching and introducing different assessment methods and learning techniques. For example, Collins et al. found decreased anxiety levels for mathematics courses that used mastery-based testing [4]. Mastery-based testing is an assessment scheme that gives students multiple attempts to demonstrate full "mastery" of problems derived from clear course concepts. In [6], Boaler found that active, heterogeneous high school classrooms led to positive beliefs about relational equity and mathematics. Further exploration of active learning techniques in the classroom found positive impact on student learning in STEM classrooms [12, 15].

4 Study Design

This study was conducted at Lewis University, a four-year, private, Catholic university located outside of Chicago. The enrollment is approximately 6,800 with an undergraduate population of around 4,500. Lewis is a primarily teaching-focused university, and many of its students are from the Chicago and Joliet areas. Because of its increasingly diverse population, Lewis is also an Emerging Hispanic-Serving Institution that services a 34% minority population.

In 2019, Lewis University rolled out a new revised General Education Plan. Prior to 2019, the mathematics general education requirement could be satisfied by Finite Mathematics, Introductory Statistics, College Mathematics, or one of several calculus options. During this transition, we introduced a new mathematics course, "Win, Lose, or Draw," which could also satisfy the math general education requirement. Our initial study compares the attitudes of students in Win, Lose, or Draw to those of the students in College Mathematics over a one-year period. Specifically, the following questions guided our research:

1. Is there a difference in math anxiety levels between students in these courses?
2. Do students in these courses reflect different mindsets of learning?
3. Is there a difference between students' confidence levels in their mathematical ability?
4. What are the opinions regarding effective teaching techniques in mathematics courses for students in these courses and do they differ?

We also tracked the changes, if any, of student perceptions over the entirety of the semester in each of the courses.

4.1 Courses Examined

College Mathematics covers many of the same topics as a typical discrete mathematics course, but at a less exhaustive and rigorous level. Topics include set theory, counting, probability, and statistics. This course has been taught at Lewis University in a traditional lecture-based setting by experienced mathematics adjuncts for the past 20 years.

Created by Dr. Karen Holmes of Butler University, Win, Lose, or Draw is an analytical-reasoning course that covers set theory, counting, and probability. Each topic is motivated by games, and there are many active learning components including collaborative, cooperative, and problem-based learning. While class time is structured around group work, the majority of the course assessment is individual. In the classroom, students work together in groups of four on problems in their interactive workbook, which serves as their course textbook. What they do not finish in class, they take home as homework, and each student must submit his or her own workbook as a homework grade. All tests and quizzes are taken individually. At the end of the semester, students work on a group project that they present to the class. Since this course was adopted by Lewis in the fall of 2018, all instructors teaching this course were teaching it for the first time. The student population of these two courses are the same; the majority of students enrolled are either nursing or humanities majors.

4.2. Survey Design

To perform our analysis, we created and administered surveys to students at three stages in the semester: pre-surveys (given on the first day of classes), mid-surveys (given in the middle of the semester), and post-surveys (given on the last day of classes). The surveys asked students how much they agreed or disagreed with statements involving math intelligence, confidence in their mathematical ability, techniques that help them learn mathematics, and enjoyment in doing mathematics. Students were also asked to rate their current anxiety level with regard to nine aspects of enrollment in a math course as well as give an overall anxiety rating compared to that of previous math courses. Students were assigned an identification number to preserve their anonymity, but also to allow us to track individual changes in feelings over the course of the semester. A complete version of our survey is available upon request. The questions were adapted from the Mathematics Self-Efficacy and Anxiety Questionnaire (MSEAQ), which was found to be highly reliable and relatively valid [29]. In total, we surveyed 126 students from eight sections of College Mathematics and 60 students from four sections of Win, Lose, or Draw. Note that there were 191 students enrolled in College Mathematics at the beginning of the respective semesters, which means our response rate was 65%. Similarly, there were 66 students enrolled in Win, Lose, or Draw at the beginning of the respective semesters, which gives us a response rate of 91%. The lower response percentage for College Mathematics could be at least partially attributed to our totals not accounting for any students who dropped the course.

5 Results

Our analysis considered data from the pre-survey and post-survey results gathered from students enrolled in College Mathematics and Win, Lose, or Draw. We did not include the mid-survey results in this analysis because there was a low response rate. In this section, we present our initial findings from comparing the two course types, organized into the following categories based on our research questions: effective learning styles in the classroom, mathematical confidence level, enjoyment of course as well as mathematics in general, and mathematics anxiety level. On our pre-surveys and post-surveys, students were provided with four statements related to how students learn, four statements related to mathematical confidence, and four statements related to enjoyment. Responses to each of these questions could range from 1 to 5, with a 1 signifying "strongly disagree" and a 5 signifying "strongly agree."

We present the resulting means and standard deviations of two questions related to mathematical learning styles in Tables 1 and 2. Table 1 indicates that while there was no significant difference in how students valued active learning, students did seem to believe that it was helpful to their studies. Table 2 gives the results of students' overall perceptions of lecture-style teaching. Coming into the course, students in Win, Lose, or Draw indicated on their pre-surveys that they found lecture significantly less helpful than their College Mathematics counterparts, with a t-test providing a p-value of 0.0060. In the post-survey, however, both types of students valued lecture-based learning similarly. In fact, Win, Lose, or Draw students seemed to find lecture more helpful after taking their course, with a t-test providing a p-value of 0.0306. The exact breakdown of the proportions of Win, Lose, or Draw (WLD) students' responses to this statement about the usefulness of lecturing is illustrated in Figure 1.

Course	Pre-survey mean (sd)	Post-survey mean (sd)
Win, Lose, or Draw	3.91 (0.73)	4.06 (0.66)
College Mathematics	3.79 (0.82)	3.84 (1.00)

Table 1: Average and standard deviation of pre-survey and post-survey responses to the statement: *I generally find interactive learning activities helpful to my studies.*

Course	Pre-survey mean (sd)	Post-survey mean (sd)
Win, Lose, or Draw	3.03 (1.12)	3.51 (1.02)
College Mathematics	3.31 (1.23)	3.53 (1.17)

Table 2: Average and standard deviation of pre-survey and post-survey responses to the statement: *Listening to a lecture is helpful for learning mathematics.*

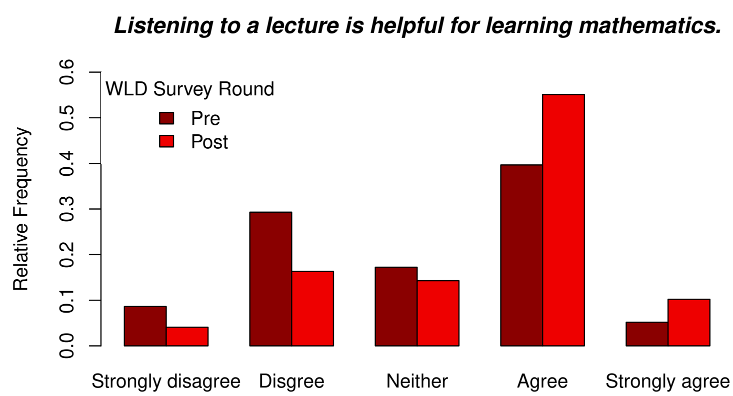


Figure 1: Breakdown of Win, Lose, or Draw (WLD) students' responses, divided by survey round, to the statement: *Listening to a lecture is helpful for learning mathematics.*

Students' perceptions of growth mindset and mathematical confidence are summarized in Tables 3 and 4, respectively. There was no significant difference in the results between the classes for the statement referenced in Table 3 about being able to change one's mathematical intelligence, nor when comparing the pre-survey responses to the post-survey responses within classes. Table 3 indicates students seemed to disagree with this statement with little variation, which indicates they did feel they could change how well they performed.

In the pre-survey, there was not a significant difference between Win, Lose, or Draw and College Mathematics for the statement referenced in Table 4 about mathematical confidence. However, in the post-survey College Mathematics students were significantly more confident than Win, Lose, or Draw students, with a t-test obtaining a p-value of 0.006. We can see how much more confident the College Mathematics (CM) students were by the end of the semester in Figure 2, which illustrates the breakdown of post-survey responses to this statement. While the average for Win, Lose, or Draw seemed to go down, it did not go down significantly, with a p-value of 0.1343. Similarly, the average for College Mathematics seemed to go up but also not significantly, with a p-value of 0.1075.

Course	**Pre-survey mean (sd)**	**Post-survey mean (sd)**
Win, Lose, or Draw	2.10 (0.86)	2.08 (0.73)
College Mathematics	2.15 (0.95)	2.28 (1.08)

Table 3: Average and standard deviation of pre-survey and post-survey responses to the statement: *To be honest, you can't really change how intelligent you are in mathematics.*

Course	**Pre-survey mean (sd)**	**Post-survey mean (sd)**
Win, Lose, or Draw	2.92 (1.37)	2.71 (1.14)
College Mathematics	2.89 (1.12)	3.21 (1.18)

Table 4: Average and standard deviation of pre-survey and post-survey responses to the statement: *I feel confident when doing mathematics.*

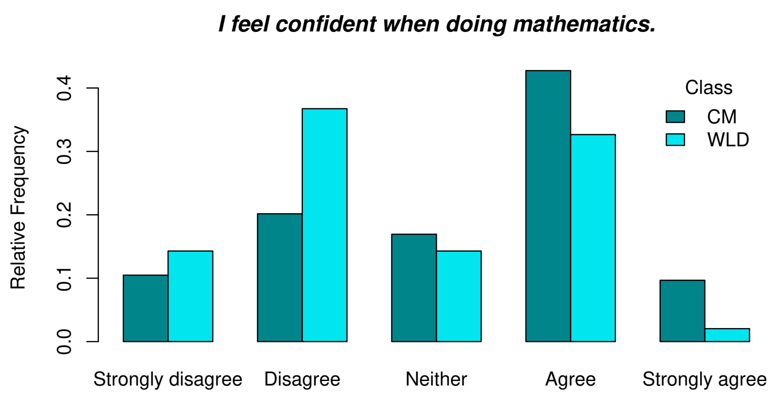


Figure 2: Breakdown of students' post-survey responses, divided by course, to the statement: *I feel confident when doing mathematics.*

We present the means and standard deviations of responses to two questions relating to enjoyment in Tables 5 and 6. Overall, students did not seem to agree or disagree that they enjoyed mathematics in general (Table 5). Between the two classes, t-tests did not find a significant difference between student enjoyment in College Mathematics and Win, Lose, or Draw. However, a t-test did indicate that College Mathematics students had higher enjoyment after taking their class, with a p-value of 0.0102.

Students enrolled in Win, Lose, or Draw were significantly more excited about their class in the pre-survey than their College Mathematics counterparts, with a t-test resulting with a p-value of

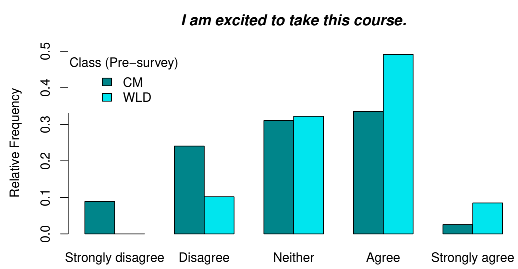
0.0040 (Figure 3a and Table 6). In the post-survey, however, both classes seemed to enjoy their course just as much as each other (Figure 3b and Table 6). While both courses went up in average, College Mathematics had significant results, with the t-test providing a p-value of 2.305×10^{-10} and with the breakdown of responses shown in Figure 4a. On the other hand, the t-test conducted for Win, Lose, or Draw did not have significant results, with a p-value of 0.0759 (Figure 4b).

Course	Pre-survey mean (sd)	Post-survey mean (sd)
Win, Lose, or Draw	2.83 (1.28)	2.96 (1.32)
College Mathematics	2.82 (1.22)	3.22 (1.20)

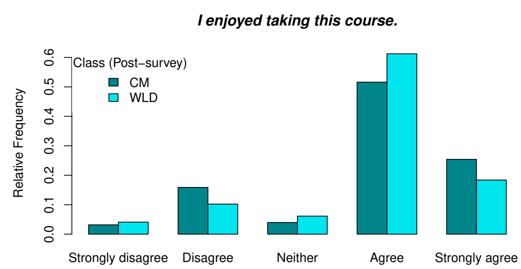
Table 5: Average and standard deviation of pre-survey and post-survey responses to the statement: *I enjoy mathematics.*

Course	Pre-survey mean (sd)	Post-survey mean (sd)
Win, Lose, or Draw	3.56 (0.79)	3.80 (1.00)
College Mathematics	2.97 (1.02)	3.80 (1.09)

Table 6: Average and standard deviation of pre-survey and post-survey responses to the statement: *I am excited to take (enjoyed taking) this course.*

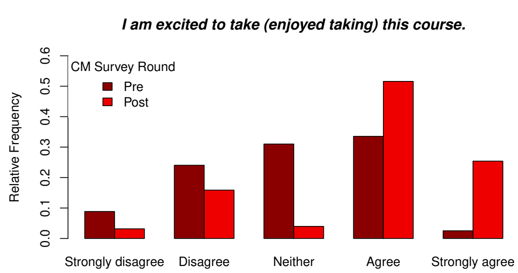


(a) Pre-survey

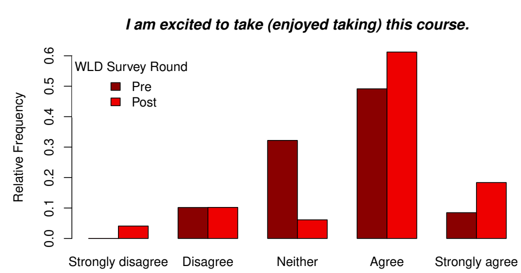


(b) Post-survey

Figure 3: Breakdown of students' responses, divided by course, to the statement *I am excited to take (enjoyed taking) this course.*



(a) College Math (CM)



(b) Win, Lose, or Draw (WLD)

Figure 4: Breakdown of students' responses, divided by survey round, to the statement: *I am excited to take (enjoyed taking) this course.*

On the mid-surveys and post-surveys, students were asked to finish the following two statements relating to anxiety with the answer choice that best fits their experience:

1. *At this point in the semester my anxiety level has . . .*
   decreased, *stayed the same,* *increased,* *I have no anxiety,* *no response*
2. *Compared to other math courses, my anxiety level in this course is . . .*
   lower, *about the same,* *higher,* *I have no anxiety in math courses,* *no response.*

For our initial analysis, we focused on students' responses to these two statements on the post-survey only. We grouped the two questions together and analyzed their responses as a whole since the response choices were similar: "decreased" is grouped with "lower", "increased" is grouped with "higher", etc. Our main question was whether or not anxiety levels differed depending on the class being taken. To answer this, we ran a Chi-Square Test of Independence with the assumption that anxiety levels were independent of the class. The data we used included students' responses to both of the above anxiety statements.

This test resulted in a p-value of 0.058, so there was not enough evidence to conclude that anxiety levels depended on the class. However, due to extenuating circumstances, one College Mathematics professor had to leave the country with two weeks left in the semester and finished the course online. If we complete a Test of Independence without including this outlying College Mathematics section, we obtain a p-value of 0.003. This is enough evidence to conclude that anxiety levels did, in fact, depend on the class being taken. To further explore this, we ran a t-test to see if the mean anxiety levels, of those who had anxiety, were higher for those in Win, Lose, or Draw than for those in College Mathematics. This test resulted in a p-value of 0.0005, so we have strong evidence to conclude that mean anxiety levels are higher for those in Win, Lose, or Draw than for those in College Mathematics. Figure 5 shows the proportions of each response, divided by class, not including the previously mentioned College Mathematics section.

6 Discussion

Our initial findings indicate that College Mathematics, our traditional lecture-based course, seems to be better for the confidence levels of students when compared to Win, Lose, or Draw, our active learning-based course. One reason for this may be due to the experience level that our instructors had in teaching these courses. College Mathematics has been taught in a traditional lecture style for many years, and the instructors have been able to cultivate their assignments and notes to effectively teach this course. On the other hand, Win, Lose, or Draw is a new course to instructors. Consequently, instructors have not yet had the opportunity to adjust the material to their own teaching styles. Any lack of confidence displayed by the instructors of Win, Lose, or Draw may have translated to the students. We will track future data to determine if repeat instructors will have more confident students. Additionally, students may be more accustomed to lecture-based classes, especially in their previous math courses. Since Win, Lose, or Draw is based on active learning, students may feel a sense of discomfort while trying a different method for the first time. The open-ended methodology of interactive learning courses may also cause students to doubt their own abilities, resulting in lower confidence levels.

When comparing learning styles, students from both types of courses found active learning helpful while they were relatively neutral about lecture-based learning. This seems to contradict our

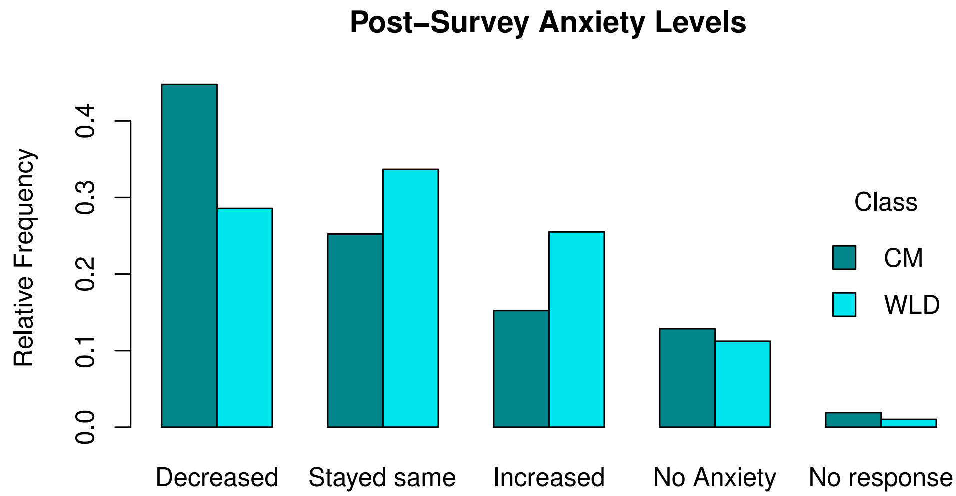


Figure 5: Breakdown of students' post-survey responses (excluding those in the "outlying" College Mathematics section), divided by course, to the two questions about anxiety levels, where "decreased" is grouped with "lower", "increased" is grouped with "higher", etc.

results on confidence and anxiety, as College Mathematics students were confident and less anxious. Perhaps students do not know what is meant by "interactive learning."

In the future, we may wish to ask about specific methods used in these styles of learning. For example, asking students to evaluate the statement, *I find that discovering new concepts while working on problems is helpful for learning mathematics*, may be better than asking them to evaluate the statement, *I generally find interactive learning activities helpful to my studies.*

Overall, our findings indicate the students enjoyed their courses, and the levels of enjoyment were similar. Students seemed neutral in their attitudes towards mathematics otherwise. To further investigate enjoyment levels, we could ask about specific course topics. For example, both courses cover probability, so we might include the statement, *I enjoyed learning about probability in this course.*

In our analysis of anxiety levels, we saw no difference between the two classes unless we removed what we considered to be an outlier section. In that case, we observed that anxiety levels were overall higher for those in Win, Lose, or Draw than for those in College Mathematics.

7 Conclusion and Future Work

As we continue to study the impacts of active learning in our general education courses, our future endeavors include the following:

- **Modify future surveys.** For our initial round of surveys, we passed out paper forms that had check-boxes and places for students to write. This led to a number of errors in skipped questions. In addition, one question asked students to perform a ranking; however, students instead gave each individual response a rating. A number of these problems will be resolved when we move to an online version of the survey in Fall 2019.

- **Track students pre and post.** We provided students with a way to create anonymous, individualized tracking numbers. In the future, our goal is to use these in order to track student changes throughout the semester.
- **Continue to add data, especially as instructors gain experience teaching Win, Lose, or Draw.** A larger sample of students will improve our results, as we will be able to more easily identify significant results between the courses. Additionally, many of our instructors teaching College Mathematics were familiar with how they expected the class to run but our Win, Lose, or Draw instructors were not. As our instructors gain confidence with Win, Lose, or Draw, we may see some results change.
- **Add analysis of another new math general education course.** Starting in Fall 2019, we plan to offer another general education course, Storytelling with Data. This multidisciplinary course follows Wesleyan University's Passion-Driven Statistics curriculum (available at (https://passiondrivenstatistics.com/) and follows a project-based approach in which students work with existing data covering health, biology, government, business, education, etc. to conduct data analysis on a research topic of their own choosing. With this new course, we will expand our study to gauge student disposition and attitudes in a course utilizing project-based learning.
- **Explore trends for teachers who teach both courses.** In our initial study, only one professor taught sections of both College Mathematics and Win, Lose, or Draw. Going forward, we would like to see more instructors teach both courses. This will reduce confounding variables and allow us to more thoroughly assess the differences between students' attitudes in the two courses. We did take a look at the data collected from the one instructor who taught both types of courses during our initial period, but with our small sample sizes, we did not find any significant results.

Our initial findings in this study motivate further research questions concerning student disposition and attitudes in our lecture-based College Mathematics course compared with our active learning-based Win, Lose, or Draw course. Our analysis did not indicate any significant difference in students' enjoyment of their math course, their math anxiety level, and their disposition of effective learning styles. In both courses, students indicated a growth-mindset view of learning when asked about math intelligence. Also, student enjoyment of mathematics in general was increased in both courses when considering pre- and post-survey data. These results speak highly of our mathematics general education courses, which is encouraging for our department. Our analysis did find that students in Win, Lose, or Draw experienced higher anxiety levels and lower confidence levels when compared to students in College Mathematics, which was surprising. As instructors become more familiar with the Win, Lose, or Draw course, it will be interesting to see if there are any changes in this finding. We predict the differences of student disposition and attitudes towards effective learning styles, math confidence, and math anxiety will become more pronounced as we continue to collect data over time as well as expand our study to include our project-based learning course.

References

[1] Ahmet Akin and Izzet N Kurbanoglu. The relationships between math anxiety, math attitudes, and self-efficacy: a structural equation model. *Studia Psychologica*, 53(3):263, 2011.

[2] Susan M Awbrey. General education reform as organizational change: integrating cultural and structural change. *The Journal of General Education*, pages 1–21, 2005.

[3] Adeneye OA Awofala. Effect of personalisation of instuction on students' anxiety in mathematical word problems in nigeria. *Bulgarian Journal of Science & Education Policy*, 11(1), 2017.

[4] J B. Collins, Amanda Harsy, Jarod Hart, Katie Anne Haymaker, Alyssa Marie Hoofnagle, Mike Kuyper Janssen, Jessica Stewart Kelly, Austin Tyler Mohr, and Jessica OShaughnessy. Mastery-based testing in uundergraduate mathematics courses. *PRIMUS*, pages 1–20, 2019.

[5] Sandeep Bhowmick, Joyce A Young, Paul W Clark, and Nandini Bhowmick. Marketing students' mathematics performance: the mediating role of math anxiety on math self-concept and math self-efficacy. *Journal of Higher Education Theory and Practice*, 17(9), 2017.

[6] Jo Boaler. How a detracked mathematics approach promoted respect, responsibility, and high achievement. *Theory into Practice*, 45(1):40–46, 2006.

[7] Steven Brint, Kristopher Proctor, Scott Patrick Murphy, Lori Turk-Bicakci, and Robert A Hanneman. General education models: continuity and change in the us undergraduate curriculum, 1975–2000. *The Journal of Higher Education*, 80(6):605–642, 2009.

[8] Jerome S. Bruner. On learning mathematics. *The Mathematics Teacher*, 53(8):610–619, 1960.

[9] J.S. Bruner and U.P.J. Bruner. *Toward a theory of instruction*. Belknap Press Series. Belknap Press of Harvard University, 1966.

[10] Isabelle D Cherney. The effects of active learning on students' memories for course content. *Active Learning in Higher Education*, 9(2):152–171, 2008.

[11] Ann D. Dowker, Amar Deep Sarkar, and Chung Yen Looi. Mathematics anxiety: what have we learned in 60 years? *Frontiers in Psychology*, 7:508, 2016.

[12] Jerome Epstein. The calculus concept inventory-measurement of the effect of teaching methodology in mathematics. *Notices of the American Mathematical Society*, 60(8):1018–1027, 2013.

[13] Ann S Ferren. General education reform and the computer revolution. *The Journal of General Education*, pages 164–177, 1993.

[14] OECD-The Organisation for Economic Co-operation and Development. *PISA 2012 results: ready to learn (volume III): students' engagement, drive and self-beliefs*. OECD Publishing, Paris, 2013.

[15] Scott Freeman, Sarah L Eddy, Miles McDonough, Michelle K Smith, Nnadozie Okoroafor, Hannah Jordt, and Mary Pat Wenderoth. Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 111(23):8410–8415, 2014.

[16] Nejla Gurefe and Orkide Bakalim. Mathematics anxiety, perceived mathematics self-efficacy and learned helplessness in mathematics in faculty of education students. *International Online Journal of Educational Sciences*, 10(3):147 – 161, 2018.

[17] Eric W Hafner. *The relationship between math anxiety, math self-efficacy, and achievement among a sample of eighth grade students*. Capella University, 2008.

[18] Paul Hager, Ray Sleet, Peter Logan, and Mal Hooper. Teaching critical thinking in undergraduate science courses. *Science & Education*, 12(3):303–313, 2003.

[19] Diane F Halpern. Teaching for critical thinking: helping college students develop the skills and dispositions of a critical thinker. *New Directions for Teaching and Learning*, 1999(80):69–74, 1999.

[20] HM Havenga. Project-based learning in higher education: exploring programming students' development towards self-directedness. *South African Journal of Higher Education*, 29(4):135–157, 2015.

[21] Peter Hilton. Math anxiety: some suggested causes and cures. *The Two-Year College Mathematics Journal*, 11(3):174–188, 1980.

[22] Xiaoxia Huang, Jie Zhang, and Laura Hudson. Impact of math self-efficacy, math anxiety, and growth mindset on math and science career interest for middle school students: the gender moderating effect. *European Journal of Psychology of Education*, 34(3):621–640, 2019.

[23] Kyoungna Kim, Priya Sharma, Susan M Land, and Kevin P Furlong. Effects of active learning on enhancing student critical thinking in an undergraduate general science course. *Innovative Higher Education*, 38(3):223–235, 2013.

[24] Marina Kogan and Sandra L Laursen. Assessing long-term effects of inquiry-based learning: a case study from college mathematics. *Innovative Higher Education*, 39(3):183–199, 2014.

[25] Athanasios Kyriazis, Sarantos Psycharis, and Konstantinos Korres. Discovery learning and the computational experiment in higher mathematics and science education: a combined approach. *International Journal of Emerging Technologies in Learning (iJET)*, 4(4):25–34, 2009.

[26] David A Lake. Student performance and perceptions of a lecture-based course compared with the same course utilizing group discussion. *Physical Therapy*, 81(3):896–902, 03 2001.

[27] Silke Luttenberger, Sigrid Wimmer, and Manuela Paechter. Spotlight on math anxiety. *Psychology Research and Behavior Management*, 11:311, 2018.

[28] Patricia L. Machemer and Pat Crawford. Student perceptions of active learning in a large cross-disciplinary classroom. *Active Learning in Higher Education*, 8(1):9–30, 2007.

[29] Diana Kathleen May. *Mathematics self-efficacy and anxiety questionnaire*. PhD thesis, University of Georgia, 2009.

[30] Frank C Richardson and Richard M Suinn. The mathematics anxiety rating scale: psychometric data. *Journal of counseling Psychology*, 19(6):551, 1972.

[31] Marlene Scardamalia and Carl Bereiter. New cultures of schooling. *International Perspectives on the Design of Technology-Supported Learning Environments*, 149, 1996.

[32] D.H. Schunk and F Pajares. Self-efficacy theory. *Handbook of Motivation at School*, pages 35–54, 01 2009.

[33] M Scriven and R Paul. Defining critical thinking: a draft statement for the national council for excellence in critical thinking. retrieved december 8, 2009 from the foundation for critical thinking, 1987.

[34] Kenny Skagerlund, Rickard Östergren, Daniel Västfjäll, and Ulf Träff. How does mathematics anxiety impair mathematical abilities? investigating the link between math anxiety, working memory, and number processing. *PloS one*, 14(1):e0211283, 2019.

[35] C. Veronica Smith and LeAnn Cardaciotto. Is active learning like broccoli? student perceptions of active learning in large lecture classes. *Journal of the Scholarship of Teaching and Learning*, 11(1):53–61, Feb. 2012.

[36] Nur Sırmacı. Anxiety and attitudes of university students towards mathematics: erzurum sample. *Education and Science*, 32(145), 2010.

[37] Lev Semenovich Vygotsky. *Mind in society: the development of higher psychological processes*. Harvard University Press, 1980.

[38] John N Ward. Group-study versus lecture-demonstration method in physical science instruction for general education college students. *The Journal of Experimental Education*, 24(3):197–210, 1956.

[39] Aman Yadav, Dipendra Subedi, Mary A. Lundeberg, and Charles F. Bunting. Problem-based learning: influence on students' learning in an electrical engineering course. *Journal of Engineering Education*, 100(2):253–280, 04 2011. Copyright - Copyright American Society for Engineering Education Apr 2011; Document feature - Graphs; Tables; ; Last updated - 2017-11-18; CODEN - JEEDEQ.

Lagrange's Interpolation, Chinese Remainder, and Linear Equations

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Abstract

We used the Gram-Schmidt orthogonalization algorithm to construct a solution to a system of linear equation. Our approach was inspired by the construction of solutions to Lagrange's interpolation polynomial problem or to a system of linear congruences as in the Chinese Remainder Theorem.

1 Introduction

The standard approach for solving a system of m linear equations is to use Gaussian elimination by means of elementary row operations to get a reduced system of equations that is equivalent to the original one. In this paper we imitate the construction of solutions to the Lagrange's interpolation polynomial problem or to a system of linear congruences as in the Chinese Remainder Theorem. We use the Gram-Schmidt orthogonalization algorithm to construct a reduced system of equations that is equivalent to the given one. At each step of the algorithm one can decide if the system is inconsistent or if a given linear equation is redundant in the system. If neither of these cases is true we, construct a new equation whose coefficients are linear combinations of the previous ones. See Algorithm 5 for the details. The algorithm returns either, (a) the system has no solution or (b) a set of orthogonal vectors which are the rows of the reduced matrix. A linear combination of these orthogonal vectors is a solution of the original system of linear equations.

2 Lagrange's Interpolation

Theorem 1. *Let*

$$\{(x_1, y_1), \ldots, (x_{k-1}, y_{k-1}), (x_k, y_k)\}$$

be a set of k points in $\mathbb{R}^2$. Set $P(x) = \prod_{i=1}^{k}(x - x_i)$ and for $1 \leq i \leq k$ let $P_i(x) = \frac{P(x)}{(x-x_i)}$ and $p_i(x) = \frac{P_i(x)}{P(x_i)}$. Then, for $1 \leq i, j \leq k$ we have $p_i(x_j) = \delta_{ij}$ and the polynomial

$$p(x) = y_1p_1(x) + y_2p_2(x) + \cdots + y_{k-1}p_{k-1}(x) + y_kp_k(x)$$

of degree less than or equal to $k - 1$ is a solution to the interpolation problem. See [2] Theorem 3.2 on page 109.

3 Chinese Remainder Theorem

Theorem 2. *Let n_1, n_2, ..., n_{k-1}, n_k be positive integers such the greatest common divisor* $\gcd(n_i, n_j) = 1$ *for* $i \neq j$, $1 \leq i, j \leq k$. *Let*

$$x \equiv a_1 \bmod n_1,\ x \equiv a_2 \bmod n_2,\ \cdots,\ x \equiv a_{k-1} \bmod n_{k-1},\ x \equiv a_k \bmod n_k$$

be a system of congruences. Set $N = \prod_{i=1}^{k} n_i$ *and* $N_i = \frac{N}{n_i}$. *Since* $\gcd(N_i, n_i) = 1$, *there exist integers* c_i *such that* $c_i N_i \equiv 1 \bmod n_i$ *for* $1 \leq i \leq k$ *and* $c_i N_i \equiv 0 \bmod n_j$ *for* $i \neq j$ *and* $1 \leq i, j \leq k$. *Then,*

$$x = a_1 c_1 N_1 + a_2 c_1 N_2 + \cdots + a_{k-1} c_{k-1} N_{k-1} + a_k c_k N_k$$

is a solution to the system of linear congruences. See [1] Proposition I.3.3 on page 21.

Let us take another look at Langrange's interpolation problem. If $p(x)$ is a polynomial in $\mathbb{R}[x]$ then $p(x_i) = y_i$ is equivalent to the existence of a polynomial $q_i(x)$ satisfying the equation

$$p(x) = q_i(x) \cdot (x - x_i) + y_i.$$

This last equation is equivalent to the congruence

$$p(x) \equiv y_i \bmod (x - x_i).$$

Therefore, Langrange's interpolation problem is equivalent to solving the set of congruences $p(x) \equiv y_i \bmod (x - x_i)$, $0 \leq i \leq k$ over the ring of polynomials $\mathbb{R}[x]$. Since $x_i \neq x_j$ for $i \neq j$, the linear polynomials $x - x_i$ and $x - x_j$ are relatively prime. This implies that the congruences $p(x) \equiv y_i \bmod (x - x_i)$, $0 \leq i \leq k$ have a solution. This solution agrees with the solution constructed in the Chinese Remainder Theorem over the ring of polynomials $\mathbb{R}[x]$.

4 Linear Equations

Consider the system of linear equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Let $\mathbf{a}_i = [a_{i1}, a_{i2}, \ldots, a_{i(n-1)}, a_{in}]$, $\mathbf{x} = [x_1, x_2, \ldots, x_{n-1}, x_n]$ and $\mathbf{a}_i \bullet \mathbf{x}$ be the dot product of $\mathbf{a}_i$ and $\mathbf{x}$. Then, the system can be written as

$$\mathbf{a}_1 \bullet \mathbf{x} = b_1,\ \mathbf{a}_2 \bullet \mathbf{x} = b_2,\ \ldots,\ \mathbf{a}_m \bullet \mathbf{x} = b_m.$$

Definition 3. *Let V be a subspace of $\mathbb{R}^n$. The orthogonal complement of V in $\mathbb{R}^n$, denoted $V^\perp$ is the subspace*

$$V^\perp = \{\mathbf{w} \text{ in } \mathbb{R}^n \text{ such that } \mathbf{v} \bullet \mathbf{w} = 0 \text{ for all } \mathbf{v} \text{ in } V\}.$$

Lemma 1. *Consider the system of linear equations*

$$\mathbf{a}_1 \bullet \mathbf{x} = b_1,\ \mathbf{a}_2 \bullet \mathbf{x} = b_2,\ \ldots,\ \mathbf{a}_m \bullet \mathbf{x} = b_m.$$

Let $\mathbf{0}$ *denote the zero vector in* $\mathbb{R}^n$*. If* $\mathbf{a}_i \neq \mathbf{0}$ *for* $1 \leq i \leq m$ *and* $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ *for* $i \neq j$, $1 \leq i, j \leq m$ *then the vector*

$$\mathbf{x} = \frac{b_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{b_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{b_{m-1}}{\mathbf{a}_{m-1} \bullet \mathbf{a}_{m-1}}\mathbf{a}_{m-1} + \frac{b_m}{\mathbf{a}_m \bullet \mathbf{a}_m}\mathbf{a}_m$$

is well defined since $\mathbf{a}_i \bullet \mathbf{a}_i \neq 0$ *and is a solution of the system of equations. Moreover, if* $\mathbf{y}$ *is another solution to the system of equation then* $\mathbf{x} - \mathbf{y}$ *is in* $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}^{\perp}$*.*

Proof. Clearly $\mathbf{x}$ is in $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$. Since $\mathbf{a}_i \bullet \mathbf{a}_i \neq 0$ for $1 \leq i \leq m$ and $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ for $i \neq j$, $1 \leq i, j \leq m$ we have

$$\begin{aligned}
\mathbf{x} \bullet \mathbf{a}_i &= \left(\frac{b_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{b_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{b_{m-1}}{\mathbf{a}_{m-1} \bullet \mathbf{a}_{m-1}}\mathbf{a}_{m-1} + \frac{b_m}{\mathbf{a}_m \bullet \mathbf{a}_m}\mathbf{a}_m\right) \bullet \mathbf{a}_i \\
&= \frac{b_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 \bullet \mathbf{a}_i + \frac{b_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 \bullet \mathbf{a}_i + \cdots + \frac{b_i}{\mathbf{a}_i \bullet \mathbf{a}_i}\mathbf{a}_i \bullet \mathbf{a}_i + \ldots + \\
&\quad + \frac{b_{m-1}}{\mathbf{a}_{m-1} \bullet \mathbf{a}_{m-1}}\mathbf{a}_{m-1} \bullet \mathbf{a}_i + \frac{b_m}{\mathbf{a}_m \bullet \mathbf{a}_m}\mathbf{a}_m \bullet \mathbf{a}_i \\
&= b_i.
\end{aligned}$$

This shows that $\mathbf{x}$ is a solution to the system of equations. Assume that

$$\begin{aligned}
\mathbf{x} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_m\mathbf{a}_m \text{ and} \\
\mathbf{y} &= y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \ldots + y_m\mathbf{a}_m
\end{aligned}$$

are two solutions of the given system of equations. Linearity of the dot product implies that $(\mathbf{x} - \mathbf{y}) \bullet \mathbf{a}_i = 0$ for $1 \leq i \leq m$. It follows that $\mathbf{x} - \mathbf{y}$ is in $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}^{\perp}$. □

Definition 4. *Let* $A_r = \{\mathbf{a}_1,\ \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ *be a set of nonzero vectors in* $\mathbb{R}^n$ *such that* $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ *for* $i \neq j$ *and* $B_r = \{b_1, b_2, \ldots, b_r\}$ *be a set of real numbers. Let* $\mathbf{a}$ *be a vector in* $\mathbb{R}^n$ *and* b *be a real number. Define the operators*

$$\mathbf{P}(\mathbf{a}, A_r) = \mathbf{a} - \left(\frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}\mathbf{a}_{r-1} + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}\mathbf{a}_r\right)$$

and

$$\mathbf{p}(b, B_r, A_r) = b - \left(\frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}b_1 + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}b_2 + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}b_{r-1} + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}b_r\right).$$

We observe that $\mathbf{P}(\mathbf{a}, A_r)$ is the orthogonal projection of $\mathbf{a}$ onto $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ and that $\mathbf{p}(b, B_r, A_r)$ will help us to keep track of the changes in the last column of the augmented matrix associated to the system of equations.

Lemma 2. *Let* $A_r = \{\mathbf{a}_1,\ \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ *be a set of nonzero vectors in* $\mathbb{R}^n$ *such that* $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ *for* $i \neq j$*. Let* $\mathbf{a}$ *be a vector in* $\mathbb{R}^n$ *then,*

1. $\mathbf{P}(\mathbf{a}, A_r) = \mathbf{0}$ *if and only if* $\mathbf{P}(\mathbf{a}, A_r)$ *is in* $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$

2. $\mathbf{P}(\mathbf{a}, A_r) \bullet \mathbf{a}_i = 0$ *for* $1 \le i \le r$. *Therefore, if* $\mathbf{P}(\mathbf{a}, A_r) \neq \mathbf{0}$ *then* $\mathbf{P}(\mathbf{a}, A_r)$ *is orthogonal to* $\mathbf{a}_i$ *for* $1 \le i \le r$.

Proof. Let us start with

1. If $\mathbf{P}(\mathbf{a}, A_r) = \mathbf{0}$ then

$$\mathbf{a} = \frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}\mathbf{a}_{r-1} + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}\mathbf{a}_r.$$

and $\mathbf{a}$ is in span$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$. On the other hand, if $\mathbf{a}$ is in span$\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ then there exist real numbers $a_1, a_2, \ldots, a_r$ such that $\mathbf{a} = a_1\mathbf{a}_1 + a_2\mathbf{a}_2 + \cdots + a_r\mathbf{a}_r$. Since $\mathbf{a}_i \bullet \mathbf{a}_i \neq 0$ and $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ for $i \neq j$ we have

$$\mathbf{a} \bullet \mathbf{a}_i = (a_1\mathbf{a}_1 + a_2\mathbf{a}_2 + \cdots + a_r\mathbf{a}_r) \bullet \mathbf{a}_i = a_i\,(\mathbf{a}_i \bullet \mathbf{a}_i)\,.$$

This shows that

$$a_i = \frac{\mathbf{a} \bullet \mathbf{a}_i}{\mathbf{a}_i \bullet \mathbf{a}_i}.$$

Therefore,

$$\mathbf{a} = \frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}\mathbf{a}_{r-1} + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}\mathbf{a}_r$$

and it follows that $\mathbf{P}(\mathbf{a}, A_r) = \mathbf{0}$.

2. If $\mathbf{P}(\mathbf{a}, A_r) = \mathbf{0}$ then $\mathbf{P}(\mathbf{a}, A_r) \bullet \mathbf{a}_i = 0$ for $1 \le i \le r$. So, assume $\mathbf{P}(\mathbf{a}, A_r) \neq \mathbf{0}$. Since $\mathbf{a}_i \bullet \mathbf{a}_i \neq 0$ and $\mathbf{a}_i \bullet \mathbf{a}_j = 0$ for $j \neq i$, $1 \le i, j \le r$, we have,

$$\begin{aligned} \mathbf{P}(\mathbf{a}, A_r) \bullet \mathbf{a}_i &= \left(\mathbf{a} - \frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}\mathbf{a}_{r-1} + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}\mathbf{a}_r\right) \bullet \mathbf{a}_i \\ &= \mathbf{a} \bullet \mathbf{a}_i - \frac{\mathbf{a} \bullet \mathbf{a}_1}{\mathbf{a}_1 \bullet \mathbf{a}_1}\mathbf{a}_1 \bullet \mathbf{a}_i + \frac{\mathbf{a} \bullet \mathbf{a}_2}{\mathbf{a}_2 \bullet \mathbf{a}_2}\mathbf{a}_2 \bullet \mathbf{a}_i + \cdots + \frac{\mathbf{a} \bullet \mathbf{a}_i}{\mathbf{a}_i \bullet \mathbf{a}_i}\mathbf{a}_i \bullet \mathbf{a}_i + \ldots + \\ &+ \frac{\mathbf{a} \bullet \mathbf{a}_{r-1}}{\mathbf{a}_{r-1} \bullet \mathbf{a}_{r-1}}\mathbf{a}_{r-1} \bullet \mathbf{a}_i + \frac{\mathbf{a} \bullet \mathbf{a}_r}{\mathbf{a}_r \bullet \mathbf{a}_r}\mathbf{a}_r \bullet \mathbf{a}_i \\ &= \mathbf{a} \bullet \mathbf{a}_i - \mathbf{a} \bullet \mathbf{a}_i = \mathbf{0}. \end{aligned}$$

□

Algorithm 5 (Orthogonal Row Reduction)**.** *Suppose we are given a system of linear equations*

$$\mathbf{a}_1 \bullet \mathbf{x} = b_1,\ \mathbf{a}_2 \bullet \mathbf{x} = b_2,\ \ldots\ ,\ \mathbf{a}_m \bullet \mathbf{x} = b_m.$$

1. *Let* $\bar{\mathbf{a}}_1 = \mathbf{a}_1$, $A_1 = \{\bar{\mathbf{a}}_1\}$, $\bar{b}_1 = b_1$, *and* $B_1 = \{\bar{b}_1\}$.
2. *Let* $\bar{\mathbf{a}}_2 = \mathbf{P}(\mathbf{a}_2, A_1)$ *and* $\bar{b}_2 = \mathbf{p}(b_2, B_1, A_1)$
   (a) *If* $\bar{\mathbf{a}}_2 = \mathbf{0}$ *and* $\bar{b}_2 \neq 0$ *then the system of equations has no solution.*
   (b) *If* $\bar{\mathbf{a}}_2 = \mathbf{0}$ *and* $\bar{b}_2 = 0$ *then by Lemma 2 part 1* $\mathbf{a}_2$ *is a multiple of* $\mathbf{a}_1$ *and the equation* $\mathbf{a}_2 \bullet \mathbf{x} = b_2$ *is redundant and can be deleted.*
   (c) *If* $\bar{\mathbf{a}}_2 \neq \mathbf{0}$ *set* $A_2 = \{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ *and* $B_2 = \{\bar{b}_1, \bar{b}_2\}$

3. *Assume that* $A_i = \{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_i\}$ *and* $B_i = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_i\}$ *have been defined.* *Let* $\bar{\mathbf{a}}_{i+1} = \mathbf{P}(\mathbf{a}_{i+1}, A_i)$ *and* $\bar{b}_{i+1} = \mathbf{p}(b_{i+1}, B_i, A_i)$.
   (a) *If* $\bar{\mathbf{a}}_{i+1} = \mathbf{0}$ *and* $\bar{b}_{i+1} \neq 0$ *then the system of equations has no solution.*
   (b) *If* $\bar{\mathbf{a}}_{i+1} = \mathbf{0}$ *and* $\bar{b}_{i+1} = 0$ *then by Lemma 2 part 1* $\mathbf{a}_{i+1}$ *is a linear combinations of the vectors in* A_i *and since each vector* $\bar{\mathbf{a}}_j$ *in* A_i *is a linear combination of the vectors* $\mathbf{a}_1$, $\mathbf{a}_2$, $\ldots$, $\mathbf{a}_i$ *the equation* $\mathbf{a}_{i+1} \bullet \mathbf{x} = b_{i+1}$ *is redundant and can be deleted.*
   (c) *If* $\bar{\mathbf{a}}_{i+1} \neq \mathbf{0}$ *set* $A_{i+1} = \{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_{i+1}\}$ *and* $B_2 = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{i+1}\}$

Continuing in this manner we either conclude that the system has no solution or we get a reduced system of equations

$$\bar{\mathbf{a}}_1 \bullet \mathbf{x} = \bar{b}_1,\ \bar{\mathbf{a}}_2 \bullet \mathbf{x} = \bar{b}_2,\ \ldots,\ \bar{\mathbf{a}}_r \bullet \mathbf{x} = \bar{b}_r$$

which by Lemma 1 has solution

$$\mathbf{x} = \frac{\bar{b}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{\mathbf{a}}_1 + \frac{\bar{b}_2}{\bar{\mathbf{a}}_2 \bullet \bar{\mathbf{a}}_2}\bar{\mathbf{a}}_2 + \cdots + \frac{\bar{b}_{r-1}}{\bar{\mathbf{a}}_{r-1} \bullet \bar{\mathbf{a}}_{r-1}}\bar{\mathbf{a}}_{r-1} + \frac{\bar{b}_r}{\bar{\mathbf{a}}_r \bullet \bar{\mathbf{a}}_r}\bar{\mathbf{a}}_r.$$

We claim that $\mathbf{x}$ *is also a solution of the original system of equations.*

Proof. We will proceed using the strong principle of induction.

1. If $n = 1$ then $\mathbf{x} \bullet \mathbf{a}_1 = \mathbf{x} \bullet \bar{\mathbf{a}}_1 = \bar{b}_1 = b_1$.
2. Assume that $\mathbf{x} \bullet \mathbf{a}_i = b_i$ for $1 \leq i \leq k$.
3. Now we will prove that $\mathbf{x} \bullet \mathbf{a}_{k+1} = b_{k+1}$. Since

$$\bar{\mathbf{a}}_{k+1} = \mathbf{P}(\mathbf{a}_{k+1}, A_k) = \mathbf{a}_{k+1} - \left(\frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{\mathbf{a}}_1 + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{\mathbf{a}}_{k-1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{\mathbf{a}}_k \right)$$

we have

$$\mathbf{a}_{k+1} = \bar{\mathbf{a}}_{k+1} + \left(\frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{\mathbf{a}}_1 + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{\mathbf{a}}_{k-1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{\mathbf{a}}_k \right).$$

Therefore,

$$\begin{aligned}
\mathbf{x} \bullet \mathbf{a}_{k+1} &= \mathbf{x} \bullet \left(\bar{\mathbf{a}}_{k+1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{\mathbf{a}}_1 + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{\mathbf{a}}_{k-1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{\mathbf{a}}_k \right) \\
&= \mathbf{x} \bullet \bar{\mathbf{a}}_{k+1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{\mathbf{a}}_1 \bullet \mathbf{x} + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{\mathbf{a}}_{k-1} \bullet \mathbf{x} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{\mathbf{a}}_k \bullet \mathbf{x} \\
&= \bar{b}_{k+1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{b}_1 + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{b}_{k-1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{b}_k \\
&= b_{k+1}.
\end{aligned}$$

The last equality follows from

$$\bar{b}_{k+1} = \mathbf{p}(\mathbf{a}_{k+1}, A_k, B_k) = b_{k+1} - \left(\frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_1}{\bar{\mathbf{a}}_1 \bullet \bar{\mathbf{a}}_1}\bar{b}_1 + \cdots + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_{k-1}}{\bar{\mathbf{a}}_{k-1} \bullet \bar{\mathbf{a}}_{k-1}}\bar{b}_{k-1} + \frac{\mathbf{a}_{k+1} \bullet \bar{\mathbf{a}}_k}{\bar{\mathbf{a}}_k \bullet \bar{\mathbf{a}}_k}\bar{b}_k \right)$$

□

5 Conclusion

In this paper we have given an alternative method for row reducing the augmented matrix associated to a system of linear equations by using the Gram-Schmidt orthogonalization algorithm. This approach was inspired by the construction of solutions to the Lagrange's interpolation problem and to system of congruences as in the Chinese Remainder Theorem. We believe that this algorithm could be presented at the end of a linear algebra course as an application of the standard inner product on $\mathbb{R}^n$.

References

[1] Neal Koblitz, *A course in number theory and cryptography.* Springer, Second edition. 1994.

[2] Richard L. Burden and J. Douglas Faires, *Numerical analysis.* Brooks/Cole, Seventh edition. 2001.

[3] Gilbert Strang, *Linear algebra and its applications.* Harcourt Brace Jovanovich, Publishers. Third edition. 1988.

Factors that Motivate Students to Learn Mathematics

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Abstract

What motivates some students to want to learn mathematics while others are not similarly motivated? Are these factors intrinsic, extrinsic, or a combination of both? To answer these questions, we adapted a survey originally developed by Tapia (1996) and later shortened by Lim and Chapman (2015). We administered the survey to multiple middle schools, high schools, colleges, and universities. We obtained a total of 645 surveys. We offer an analysis of these data, including both descriptive statistics and confidence intervals. In addition to Likert scale items from the original survey, we explore a variety of qualitative data from a free response item. For the college and university sample, we also provide comparisons among students majoring

in mathematics or mathematics education, those majoring in elementary education, and those with a variety of other majors.

1 Introduction

What motivates someone to want to learn mathematics concepts and to persist even in the face of difficulties and obstacles? Are these motivational factors primarily extrinsic to the person and the mathematical concepts, such as earning a good grade, pleasing a parent, or applying the concept to a future career? Or are the factors more intrinsic in nature, such as enjoyment of math itself or enjoying the problem solving process? How does someone decide to pursue a career in the mathematical sciences? Although all of these questions are both interesting and important, we used the first two questions to design a research study. By identifying and classifying factors that motivate students to learn and persist in mathematics, we hope to provide insights for teachers, parents, and others who have a stake in the ultimate success of middle school, high school, and college or university students in the area of mathematics. This, in turn, may shed some light on the third question raised above.

2 Background and Research Questions

Prior research has explored the role of motivation in a variety of contexts, including mathematics. Tapia (1996) created a lengthy survey instrument designed to highlight which factor or factors most impact a person's attitudes toward mathematics. Lim and Chapman (2015) shortened this survey in a research study that explored sources of math anxiety as well as factors that motivate students to learn mathematics. The stakes of this research are high since many professions and many facets of our everyday lives assume a strong background in quantitative reasoning in general and the application of numerical and spatial reasoning skills in particular. Indeed, professional organizations have called for increased focus on conceptual reasoning and problem solving within the K-12 mathematics curriculum (cf., Common Core State Standards Initiative, 2011; NCTM 1989; 2000; 2014).

Other researchers have studied issues closely related to motivation and mathematics learning. Eggleton (2017) describes the importance of success as a motivational tool for students to persist in the face of cognitive obstacles and difficulties in mathematics. King (2019) addresses the same issue and emphasizes helping students to "tackle challenging problems and be creative problem solvers" and expecting them to "work hard, persevere, and learn from the mistakes they may make" (p.507). Kinser-Traut (2019) highlights the importance of connecting mathematics to other subjects, to future careers, and to everyday life. Finally, Wilkerson (2015) details ways that students value mathematics and calls for teachers to provide their students with a rationale for learning mathematics, one that uncovers mathematical patterns in God's Creation as well as ways to serve others in society through the application of statistical, numerical, and spatial reasoning.

In the present study, we seek to identify factors that motivate students to want to learn mathematics and to persist when confronted with difficult mathematical concepts. Some of these factors are intrinsic in that the source is either in the student herself or in the nature of the mathematics. Other factors are more extrinsic, such as a course grade, a future career, connections to other disciplines or to everyday life, or a person such as a teacher, a parent, or a friend. Based upon this framework and existing research, we designed a research study to address the following questions:

- Question 1: What similarities and differences exist among the primary motivation factor to learn mathematics for students at the middle school, the high school, and the college and university levels?
- Question 2: What is the preponderance of intrinsic versus extrinsic motivation factors for students at these grade levels?
- Question 3: Which groups or subgroups (e.g., individual schools, students with common motivation factor, college/university major, etc.) demonstrate unusually high or low Likert scale responses for specific items? What do these results tell us about how these students approach mathematics?

3 Research Methodology

At Trinity Christian College, all mathematics majors complete a research project as part of the senior capstone seminar. Ben Gliesmann approached his seminar professor, Dave Klanderman, to propose a study that explored different factors that motivate students to learn mathematics. This idea originated from both Ben's own unique path to becoming a secondary mathematics teacher as well as the preparation for his upcoming student teaching internship and his desire to motivate his own students to want to learn mathematics. After researching the topic, Ben identified the lengthy survey published by Tapia and the subsequent shortened version by Lim and Chapman. With input from his professor, Ben ultimately decided to adapt the shortened version of Lim and Chapman by adding five items at the end to produce a total of 20 Likert-scale items (see Appendix for survey items). To reduce the likelihood of survey fatigue in which a person eventually falls into a pattern of giving the same response to every item, the logic for items 6 through 10 is reversed. This means that the optimal response for these "negatively-stated" items was "strongly disagree," whereas we hoped to document a preponderance of "agree" and "strongly agree" responses for the remaining 15 items in the survey. At the very end of the survey, students were asked to identify the single "person, place, or thing" that most motivates them to learn mathematics. All surveys were completed anonymously. However, for students at the college or university level, the academic major was collected according to three subgroups: mathematics or mathematics education, elementary education, and all other majors. The last category was necessary because the number of students in any particular major was too small to allow further analysis. This additional piece of data later allowed for analyses of subgroups in the college or university samples.

As we explored potential samples for the survey, we identified the need to include students at multiple educational levels. Specifically, we hypothesized that middle school students, high school students, and students at the college and university levels would exhibit different underlying factors that motivate them to learn mathematics. In part, the specific schools included in this study are a convenience sample. However, we took steps to include a cross section of students from each educational setting with the goal of producing an overall sample that is representative of students at these levels. At the middle school level, we surveyed a total of 245 students by including all available students in multiple sections. At the high school level, we surveyed 163 students from a variety of different mathematics courses. At the college and university levels, we surveyed a total of 237 students enrolled in both general education mathematics courses such as statistics and content courses for elementary education majors as well as courses taken primarily by students majoring in mathematics.

Furthermore, we chose to include both public and private schools at each of the three major educational levels. While we hypothesized that the type of school setting might affect the nature of the motivation factors, we were interested to explore both the similarities and differences that might exist. Overall, four middle schools (two public and two private), two high schools (one public and one private), and three colleges or universities (one public and two private) were included in the sample. Our research team expanded by three members in an effort to reach this wider audience of students in a variety of educational settings. Each team member applied for permission to administer the survey through an appropriate administrative channel, including institutional review boards, school principals, and classroom teachers. In addition, all students under the age of 18 needed to obtain a signed permission form from a parent or guardian before the survey could be administered.

Once the data were collected, a variety of descriptive methods were used to analyze the data. First, the Likert scale responses were converted to a numerical scale, with 1 denoting "strongly disagree" and 5 denoting "strongly agree." For each of these 20 items, mean values were determined for the entire sample, for each school setting, and for the academic major subsamples mentioned earlier.

For the free response item ("What is the one person, place, or thing that motivates you to learn mathematics?"), a more qualitative analysis was used. Different categories emerged and were eventually grouped as follows:

- Teacher – this includes both the general term and specific names of teachers ranging from first grade through university
- Family and Friends – this includes both "my mom" and "my dad," but also incorporated other family members (siblings, aunts and uncles, grandparents, etc.) and friends
- Future/Job/Application – this includes references to a career goal, future financial and related activities, and other applications in the surrounding world
- Grades – some students identified the grade on a test or assignment or the course grade as the primary motivation
- Intrinsic – some students identified features of the mathematical concepts, the personal challenge and satisfaction of solving difficult problems, or other related factors
- Other – some of the responses were either left blank or were essentially unique (e.g. Albert Einstein, music, Jesus, etc.)

We used these free response categories to identify subgroups within different subsamples. Means for the 20 Likert scale items were then computed for the subgroups. In an effort to highlight unusual response patterns, we decided to identify school samples and subsamples which were outliers, as defined by means that were at least 1.0 higher or lower than the overall item mean. We later used 95% confidence intervals for the mean to discover how each educational level (middle school, high school, college/university) compared to the overall sample. Finally, the total number of responses at the college/university level that were classified as mathematics or mathematics education majors were sufficiently large to allow interval estimation.

4 Results

Overall, 645 students completed the survey, including 245 students from a total of four different middle schools, 163 students from two high schools, and 237 students from a total of three different colleges and universities. In this section, we provide an overview of the findings from these surveys. Figure 1 provides a summary of the mean scores for the entire sample, for each of the three different educational level settings, and for those students who identified as mathematics or mathematics education majors. As a reminder, a score of 1 represents "strongly disagree," a score of 2 represents "disagree," a score of 3 represents "neutral," a score of 4 represents "agree," and a score of 5 represents "strongly agree."

As seen in the table below, the combined sample showed mean responses in the "neutral" range (between 2.5 and 3.5) with some exceptions. Items 11 through 15 were in the "agree" range, meaning that the students tended to agree that math is worthwhile, important to everyday life, important to study, important for the future, and helpful in life. A similarly high mean for item 16 indicates that students tend to agree that grades are an important motivational factor for learning mathematics. Two of the 20 items had overall mean scores in the "disagree" range (less than 2.5 but above 1.5). One of these was Item 8, indicating that even thinking about doing a math problem is a major cause of anxiety. The other item in this range was item 17, perhaps an indicator that most students are not familiar with non-textbook literature related to mathematics or that those who are familiar with this genre do not find reading such books to be enjoyable.

Looking more closely at the mean values for the subsamples listed in Figure 1, the reader will notice that some of the values are printed in boldface. These indicate cases where the mean of the subsample and the mean of the combined sample for that item were determined to be different based on non-overlapping 95% confidence intervals (CIs). For example, for item number 1, the 95% CI for the combined sample would be (3.31, 3.49). The 95% CI for the college and university subsample would be (3.32, 3.68) which overlaps with the combined sample CI. By contrast, the 95% CI for the mathematics and mathematics education subsample would be (4.07, 4.73) which does not overlap with the combined sample CI and hence is statistically different.

Applying this analysis, we see that the middle school subsample is very similar to the combined sample. In fact, the only item with a statistical difference is item 9, indicating that, on average, middle school students experience confusion in math class more often than the combined sample. Similarly, the college and university subsample means are very similar to those of the combined sample. The only item showing a difference was item 19, perhaps an indication that connections between math and other content areas are more important to students at the college and university level.

Shifting the focus to the high school subsample, it is interesting that all five of the survey items that anticipated a response of disagreement showed lower means for high school students than for the combined sample. Perhaps this education level is the sweet spot for avoiding negative experiences with mathematics. However, another interpretation might be the impact of particular teachers since all of these high school students were enrolled in a math course taught by one of two teachers.

Although perhaps unsurprising, it is nonetheless worth mentioning that the subsample of the mathematics and mathematics education majors were very different than the combined sample. In fact, the means for this subsample were significantly higher than the combined sample for all 15 of the positively-stated items. In addition, the means of this subsample were significantly lower than the

Item Number (Brief Summary of Item – see Appendix for full wording)	Combined Sample (N=645) MoE=0.09	Middle School (n = 245) MoE=0.15	High School (n = 163) MoE=0.18	College or University (n = 237) MoE=0.15	Math or Math Educ. (n=52*) MoE=0.33	Elementary Education (n=32*) MoE=0.42*
1 Enjoy math	3.4	3.3	3.5	3.5	**4.4**	**2.8**
2 Like to solve math problems	3.4	3.2	3.6	3.5	**4.3**	2.9
3 Really like math	3.2	3.1	3.3	3.3	**4.4**	**2.7**
4 Happiest when in math class	2.7	2.8	2.8	2.6	**3.7**	**2.0**
5 Math is interesting	3.5	3.4	3.4	3.6	**4.5**	3.2
6 Studying math makes me nervous	2.8	2.9	**2.4**	3.0	2.5	**3.5**
7 Always under strain in math class	2.6	2.8	**2.3**	2.5	2.2	3.0
8 Thinking about doing a math problem makes me nervous	2.1	2.3	**1.8**	2.1	**1.5**	2.5
9 Always confused in math class	2.6	**2.9**	**2.2**	2.5	**2.1**	3.0
10 Insecure when attempting math	2.5	2.6	**2.0**	2.6	**2.0**	**3.0**
11 Math is worthwhile	3.9	3.8	3.9	4.1	**4.6**	3.9
12 Math is important in everyday life	3.9	4.0	3.9	3.9	**4.4**	3.8
13 Math is important subject to study	3.8	3.8	3.8	3.7	**4.2**	3.6
14 Math is important for future	3.7	3.7	3.7	3.6	**4.2**	3.5
15 Math background helps me in life	4.0	4.1	3.8	4.1	**4.6**	3.7
16 Getting good math grades motivates me	4.0	3.8	4.0	4.1	**4.4**	4.1
17 Enjoy math reading	2.4	2.3	2.6	2.4	**3.1**	**1.9**
18 People motivate me to learn math	3.1	3.0	3.1	3.2	**3.8**	3.0
19 Math connections motivate me	3.4	3.2	3.3	**3.7**	**4.1**	3.4
20 Difficult math motivates me	2.9	2.9	3.0	2.9	**3.6**	**2.3**

Figure 1: Item Mean Scores:
∗ A total of 52 of the students in the college/university sample are included in the math and math education subsample. Similarly, 32 of the students in the college/university sample are included in the elementary education subsample.
∗∗ Constructing 95% confidence intervals (CIs) for the mean produces the stated margin of error (MoE) in each case. Mean values listed in **boldface** denote items where the 95% CIs for the indicated subsample and the overall sample do not overlap.

combined sample for three of the negatively-stated items. Thus, college and university students with a major in mathematics or mathematics education tend to provide more optimal responses to these survey items. Whether this finding is correlational or causational cannot be determined by this study, although it may help to explain why these students select these majors.

On the other hand, the results for the subsample of elementary education majors is sobering. For nearly all of the 20 items, the means scores for this relatively small subsample (n=32) were less optimal than for the combined sample of all students. This includes several items related to enjoying or liking mathematics where these future teachers had significantly lower means as well as multiple items related to math anxiety which had significantly higher means. It is also notable that this subgroup does not seem to enjoy math-related books nor experiences with difficult mathematics. Given the impact that these elementary school teachers have on students and their dispositions towards mathematics, these findings, even based upon a relatively small sample, should raise alarm bells among the colleges and universities that prepare future teachers.

We next turn our attention to the qualitative data provided in item 21. Figure 2 below presents a word cloud for the combined sample in which the font size of the response category is relative to its frequency.

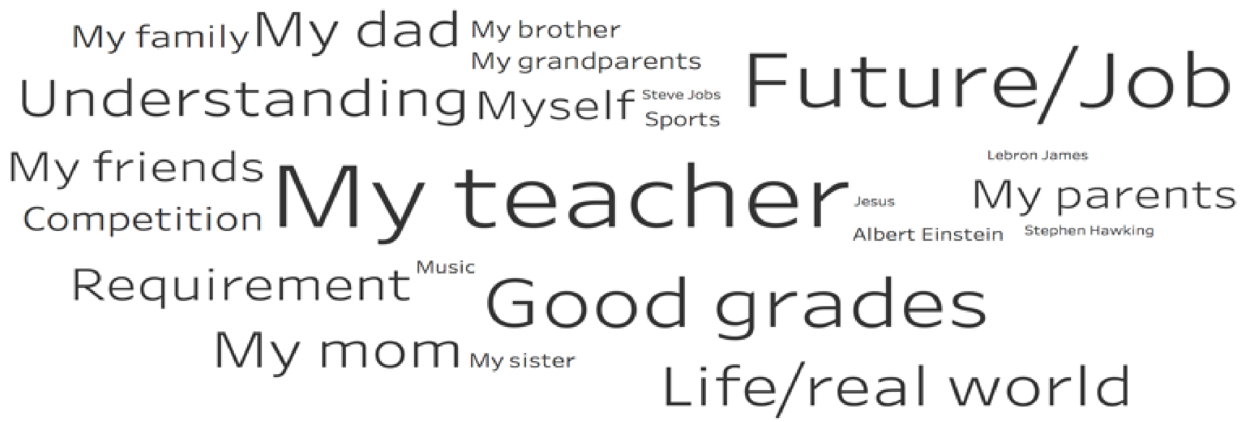

Figure 2: Word Cloud for Item #21

A few trends follow from the above figure. First, the teacher is the most commonly cited motivation factor. It is also true that if categories related to family and friends are grouped together (e.g., My mom, My dad, My parents, My friends, etc.), then this combined category occurs even more often than the teacher. Notice also that "My Future/Job" is also frequently mentioned by these students. By contrast, other categories occur less frequently and a few, such as "music" or specific individuals such as "Jesus," "Albert Einstein," or "Stephen Hawking" occur only a few times. Overall, responses classified as "friends or family" constitute 25% of the total, responses grouped under the heading "future, job, or application" comprise 23% of the total, and approximately 20% of the responses identified the teacher as the most important motivational factor. Achieving good grades (12%) and intrinsic motivational factors (10%) occurred somewhat less often.

Based upon these responses for Item 21, we conducted two additional sets of analyses. First, we compared responses from the three different educational levels and produced bar graphs to summarize the findings in each case. They are offered in the pages below as Figures 3, 4, and 5. Note that responses that could not be easily categorized or were left blank are excluded. Collectively, these "other" responses constitute 10% of the total data set.

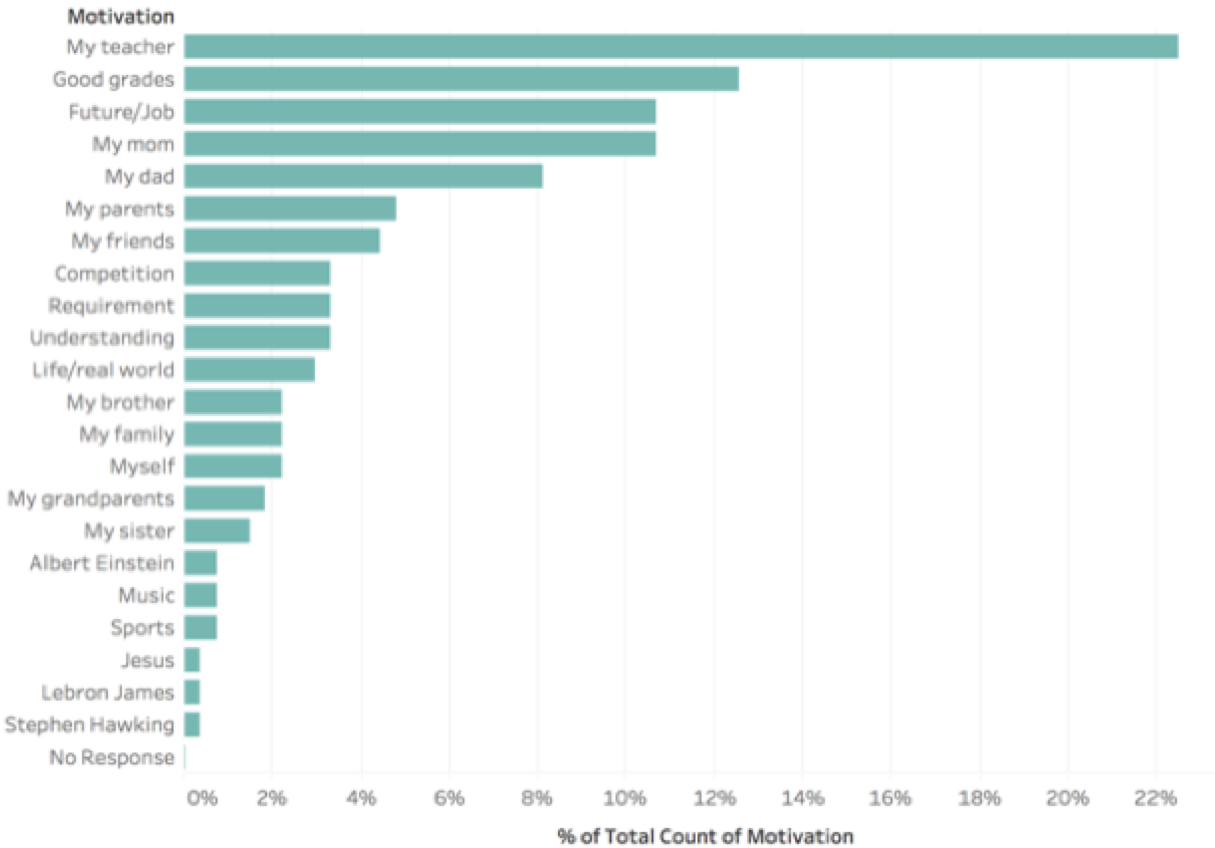

Figure 3: Summary of responses for item 21 at the Middle School Level

Comparing these three bar charts, it is interesting that the top three categories are the same at each educational level. However, the importance of “Future/Job” as a motivating factor increases from 3rd place (middle school) to 2nd place (high school) to 1st place (college/university). This confirms our suspicion that students become more and more career-focused as they progress academically. Once again, it is important to point out that there are multiple response categories that could be grouped into the larger category of “family and friends.” However, though achieving good grades is not the most frequent motivator at any of the educational levels, neither is it irrelevant.

Finally, we used the broad categories discussed earlier for item 21 (teacher, family and friends, future/job/application, grades, intrinsic) to create subsamples at each educational level and for each individual school site. In some cases, the resulting subsample sizes were quite small, so we will only highlight cases where the subsample mean was at least 1.0 higher or lower than the mean of the combined sample, indicating a mean response that was at least one category different from the total group. Figure 6 provides a summary of these findings. The following abbreviations are used: IN-MS (Indiana Public Middle School), IL-MS (Illinois Public Middle School), P-MS (Illinois Private Middle School), P-HS (Texas Private High School), IL-HS (Illinois Public High School), U (Midwest Public University), P-U (Midwest Private University), P-C (Midwest Private College). Items left blank indicate an absence of subsamples with responses outside of the mean.

Based upon the table in Figure 6, several trends emerge. First, students who are motivated by intrinsic factors tend to provide more optimal responses. In this case, optimal means higher values for the positively-worded items and lower values for the negatively-worded items. Notice that all

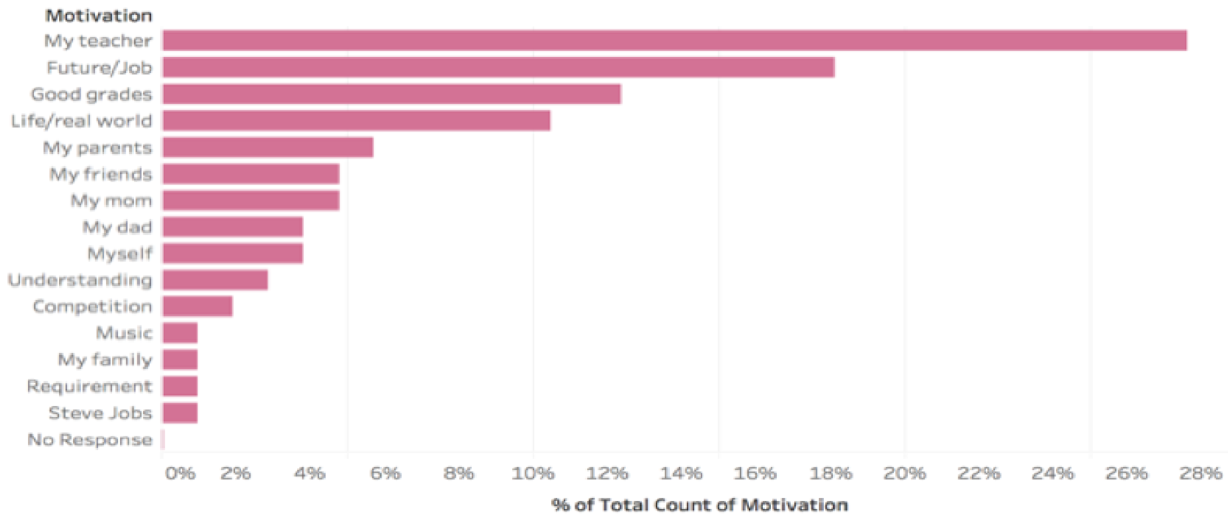

Figure 4: Summary of responses for item 21 at the High School Level

three middle school samples and the public university sample appear multiple times in the table for which those "intrinsically motivated" have unusually high means (or unusually low means for the negatively-stated items). Second, students who identify teachers as the key motivational factor exhibit more optimal means for at least one school subsample at each educational level. The exception to this finding was the public university subsample for which those indicating "teachers" provided less optimal responses. Third, the subsample from the Illinois public school typically provided more optimal responses whereas the subsample from the public university typically provided less optimal responses. Finally, with one exception for item 7, students indicating "grades" as the primary motivational factor tended to exhibit less optimal responses.

5 Conclusions

Based upon the results presented in the previous section, we draw several important conclusions. Recall that we originally hoped to provide insight on the following questions:

1. What similarities and differences exist among the primary motivation factor to learn mathematics for students at various grade levels?
2. What is the preponderance of intrinsic versus extrinsic motivation factors for these students?
3. Which groups or subgroups demonstrate unusually high or low Likert scale responses for specific items, and what do these results say about how these students approach mathematics?

First, related to the second research question, there is a relative paucity of students at all three grade levels who claim to be motivated by intrinsic factors. At the same time, those students who are intrinsically motivated tend to give the most optimal responses for nearly every item in the survey. For teachers, it may prove useful to highlight intrinsic reasons to study mathematics and to identify those students who find this factor most helpful and who may benefit from differentiated instruction that prompts them to dig deeper into mathematical topics.

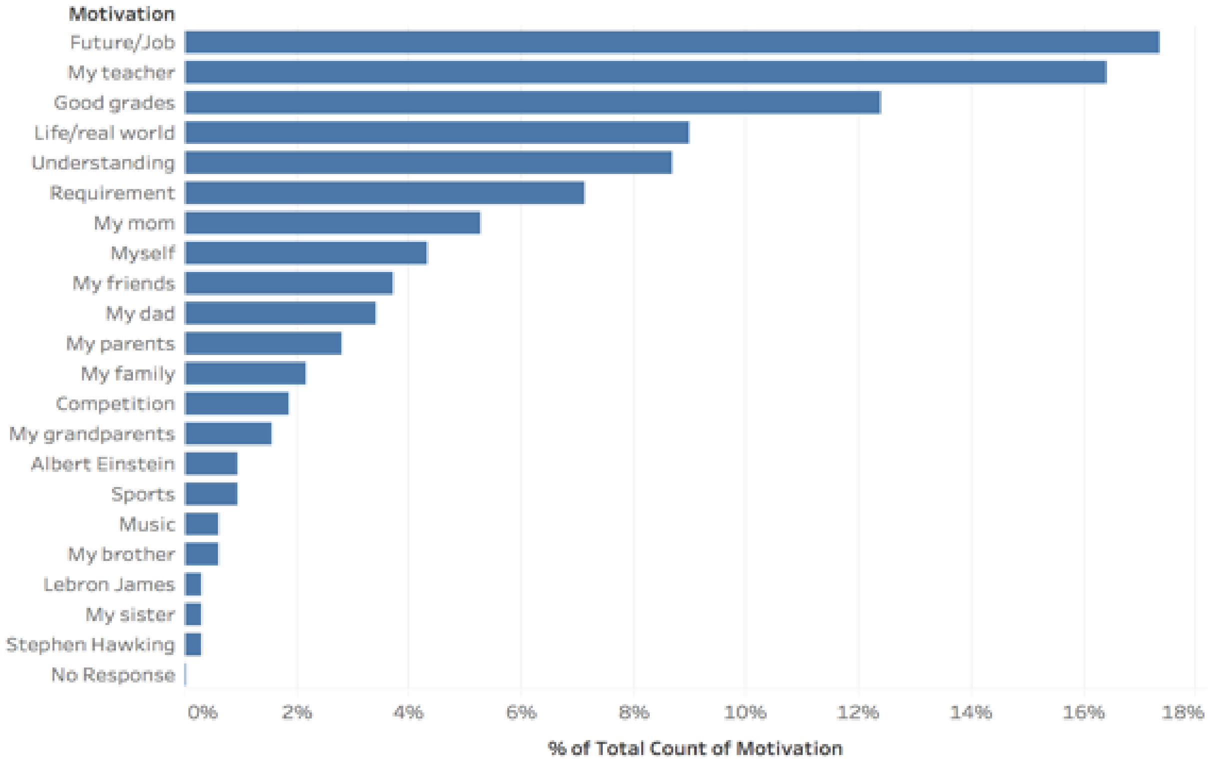

Figure 5: Summary of responses for item 21 at the College/University Level

A second similar theme relates to the third research question and emerged from the subsample of college and university students who indicated a mathematics or mathematics education major. While this result is hardly surprising, it does remind college and university mathematics professors that these students often select such a major because of positive experiences in the mathematics classroom over multiple years. Efforts to excite students at the middle grades, whether through enrichment tasks, local competitions, or other initiatives, often bear fruit later at the college and university levels. A related theme is the consistently less optimal responses given by the subsample of college and university students who indicated an elementary education major. This result confirms other studies (e.g. Vinson, 2001; Uusimaki and Nason, 2004; Malinsky, Ross, Pannells, and McJunkin, 2006) that have documented math anxiety within this group and underscores the need to help these future teachers to overcome their negative feelings towards mathematics to avoid perpetuating these attitudes to future generations of students. It is interesting that the levels of anxiety were actually lower for the middle school students in this sample, although the impact of a few middle school teachers with both deep content knowledge and the ability to support learners is probably relevant.

A third conclusion relates to the first research question and emerges from the finding that teachers, along with the combined category of family and friends, are consistently primary sources of motivation for students, especially at the middle school and high school levels. This serves to remind those of us who teach at these levels, or perhaps those of us who help to prepare future teachers for these grade levels, of the key role that classroom teachers play in motivating students to persist and achieve in mathematics. On a related matter, the regular communication between teachers and parents or guardians, whether during a face-to-face conference, a phone call, or an exchange of written communications, serves as a critical moment in the overall effort to keep students motivated

Item Number	Combined sample means	Unusually high subsample means	Unusually low subsample means
1 Enjoy math	3.4	IN-MS "Intrinsic" (4.5) P-C "Teachers" (4.6)	U "Grades" (2.0)
2 Like to solve math problems	3.4	P-C "Teachers" (4.5)	U "Future Job Applications" (2.4)
3 Really like math	3.2	IL-MS "Teachers" (4.3) IL-HS "Teachers" (4.3) IN-MS "Intrinsic" (4.3) P-C "Teachers" (4.5)	U "Future Job Applications" (2.0)
4 Happiest when in math class	2.7	IL-MS "Teachers" (4.1) IL-MS "Family and Friends" (3.9) IL-MS "Future Job Applications" (3.7) IL-MS "Intrinsic" (3.7) P-MS "Intrinsic" (4.0) IN-MS "Intrinsic" (3.8) P-C "Teachers" (3.9)	
5 Math is interesting	3.5	P-C "Teachers" (4.5)	U "Future Job Applications" (1.9) U "Grades" (2.0)
6 Studying math makes me nervous	2.8	**U "Future Job Application" (3.9)***	**IN-MS "Intrinsic" (1.8)**
7 Always under strain in math class	2.6		**P-MS "Future Job Application" (1.8)** **P-MS "Grades" (1.8)** **IN-MS "Intrinsic" (1.5)**
8 Thinking about math makes nervous	2.1		
9 Always confused in math class	2.6	**IN-MS "Grades" (3.9)**	**P-MS "Intrinsic" (1.5)**
10 Insecure when attempting math	2.5	**IN-MS "Grades" (3.5)**	None
11 Math is worthwhile	3.9		U "Teachers" (2.8)
12 Math is important in everyday life	3.9		
13 Math is important subject to study	3.8		
14 Math is important for future	3.7		IL-MS "Intrinsic" (2.0)
15 Math background helps me in life	4.0		
16 Getting good math grades motivates me	4.0		U "Teachers" (2.8)
17 Enjoy math reading	2.4	IL-MS "Future Job Application" (3.7) IL-HS "Teachers" (3.7)	
18 People motivate me to learn math	3.1	U "Intrinsic" (4.4)	
19 Math connections motivate me	3.4		U "Teachers" (2.3)
20 Difficult math motivates me	2.9	P-MS "Intrinsic" (4.8) U "Intrinsic" (4.2)	

Figure 6: Unusually high or low subsample means
∗ For items 6-10, the entries are in **boldface** to remind the reader that these are "negatively-stated," meaning that lower values are more optimal.

to continue learning mathematics. Taken together, these groups of adults constitute the majority of the sources of motivation for both middle school and high school students.

Finally, and also related to the first research question, it is interesting to note the relative importance of mathematics in future careers, everyday life activities, and patterns in the surrounding world. The fact that these connections are a major source of motivation for students at all educational levels should prompt teachers to continue looking for real-life applications in the learning activities they create for students. Indeed, more students are likely to pursue careers in science, technology, engineering, and mathematics (STEM), or perhaps also art (expanding the acronym to STEAM), if teachers make a conscious effort to help students discover the myriad connections among the disciplines.

6 Limitations of the Study

Although our research includes multiple schools at three different educational levels, we must acknowledge several limitations of the study. First, we used a convenience sample rather than a random sample, so our conclusions are only generalizable to the extent that our selected schools form a combined sample that is representative of typical students at these educational levels. Second, we provide a descriptive snapshot for these students rather than a longitudinal study. Although it appears that some factors, such as the role of a future career, appear to become more important as a student progresses academically, it would be more compelling to survey the same group of students at multiple points along their academic journey. Finally, a longer survey with multiple items linked to each potential motivational factor would better address validity and reliability concerns.

7 Implications for Future Research

After completing this research study, several potential avenues for future research have emerged. First, several colleagues in the Association of Christians in the Mathematical Sciences have responded favorably to a recent overview of our study and have asked to extend the data collection to their own colleges and universities. It is possible that an updated version of this research may be presented that encompasses this larger set of survey data. Second, the finding that the importance of the future uses of mathematics in a career or everyday life appears to increase as one progresses from middle school to high school to college might lead to two potential extensions: a longitudinal study might be able to document this growth over time; a study of adults in the workplace might document an even higher importance placed upon the applicability of mathematics in various careers and in other life activities. Third, a future survey of high school students might allow the comparison of senior students enrolled in advanced placement mathematics courses such as calculus and statistics with other seniors who are not enrolled in a mathematics course. Any of these future studies could deepen our understanding of the different factors that both motivate people to learn mathematics as well as to think mathematically in settings outside the classroom.

8 Summary

We entered into this research study based upon the motivation of a single undergraduate mathematics education major. Our findings point to a complex web of factors which, taken as a totality, help to motivate middle school, high school, and college or university students to study mathematics, to persist at solving mathematical problems, and to pursue careers that make use of mathematical reasoning.

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9 Appendix

(For Items 1-20, choose strongly agree, agree, neutral, disagree, or strongly disagree):

1. I have usually enjoyed studying mathematics in school
2. I like to solve new problems in mathematics
3. I really like mathematics
4. I am happier in mathematics class than in any other class
5. Mathematics is a very interesting subject
6. Studying mathematics makes me feel nervous
7. I am always under a terrible strain in mathematics class
8. It makes me nervous to even think about having to do a mathematics problem
9. I always feel confused in mathematics class
10. I feel a sense of insecurity when attempting mathematics
11. Mathematics is a very worthwhile and necessary subject
12. Mathematics is important in everyday life
13. Mathematics is one of the most important subjects for people to study
14. Mathematics lessons would be very helpful no matter what I decide to study in the future
15. A strong mathematics background could help me in my life
16. Getting good grades motivate me to do better in mathematics
17. I enjoy reading about mathematics (non-textbook readings)
18. The people around me motivate me to do better at mathematics
19. Having mathematics connect to other content areas and the world around me, motivates me to learn mathematics
20. I am motivated by how difficult math is
21. (free response): What is the one person, place, or thing that motivates you to learn mathematics?

References

[1] Common Core State Standards Initiative. (2011). Common Core State Standards for Mathematics.

[2] Eggleton, Patrick. (2017). "Developing the Underutilized Mathematical Strengths of Students." ACMS 21st Biennial Conference Proceedings. Charleston Southern University, 71-76.

[3] King, Alessandra. (2019). "Kindling the Fire: Why I Do What I Do." Mathematics Teacher, 112(7), 504-508.

[4] Kinser-Traut, Jennifer. (2019). "Why Math?" Mathematics Teacher, 112(7), 526-530.

[5] Lim, Siew Yee and Chapman, Elaine. (2015). "Effects of using history as a tool to teach mathematics on students' attitudes, anxiety, motivation and achievement in grade 11 classrooms." Educational Studies in Mathematics. 90(2), 189-212.

[6] Malinsky, Marci, Ross, Ann, Pannells, Tammy, and McJunkin, Mark. (2006). Math Anxiety in Pre-Service Elementary School Teachers. Education. 127(2), 274-279.

[7] National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: The Council.

[8] National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics. Reston, VA: The Council.

[9] National Council of Teachers of Mathematics (2014). Principles to actions: ensuring mathematical success for all. Reston, VA: The Council.

[10] Tapia, Martha. (1996). The Attitudes toward Mathematics instrument. Paper presented at the Annual Meeting of the Mid-South Educational Research Association. Tuscaloosa, AL: ERIC.

[11] Uusimaki, Liisa, and Nason, Rod. (2004). Causes Underlying Pre-Service Teachers' Negative Beliefs and Anxieties About Mathematics. Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 369-376.

[12] Vinson, Beth. (2001). A Comparison of Preservice Teachers' Mathematics Anxiety Before and After a Methods Class Emphasizing Manipulatives. Early Childhood Education Journal. 29(2), 89-94.

[13] Wilkerson, Joshua. (2015). "Cultivating Mathematical Affectations: The Influence of Christian Faith on Mathematics Pedagogy." Perspectives on Science and Christian Faith, 67(2), 111-123.

Teaching Mathematics at an African University–My Experience

Kathleen Lewis (University of the Gambia)

Kathy Lewis (B.A., University of Rochester; Ph.D., University of Chicago) taught at Wellesley College, Suffolk University, and SUNY Oswego. After her husband Gaunce died in 2006, Kathy felt called to look into opportunities serving God overseas. The mission agency WEC International recruited her to teach at the University of the Gambia in the Republic of the Gambia, West Africa. She retired from that university in 2018.

Abstract

After taking early retirement from one of the State University of New York colleges, I taught for nine years at the University of the Gambia, in the Republic of the Gambia, West Africa. In this paper, I will discuss my experiences there and talk about how other faculty members can help build the educational system of a developing country.

The University of the Gambia was founded in 1999. Until then, there had been no university in the small West African country. Students who wished to get a university degree typically went to Sierra Leone to study, since that was the nearest country that also had an English-language educational system. (There were programs in the Gambia to train teachers for elementary school and junior high school positions, but even to become a certified high school teacher one had to travel to another country.) When the civil wars broke out both in Sierra Leone and Liberia, studying in both of those countries became impossible and the other English-speaking countries in West Africa were too far away to be reasonable choices for most potential students. At that point, a Canadian university helped the country start its own university and many of the first courses were taught by faculty members from Canadian schools. The university tried to recruit long-term faculty from overseas, but offered such low salaries (even by African standards) that very few qualified people came. So the lower-level courses were mostly taught by minimally qualified Gambians and the upper-level courses were only taught when a visiting faculty member from another country came for a semester or a year. Students might have to wait several years for some of the courses they needed to take.

I first taught at the University of the Gambia at a visiting lecturer in 2008-09, on leave from my university in the United States. Since I had taught several of the upper-level courses that students were waiting to take, I was very warmly welcomed by both the students and the faculty. I was paid the standard salary for my rank, but was given free housing in a university-owned house. After that first year, I went back to the United States, retired from my job there and returned to Gambia in 2010. At this point, I was a regular, rather than visiting, faculty member and fairly quickly became de facto head of mathematics. (Officially, mathematics was part of the division of science, which had a single head, but he was very willing to leave all of the mathematics business to me.)

1 The successes

I was able to introduce several new courses to the curriculum. Since I taught most of the more advanced courses, using American textbooks and an American syllabus, my best students were prepared for graduate study in Europe or North America. Several of my students have already earned Master's degrees and have returned to the university to teach. Several more are now in graduate programs.

2 The failures

I was unable to improve the dire working conditions of my staff, since the administration was not cooperative. So several of my best young colleagues left the university and got jobs elsewhere in the Gambia. Others went overseas for graduate study and will probably never return. Several of my brightest former students who got jobs with the university after they graduated were then forced by the university to enroll in bad graduate programs where they earned master's degrees without learning what they should have. Other good students are still trying unsuccessfully to gain admission to graduate schools in Europe or North America, since many schools don't take degrees from African universities seriously.

3 What can educators do to help at places like the University of the Gambia?

1. Go long-term to teach.

   Unless you do this as a second career, as I did, you would probably need to go under the auspices of a missions agency and raise funds to supplement your university salary. If you do this, you need to be careful that the agency doesn't demand so much of your time and energy that you can't do a good job with your classes. There were Christian faculty at the University of the Gambia who saw teaching at the university merely as a visa platform or a way to pay for their "real work" as missionaries, and they hurt the reputation of Christianity on campus. (Gambia is a 95% Muslim country, so most of my colleagues were Muslims.) The workload is quite heavy, so you are unlikely to have very much time to do research. If you plan to eventually return to an academic job in the U.S., you need to think about that issue.

2. Go for a semester or a year.

   Four visiting faculty members from the United States and the Netherlands came to teach mathematics for a semester or a year while I was in Gambia. Some came as a sabbatical from their regular university jobs and some came as part of an exchange program between their universities and the University of the Gambia. (Exchange programs typically sent a group of students and one supervising faculty member, who might be in any department. We were fortunate to have two of them in our area.) Each of them taught somewhere between two and four courses during the semester. When I knew in advance who was coming, I tried to assign an upper-level elective in that person's field as one of their courses. This allowed me to offer courses I normally could not have offered. It also meant that the math majors had exposure to another teaching style, since they otherwise had most of their upper-level courses with me. Several of these instructors also agreed to write letters of recommendation for graduate school for the good students in their classes. This was especially useful because a letter from an American professor carries much more weight than a letter from an African instructor.

3. Send books to a faculty member that you know in a developing country.

   My students couldn't afford to buy books and the university library had very little. One of my friends in the US went to her colleagues and got donations of old-edition desk copies when new editions appeared. So I had a small lending library for my students. I do not recommend sending books to a university without directing them to an individual teacher. If you do that, the books may just end up being dumped into a storeroom somewhere.

4. If you teach at a university with a graduate program in math, encourage your graduate committee to consider applicants from developing countries seriously.

   In many cases, these students are dismissed without serious consideration because their undergraduate institution is an unknown quantity.

4 Conclusion

There are plenty of opportunities for first-world faculty members to help universities in developing countries. The students in those countries are just as smart as our students, and just as deserving of getting a good education. As Christians, we serve the God of the entire world and should be even more motivated than our non-Christian colleagues to reach out to them. I hope that you will pray and consider how you or your college can help.

Faith, Mathematics, and Science:
The Priority of Scripture in the Pursuit and Acquisition of Truth

Bob Mallison (Indiana Wesleyan University)

Bob Mallison (B.S., Marion College; Ph.D., Purdue) was an actuary for 20 years before joining the faculty of Indiana Wesleyan University. His interests include continued involvement in the actuarial profession as coordinator of the Minor in Actuarial Science at Indiana Wesleyan. Current research interests include resolving the (illusory) conflict between faith and science. He is active in his local church as worship leader and board member.

Abstract

In this paper, we present several arguments to support a literal, or natural, interpretation of Scripture. Even though Mathematics and Science have made vast contributions to the body of knowledge, we conclude that *scientific information does not have adequate authority to require rejection of a literal interpretation of Scripture.*

We proceed from a decidedly Christian perspective including the convictions that God created an orderly universe (and that studying nature provides valuable information about Him), and that God inspired His Word (and that the Bible, even *more clearly*, expresses information about Him).

We discuss some of the essential tools used by mathematicians and scientists for the discovery of truth - namely, models. We examine some valuable models from history, and briefly discuss that as additional scientific information became available, the models required refinement, and sometimes replacement. The Bible, on the other hand, is perfect and needs no corrections.

We also briefly consider the following items:

1. The nature of higher dimensions and how a literal reading of the Bible can be supported by understanding some of the implications of higher dimensions;
2. The role of hermeneutics in Biblical interpretation (and scientific interpretation) regarding reconciliation of science with the Bible;
3. Finally, we speculate about possible implications for the frequent use of the phrase "[God] stretched out the heavens."

We conclude by summarizing the results and recognizing that as we study mathematics and science, along with God's Word, we can know the truth. In fact, God's plan for each of us is to know Him Who is the Truth.

1 Introduction

"*Sanctify them by the truth; your word is truth.*" *(John 17:17)*

1.1 Where are We Going?

In this paper we will examine some approaches for identifying truth as well as some issues involved in recognizing reliable sources of information. We will proceed from a decidedly Christian perspective including the convictions that God created an orderly universe, that studying nature provides valuable information about Him, and that His Word, the Bible, even *more clearly*, expresses information about Him.

Our approach is intended to be consistent with that of Alister McGrath as described by Bradley and Howell:

> Recently the theologian and scientist Alister McGrath has offered a new approach to natural theology. He suggests accepting special revelation as providing an interpretive framework through which nature can be seen (Bradley and Howell, 2011, p. 11).

With a slightly different emphasis, we would say that the special revelation of scripture can, and should, inform and modify the information we receive through general revelation (not the other way around).

We will begin by briefly discussing "truth" and "knowledge," and then examine a specific cosmological model involving the existence of higher dimensions. We will observe that for some passages, adopting a literal interpretation of Scripture appears to be problematic, until we understand the nature of higher dimensions. As we will see throughout the paper, as more information becomes available, the reliability of Scripture is repeatedly confirmed.

We will continue by discussing some of the essential tools used by mathematicians and scientists for the discovery of truth - namely, models. We will examine some valuable models from history, and briefly discuss that as additional scientific information became available, the models required refinement, and sometimes replacement. We will conclude that when mathematical or scientific models lead to inconsistencies with other reliable sources of information (e.g. the Bible), we should be hesitant about rejecting the other sources (because models are subject to change).

Next, we will identify various implications for teaching and learning mathematics, as well as some implications for reconciling science with Scripture. All of these observations also have implications for worship as we understand a little more about God's nature, the world of nature He created, and His desire to help us as we pursue and discover truth.

Sometimes, the reconciliation of scientific information with Scriptural information involves hermeneutics; so we will briefly examine the role of Biblical interpretation in Chapter 5 of the paper. Ultimately, one of the primary goals of this paper is to defend the literal interpretation of Scripture as the best interpretation (or at least the best place to begin the process of Biblical interpretation). While other sources of knowledge are very valuable, the Bible is the only perfect and flawless source of information we have.

We conclude by summarizing the results and recognizing that as we study mathematics and science, along with God's Word, we can know the truth. In fact, God's plan for each of us is to know Him Who is the Truth.

1.2 What is Truth?

The first chapter of John's Gospel talks about the Word and truth:

> In the beginning was the Word, and the Word was with God, and the Word was God.

> ... The Word became flesh and made his dwelling among us. We have seen his glory, the glory of the One and Only, who came from the Father, full of grace and truth (John 1:1, 14).

Then, near the end of the Gospel, Jesus and Pilate discussed kingdoms and truth, and Pilate uttered his famous question: "What is truth?" (John 18:38)

Part of the goal of education is the acquisition of knowledge and the discovery of truth. But before truth can be discovered, we need to understand how to identify it when we find it.

Geisler and Feinberg (2002) identify four theories of truth: the coherence theory, the pragmatic theory, the performative theory, and the correspondence theory. They conclude that only the fourth is consistent with Christian faith:

> We have argued that the first three theories of truth are inadequate, that the correspondence theory alone is sufficient. As Christians, we cannot accept any theory of truth which results in either relativism or agnosticism. The Bible clearly declares that man can know the truth, and will be held responsible for such knowledge. [See, for example, John 8:32-36 and Romans 1:18-22.] (Geisler and Feinberg, 2002, p. 250)

And as they explain previously, "The correspondence theory of truth holds that truth consists in some form of correspondence between a belief or a sentence and a fact or a state of affairs" (Geisler and Feinberg, p. 247).

The correspondence theory of truth is the theory which is assumed in the study of science. When we study science and mathematics, we are trying to discover facts, and we are trying to formulate sentences which correspond to those facts.

If we approach the subject of truth from a Christian perspective, we can accept that God created an orderly universe and that there are certain true statements which can be made about the universe. In particular, we can agree with St. Augustine that "All truth is God's truth"; so if we accurately interpret scripture and accurately interpret the scientific information, we should find beautiful harmony among the various sources of information about the universe.

1.3 How Do We Know?

If we "will be held responsible" for knowledge of the truth, how do we acquire that knowledge? This is one of the primary questions in the philosophical discipline of *Epistemology.* Willem Van De Merwe summarizes and explains several approaches to knowledge:

> Knowledge can come from various sources, including the ones listed in the following:
>
> 1. Observations - passively acquired data. (Empiricism)
> 2. Experiments - actively acquired data. (Empiricism)
> 3. Reason - inductive or deductive mental processes leading to new knowledge. (Rationalism)

4. Scriptures - accepted written stories and proclamations containing knowledge and truth statements, which are not obtained empirically or rationally. (Religion)
5. Experiences - caused by events that happen and that affect our awareness or our "feelings." (Existentialism)
6. Testimony of others. (Authoritarianism)
7. Traditions - transmitted generally accepted ways of doing or interpreting things within a cultural setting. (Authoritarianism)
8. Revelation - knowledge passively acquired either by everyone generally [*general revelation*], or specifically only by one individual or a select group of individuals [*special revelation*]. (Religion)
9. Faith - knowledge actively acquired through a spiritual process without recourse to reason. (Fideism)
10. Intuition/Imagination/Inspiration.

(Van De Merwe, 2014, p. 17-18)

In the following pages, we will discuss several of these approaches as well as some of their applications and limitations. *Two primary goals of the following discussion are to identify the Bible as the best and most perfect source of knowledge, and to recognize that adopting a literal interpretation of Scripture is both scholarly and defensible.*

1.4 Faith and Reason

In Isaiah 1:18, God invites us to dialogue - "Come now, let us reason together." When God created man in His image, He gave us the ability to think. I find it very interesting that in John 1, Jesus is described as the Word - *Logos* - which carries the connotation of "*mind of God*." I believe this reference to The Word reflects (at least partially) how Jesus embodies both divinity and humanity.

Hebrews 11 talks a lot about faith. Verse six says that "without faith it is impossible to please God." A popular (secular) understanding of "faith" is the act of believing something, even if you know it is false. Actually, this type of thinking would better be described as "blind faith," like the scientist who knows spontaneous generation is impossible, but because of naturalistic presuppositions, is forced to believe it. Another (more appropriate) understanding of "faith," is the ability to accept certain truths when evidence is compelling, but not conclusive. If you cannot prove something conclusively (mathematically or logically), but there is supporting evidence, you can take a step of faith to accept the position. Another view of "faith" (from C.S. Lewis - Mere Christianity Book III Chapter 11) is the ability to hold onto something the mind once accepted, even when the position is inconvenient.

So what about mathematics? At the turn of the twentieth century, there was an effort to put all of mathematics and science on a firm logical foundation. Beginning with axioms, the goal was to prove (mathematically) everything that was known, and develop a system of logic by which everything could be proven. These efforts seemed to be yielding good progress, but there was always some unforeseen complication; a complete mathematical structure proved to be elusive. In 1931, the Austrian mathematician Kurt Gödel dashed the hopes for a complete mathematical system when he published his celebrated "Incompleteness Theorem." In short, he proved that there is no (adequately powerful) complete system of mathematical logic; if such a system could be developed, it would prove not only all true things to be true, it would also prove all false things to be true. To remain consistent, a system of mathematical logic must be incomplete. In other words, there will always be some true things which cannot be proven.

A common secularist mantra is to accept nothing based on faith alone; accept only those propositions which can be proven. In light of Gödel's Incompleteness Theorem, such thinking will be inadequate for certain truths; in short, "faith" is necessary for the discovery of certain truths. My hope and prayer is that people will be able to recognize the role of faith in the acquisition of truth, and in the receiving of The Truth (John 14:6).

2 The Origin of the Universe - Genesis 1:1

"By faith we understand that the universe was formed at God's command, so that what is seen was not made out of what was visible." (Hebrews 11:3)

2.1 The Beginning

When did God create the heavens and the earth? "In the beginning!" (Genesis 1:1). This truly was the beginning of our universe - our three-dimensional space-time continuum. Augustine also understood that time had a beginning, as noted by Bradley and Howell:

> Augustine spent many years reflecting on the book of Genesis. He concluded that the opening words of the Bible, "In the beginning ..." refer to the beginning of time (Bradley and Howell, 2011, p. 224).

With the beginning of the universe, there was also the beginning of Science. When God created the universe, He implemented many scientific and mathematical laws, which demonstrates in nature something that we already know from scripture - God is a God of order (see, for example, I Corinthians 14:33; also, Psalm 104 is a celebration of God's creativity and order in nature).

But what about the beginning of Mathematics; or did Mathematics have a beginning? If we consider the nature of God (the Trinity), we see that the concepts of "oneness" and "manyness" (and the concept of "number") seem to exist before the creation of the universe (see Poythress, Chapter 2). An important philosophical question, which is still debated by mathematicians, is whether mathematics is *invented* or *discovered* (see Bradley and Howell, Chapter 10). As image-bearers of God, we certainly have creative abilities; but since God is omniscient, we must conclude that He knows everything, and thus, that He knows all of mathematics. But we have the opportunity to experience joy and excitement as we discover the mathematical laws He created.

2.2 A New Dimension

As mentioned above, when God created the heavens and the earth, our three-dimensional space-time continuum came into being. Before Genesis 1:1, the three-dimensional universe did not exist.

There are several biblical passages which indicate that there is a realm beyond our three-dimensional universe - a realm with more dimensions; perhaps this can be understood as the spiritual realm. Many of the verses are included at the end of this chapter; one example is:

> . . . so that Christ may dwell in your hearts through faith. And I pray that you, being rooted and established in love, may have power, together with all the saints, to grasp how wide and long and high and deep is the love of Christ (Ephesians 3:17-18).

When Paul was describing the love of Christ, he could have easily accomplished his objective by using three dimensions - say length, width and height - but he chose to use four dimensions. This is certainly not a proof that Paul was referring to higher dimensions, and some scholars readily point out that ancient Jewish writers often used *hyperbole* - intentional exaggeration for emphasis; but this passage could also hint at the possibility that we should try to understand the nature of higher dimensions. As a mathematician, I can readily accept the theoretical existence of higher dimensions (even if it is difficult to try to visualize higher dimensions). When I was first introduced to the concept of higher dimensions by my former mathematics professor, Dr. Donald H. Porter, he suggested examining how our three-dimensional world might relate to a two-dimensional world.

2.3 Two-Dimensional Nature

A two-dimensional world is a flat surface. (Strictly speaking, a two-dimensional world need not be"flat," but is approximately flat locally.) The book *Flatland*, [2], by Edwin A. Abbott gives a rather detailed description of the nature of a two-dimensional world. There is no concept of "up" in this world; there is only length and width - no height. Inhabitants, as well as all objects, are plane figures such as circles, squares and triangles. A room in a house in this world might be in the shape of a square, with a swinging line segment for a door. Now suppose that in a certain room there is a two-dimensional "person." Since we are in a three-dimensional realm, he has no perception of us; we do not exist in his world because to him, there is no concept of "up" (and he would need to look up to see us). We might place our hand very close to his body, and even though we might be closer to him than anything in his two-dimensional world, he has no realization of our presence.

Now if we reach down and touch the inside of his room, he would perceive a blot. He would probably be startled, for from his point of view, the blot appeared from nowhere. If we now remove our finger, the blot would immediately disappear; or if we move our finger across the boundary of his room, the blot would seem to have moved through a wall. Also, if there were something wrong inside his body, we could see the problem, reach inside, and fix the problem. From his perspective, this might look like a miracle, but from our point of view, everything is quite natural.

Finally, if there were two people in the two-dimensional world, one in the room and one outside the room, we could see both of them from our position in three-dimensions, but they would not be able to see each other; and if they were somehow allowed to perceive three-dimensions, they would both be able see us, even though they would still be unable to see each other. And if we could communicate with them, we could inform them that we are able to "see all around them" even though their line of vision is in only one direction. If they were to try to describe this ability of ours, they might say that we are covered with eyes all around. This idea may cause us to recall the images in the Biblical books of Ezekiel (chapter 10) and Revelation (chapter 4) about some of the creatures covered with eyes.

2.4 Applications to Spiritual Phenomena

Considering the relationship between two-dimensions and three-dimensions, we can anticipate the possibilities of certain unusual experiences if, in fact, the spiritual realm can be characterized as four-dimensional (or more than four). For example, Jesus appeared to walk through walls after his resurrection (John 20). He appeared (and disappeared) miraculously after talking with the disciples on the Emmaus road (Luke 24). Moses and Elijah appeared (and disappeared) on the Mount of Transfiguration (Matthew 17). There are many examples of similar phenomena in both the Old and New Testaments. And there are other implications.

An interesting result from Einstein's General Theory of Relativity and higher dimensions is the theoretical possibility of things called "wormholes," discussed by Hawking and Mlodinow (Hawking and Mlodinov, 2005, Chapter 10). Briefly, wormholes allow for shortcuts through space which can be available because of the curvature of space. On pages 110 and 139 of their book, they provide pictures of what a wormhole may look like. The interesting thing is that these pictures resemble something like a funnel or a vortex. This may cause us to remember (II Kings 2:11) that when Elijah was taken away, he "went up to heaven in a whirlwind." This does not prove anything, but the similarity is interesting.

In the above-mentioned verses from Ephesians, we included both verses seventeen and eighteen. We discussed verse eighteen in some detail - the reference to four dimensions. Verse seventeen talks about Christ dwelling in our heart by faith. As children, we are willing to talk about Jesus living in our heart; but when we get older (and more mature), we come to realize that this kind of talk is only symbolic - we don't really think there is a little person inside our physical "blood-pumping organ." But if we accept the possibility that the spiritual realm is a higher dimensional realm, then it is possible that Jesus literally lives inside every true born-again believer. I am convinced that Jesus literally lives in my heart. Some might consider this idea simple-minded or immature, but the math and the Scripture agree that a literal interpretation might be acceptable for Ephesians 3:17. As we said in Section 1.1, understanding the nature of higher dimensions can sometimes support a literal reading of the Bible.

2.5 Summary

As I continue to study the Bible, and as I continue to study Mathematics and Science, I continue to be overwhelmed by the beauty of creation and by the majesty of our Creator, Savior, and Lord. The following passages from the Bible may suggest the existence of higher dimensions, some directly, some indirectly, and studying these passages can possibly give a new perspective on some old stories from the Bible. Again we see that even when Scripture seems to violate known scientific laws, a figurative or symbolic interpretation is not necessary; indeed, a literal interpretation may be the best interpretation.

1. Genesis 5:21-24 – Enoch miraculously taken away to another dimension (?)
2. II Kings 2:11 – Elijah taken away in a whirlwind to another dimension (?)
3. II Kings 6:15-17 – Elisha's servant's eyes opened to see higher dimensions (?)
4. Ezekiel 10:12 – Creatures completely covered with eyes

5. Daniel 10:1-21 – Daniel (but not his companions) saw into higher dimensions (?)
6. Matthew 17:1-5 – Moses and Elijah arrive from and depart to higher dimensions (?)
7. Luke 24:13-31 – Jesus vanishes to another dimension (?)
8. John 6:21 – The boat *immediately* reached the shore
9. John 20:19, 26 – Jesus walks through walls from another dimension (?)
10. Acts 1:9 – Jesus taken away to another dimension (?)
11. Acts 8:39-40 – Philip transported (through a wormhole?)
12. Acts 9:3-7 – Saul (but not his companions) see into another dimension (?)
13. Acts 12:5-11 – Peter's deliverance through another dimension (?)
14. II Corinthians 12:2-4 – Paul's vision of the third heaven (another dimension?)
15. Ephesians 3:17-18 – Explicit reference to four dimensions
16. Ephesians 6:12 – Spiritual warfare (in another dimension?)
17. Hebrews 11:5-6 – Reference (again) to Enoch
18. Revelation 1:7 – Every eye will see him (through another dimension?)
19. Revelation 4:6-8 – Creatures covered with eyes

3 Models - Uses and Limitations

"The kingdom of heaven is like ..." (Matthew 13:24, 31, 33, 44, 45, 47)

3.1 Introduction

In studying the physical world, scientific and mathematical models are developed to represent various phenomena. As a familiar example, many people may have memories of assembling a model car or model airplane. A model is not identical to the "real thing," but a good model can fairly represent the genuine article; and, with more details, the representation can be made more accurate. But we must never confuse the model with the phenomenon being represented (we will call this the *confusion fallacy*). The model is just a model, not the thing itself.

A useful mathematical model from calculus and physics is based on the formula

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0.$$

Using this model, we can determine how long it will take for a rock to hit the ground if it is dropped from a height of, say, 64 feet. This model works very well in most typical situations. But if we drop the rock from a height of 64,000 feet, the model will not give a precisely correct answer. As the velocity increases (while the rock is falling) the wind resistance becomes a significant factor. And so the model above (which ignores wind resistance) is no longer as useful. Models can be very valuable tools for a scientist or mathematician, but when extreme situations are being analyzed, models must sometimes be enhanced, or even completely discarded and replaced.

In this chapter, we will briefly discuss three historical models which seemed to provide very good explanations for the phenomena they represented, but when additional evidence became available, they were found to be deficient and required refinement, or even replacement. We can learn from the words of the 20th Century statistician G. E. P. Box: “All models are wrong; some are useful” (Cannon, et al., 2019, p. 2).

We will look at Ptolemy’s model of the universe, Newton’s laws of motion, and Darwin’s theory of evolution by Natural Selection.

3.2 Ptolemy’s Model of a Geocentric Universe

From antiquity, people have noticed that each new day, the sun rises in the East, is nearly straight up at noon, and sets in the West. Ptolemy’s model was accepted as the correct representation of the universe from the second century until the middle of the sixteenth century. Based on observations and confirmed by detailed calculations, the model proved to be very useful for describing planetary motion and predicting future motion. As Alister McGrath summarizes, the model is based on the following assumptions:

1. The earth is at the center of the universe;
2. All heavenly bodies rotate in circular paths around the earth;
3. These rotations take the form of motion in a circle, the center of which in turn moves in another circle. This central idea, which was originally due to Hipparchus, is based on the idea of epicycles - that is, circular motion imposed upon circular motion (McGrath, 2010, p. 18).

As mentioned above, this model was very useful, and in most cases gave reliable results. But as more information became available based on additional (extreme) observations, the model was no longer feasible. Because of the *confusion fallacy*, there was significant difficulty in refining the model. Some scholars point to this issue as a classic example of the conflict between faith and science. And while many people acknowledge the culpability of the church in this matter, McGrath (p. 22) explains that the conflict was more between competing political/ecclesiastical forces (and some philosophical differences about Biblical interpretation) than between faith and science. The existing intellectual establishment, with its power structure, resisted change. But with continued accumulation of observed data, the geocentric model, which persisted for more than 1000 years, was eventually discarded and replaced by a heliocentric model for our solar system.

3.3 Newton’s Models of Motion

Many people point to the work of Sir Isaac Newton as the beginning of the Scientific Revolution. Newton’s famous laws led to an explosion of scientific knowledge in the seventeenth and eighteenth centuries. His laws were based on some very natural (and quite obvious?) assumptions about the nature of the universe. Pearcy and Thaxton summarize as follows:

> Absolute space remains, according to its nature and without relation to an external object, always constant and fixed.
>
> Absolute, true, mathematical time passes continually, and by virtue of its nature flows uniformly and without regard to any external object whatsoever (Pearcy and Thaxton, 1994, p. 167).

Newton's laws were so powerful that some have suggested that they led to the Industrial Revolution. They were very accurate whenever and wherever they were tested. His model was based on observation, confirmed by further observations and detailed calculations, and accurate for predicting future phenomena.

Newton's model persisted for more than 300 years, and again, the *confusion fallacy* emerged. The accuracy was so striking that a mechanistic model of the universe soon became the prevailing cosmological model, and this led to the rise of Deism as a theology and rationalism as a philosophy. Deism and rationalism had already been around for a while, but the predictive power of Newton's laws gave them intellectual support. As McGrath observes: "... Deism owed its growing intellectual acceptance in part to the success of the Newtonian mechanical view of the world" (McGrath, p. 31).

However, as extreme data (light years away) became available, some inadequacies became apparent in Newton's models. The (long-standing, self-evident, intuitively obvious) assumption about the Euclidean nature of the universe came into question. As Poythress notes:

> Albert Einstein's general theory of relativity postulated that space (together with time, which is treated as a fourth dimension not strictly isolatable from the spatial dimensions) is curved, not Euclidean (Poythress, 2015, p. 138).

Einstein's theory, also based on observations and confirmed by calculations and predictions, postulated that the presence of gravitational fields actually curved space. The underlying assumptions supporting Newton's model (which was so successful throughout the Industrial Revolution) were substantially refined and replaced by Einstein's theories of relativity.

3.4 Darwin's Model of Evolution by Means of Natural Selection

With the success of Newton's model and the subsequent rise of rationalism as a dominant philosophy, *naturalism* gradually became the framework for scientific investigation. At the beginning of the scientific revolution, most scientists were Christians; but as Anderson observes:

> By the nineteenth century, secular trends began to change the perspective of scientists. This culminated with the publication of *On the Origin of Species* by Charles Darwin. His theory of evolution provided the foundation needed by naturalism to explain the world without God (Anderson, 2008, p. 15-16).

A simplified summary of Darwin's model begins by observing that many forms of life (e.g. plants, animals, people) have the ability to adapt to changing environmental conditions. (This is commonly called *microevolution.*) With enough time and enough environmental changes, the adaptations can be very dramatic. In fact, if individual members of the same species became separated and evolved separately under different environmental conditions, there could emerge two brand new species - different from each other and different from the original species from which they descended. Thus we have the claim that humans and chimpanzees have a common ancestor. (This is commonly called *macroevolution.*)

Behe provides a more detailed description:

> Like many great ideas, Darwin's is elegantly simple. He observed that there is variation in all species: some members are bigger, some smaller, some faster, some lighter in color, and so forth. He reasoned that since limited food supplies could not support all organisms that are born, the ones whose chance variations gave them an advantage in the struggle for life would tend to survive and reproduce, outcompeting the less favored ones. [This phenomenon is known as "survival of the fittest."] If the variation were inherited, then the characteristics of the species would change over time; over great periods, great changes might occur (Behe, 1996, p. 3-4).

Darwin's model was based on observations, and confirmed by additional observations and calculations. Even though it was not universally accepted, the scientific establishment, which consisted primarily of naturalists, eventually embraced the theory, even to the point of considering it an established scientific fact.

Its acceptance in the scientific community should not be interpreted as meaning there were no challenges for the model. There were at least two significant problems with the model - (1) the fossil record and (2) the human eye (as well as eyes of other creatures). The problem with the fossil record is the lack of transitional forms; if evolution has been happening for millions of years, we should be able to see fossil evidence of how individual species evolved, little by little. But what we find in the fossil record are many fossils of organisms with no apparent relationship to other organisms. This led to the popular scientific problem of "the missing link" (actually, a whole lot of links).

To address these "missing links," a new theory was developed by Stephen Jay Gould and Niles Eldredge. As Anderson summarizes:

> Their theory, known as punctuated equilibrium, proposed that biological change occurred in isolated populations. During these periods of rapid evolutionary changes in small isolated populations, virtually no organisms would show up in the fossil record because their numbers were small and geographically isolated. Unlike the previous views of neo-Darwinian evolution, punctuated equilibrium predicts that biological change takes place in larger, more discrete jumps, and these would effectively be hidden from the fossil record (Anderson, p. 30-31).

It is interesting that the very evidence most people point to for support of Darwin's theory of evolution (the fossil record) is the problem which needs to be "explained away" by newer versions of neo-Darwinian evolution. This may remind us of the many complicated explanations which were required to try to hold on to Ptolemy's geocentric model of the universe (see McGrath, p.18).

The second challenge for the model is the complexity of the human eye. Darwin himself was perplexed about how his model could explain the evolution of the eye. He had faith that with enough future research, his model could explain the evolution of the eye; but as Anderson points out:

> Charles Darwin acknowledged in Origin of Species: "If it could be demonstrated that any complex organ existed, which could not possibly have been formed by numerous successive, slight modifications, my theory would absolutely break down." Darwin went on to add that he could find no such case that would refute his theory (Anderson, p. 82).

When Darwin developed his model, very little was understood about the structure of cells (compared to the current understanding). Behe (p. 6-10) discusses black boxes, devices whose inner workings are mysterious or incomprehensible. In Darwin's time, the cell was a black box, and because of limitations in technology, microscopes were not able to penetrate the mysteries of the cell. Behe notes: "The black box of the cell could not be opened without further technological improvements" (Behe, p. 10).

Perhaps Darwin should be excused for not anticipating the significance of the discoveries that would come with the technological advances in the twentieth century. The black box of the cell was opened, and what was inside led Behe to introduce the concept of *irreducible complexity*:

> By *irreducibly complex* I mean a single system composed of several well-matched, interacting parts that contribute to the basic function, wherein the removal of any one of the parts causes the system to effectively cease functioning. An irreducibly complex system cannot be produced directly (that is, by continuously improving the initial function, which continues to work by the same mechanism) by slight, successive modifications of a precursor system, because any precursor to an irreducibly complex system that is missing a part is by definition nonfunctional (Behe, p. 39).

Based on Darwin's own words, the irreducible complexity of the cell causes his theory to "absolutely break down." As with previous models, the scientific establishment (with naturalistic presuppositions) is very resistant to changing models. A new model based on *intelligent design* is gaining momentum and as further research continues in the area of information theory, Darwin's model (based on Natural Selection) could possibly be enhanced or replaced by a new model based on Intelligent Design.

We should not conclude that Intelligent Design *proves* that evolution never happened; some proponents of Intelligent Design believe that evolution *did* occur. What Intelligent Design *does* conclude is that certain features in creation (like cells) could not have evolved by means of Natural Selection. Because of irreducible complexity, a Designer was needed. What we can say is that with further detailed analysis of extreme data (microscopically small), Darwin's model of evolution by Natural Selection is seen to be inadequate for the understanding of life; it is a model which needs to be refined (and perhaps replaced).

3.5 Summary

Models can be very valuable tools for scientists and mathematicians, but a model should never be equated with the actual phenomenon being modeled (remember the words of Box quoted in Section 3.1). Models are imperfect and frequently require revision (or replacement). When a scientific model seems to contradict a reliable source of information (such as the Bible), we should be hesitant in rejecting the other "reliable source," because we have learned from experience that the scientific model will be refined (and the Bible will stand forever - I Peter 1:24-25).

4 Implications for Teaching, Faith and Learning

"All Scripture is God-breathed and is useful for teaching, rebuking, correcting and training in righteousness, so that the man of God may be thoroughly equipped for every good work." (II Timothy 3:16-17)

4.1 Introduction

Mathematics is foundational in every area of life from financial management to computer security to project management and on and on, especially in the study of the natural sciences. A popular (though somewhat irreverent) adage says:

What is Biology? Applied Chemistry!
What is Chemistry? Applied Physics!
What is Physics? Applied Mathematics!
What is Mathematics? God?

Bradley and Howell summarize Galileo's five properties of mathematics:

1. God has written the book of nature - which is the object of natural philosophy - in the language of mathematics.
2. Man can learn this language.
3. Man can "apply it to the study of nature" due to its logical structure.
4. Handled with care, this language cannot err or go astray.
5. This language is "not only the most certain epistemological tool, but" in fact is "the most perfect one capable of elevating the mind to divine knowledge" [but see Section 4.4 below] (Bradley and Howell, p. 23).

Since mathematics is truly the foundation of the natural sciences, there is the possibility for the temptation of a certain kind of arrogance for a mathematician. According to Pearcey and Thaxton:

> By the end of the eighteenth century, mathematics had become an idol. In the scholarly world it was a matter of faith that the universe was a perfectly running perpetual-motion machine - a view that eliminated the need for God to do anything except perhaps start

> it all off. In epistemology it became likewise a matter of faith that the axiomatic method led to universal and absolute truth - a view that eliminated the need for divine revelation (Pearcey and Thaxton, p. 137).

However, the Incompleteness Theorem of Kurt Gödel, mentioned earlier in Section 1.4, demonstrates that the mathematical foundation has some cracks. The cracks are so substantial that Pearcey and Thaxton entitled the seventh chapter in [17] “The Idol Falls.” Mathematicians should maintain an attitude of humility in the studying, learning and teaching of mathematics.

4.2 More About Models

As mentioned above, models can be very useful in representing various phenomena in the physical sciences. Models are also used in other disciplines such as economics, sociology, psychology, philosophy and even theology. The writer of the letter to the Hebrews seems to indicate that the old covenant is a model of a reality which is revealed by Jesus. The language used in Chapters 8-10 clearly implies that the old covenant is a model of a true reality revealed in the new covenant. The writer of Hebrews is trying to help his readers avoid the *confusion fallacy* mentioned above - confusing the model with real thing. And it is possible that one of the problems for the Pharisees during Jesus’s earthly ministry was that they were holding on to the model (the old covenant) and missing the real thing (Jesus; see John 5:39-47).

4.3 Models and the Authority of Scripture

Because a model is merely a representation of the genuine article, we must guard against the *confusion fallacy*; and we must also be careful about handling situations where the implications of the model appear to be inconsistent with other sources of information.

For example, an interesting (and somewhat troubling) situation occurs when a new scientific model is developed, which seems to describe a certain phenomenon very well, but when generalized to broader implications, provides results which are inconsistent with formerly reliable sources of information (like the Bible). Occasionally (in the past) theologians would employ creative hermeneutics to reinterpret (and perhaps misinterpret) Biblical passages to make scripture appear to be more consistent with the latest scientific *model de jour* while ignoring (or denying) the clear meaning of Scripture.

One possible response to science is to consider certain historical events as allegorical. For example, some theologians reject an historical interpretation for the Book of Jonah. As Grant Jeffrey notes:

> The question all readers must confront is this: Are we to understand the account of Jonah and the great fish literally as an historical account? Or are we to interpret this biblical story as a simple myth or symbolic truth that is not based on the historic truthfulness of the underlying story?
>
> Skeptics usually reject the story of Jonah and the great fish out of hand on the basis that they believe no known sea creature could possibly swallow a man whole, and the survival of such a man for several days is simply beyond the realm of possibility or of

human experience. However, the research ... reveals that both these assumptions are false (Jeffrey, 2002, p. 103-104).

He then goes on to identify (p. 109-110) "two documented historical accounts of people who were swallowed by whales and large fish and then survived the remarkable experience."

So we see that additional scientific research, instead of strengthening the case against an historical interpretation of the Bible, actually confirms the historical interpretation. As Jeffrey observed elsewhere:

> Only fifty years ago many disbelieving scholars totally rejected the historical accuracy of the Bible because they claimed that the Scriptures talked about numerous kings and individuals that could not be confirmed from any other historical or archaeological records. Recent discoveries, however, have shown that they should not have abandoned their faith in the Word of God so easily. If they had only trusted in the truthfulness of the Bible or waited a little longer they would have been rewarded with the recent archaeological discoveries that confirm many biblical details, events and personalities (Jeffrey, 1996, p. 71-72).

He then discussed recent archaeological discoveries confirming some biblical information (e.g. David's kingdom, the walls of Jericho) which were previously considered fictitious by secular scholars.

The point to understand is that the Bible is reliable and can be trusted. We must remember two important facts: 1) Scientific models are imperfect and require frequent revision (and sometimes replacement), and 2) God's Word is perfect and will stand firm forever (see, for example, Matthew 5:18; 24:35; II Timothy 3:16-17; I Peter 1:24-25). There may be many scientific, philosophical and theological reasons for holding various different interpretations of scripture passages; but our hermeneutics should not be dictated by apparent scientific facts based on imperfect man-made models. A different (and possibly superior) approach, as we suggested in Section 1.1, is to interpret the scientific models in light of what we know from Scripture (not the other way around).

4.4 Implications for Humility in Teaching Scientific "Facts"

Returning to Galileo's five properties of mathematics as summarized by Bradley and Howell, we can accept the first three without reservation:

1. God has written the book of nature - which is the object of natural philosophy - in the language of mathematics.
2. Man can learn this language. This is why we study Mathematics.
3. Man can "apply it to the study of nature" due to its logical structure. This is why we study Science.

The fourth can be accepted if we emphasize the phrase "Handled with care":

4. Handled with care, this language cannot err or go astray.

As we have seen, scientific models are not completely reliable and frequently require revision and/or replacement. This is not a bad thing. Because of frequent revisions, scientific models can be improved and more accurately describe the modeled phenomena. But we must maintain an attitude of humility as we present these models to students.

God has revealed Himself to us generally through nature and the natural laws that govern our universe; and we can learn about Him by studying mathematics and science. He has revealed Himself more clearly and more fully through His Word. So this leads to a departure from Galileo. His fifth property of mathematics was

5. This language is not only the most certain epistemological tool, but in fact is the most perfect one capable of elevating the mind to divine knowledge.

As the psalmist said: "To all perfection I see a limit; but your commands are boundless" (Psalm 119:96).

As valuable as mathematics is, it is not the "most perfect" for "elevating the mind to divine knowledge." And as the writer of Hebrews indicated, the most complete revelation of God is found in Jesus Christ:

> In the past God spoke to our forefathers through the prophets at many times and in various ways, but in these last days he has spoken to us by his Son, whom he appointed heir of all things, and through whom he made the universe. The Son is the radiance of God's glory and the exact representation of his being, sustaining all things by his powerful word. After he had provided purification for sins, he sat down at the right hand of the Majesty in heaven (Hebrews 1:1-3).

Mathematics and science can provide valuable information about nature and about God, but the Bible provides more specific, and more perfect information; and the clearest revelation is found in Jesus. These observations should help us as we evaluate and prioritize the various sources of information and the resulting conclusions.

4.5 A Brief Personal Note

When teaching mathematics courses, I always begin the semester by reading and discussing Genesis 1:26-28, and then I remark that the study of mathematics is part of the task of subduing the earth. And to help facilitate an attitude of worship as we study mathematics during the semester, I frequently share a devotional and/or lead a worship song at the beginning of class. I also frequently remind the students that God's Word is the only perfectly reliable source of absolute truth. This does not diminish the importance of studying mathematics and science; but it does provide a reliable framework in which to study the "book of nature" and interpret our conclusions.

5 More Observations About Hermeneutics

"Ah, Sovereign LORD! They are saying of me,

'Isn't he just telling parables?' " (Ezekiel 20:49)

5.1 Hermeneutics - Biblical and Scientific

A current topic of discussion is the age of the universe. There is clear scientific evidence (based on currently accepted scientific models, based on currently accepted assumptions about the nature of space and time) that the age of the universe is between 13 billion and 14 billion years. This estimate is largely based on our understanding of the speed of light and observations of distant galaxies (and how long it takes for the light from the stars to reach earth). There is also clear Biblical evidence that the age of the universe is less than 10,000 years. This estimate is largely based on a literal reading of the first eleven chapters of Genesis. (There are several additional arguments in favor of each estimate.) The challenge is to reconcile the vast difference in these apparent age estimates. Various approaches have been suggested for addressing the conflict. Those who hold to a more literal interpretation of the Bible observe that God could "fill in" the light beams between the stars and the earth (after all, the stars would be useless if the light couldn't be seen). Those who hold to the older age estimate based on the scientific assumptions point out that the Hebrew word for "day," *yom*, need not refer to a literal twenty-four hour day; in fact, the first three days were certainly not "solar" days since the sun was not created until the fourth day. An important observation in this discussion is that both the Biblical information and the scientific information are subject to interpretation.

Biblical hermeneutics is a fascinating and valuable discipline and a useful tool for understanding the depths of the treasures of Scripture. Klein, Blomberg and Hubbard discuss the importance of Biblical interpretation:

> *Hermeneutics* describes the task of explaining the meaning of the Scriptures.... Interpretation is neither an art nor a science; it is both a science and an art.... Hermeneutics provides a strategy that will enable us to understand what an author or speaker intended to communicate (Klein, Blomberg and Hubbard, 2004, p. 4-6).

They also discuss some theories about the location of meaning in a text: in the text itself, in the mind of the reader, or perhaps some combination of the text and the reader.

They then identify four gaps which hermeneutics attempts to bridge (p. 13-16): (1) Distance of Time, (2) Cultural Distance, (3) Geographical Distance and (4) Distance of Language. By better understanding the various components of context (time, culture, geography and language), we can experience new and fresh depths of understanding of God's Word.

John Wesley's view of Scriptural interpretation is summarized by Weeter (2007, p. 194):

> Wesley delineates this principle quite clearly when he states in a letter to Samuel Furley in 1755:
>
> > "The general rule of interpreting Scripture is this: the literal sense of every text is to be taken if it be not contrary to some other texts. But in that case, the obscure text is to be interpreted by those which speak more plainly."

Weeter goes on to clarify (p. 195) that even though Wesley clearly advocated interpreting Scripture literally, he acknowledged that the Bible contains figurative language and symbolic passages; but his position was that we should begin with the literal meaning as a foundation, and then identify the spiritual meaning, the application.

To summarize then, John Wesley advocated for a literal interpretation of Scripture unless such interpretation causes a contradiction with a more clearly understood passage.

5.2 Biblical Examples Incorrectly Rejecting Literal Interpretation

At the beginning of John Chapter 3, we are introduced to Nicodemus. Nicodemus acknowledged that Jesus came from God. But later in Chapter 7, he was faced with a dilemma: Jesus came from Galilee, but Scripture clearly taught that the Messiah would come from Bethlehem (Micah 5:2).

To resolve his dilemma he (apparently) chose to interpret Micah 5:2 figuratively; since Jesus was a descendant of David, and David came from Bethlehem, we could say that Jesus came from Bethlehem (even though he actually came from Galilee).

But Nicodemus would have had no dilemma if he had done more research (like Matthew and Luke did) and found that Jesus was, in fact, born in Bethlehem. But because of his lack of information, he incorrectly adopted a figurative interpretation of Micah 5:2 when a literal interpretation would have been correct.

So also, we should be hesitant to reject a literal interpretation, because even though we may believe our information to be complete, it may not be.

Another example is Peter. Peter is one of the most fascinating characters in the Bible. He sometimes experienced significant victories and made great declarations, and he sometimes experienced significant failures and made great blunders. One of his finest moments is recorded in Matthew 16:16: "You are the Christ, the Son of the living God." Jesus confirmed that Peter was correct in his declaration. Then (v. 21) Jesus described his upcoming suffering and death. This caused a problem for Peter; he knew Jesus was the Messiah, but now Jesus is talking about suffering and death. Jesus often spoke in parables; perhaps this is one of those times. He certainly cannot be speaking literally. Peter's perplexity became so severe that he actually rebuked Jesus; and Jesus's reply is found in verse 23: "Get behind me, Satan! You are a stumbling block to me; you do not have in mind the things of God, but the things of men." Because of Peter's lack of understanding, he adopted a figurative interpretation when a literal interpretation was appropriate. Eventually (after Pentecost) Peter understood that Jesus's suffering and death were part of God's plan for redemption (see Acts 2:23). So again we see that additional information supports a literal interpretation.

5.3 Conclusion

Just as Scripture is subject to interpretation, so also are scientific results. As mentioned above (see [13]), meaning may be (at least partly) created by the reader of the text. So also in scientific investigations, the researcher projects meaning upon the results of the scientific study, such meaning largely influenced by the worldview of the researcher. In light of these observations, much care (even hesitance) should be used in adopting a non-literal interpretation of Scripture if the only reason for such interpretation is the result of a scientific study, or a lack of available information. As we have seen, very often additional information supports a literal view of Scripture. God can be trusted to mean what He says.

As a possible caveat, we should acknowledge that what we understand as "literal" may change from time-to-time with shifts in cultural attitudes or prevailing worldviews. In addition to what we mentioned in Section 4.1, an attitude of intellectual humility is necessary in science and in Biblical interpretation. The point we wish to emphasize is that if a certain passage of scripture seems to require a symbolic or figurative interpretation, it is possible (perhaps highly likely) that more information will indicate that a literal interpretation is appropriate.

6 The Nature of Space - Toward a New Model

"The heavens declare the glory of God;
the skies proclaim the work of his hands." (Psalm 19:1)

6.1 How Old is the Universe?

In this section, we return to the question we addressed in Section 5.1. As we mentioned, there is scientific evidence to support either a young universe (thousands of years) or an old universe (billions of years). We will certainly not resolve this issue, but we may make a small contribution to the discussion. The approach we suggest may be considered somewhat unique in that we will begin with what the Bible says and then propose a different cosmological model. But first, we will review some concepts from topology.

6.2 Topological Considerations

We will begin with some basic Analytic Geometry. A circle is defined as the set of points in a plane which are equidistant from a given point, called the *center*. Algebraically, in the *Cartesian Plane*, we define the *unit circle*, centered at the *origin*, by the equation

$$x^2 + y^2 = 1.$$

The unit circle is a *one-dimensional* concept embedded in a *two-dimensional* concept - the plane.

We can compare this with the *unit disk*, centered at the origin, in the plane, which is given by the inequality

$$x^2 + y^2 \leq 1.$$

The unit disk is a two-dimensional concept. The unit circle is the one-dimensional *boundary* of the two-dimensional unit disk.

Let's move up one dimension. The *unit sphere* is defined by the equation

$$x^2 + y^2 + z^2 = 1.$$

The *unit sphere* is a two-dimensional concept embedded in three-dimensional space. We can compare this with the *unit ball* which is given by the inequality

$$x^2 + y^2 + z^2 \leq 1.$$

which is a three-dimensional concept. The unit sphere is the two-dimensional boundary of the three-dimensional unit ball.

We can continue with examples like this for higher dimensions, but the concepts and the terminology quickly become quite complicated. The study of *manifolds* in Topology addresses many properties of these ideas. The $(n-1)$-dimensional surface of an n-dimensional object is an example of a manifold. We will not go further into the details here except to note that modern cosmological models consider our universe to be like a three-dimensional surface (manifold) in four-dimensional space. We will keep this in mind as we consider some features of a balloon in the shape of a sphere.

6.3 An Expanding Sphere

Consider a balloon in the shape of a sphere. Suppose there is a snail moving along the surface of the balloon. This snail is rather fast for snails - it can move at the high rate of one inch per minute. This velocity is well-established and cannot be exceeded. Now suppose the balloon begins expanding very quickly. A few minutes later, we observe that the snail has moved a total of ten inches from its original position. We conclude that ten minutes have passed, because the "speed of snail" is one inch per minute, and that speed cannot be exceeded. However, an independent observer has informed us that only two minutes have passed. Who do we trust: our knowledge of basic physics, or the independent observer? Because of the expansion of the balloon, our ability to draw conclusions from our observations has been compromised. We know the "speed of snail" and we can see how far the snail has traveled; but the expansion of the balloon has complicated our calculation of elapsed time. The snail was moving at one inch per minute, but because the *medium* (through which the snail was traveling) was stretching, the *apparent* elapsed time was much greater than the *actual* elapsed time. If we are not aware of the expansion of the balloon, we need help from the independent observer to help us determine the actual elapsed time.

6.4 An Expanding Universe

Modern cosmological models accept that our universe is expanding, and that it has been expanding since the moment of the Big Bang. We also understand from Einstein's Theory of Relativity that space can be bent, or curved, by strong gravitational forces. So a possible question to consider is: "Can space be *stretched*?" And is it possible that the complications we encountered with the snail could also complicate our calculations about elapsed time?

There are several Bible verses referring to God *stretching out the heavens*. A few are:

1. Isaiah 42:5
2. Isaiah 44:24
3. Isaiah 45:12

4. Jeremiah 10:12
5. Jeremiah 51:15

We acknowledge that the interpretation we present here for *stretching out the heavens* is not traditional, and we could be accused of speculation and creative hermeneutics; however, since modern cosmological models recognize that the universe is expanding, and since models also recognize that our universe can be considered a three-dimensional manifold in four-dimensional space, the idea of space being *stretched* does not appear to be far-fetched.

If space can be stretched, and if space was, in fact, stretched very rapidly at the moment of the Big Bang, it is likely that our estimates of the time elapsed since the Big Bang would be greatly exaggerated. We need guidance from an Independent Observer if we want to know how long ago the Big Bang happened, and we have One!

6.5 Conclusion

Much more could be said about the relationships between *time* and *expanding space*. The idea that time is *relative* is certainly not new here. At velocities close to the speed of light, the measurement of time can be distorted. The question posed here is whether the stretching of space may also distort the measurement of time. Before the implications of the ideas in this section can be adopted, more research is needed. I am not suggesting that the ideas presented here will resolve all of the apparent conflicts about the age of the universe. But I do want to emphasize that the Final and Perfect Cosmological Model does not yet exist. The search for a Grand Theory of Everything, unifying Quantum Field Theory and General Relativity, has been elusive. And while we should all maintain an attitude of intellectual humility, I would like to suggest (emphatically) that as models get better and better, we may find that a literal reading of the Bible was the best interpretation all along.

7 Conclusion - The Pursuit of Truth

"Then you will know the truth, and the truth will set you free." (John 8:32)

7.1 We Can Know the Truth

Mathematics and science are valuable resources for understanding how the universe works. We should not approach these subjects with fear and trepidation, fearful that they will undermine the authority of Scripture. We do not need to be afraid of the truth. But neither should we bow at the altar of methodological naturalism for the only proper interpretations of scientific information. Rightly interpreted, scientific results and Scripture will be in harmony; all truth is God's truth.

We study mathematics and science, and we obtain valuable information. The scientific revolution, the industrial revolution, and more recently, the information revolution all happened because old scientific models were replaced by newer models with new underlying assumptions - all because somebody studied mathematics and science. But as the past is a clue to the future, these present models, with their assumptions (and with all the success they generated) will eventually be replaced by newer, more complete, models. But we should never expect to acquire a completely perfect scientific model. The only perfect source of information is the Bible.

As we mentioned in the Introduction (Section 1.4), God is extending a wonderful invitation: "Come now, let us reason together." We have many resources available as we endeavor to discover the truth. One of those resources is our God-given ability to reason - and to learn mathematics and science. But the greatest resources God has given are His Word and His Son. As we diligently pursue the Truth, He has promised that He will help us, and He has guaranteed our success.

"You will seek me and find me when you seek me with all your heart." (Jeremiah 29:13)

References

[1] *The Holy Bible: New International Version* (1984). Grand Rapids, MI: Zondervan.

[2] Abbott, Edwin A. (1952). *Flatland*. New York, NY: Dover.

[3] Anderson, Kerby (2008). *A Biblical Point of View on Intelligent Design*. Eugene OR: Harvest House.

[4] Bradley, James & Russell Howell (Eds.). (2011). *Mathematics Through the Eyes of Faith*. New York, NY: HarperCollins.

[5] Behe, Michael J. (1996). *Darwin's Black Box*. New York, NY: Free Press

[6] Cannon, Ann R., Cobb, G. W., Hartlaub, B. A., Legler, J. M., Lock, R. H., Moore, T. L., ... Witmer, J. A. (2019). *STAT2: Modeling with Regression and ANOVA*. New York, NY: MacMillan Learning

[7] Dembski, William A. (2004). *The Design Revolution*. Downers Grove, IL: Intervarsity.

[8] Geisler, Norman L. and Feinberg, P. D. (2002). *Introduction to Philosophy*, Grand Rapids, MI: Baker.

[9] Hawking, Stephen and Mlodinow, L. (2005). *A Briefer History of Time*. New York, NY: Bantam

[10] Howell, Russell and Bradley, J. (2005). *Mathematics in a Postmodern Age*. Grand Rapids, MI: Eerdmans

[11] Jeffrey, Grant (1996). *The Signature of God*. Toronto, ON: Frontier Research Publications

[12] Jeffrey, Grant (2002). *Unveiling Mysteries of the Bible*. Toronto, ON: Frontier Research Publications

[13] Klein, William W., Blomberg, C. L., and Hubbard, R. L. (2004). *Introduction to Biblical Interpretation*. Nashville, TN: Thomas Nelson

[14] Lennox, John C. (2011). *God and Stephen Hawking*. Oxford, England: Lion Books

[15] Lewis, C. S. (1952). *Mere Christianity*. New York, NY: Macmillan Publishing Company

[16] McGrath, Alister E. (2010). *Science & Religion - A New Introduction*. Malden, MA: Wiley

[17] Pearcey, Nancy R. and Thaxton, C. B. (1994). *The Soul of Science*. Wheaton, IL: Crossway

[18] Poythress, Vern (2015). *Redeeming Mathematics*. Wheaton, IL: Crossway

[19] Rucker, Rudolf v.B. (1977). *Geometry, Relativity and the Fourth Dimension*. New York, NY: Dover

[20] Van de Merwe, Willem, (2014). *Impacts of Science on Faith and Society (Draft)*. Unpublished

[21] Weeter, Mark L. (2007). *John Wesley's View and Use of Scripture*. Eugene, OR: Wipf & Stock

[22] Wells, Jonathan (2000). *Icons of Evolution*. Washington, D.C.: Regnery.

Addressing Challenges in Creating Math Presentations

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Abstract

For creators of mathematical content, all slide presentation composition platforms have at least one significant drawback. These can include complex learning curves, inefficient user interfaces, restrictive font selection, limited screen reader compatibility, and sometimes complete lack of math support.

We present various workarounds and solutions for the most popular platforms, including the authors' recently released software tools, SymbOffice and the TeX Math Here add-on for Chrome and Firefox. As a unified solution for math composition in almost any online platform or desktop app, TeX Math Here also has uses that extend beyond just math-centric presentations.

1 Introduction and Context

This paper is part of an ongoing research project addressing various issues that plague the electronic composition of math today. The project's initial focus was creating math slides in PowerPoint, and focus has extended to improving both the math capabilities of Web platforms in general and compatibility with screen readers. This project has also been a viable avenue for highly productive undergraduate research, including both software and peer-reviewed paper publication.

Due to its highly symbolic nature, the typesetting of mathematics has historically been a challenge. As the dawn of the computer age initiated the transition to a digital society, solutions to these challenges were soon forthcoming. Donald Knuth's groundbreaking 1970's research resulted in the revolutionary TeX system [1] whose syntax for electronically typesetting mathematics was so robust, yet intuitive, it is still used today. A decade later, Leslie Lamport's enhancement of TeX (dubbed LaTeX) ultimately became "the *lingua franca* of the scientific world" [2]. These solutions (which, moving forward, will largely be treated as one entity) have now served mathematicians well for decades.

Unfortunately, limitations to this system do still exist.

- As a code-based implementation, a moderately steep learning curve results, especially for fairly standard tasks like page formatting and typesetting non-mathematical text. This learning curve often intimidates potential new users to the point of simple avoidance.
- Because writing mathematical documents was the initial focus of TeX/LaTeX, branching out to other applications (such as presentation slides, spreadsheets, or web platforms) generally requires even more coding knowledge, and thus, greater challenge.

- Because the most common output of TEX/LATEX is PDF files, little flexibility for use in other platforms exists, at least by default.

Shifting focus specifically to presentation slides, several goals are desirable from the start.

- Ease of creation
- Accessibility for both the readers and content creators
- Robust capabilities
- Consistent appearance

For those interested, a further unpacking of these goals is presented in [3].

A Note on Scope
It would be impossible to grant full consideration to every available slide creation platform, including the rather obscure, and maintain a manageable scope for this paper. As such, we will clearly define how we will narrow our focus.

- **Widely adopted tools:** It is difficult to determine the exact market share of the various presentation software composition platforms, but inferences are possible. PowerPoint did claim 95% market share at one time [4]. Even if that number has declined in more recent years, it still held at least a 10:1 advantage over a prominent standalone competitor (Prezi) as recently as 2016 [5]. Also, if considering usage as part of a larger office suite, only two options have a significant market share: Microsoft Office 365 (44.44% combined from online and local installation usage) and Google's GSuite (55.37%) [6]. Therefore, PowerPoint and Google Slides will be the only prominent platforms explored.
- **Free tools:** In an effort to facilitate accessibility for content creators, we will focus only on add-ons for the GSuite and Office 365 platforms that are free; though it should be noted that even prominent commercial products (such as MathType) still suffer from many of the drawbacks we are trying to address.

We realize this narrowing of scope results in the exclusion from extensive discussion of several potentially useful solutions. These would include (but are not limited to) LibreOffice, reveal.js, stack.js, and Beamer. Depending on one's comfort level with learning new platforms and/or new coding processes, as well as any institutional requirements for accessibility, these tools could provide beneficial alternatives for some readers.

1.1 General Deficiencies in Current Popular Platforms

Two significant issues affect the major platforms to varying degrees. Additional platform-specific challenges exist as well and are discussed in Section 1.2.

- **Font selection:** Most computer users are accustomed to switching fonts simply by choosing a new one from some drop down menu. For math content, font switching is a much more involved process, if it is possible at all. Considering that default fonts for text often differ significantly from default fonts for math, creating a consistent appearance between the fonts can be extremely challenging.

- **Screen reader compatibility:** Accessibility is becoming an increasingly important consideration when developing course materials, particularly as it relates to accommodation for disability. Whether due to the presence of students requiring such accommodations or universities proactively adopting institutional requirements for such, screen readers play a significant role in assisting students with visual impairments. While reading most traditional text is relatively straightforward, the same cannot be said for most math content. The most widely accepted accessibility solution for math is MathML; however, MathML code is inherently complicated.[1] Its composition requires yet another platform, and integration with Office 365 and GSuite is very poor anyway. Image alt text is far easier to implement while remaining a perfectly viable solution for screen reader compatibility. Unfortunately, embedding either of these solutions into a PDF (such as those created by LaTeX or Beamer) is still prohibitively difficult; while, at the same time, despite Microsoft's claims of screen reader compatibility with their native math content, both [8] and our own experiences would seem to indicate at least some debate of this point still exists. Lack of accessibility is a significant concern that casts at least some shadow over all current widely adopted implementations.

1.2 Platform-Specific Deficiencies

In addition to the above issues, the major slide creation platforms exhibit other drawbacks as well.

- **Google Slides and PowerPoint Online:** As of this writing, neither of these platforms offers any native math capabilities.
- **Local PowerPoint installation:** For tasks dominated by typing (such as slide creation), interaction with a graphical user interface (GUI) is inherently inefficient and greatly contributes to propensity for user input error [9,10]. Because PowerPoint's math composition is, in general, GUI-driven, these issues are thus present. Going beyond the font selection issues plaguing all platforms, math font selection in PowerPoint is so rigid that creating a consistent visual appearance is nearly impossible (see [3] for examples).

2 A Preliminary Solution

An early result of this project, [3] presents a means of addressing (if not fully meeting) the four initial goals mentioned in Section 1 (ease of creation, accessibility for both the preparer and the reader, robust capabilities, and a consistent appearance). In summary, that solution used a local Windows PowerPoint installation, a free third-party PowerPoint add-in named IguanaTeX [11], and a pair of free fonts (Computer Modern's CMU Serif and Design Science's Euclid Symbol).

IguanaTeX, originally created by Zvika Ben-Haim and currently maintained by Jonathan Leroux, offers two features that really distinguish it from the many other similar offerings that exist (including Microsoft's own math tools).

[1]For example, Presentation MathML requires 56 characters spread across eight lines of code just to produce the extremely simple expression $x + 1$ [7].

- **Direct connectivity in PowerPoint:** There are multiple TEX-based solutions for Microsoft Office (such as TEXsword, LATEX To Word Equation, and Microsoft's own Math Environment); however, attention is predominantly focused on Word. IguanaTeX integrates easily and directly into PowerPoint, a feat with which even Microsoft's own TEX-based tools struggle significantly [12].
- **Editability of TEX-based expressions:** Once the mathematical expression is created from the TEX, it is exceedingly helpful if it can be edited for purposes of both error correction and the relatively quick construction of long derivations using copy/paste/edit. IguanaTEX (unlike some of the alternatives) fully supports editing the TEX code of an existing expression.

While the use of IguanaTEX provided an excellent foundation for meeting our goals, several shortcomings with the solution presented in [3] became apparent with use.

Limitations

- **Inline math symbols:** Because IguanaTEX generates images, placing such inline presents typesetting challenges when editing the surrounding text. We would prefer to easily type symbols if possible.
- **Font and color selection:** While the appearance was consistent (one font used for all math and text), changing that font to another font, or changing its color, was exceedingly difficult and unintuitive, especially to a novice.
- **Screen reader compatibility:** The initial solution addressed one of the many definitions of "accessibility" by providing a no-cost solution for content creators. However, screen reader compatibility was still not present.
- **Platform restrictions:** The solution existed only for a local PowerPoint installation on Microsoft Windows. Though the LATEXiT PowerPoint add-in should suffice for a local PowerPoint installation on Mac, other platforms were not addressed.
- **Local installation of LATEX (and Ghostscript, Image Magick, etc.):** IguanaTEX relies on a local installation (and configuration) of LATEX. Additionally, most common usages of IguanaTEX would also require the installation and configuration of Ghostscript and/or ImageMagick. Other configurations require yet more. When considered as a whole, this process is very user-*un*friendly, and proves prohibitively onerous for many novice users.

The primary focus of the remainder of this paper will be addressing these shortcomings. Section 3 will do so within the PowerPoint/IguanaTEX environment (using workarounds and a new tool named SymbOffice). Section 4 will present another new tool (TEX Math Here) useful for math composition that extends to other platforms, including those with little or no current math support (Office 365 Online, Google Slides, etc.).

3 Enhancing the Preliminary Solution

Because the preliminary solution in Section 2 provides a good starting point, we desired to address as many of its limitations as possible, while also instantly recognizing that certain issues (such as the significant platform restrictions and local LaTeX installation requirements) would forever be beyond our control.

Balancing these considerations prompted an approach that focused more heavily on finding quick LaTeX and PowerPoint-based workarounds, even when enhancement of IguanaTeX was theoretically possible and could have resulted in a far more polished user experience. The appropriateness of this more limited approach to development was later confirmed when an update to one of IguanaTeX's dependencies (Ghostscript, specifically) created the false perception of multiple bugs in IguanaTeX.

Under this more limited approach, some software development did still occur, with the result being a useful macro for Microsoft Office named SymbOffice. Additionally, by ultimately avoiding modification of IguanaTeX itself, resources were freed up, thus leading to the development of TeX Math Here (see Section 4).

3.1 Inline Math Symbols

When one types typical American English text using a standard keyboard, generally all of the symbols one could possibly need are directly accessible via the keyboard. However, the same is not true for math and its hundreds of symbols that could be of interest (for example, $\div$ or π). While these symbols are included in almost all fonts via the Unicode Standard, the text encoding system that has had almost universal acceptance internationally [13], many of the symbols that are used in mathematics are not directly present on keyboards due to space constraints. Thus, while the characters exist, the historical challenge has been in accessing them.

The access method that is likely most familiar to most users is some form of symbol menu, which is opened (and searched through) every time a math symbol is to be inserted by the user. Of course, this GUI-based approach has the inefficiency issues mentioned earlier [9, 10].

A more generalized method for inserting Unicode symbols exists: keyboard codes, which are built into all major operating systems to allow the full Unicode character set to be used. Windows uses a set of alt-codes, which are a combination of holding down the Alt key and then typing a numerical code on the numpad that corresponds to the desired symbol. MacOS uses a similar system with its Option key and alphanumeric codes. While Linux also provides similar functionality, its implementation varies based on distribution and user configuration. Problematically for all platforms, these codes are neither elegant nor effective in their solution when needed on a regular basis. The associated codes are unintuitive to memorize (e.g., π is 227) and attempting to remember the appropriate code can break a user's workflow just as GUI use would. This is particularly an issue when considering the vast number of mathematical symbols that one might need.

TeX (specifically Knuth's Plain TeX) addressed this issue by assigning short, relatively intuitive names (prefaced by a backslash) to just about all of the symbols one could want (for example, `\pi` yields π and `\pm` yields $\pm$). Because these symbol names are far more intuitive and easy to learn, this methodology provides a useful framework for symbol access in any platform, provided it can be implemented.

SymbOffice
By repurposing existing Microsoft Office functionality, we can implement the use of TeX symbol names with relative ease. While most people would typically use autocorrect as a helpful tool for

correcting spelling errors, Microsoft allows user-defined entries into the autocorrect table, and these entries can be populated in such a manner that text strings (specifically, the TEX symbol names) autocorrect to corresponding symbols. Really, only two significant challenges must be overcome.

First, desired Unicode symbols must be entered into the autocorrect table, a process that would typically require knowledge of the numeric codes for all such symbols. Second, additions to the autocorrect table are typically entered one by one, as there is no built-in method to save or restore a custom list. With the sheer number of math symbols one could desire, this process would quickly become quite tedious for some quantity of symbols, as well as necessary on each computer one may desire to use.

SymbOffice [14] is a freely downloadable, macro-enabled Word document that automates much of this setup for a local Microsoft Office installation. It contains a VBA script that, when run, adds over 200 math symbols to the autocorrect table. Because of the tight integration of Microsoft Office's various products, these additions to Word's autocorrect table are automatically transferred (upon Office restart) to Excel, PowerPoint, and Outlook as well. This means inline math symbols can be created in PowerPoint simply by typing the corresponding TEX name followed by a space or punctuation mark. The conversion takes place automatically and is clearly visible to the user.

Because Microsoft's primary objective for autocorrect differs from ours, one peculiarity required a workaround. Symbols that, in TEX, have names that are capitalized (most notably, the capital Greek letters, but also various double arrows) must append the suffix "cap" to see the desired behavior in Microsoft Office. This is necessary because autocorrect, in its primary role as spelling tool, makes no distinction between different capitalizations of the same word. As an example, using SymbOffice (and unlike TEX), both `\Gamma` and `\gamma` would yield γ. If one desires Γ when using SymbOffice, they would need to type `\gammacap` (or `\Gammacap`).

There are several options additional symbols that are not currently included.

- Add the symbol using the standard method for adding autocorrect entries [15].
- Edit the SymbOffice macro by adding a new row to the file's symbol table, populating it with the desired auto correction, and re-running the macro.
- Make requests for symbol inclusion in future updates via the www.mathaddons.com website (or its linked GitHub repository).

3.2 Font and Color Selection

In a recent version of IguanaTEX, functionality was introduced to automatically import the font size information from the user's current text box to IguanaTEX for equation sizing purposes. It was immediately apparent that having similar functionality for the text box's font selection and font color would also be highly desirable. While implementing such technically is possible (thanks to IguanaTEX's open source nature and Microsoft's VBA functionality), significant differences in the way these parameters are handled by LATEX exist. Namely, font size is handled in the command line, while font selection and color are handled in the code's preamble. This difference prevented the solution from being as straightforward as it was for font size, as GUI changes would also be necessitated by the addition of any such feature.

Still, changing these properties remains possible using a code-based solution:

1. Download and install the `mathspec` and `xcolor` LaTeX packages. Doing so in advance will avoid otherwise likely PowerPoint/IguanaTeX program crashes. These occur because if one's LaTeX installation is configured to prompt prior to downloading and installing new packages (as is MiKTeX's default), IguanaTeX suppresses the prompt and IguanaTeX will crash due to timeout. In this scenario, resolution would then only be possible with reconfiguration of LaTeX to download automatically.

2. Change the LaTeX engine used by IguanaTeX to `xelatex`.

3. Add the following code to the preamble of IguanaTeX's code

```
\usepackage{mathspec}
\usepackage{xcolor}
\setallmainfonts{Font Name Here}
\definecolor{mycolor}{RGB}{0-255, 0-255, 0-255}
```

4. Replace "`Font Name Here`" with whatever display name the desired font uses in Windows (for example, "`Times New Roman`").

5. Replace each `0-255` with a number from that range for the red, green, and blue components of the desired color, respectively.

6. Place whatever content should have the new color within the second set of curly braces in the following statement: `\textcolor{mycolor}{Colored Text or Math here}`. The content within the curly braces can include math expressions and span multiple lines.

7. Click the "Make Default" button to avoid the need to perform steps 2 and 3 again with every new expression.

Note that multiple custom colors (for example, `mycolor1`, `mycolor2`, etc.) can be defined and used similarly. Also, as an alternative to defining a custom `mycolor`, LaTeX also recognizes 68 standard color names as detailed in [16]. Figure 1 summarizes steps 2 through 7.

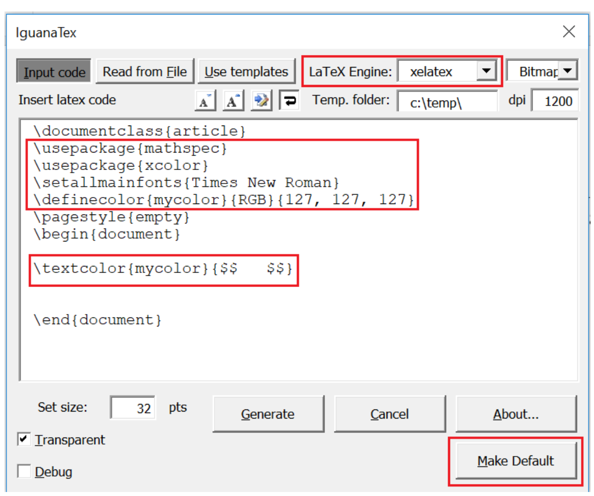

Figure 1: Screenshot of IguanaTeX highlighting the changes necessary to make the default be a math expression (enclosed by the $$'s) in a 50% gray Times New Roman font.

3.3 Screen Reader Compatibility

As a MathML-based approach would be near impossible to implement under this structure, using an image's alt text field is really the only realistic path to solution. Populating this field automatically would be ideal, but similar to the font and color selection issue, doing so necessitates changes to both the GUI and deeper functionality, thus raising issues of project scope. Similarly, a work around will be proposed instead.

However, in order for the question of *how* to populate the alt text field to even be relevant, a decision must first be made regarding with *what* it should be populated. The most straightforward solution (and the one proposed here) would be simply to populate it with the LaTeX code of the original math expression. As a strictly text-based, linear approach to math representation, LaTeX code is acknowledged by the MAA as a perfectly viable solution for facilitating screen reader compatibility [17]. Adding this compatibility manually is possible, as the following illustrates.

1. Create the image with math content using IguanaTeX.
2. Right click the image and select "Format Picture..."
3. In the new toolbar on the right, select the third picture in the top row (labeled with a ToolTip as "Size & Properties").
4. Populate the Title field, Description field, or both with the LaTeX code (or a relevant portion thereof).

Figure 2 summarizes steps 3 and 4.

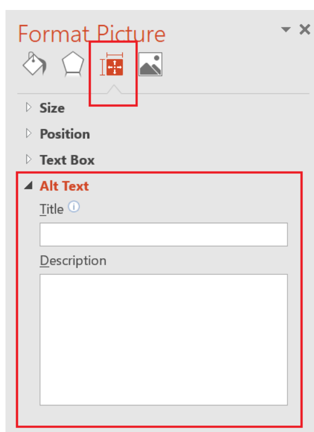

Figure 2: PowerPoint's alt text dialog with highlighting

4 TeX Math Here

As the limitations of the PowerPoint/IguanaTeX solution became more noticeable, it seemed the best approach moving forward was to attempt to build an alternative solution from the ground up. Not only does this provide a framework for implementation of all desired features, but it also addresses some longstanding issues regarding math on the Internet.

TeX Math Here [18, 19] is a browser add-on that converts TeX code to a .png image that can be pasted anywhere images are supported. In fact, even if direct image pasting is not supported by a particular platform (as is the case with Blackboard, among others), the end result can be the same by embedding in the image metadata the URL for a remotely hosted version of the image. TEX Math Here is currently available for Chrome and Firefox, and the images generated can be pasted into not only Web platforms, but into desktop apps as well. This solution introduces math capabilities to Google Slides and PowerPoint Online, and also expands beyond the boundaries of presentation tools to addresses math composition needs in most other platforms as well (Blackboard, the rest of Google Drive's office suite, Microsoft Office Online, etc.).

In addition to providing a single, unified interface that increases the number of Web platforms that one can now use to compose math content, TEX Math Here also relies on an online LATEX renderer, thus eliminating all of the LATEX installation, configuration, and dependency issues discussed earlier, with the trade-off being that an online connection is required for functionality. However, allowing for a local rendering option (with the associated benefits and drawbacks) is a goal for future release.

TEX Math Here also streamlines solutions to the issues that, in the PowerPoint/IguanaTEX solution, required the workarounds outlined in Sections 3.2 and 3.3. In TeX Math Here, the image's underlying TEX code is automatically injected into both the title and the alt text fields of the image's metadata for screen reader compatibility. Additionally, a drop down menu (albeit with rather limited choices currently) is available for choosing a different font.

As of this writing, version 0.7 (screenshot in Figure 3) is available in the Chrome Web Store. Features include support for True Color in a variety of standard math fonts, the automatic saving of font characteristics within a browser session, and keyboard shortcuts that can allow one to avoid mouse use altogether. Mozilla's recent changes to permission handling have delayed version 0.7 release on Firefox, though version 0.6 is still available in Add-Ons. The only practical drawbacks of the earlier version are the loss of color functionality and fewer choices for font selections.

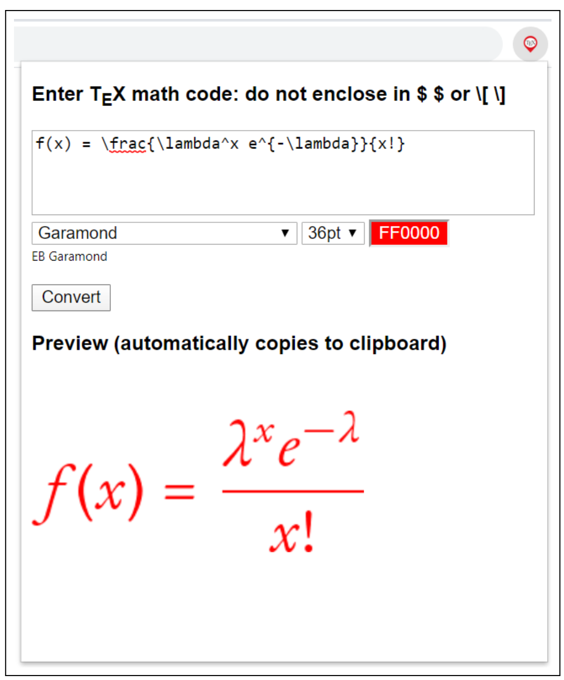

Figure 3: Screenshot of TEX Math Here converting TEX code to a math expression in red 36 point Garamond font. The expression's image is automatically copied to the clipboard for pasting elsewhere.

A Word on MathJax

A very common question raised with regard to this project has been about why MathJax [20] is not part of the solution. While MathJax is an excellent solution for writing mathematical content on the Web, it does have limitations, particularly in interfacing with other platforms. Specifically, the only image type that MathJax is capable of outputting is a scalable vector graphics (.svg) file. While .svg files have a host of advantages over other file formats (including those used by this project), overall support for .svg files is unfortunately rather limited. For example, neither Google Drive's office suite nor Microsoft Office (online or locally installed) support .svg files directly. The same holds true for both of the other output types MathJax can generate (MathML and HTML with CSS). Because of interfacing issues, such support would almost certainly need to come from Google, Microsoft, and/or MathJax developers. As such, implementing a solution based on MathJax is simply well beyond the scope of this team's capabilities.

5 Concluding Remarks

Even though much further development is needed, the early returns of this project have been most promising. Thanks to SymbOffice and some workarounds, a local PowerPoint installation can now provide a combination of efficiency, flexibility, simplicity, and accessibility that any other math presentation composition solution would be hard pressed to match. Additionally, the composition of mathematical content in almost any relevant Web platform or desktop application is now possible with a single browser add-on.

Another significant component of this project has been the ability to fully integrate undergraduates as researchers. The student co-author of this paper developed SymbOffice and the early functional prototype of TeX Math Here during the 2018-19 academic year. Two other undergraduates worked as research assistants over Summer 2019 developing the first fully featured release version of TeX Math Here and laying much groundwork for version 0.7. A fourth student has been enhancing and finalizing version 0.7, as well as configuring our own server during 2019-20.

As mathematics in particular can be a difficult subject in which to integrate undergraduate research, development of useful software tools could lower the barrier to entry for some students. This has required not only computer science knowledge, but also an introduction to a wide variety of other disciplines not typically associated with "math research," such as typography, GUI and graphic design, and accessibility.

6 Acknowledgments

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References

[1] D. Knuth, "Mathematical Typography," *Bulletin (New Series) of the American Mathematical Society*, vol. 1, no. 2, March, pp. 337–372, 1979.

[2] L. Lamport, *LaTeX: A Document Preparation System*, 2$^{\text{nd}}$ ed. Boston: Addison-Wesley, 1994.

[3] D. Schweitzer, "IguanaTEX: LATEX PowerPoint Add-in Greatly Simplifies Creating Elegant Mathematical Slides," *Proceedings of the 30th Annual International Conference on Technology in Collegiate Mathematics*, 2019. Available at https://www.pearson.com/content/dam/one-dot-com/one-dot-com/us/en/files/DavidSchweitzer-SchweitzerICTCM2018.pdf

[4] Robert Gaskins Home Page. Available at https://www.robertgaskins.com/

[5] "How Prezi's Peter Arvai Plans To Beat PowerPoint," *Forbes*. Available at https://www.forbes.com/sites/forbestreptalks/2016/06/07/how-prezis-peter-arvai-plans-to-beat-powerpoint

[6] Market Share Category: Office Suites. Available at https://www.datanyze.com/market-share/office-suites

[7] "Working With MathML," *Wolfram Language and System Documentation System.* Available at https://reference.wolfram.com/language/XML/tutorial/MathML.html

[8] S. Bahram, D. MacDonald, & CB Averitt, "Enabling Math on the Web, in Word & PDF: Emerging Solutions & Overcoming Support Problems," *30th Annual CSUN International Technology and Persons with Disabilities Conference*, 2015. Available: http://www.davidmacd.com/mathml/making-math-accessible-CSUN-2015L.docx

[9] D. M. Lane, H. A. Napier, S. C. Peres, and A. Sandor, "Hidden Costs of Graphical User Interfaces: Failure To Make the Transition from Menus and Icon Toolbars to Keyboard Shortcuts," *International Journal of Human-Computer Interaction*, vol. 18, no. 2, pp. 133–144, 2005.

[10] E. M. Altmann, J. G. Trafton, and D. Z. Hambrick, "Momentary Interruptions Can Derail the Train of Thought," *Journal of Experimental Psychology: General*, vol. 143, no. 1, pp. 215–226, 2014.

[11] IguanaTEX. Available at http://www.jonathanleroux.org/software/iguanatex/

[12] M. Sargent, "LaTeX Math in Office." Available at https://docs.microsoft.com/en-us/archive/blogs/murrays/latex-math-in-office

[13] J. M. Aliprand, "The Unicode Standard," *Library Resources & Technical Services*, vol. 44, no. 3, pp. 160–167, 2011.

[14] SymbOffice. Available at http://www.mathaddons.com/ourtools.html

[15] "Add or remove AutoCorrect entries in Word," *Microsoft Office Support.* Available at https://support.office.com/en-us/article/add-or-remove-autocorrect-entries-in-word-e7433b94-f3de-4532-9dc8-b29063a96e1f

[16] Wikibooks, *LATEX*. Available https://en.wikibooks.org/wiki/LaTeX/Colors

[17] A. Maneki, "Guidelines for Collegiate Faculty to Teach Mathematics to Blind or Visually Impaired Students," *2015 CUPM Curriculum Guide*. Available at https://www.maa.org/sites/default/files/cupm/FacultyGuidelinesForTeachingVisuallyImpairedStudents.pdf

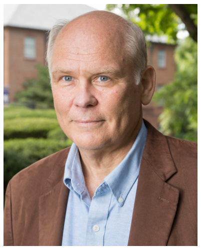
[18] TEX Math Here. Available at http://www.mathaddons.com/ourtools.html

[19] D. Schweitzer, D. Honeycutt, A. Lofgren, and D. Serban, "TEX Math Here: A Simple, Unified, and Efficient Model for Web-based Math Composition," *Electronic Journal of Mathematics and Technology*, vol. 14, no. 1, pp. 50–66, 2020.

[20] MathJax: Beautiful Math in All Browsers. Available at http://www.mathjax.org

A Unifying Project for a TEX/CAS Course

Andrew Simoson (King University)



Andrew Simoson (B.S., Oral Roberts University; Ph.D., University of Wyoming) has taught at King University since 1980. He has published prolifically with the MAA and received two Fulbright awards: to Botswana, and Tanzania. His first ACMS conference was in 1987. The many mathematicians he has met through Bob Brabenac's amazing legacy in this assembly has been a very real blessing, both in friendship and in being mentored as a professor.

Abstract

We describe a CAS and TEX usage course for mathematics majors. As a unifying project, each student selects two primes p and q with $pq < 100$, explores mathematical pq ideas, and generates associated graphs, figures, and tables for a final TEX paper. We summarize several pq explorations: students (1) render page pq from Schwartz's *You Can Count on Monsters*; (2) generate fractal images of pq; (3) generate flowers characterizing $\sqrt{pq}$; and (4) discover the primes of the ring $\mathbb{Z}[\sqrt{pq}\,]$.

1 Introduction

At King, a two-credit, semester-long course in LATEX/*Mathematica* is part of the requirements for the mathematics major. The lion-share of the student-earned grade for the course is a final paper involving various LATEX (a version of TEX) tricks and mathematical algorithms explored during the term. To provide thematic unity in this open-ended project, each student selects a unique pair of different primes $\{p, q\}$ where $pq < 100$. Because $pq = 10$ is used as a classroom example, 30 options for pq remain, ranging from $pq = 6$ to $pq = 94$. Together we explore various mathematical ideas related to pq from across the mathematical curriculum. This article outlines some of these explorations, but first we present a rationale for the course.

2 Purpose of the course

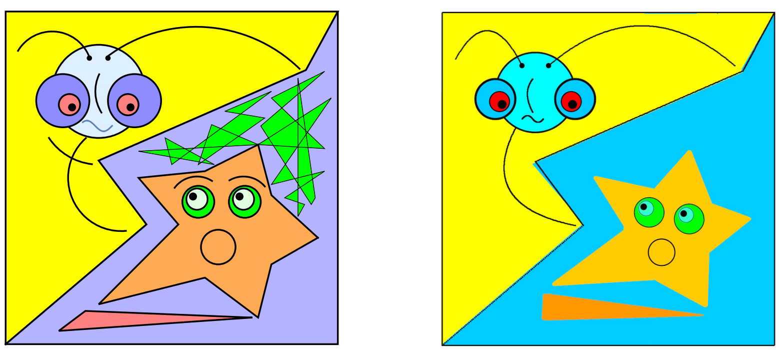
The two-fold objective of the course is to enable students to produce professional-quality mathematical type-set pages and to explore conjectures by generating examples using a computer algebra system (CAS). That is, the student learns the nuts and bolts of mathematical notation in TEX, including the generation of user-defined symbols, tables, graphic viewports, dynamic references for displayed formulas/figures/tables/citations. Students also learn coding tricks for graphics, loops, and recursion.

As a capstone course for seniors and a look-ahead course for juniors, this course explores, reviews, and foreshadows ideas from the entire undergraduate mathematics curriculum. From linear algebra and geometry, we incorporate rotations, reflections, translations, projections, and scalings; from calculus and vector analysis, we incorporate parametric equations and envelopes of families of curves; from analysis, we incorporate fractal generation as fixed points for systems of contractions on the power set of $\mathbb{R}^2$ into itself; from abstract algebra, we search for units, irreducibles, and primes in the ring $\mathbb{Z}[\sqrt{n}]$ where $n \in \mathbb{Z}^+$; and from number theory, we explore Euclid's two-parameter method for generating primitive Pythagorean triples and Gauss's quadratic reciprocity theorem and various continued fraction algorithms. When the course is over, the students should have confidence in their ability to program basic algorithms and to produce documents unhindered by a TeX-related learning curve. In early iterations of this class, students were allowed to choose a final paper topic from anything related to mathematics. However, those papers were usually disjointed weavings of the elements of the course. The *pq* approach we illustrate below allows for a unifying, distinct narrative. With a unique *pq*, each student, somewhat as a musical composer, chains together variations on his or her *pq* theme.

For those interested in adopting a course similar to this one at their institution, we outline several sample ideas to use from among a myriad of *pq* ideas.

3 Exploration 1: A kaleidoscope of geometrical shapes

The children's book *You Can Count on Monsters* by Richard Schwartz is a picture book of primes and the fundamental theorem of arithmetic [5]. From 2 through 97, the primes—called *monsters*—are rendered as distinct geometrical configurations. For instance, the prime 2 appears as an oval with two googley eyes, and 5 is a five pointed star. For each integer n, $1 \leq n \leq 100$, Schwarz has given us a full page collage of the monster primes whose collective product is n. For example, Figure 1a is a *Mathematica* quasi-reproduction of Schwartz's page for $n = 10$, involving overlays of disks, circles, circular arcs, and polygons in the coordinate plane. Although a variety of packages may be imported into LaTeX to enhance graphic capabilities within LaTeX's picture environment, the more primitive version of Schwartz's 10 appearing in Figure 1b was created with the standard picture environment from LaTeX as outlined below.



(a) A *Mathematica* rendition (b) A LaTeX rendition

Figure 1: Versions of Schwartz's image for $pq = 10$

Step 1: Use the *rule* command to create a 2 inch $\times$ 2 inch colored box as background, with its bottom left-hand corner at (0, 0) in the x-y plane.

Step 2: With a lighter color, use the *rule* command to create a series of smaller rectangles, rotated appropriately and positioned atop the background to create a second background above its lower bounding jagged edge which lies in the vicinity of the line $y = x$.

Step 3: With color *white*, use the *rule* command to give the appearance of erasing the secondary background lying outside the boundary of the initial background.

Step 4 and beyond: Monster 2 was created by scaling and layering disks and circles along with *bezier* curves involving *quadratic splines*. Monster 5's body was constructed by scaling and rotating five copies of the symbol ▼ from the AMS's symbol package, along with a disk in the star's center.

4 Exploration 2: A fractal for $pq = 10$

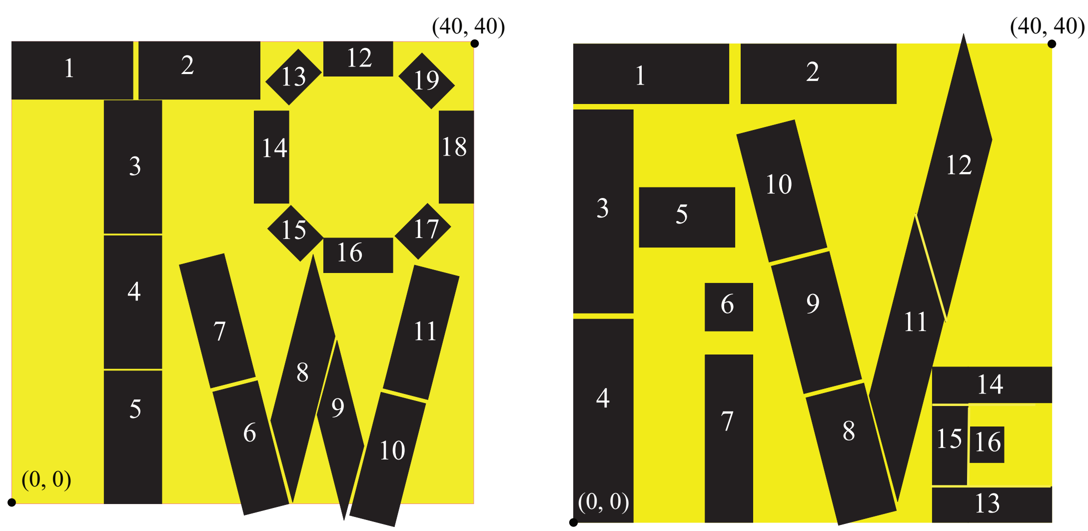
Our next exploration involves a little more mathematics. Define a *contraction* T of $\mathbb{R}^2$ as a function of the form $T(x) = Mx + b$ where M is a 2×2 real-valued matrix, x and b are 2-dimensional vectors, and $||Mx|| < ||x||$ for all $x \neq \mathbf{0}$. We use the following algorithm to construct a fractal whose fundamental building blocks are the words *two* and *five*. To do so we utilize Algorithm 1, adapted from [2, pp. 80–82].

Algorithm 1. A fixed point for a family of contractions
Let $\mathcal{F}$ be a family of integer m contractions T_i, $1 \leq i \leq m$. Then there is a unique non-empty compact set A of $\mathbb{R}^2$ where

$$A = \bigcup_{T \in \mathcal{F}} T(A).$$

Furthermore, given $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$, and $x_{n+1} = T_i(x_n)$ where i is randomly chosen from the index set 1 through m for all $n \geq 0$, then there is an integer N such that for all $n \geq N$, x_n is within ε distance of A.

To implement this algorithm, our first step is to construct two families of contractions that map the rectangle R, with lower-left vertex $(0, 0)$ and upper-right vertex $(40, 40)$, into a set slightly larger than R. The first family consists of 19 contractions, T_i, $1 \leq i \leq 19$, and the second consists of 16 contractions, F_i, $1 \leq i \leq 16$. The images of all $T_i(R)$ is a representation of the word *Two* as a collection of 19 parallelograms shown in Figure 2a. Similarly the images of all $F_i(R)$ is a representation of the word *FiVe* shown in Figure 2b.



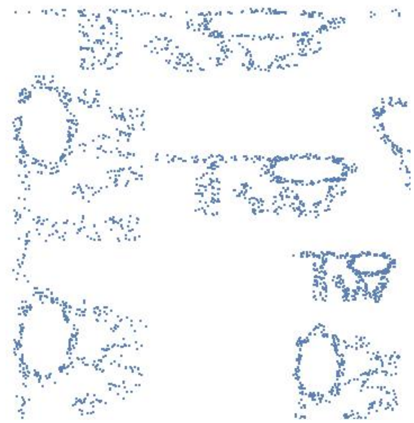
(a) Word *Two* as 19 contractions (b) Word *FiVe* as 16 contractions

Figure 2: The structure for two families of contractions

Our second step is to start with an initial vector such as $x_0 = (1, 1)$, and plot the points $x_{n+1} = T_j \circ F_i(x_n) = T_j(F_i(x_n))$ for all n up to a designated stopping point $n = M$, where, at each step, i and j are randomly chosen from the indices 1 through 19 and 1 through 16, respectively. Plotting

$M = 250$ thousand points gives Figure 3a. Observe that the stem of the letter T in the image of the word *Two* in the figure consists of three copies of *FiVe*. A close-up of the letters *Fi* in the cross-bar of T appears in Figure 3b. Of course, similar *double-minded* fractal images can be generated using any two words. For example, see [6] for an image alternating between the words *yes* and *no*.

(a) The fractal of 250 thousand points



(b) A close-up of the upper left-hand corner

Figure 3: A fractal of *two*'s and *five*'s

5 Exploration 3: A continued fraction flower

The flower in Figure 4 is a representation of the *continued fraction* for $\omega = \sqrt{pq} = \sqrt{10}$, and is similar to a flower representation of the transcendental number e as appears in [7]. This flower consists of successive *corona* layers about a single point—where a *corona* is a finite family of radially symmetric petals, and a *petal* is a scaled single loop in the polar flower $r = \cos m\theta$ with $m \in \mathbb{Q}$. In particular the corona progression for this flower is a single central circular petal, six petals, 37 petals, and 228 petals. These integers are the denominators in the successive *convergents* of the continued fraction for $\sqrt{10}$.

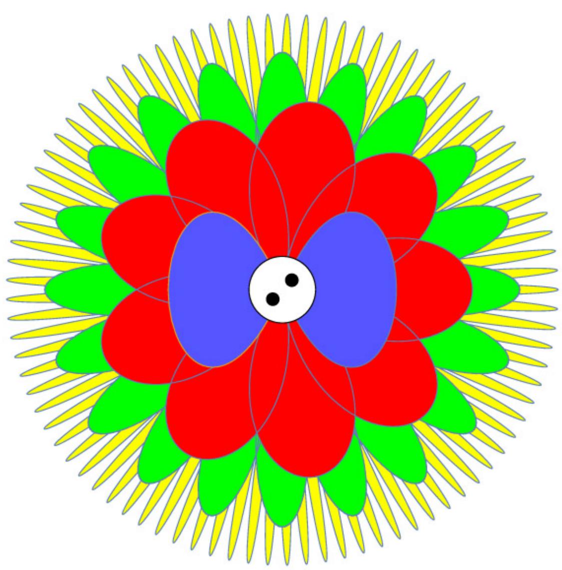


Figure 4: A flower for $\sqrt{pq} = \sqrt{10}$

To explain briefly what is meant by *convergents* of *continued fractions* let ω be a positive irrational number. The standard continued fraction algorithm produces a sequence of *partial denominators* m_i, where m_k is an integer for all integers $k \geq 0$. These partial denominators define a sequence of fractions $C_k = \frac{p_k}{q_k}$ called *convergents* that converge to ω, where p_k and q_k are relatively prime integers for all k. That is,

$$C_0 = m_0, \quad C_1 = m_0 + \frac{1}{m_1}, \quad C_2 = m_0 + \frac{1}{m_1 + \frac{1}{m_2}},$$

and so on. By convention, we write $\omega = [m_0; m_1, m_2, \ldots]$. To generate partial denominators and convergents recursively we use the next algorithm, a combination of several theorems appearing in many elementary number theory texts such as Rosen [4, pp. 485–487]. The convergents C_{-2} and C_{-1} are called *preconvergents*. The convergent C_{-1} is a symbol with numerator 1 and denominator 0.

Algorithm 2. A recursion in convergents
Let ω be a positive irrational number. Then ω's partial denominators m_k, for all integers $k \geq 0$, and ω's convergents $C_k = \frac{p_k}{q_k}$, for all $k \geq -2$, are defined by

$$\begin{array}{ll} p_{-2} = 0, & q_{-2} = 1, \\ p_{-1} = 1, & q_{-1} = 0, \\ p_k = m_k p_{k-1} + p_{k-2}, & q_k = m_k q_{k-1} + q_{k-2}, \end{array} \tag{1}$$

where $m_k = \lfloor t \rfloor$ and $t \in \mathbb{R}$ is the solution to the equation

$$\omega = \frac{tp_{k-1} + p_{k-2}}{tq_{k-1} + q_{k-2}} = tC_{k-1} \oplus C_{k-2}.$$

Furthermore, $gcd(p_n, q_n) = 1$ and $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ for all $n \geq 0$. □

Since

$$\sqrt{10} = tC_{-1} \oplus C_{-2} = \frac{t \cdot 1 + 0}{t \cdot 0 + 1} = t,$$

then $m_0 = \lfloor \sqrt{10} \rfloor = 3$ and $C_0 = 3C_{-1} \oplus C_{-2} = \frac{3}{1}$. Solving

$$tC_0 \oplus C_{-1} = \sqrt{10} = \frac{t \cdot 3 + 1}{t \cdot 1 + 0} = \frac{3t + 1}{t}$$

gives $t = \frac{1}{\sqrt{10}-3} \approx 6.16$, which means $m_1 = 6$ and $C_1 = 6C_0 \oplus C_{-1} = \frac{19}{6}$. Continuing in this vein gives $\sqrt{10} = [3; 6, 6, 6, \ldots]$, and

$$C_0 = 3, C_1 = \frac{19}{6}, C_2 = \frac{117}{37}, C_3 = \frac{721}{228}, C_4 = \frac{4443}{1405}, C_5 = \frac{27379}{8658}. \tag{2}$$

The significance of the three dots in the central circular corona is that $\lfloor \sqrt{10} \rfloor = 3$; with this information, it is a fun exercise to recover the numerators of the successive convergents for $\sqrt{10}$ from Figure 4 alone.

6 Exploration 4: The morphology of $\mathbb{Z}[\sqrt{pq}\,]$

The ring $\mathbb{Z}[\sqrt{n}\,]$ is the set of all numbers of the form $a + b\sqrt{n}$ where a and b are integers and n is a fixed positive integer. Each student's quest is to identify the units of their $\mathbb{Z}[\sqrt{pq}\,]$ (numbers having multiplicative inverses in $\mathbb{Z}[\sqrt{pq}\,]$), explore irreducibles (numbers that have no divisors except themselves and units), and search for primes (a number that must divide either x or y whenever it divides the product xy).

When $pq = 10$ the result of our search appears in Figure 5, where the irreducibles are represented by circles, primes by black disks, and units by gray disks. As useful terminology, we say that the *conjugate*, sometimes called a *surd conjugate* [1], of a number $w = a + b\sqrt{n}$, denoted $\overline{w}$, is $a - b\sqrt{n}$. The labels at points $w = a + b\sqrt{10}$ are the integers $w\overline{w}$. We outline this analysis below.

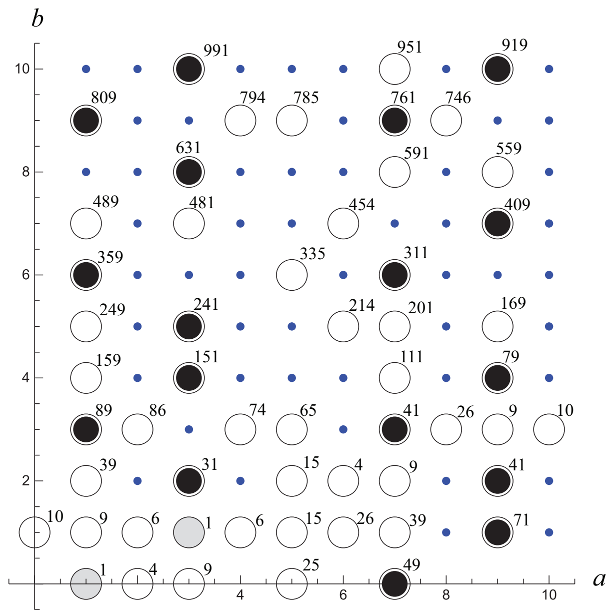


Figure 5: The morphology of $\mathbb{Z}[\sqrt{10}]$

6.1 Step 1: Identifying the units

Students find their units by adapting the following algorithm to $\mathbb{Z}[\sqrt{pq}\,]$.

Algorithm 3. Units as integer powers of $3 + \sqrt{10}$

Let $C_n = \frac{p_n}{q_n}$ be convergent n for the regular continued fraction of $\sqrt{10}$. Then for $n \geq 0$,

$$p_n + q_n\sqrt{10} = (3 + \sqrt{10}\,)^{n+1}, \quad p_n^2 - 10q_n^2 = (-1)^{n+1}, \quad \text{and} \quad p_n p_{n-1} - 10 q_n q_{n-1} = 3(-1)^n.$$

Furthermore, the only units $p + q\sqrt{10}$ with $p,\, q \geq 0$ in $\mathbb{Z}$ are $p_n + q_n\sqrt{10}$ for all $n \geq -1$.

Proof. *By Algorithm 2 and (2), $C_0 = 3C_{-1} \oplus C_{-2}$ and $C_n = 6C_{n-1} \oplus C_{n-2}$ for integers $n \geq 1$. A mathematical induction argument yields the validity of the three displayed formulas, the second of which shows that $p_n + q_n\sqrt{10}$ is a unit in $\mathbb{Z}[\sqrt{10}\,]$.*

Let a and b be positive integers. The only integers that are units are ± 1, and no number of the form $b\sqrt{10}$ is a unit. In searching for other units in $\mathbb{Z}[\sqrt{10}\,]$ we may assume that $\gcd(a,\, b) = 1$.

Suppose that $|a - b\sqrt{10}\,| > \frac{1}{2b}$. *Then* $|a^2 - 10b^2| = (a + b\sqrt{10}\,)|a - b\sqrt{10}\,| > (b\sqrt{10}\,)\frac{1}{2b} = \frac{\sqrt{10}}{2} > 1$. *So* (a, b) *is not a unit. Suppose* $|a - b\sqrt{10}\,| < \frac{1}{2b}$ *and* $a + b\sqrt{10}$ *is a unit. Since* $|\frac{a}{b} - \sqrt{10}\,| < \frac{1}{2b^2}$ *is Dirichlet's well-known criterion (see [4, pp. 499–500]) for* $\frac{a}{b}$ *to be a convergent of the continued fraction for* $\sqrt{10}$, *then the only units* $a + b\sqrt{10}$ *where* a *and* b *are non-negative integers are those for which* $a = p_n$, $b = q_n$, *and* $C_n = \frac{p_n}{q_n}$ *is a convergent for* $\sqrt{10}$. *So the units of* $\mathbb{Z}[\sqrt{10}\,]$ *are* $\pm(p_n \pm q_n\sqrt{10})$. □

6.2 Step 2: Identifying the irreducibles

We make the following observations.

Lemma 1. A few properties of $\mathbb{Z}[\sqrt{\boldsymbol{n}}\,]$
Let n *be a square free positive integer. Let* α *and* ω *be nonzero numbers in* $\mathbb{Z}[\sqrt{n}\,]$. *Let* $a, b, x, y \in \mathbb{Z}$. *Then*

i. *If* $\alpha|\omega$ *then* $(\alpha\overline{\alpha})|(\omega\overline{\omega})$.

ii. *If* $\alpha|\omega$ *and* $\alpha\overline{\alpha} = \pm(\omega\overline{\omega})$ *then* $\omega = \alpha u$, *where* u *is a unit.*

iii. *Let* $m = \alpha\omega$ *where* m *is an integer,* $\alpha = a + b\sqrt{n}$, $\omega = c + d\sqrt{n}$ *with* a, b, c, d *non-zero integers, and* $gcd(a, b) = 1 = gcd(c, d)$. *Then* α *is either* $\overline{\omega}$ *or* $-\overline{\omega}$.

Proof. For part (i), $\alpha|\omega$ means that there exists γ in $\mathbb{Z}[\sqrt{n}\,]$ with $\alpha\gamma = \omega$. Thus $\overline{\alpha\gamma} = \overline{\omega}$. So $\alpha\overline{\alpha}\gamma\overline{\gamma} = \omega\overline{\omega}$, which gives the desired conclusion. For part (ii), let $\alpha = a + b\sqrt{n}$ and $\omega = x + y\sqrt{n}$. Consider the equation $\frac{\alpha}{\omega} = \gamma$ where $\gamma = p + q\sqrt{n}$ for some numbers p, q in $\mathbb{Q}$. Since $a = px + nqy$ and $b = py + qx$ then $p = \frac{ax - nby}{x^2 - ny^2}$ and $q = \frac{-ay + bx}{x^2 - ny^2}$. Thus

$$\gamma \cdot \overline{\gamma} = p^2 - nq^2 = \left(\frac{ax - nby}{x^2 - ny^2}\right)^2 - n\left(\frac{-ay + bx}{x^2 - ny^2}\right)^2 = \frac{a^2 - nb^2}{x^2 - ny^2} = \frac{\alpha\overline{\alpha}}{\omega\overline{\omega}} = \pm 1.$$

Therefore the inverse of γ in $\mathbb{Q}[\sqrt{n}\,]$ is $\pm\overline{\gamma}$. Since $\frac{\omega}{\alpha} \in \mathbb{Z}[\sqrt{n}\,]$, then

$$\frac{\omega}{\alpha} = \frac{1}{\gamma} = \pm\overline{\gamma}.$$

So $\overline{\gamma}$ belongs to $\mathbb{Z}[\sqrt{n}\,]$, which means that γ does as well. Thus p and q are integers and γ is a unit in $\mathbb{Z}[\sqrt{n}\,]$.

For part (iii), $m = (ac + 10bd) + (ad + bc)\sqrt{n}$, which means that $m = ac + nbd$ and $ad = -bc$. Since $a = -\frac{bc}{d}$ and $\gcd(c, d) = 1$, $d|b$. Similarly, with $c = -\frac{ad}{b}$, then $b|d$. Thus $b = d$ or $b = -d$. When $b = d$ then $a = -c$, and when $b = -d$ then $a = c$. □

To search for their irreducibles, students appropriately modify this next proposition.

Proposition 2. Steps towards irreducibility
Let a, b, *and* d *be positive integers where* $d = |a^2 - 10b^2|$. *Let* $\alpha = |3a - 10b|$, *and* $\beta = |a - 3b|$, *and* $\delta = a - b\sqrt{10}$. *Then*

i. $|\alpha^2 - 10\beta^2| = d$.

ii. *If* $|a - b\sqrt{10}| > \frac{1}{2}$ *then* $0 < b < \frac{2d}{\sqrt{10}}$ *and* $0 < a < 2d - b\sqrt{10}$.

iii. *If* $|a - b\sqrt{10}| < \frac{1}{2}$ *then* $0 \leq \alpha < a$, *and* $0 \leq \beta < b$ *and* $a + b\sqrt{10} = (\alpha + \beta\sqrt{10})u$ *where* u *is a unit.*

iv. *If* p *is prime in* $\mathbb{Z}$ *and the last decimal digit of* p *is not 1 or 9, then* p *is irreducible in* $\mathbb{Z}[\sqrt{10}\,]$.

Proof. For (i),

$$|\alpha^2 - 10\beta^2| = |(3a - 10b)^2 - 10(a - 3b)^2| = | - a^2 + 10b^2| = d.$$

For (ii), $|\delta| > \frac{1}{2}$. Therefore

$$2d = 2|a^2 - 10b^2| = 2(a + b\sqrt{10}\,)|a - b\sqrt{10}\,| > a + b\sqrt{10},$$

which gives the stated bounds on a and b.

For (iii), $|\delta| < \frac{1}{2}$ and $a \geq 3$. Therefore

$$\alpha = |10b - 3a| = a\left|\frac{\sqrt{10}(b\sqrt{10} - a)}{a} + \sqrt{10} - 3\right| \leq a\left|\sqrt{10} - 3 + \frac{|\delta|\sqrt{10}}{a}\right| < a\left|\sqrt{10} - 3 + \frac{\sqrt{10}}{6}\right| < a,$$

and

$$\beta = |a - 3b| = b\left|\sqrt{10} - 3 + \frac{a - b\sqrt{10}}{b}\right| \leq b\left|\sqrt{10} - 3 + \frac{|\delta|}{b}\right| < b\left|\sqrt{10} - 3 + \frac{1}{2}\right| < b.$$

Finally, observe that $(a + b\sqrt{10}\,)(3 - \sqrt{10})) = (3a - 10b) + (-a + 3b)\sqrt{10} = \pm\alpha \pm \beta\sqrt{10}$ and that $3 - \sqrt{10}$ is a unit.

For (iv), let p be a prime in $\mathbb{Z}$. Suppose that it is reducible in $\mathbb{Z}[\sqrt{10}\,]$. By Lemma 1.iii, $p = \pm(a + b\sqrt{10}\,)(a - b\sqrt{10}\,)$ for some integers a, b, c, d. Then $p \equiv \pm(a^2 - 10b^2) \equiv \pm a^2 \bmod 10$. Since the last decimal digit of the square of an integer is either 0, 1, 4, 5, 6, or 9, then any prime in $\mathbb{Z}$ that ends in 2, 3, or 7 is irreducible in $\mathbb{Z}[\sqrt{10}\,]$. For example, 2, 3, 7, 17, and 23 are irreducible.

To see that 5 is also irreducible, suppose that 5 is reducible. Let A be the set of all solutions (a, b) to $|x^2 - 10y^2| = 5$ where $|a - b\sqrt{10}| > \frac{1}{2}$. By (ii), a necessary condition for membership in A is either $b = 1$ and $a \in \{1, 2, 4, 5, 6\}$, or $b = 2$ and $a \in \{1, 2, 3\}$. Checking all possibilities shows that $|a^2 - 10b^2| \neq 5$ for $(a, b) \in A$. So $A = \emptyset$. Suppose a solution to $|x^2 - 10y^2| = 5$ exists where $|x - y\sqrt{10}\,| < \frac{1}{2}$. Let $a_0 > 0$ and $b_0 > 0$ be that solution where b_0 is least. By (iii), let $\alpha = |3a_0 - 10b_0|$ and $\beta = |a_0 - 3b_0|$. Observe that $|\alpha^2 - 10\beta^2| = 5$ and so $\alpha + \beta\sqrt{10} \in A$, a contradiction. Therefore, 5 is irreducible in $\mathbb{Z}[\sqrt{10}\,]$. $\square$

To determine their irreducibles of the form $a + b\sqrt{10}$, $b \neq 0$, students generalize the following example.

Example 1. The number $2 + \sqrt{10}$ is irreducible in $\mathbb{Z}[\sqrt{10}\,]$.

Let $\alpha = 2 + \sqrt{10}$. *Then* $|\alpha\overline{\alpha}| = 6$. *Let* β *be a divisor of* α. *By Lemma 1.i,* $\beta\overline{\beta}|6$. *The positive integer divisors of 6 are 1, 2, 3, and 6. If* $|\beta\overline{\beta}| = 1$, *then* β *is a unit by definition. By Proposition 2.iv,* $|x^2 - 10y^2| = d$ *has no integer solutions when* d *is 2 or 3. If* $|\beta\overline{\beta}| = 6$, *by Lemma 1.ii,* $\alpha = \beta u$ *for some unit* u. *Thus* $2 + \sqrt{10}$ *is irreducible.*

6.3 Step 3: Identifying the primes

The integers 2 and 3 are not prime in $\mathbb{Z}[\sqrt{10}\,]$ because 2 and 3 both divide $6 = (-2+\sqrt{10}\,)(2+\sqrt{10}\,)$, but not $-2+\sqrt{10}$ or $2+\sqrt{10}$. Similarly, 5 is not prime because $5|65$ but neither $5+3\sqrt{10}$ nor $5-3\sqrt{10}$. However 7 is prime, as shown in the next example. We say that the positive integer $m < n$ is a *primitive quadratic residue* of positive integer n if there is a positive integer $p < n$ with $p^2 \equiv m \bmod n$, and m is a *primitive quadratic non-residue* of n if no such p exists, (see [3, p. 53]).

Example 2. The integer 7 is prime in $\mathbb{Z}[\sqrt{10}\,]$.

The primitive quadratic residues of 7 form the set $A = \{1, 2, 4\}$ and the primitive non-residues of 7 form the set $B = \{3, 5, 6\}$. Suppose that $7|\alpha\beta$ where α and β are in $\mathbb{Z}[\sqrt{10}\,]$. By the fundamental theorem of arithmetic for $\mathbb{Z}$ we may assume without loss of generality that $7|\alpha\overline{\alpha}$, and—after eliminating some trivial cases—that $\alpha = a + b\sqrt{10}$ for some positive integers $a < 7$ and $b < 7$ where $a^2 \equiv 10b^2 \bmod 7$, which means that some quadratic residue of 7 is equal to 3 times a quadratic residue of 7. But $3A \equiv B \bmod 7$. Thus there are no such numbers a and b.

Example 2 generalizes nicely: the prime number p in $\mathbb{Z}$ is also prime in $\mathbb{Z}[\sqrt{10}\,]$ if and only if 10 is not a quadratic residue of p. Showing that numbers of the form $a + b\sqrt{10}$ are prime where a and b are positive integers involves searching a somewhat large space, as we illustrate, an argument that the students may generalize.

Example 3. The number $3 + 2\sqrt{10}$ is prime in $\mathbb{Z}[\sqrt{10}\,]$.

Let $(3 + 2\sqrt{10}\,)|(p + q\sqrt{10}\,)$ where $p + q\sqrt{10} = (a + b\sqrt{10}\,)(c + d\sqrt{10}\,)$ and a, b, c, d, p, q are integers. After eliminating the trivial case, we may assume that a, b, c, d are all at least 1 and no more than 30. Since $|3^2 - 10 \cdot 2^2| = 31$, a prime in $\mathbb{Z}$, then $31|(a^2 - 10b^2)$ or $31|(c^2 - 10d^2)$. So we may also assume that $\gcd(a, b) = 1 = \gcd(c, d)$ and that either $(a^2 - 10b^2) \bmod 31 \equiv 0$ or $(c^2 - 10d^2) \bmod 31 \equiv 0$. Let $\mathcal{F}$ be the family all such sets of four numbers a, b, c, d, denoted $\{(a, b), (c, d)\}$. Using a series of four nested loops as each of these four variables range from 1 to 30, we can test all possible 30^4 sets $\{(a, b), (c, d)\}$, and so build $\mathcal{F}$, which we may view as a list of sets. In order to avoid redundancy when generating $\mathcal{F}$, we ignore any $\{(a, b), (c, d)\}$ for which $\{(c, d), (a, b)\}$ already belongs to $\mathcal{F}$. Once built, $\mathcal{F}$ has 9873 members, among which are these 13 members, the first few from $\mathcal{F}$'s beginning and the last few from $\mathcal{F}$'s ending:

$$\{(1, 2), (17, 1)\}, \{(1, 3), (3, 2)\}, \{(1, 3), (17, 1)\}, \{(1, 4), (3, 2)\}, \{(1, 4), (17, 1)\},$$
$$\{(1, 4), (20, 3)\}, \{(1, 5), (3, 2)\}, \ldots, \{(30, 29), (22, 25)\}, \{(30, 29), (23, 5)\},$$
$$\{(30, 29), (25, 27)\}, \{(30, 29), (26, 7)\}, \{(30, 29), (28, 29)\}, \{(30, 29), (29, 9)\}.$$

The first of these sets in $\mathcal{F}$ is $\{(1, 2), (17, 1)\}$, which corresponds to the product $(1 + 2\sqrt{10}\,)(17 + \sqrt{10}\,)$ which is divisible by $3 + 2\sqrt{10}$. While the former number $1 + 2\sqrt{10}$ fails to be divisible by $3 + 2\sqrt{10}$, the latter number is divisible by $3 + 2\sqrt{10}$:

$$\frac{17 + \sqrt{10}}{3 + 2\sqrt{10}} = -1 + \sqrt{10}.$$

Each member of $\mathcal{F}$ corresponds to a product of two numbers in $\mathbb{Z}[\sqrt{10}\,]$ which is divisible by $3+2\sqrt{10}$, and at least the former or the latter of the two numbers is divisible by $3+2\sqrt{10}$. Therefore, $3+2\sqrt{10}$ is prime in $\mathbb{Z}[\sqrt{10}\,]$. A similar argument shows that $3 - 2\sqrt{10}$ is also prime.

Likewise, the primeness question for each irreducible number $w = a + b\sqrt{10}$ can be determined, albeit as $|w\overline{w}|$ increases, so also does the corresponding computer verification time. For more on these ideas, see [8].

7 Conclusions

I have taught this course approximately every third semester for the last twenty plus years. For variety and interest, both on the students' part and my part, each time through the course is different. Serendipity plays a part in topic selection. For example, in the spring 2019 class, I discovered that one of my students loves drawing flowers—and so we all generated flowers similar to Figure 4; and since most of my students in the class were concurrently taking abstract algebra, the students generated figures like Figure 5 for the ring structure of their own $\mathbb{Z}[\sqrt{pq}\,]$. When at all possible, I try to incorporate material from the courses my students have either taken or are currently taking. My main reason for adopting the *pq* format as described in this paper, is that, from my experience, students often have difficulty devising open-ended, sufficiently interesting examples on their own. So why not limit the assignment scope, and allow for student creativity on the details? Will I use this same format next time around? Fortunately, there are plenty of algorithms involving pairs of integers, so the material for this format can ever remain fresh for the instructor. What about student response to the course? Writing proofs and writing code are different skills. The focus of this course is on the latter. Some students become frustrated with tracking down bugs in their CAS or LaTeX code. Yet they all seem pleased—even proud—after producing a final paper that exhibits their work at semester end. I close with a typical student's evaluation of the course.

> The course has been great and went by extremely fast. Not only have I learned how to do proper coding, but I have learned how to step back and re-evaluate situations good and bad. Most assignments we turned in were really cool because they allowed for creativity in *e*-matics.

References

[1] E. J. Barbeau, *Pell's Equation*, Springer Press, 2003.

[2] M. Barnsley, *Fractals Everywhere*, Academic Press, 1988.

[3] U. Dudley, *A Guide to Elementary Number Theory*, Mathematical Association of America, 2009.

[4] K. H. Rosen, *Elementary Number Theory and its Applications*, Addison Wesley Press, Boston, 2011.

[5] R. E. Schwartz, *You Can Count on Monsters*, American Mathematical Society, 2010.

[6] A. Simoson, A double-minded fractal, PRIMUS 19 (2009) 381–387.

[7] A. Simoson, A phyllotaxis of the irrational, *Math Horizons* 26:3 (2019) 10–13.

[8] A. Simoson, The morphology of $\mathbb{Z}[\sqrt{10}\,]$, *Mathematical Gazette*, 103:558 (2019) 442–460.

Is Mathematical Truth Time Dependent?
Comments from a Paper by Judith Grabiner

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As a mathematics major at Wheaton College, Dick was among Bob Brabenec's first group of students. Going on to receive a Ph.D. at Penn State, Dick has taught mathematics at Gordon College since 1980. He and his wife, Martha, live in Ipswich, Massachusetts, and in their free time enjoy traveling, wildlife viewing, and rooting for the Red Sox. Dick has been an active member of ACMS since the first gathering in 1977.

Abstract

Judith Grabiner's paper "Is Mathematical Truth Time Dependent?" appeared in the April, 1974 issue of the *Monthly.* Although the paper was published forty-five years ago, the topic is relevant today, especially as we consider questions about how to integrate ideas about mathematical truth with a Christian faith. Focusing on the work done in the eighteenth and nineteenth centuries to make the foundations of analysis more rigorous, Grabiner explores what it means to say that our ideas about truth in mathematics can change over time. In this paper we summarize and react to some of her ideas, including her intriguing suggestion about why this project in analysis was originally undertaken.

In his book *Soul Survivor*, Philip Yancey profiles thirteen people who, through their lives and written work, have mentored him on his Christian journey. One such mentor is the English writer G. K. Chesterton, who, as Yancey points out, was fascinated by both the problem of pain and its opposite, the problem of pleasure. As Yancey writes, "It struck me, after reading my umpteenth book on the problem of pain, that I have never even seen a book on 'the problem of pleasure.' Nor have I met a philosopher who goes around shaking his or her head in perplexity over the question of why we experience pleasure. Yet it looms a huge question; the philosophical equivalent, for atheists, to the problem of pain for Christians." [5, pp. 53, 54]. Yancey then recounts how finding an answer to the problem of pleasure helped bring Chesterton back to faith, realizing that a good God would want us to experience pleasure

As I read that passage I thought about things that have given me pleasure and mathematics is definitely on the list. For some of my students and most of my acquaintances, such a claim probably sounds very strange. They would likely include mathematics on the "pain" list rather than the "pleasure" list. That isn't to say that doing mathematics is without any pain whatever. Solving problems or understanding and proving a theorem can be accompanied by false starts, some head scratching, discouragement, and maybe even down time to just sit and ponder, any one of which is not necessarily pleasurable. But when the problem is solved or the theorem is proved, all the painful work often results in a profound sense of satisfaction and yes, pleasure. In the BBC video about Andrew Wiles and his efforts to prove Fermat's Last Theorem, Wiles shows genuine emotion as he describes the incredible beauty and satisfaction he experienced during his seven-year journey to find the proof, even as he went through some well-publicized painful times after discovering an error in the original argument. Although solving the problem made him famous, what seemed most rewarding was his pleasure in observing the intrinsic beauty of the subject matter he worked with during his seven-year journey.

Why does mathematics have this effect? Although finishing any project brings satisfaction, the pleasure I'm referring to goes deeper than that. What, for example, attracts us to continue studying mathematics long after we've learned all the algorithms we need for our daily lives? I doubt the answer is just because we can then solve more practical problems. After all, we rarely encounter practical situations where a knowledge of group theory or hyperbolic geometry is helpful. Perhaps we do it because understanding mathematics is deeply rewarding and brings a great deal of pleasure. And that pleasure comes, at least in part, from working with material that is not part of our normal experience. Doing mathematics gives us an opportunity to glimpse abstract and eternal truths. I recall having an epiphany one evening in graduate school, when it suddenly occurred to me that doing mathematics was like shoveling a sidewalk after a snowstorm. Confronted with a blanket of snow that covers everything, our job is to determine how to find and uncover the sidewalk that lies underneath the snow.

Over the years I've taken comfort that my sense of discovery is shared by others in the community. For instance, Andrew Wiles describes the process of doing mathematics with the metaphor of entering a dark room and searching for the light switch. Once that is found and he knows where all the furniture is, he moves on to the next room. Barry Mazur offers the following as part of his response to the big question of whether mathematics is invented or discovered. Mazur states: "The bizarre aspect of the mathematical experience—and this is what gives such fierce energy to The Question—is that one feels (I feel) that mathematical ideas can be hunted down, and in a way that is essentially different from, say, the way I am currently hunting the next word to write to finish this sentence" [3, p. 10]. At the conclusion of that paper Mazur discusses the morass of being caught up in philosophical arguments. But then returns to the practice of doing mathematics: "Happily I soon snap out of it and remember again the remarkable sense of independence—autonomy even—of mathematical concepts, and the transcendental quality, the uniqueness—and the passion—of doing mathematics. I resolve then that (Plato or Anti-Plato) whatever I come to believe about The Question, my belief must thoroughly respect and not ignore all this" [3, p. 21].

Seeing these and similar testimonies, I was convinced that my ideas were not out of the mainstream and saw myself as venturing forth, confident in the fact that my job was to understand and identify the truth. Of course, doing so requires an assumption that there is something to search for, some content that we refer to as truth.

Feeling secure about this search, I was shaken a bit some years ago when I encountered Judith Grabiner's paper *Is Mathematical Truth Time Dependent?*. Grabiner, an eminent mathematical historian, has written numerous papers in the history of mathematics and received both the Ford and Allendorfer Awards for her skill as an expository writer. Over the years I've come back to this paper on several occasions, always intrigued by the title. If there is such a thing as mathematical truth forming the basis for our work, how could it be dependent on anything, and how could it change with time? Even though this paper first appeared in the *Monthly* in April 1974, I think it is worth considering today, so I would like to share some of her ideas as well as remarks of my own. In doing so, I am assuming that even though her paper appeared some time ago, her ideas are still worth discussing and are not themselves time-dependent.

Grabiner begins by acknowledging that when confronted by the question of whether mathematical truth is time dependent, our first impulse is to answer no. So far so good! She describes how our discipline is unlike other sciences, which periodically undergo radical change as accommodations are made to theories. Although mathematicians generally do not tear down, but add to, structures that are already in place, there are upheavals in mathematics. Examples include the axiomatization of geometry, changing it from an experimental science into an intellectual pursuit, or the discovery of non-Euclidean geometries or non-commutative algebras, which Grabiner claims helped us realize that mathematics is mainly a study of abstract systems. She states "These were revolutions in thought which changed mathematicians' views about the nature of mathematical truth, and what could or should be proved" [2, p. 355].

In her paper Grabiner focuses on the revolution of developing a rigorous foundation for the calculus which occurred in the late eighteenth on through the nineteenth century. Describing this change as "a rejection of the mathematics of powerful techniques and novel results in favor of the mathematics of clear definitions and rigorous proofs" [2, p. 355], she begins by offering an eighteenth century example, showing how Euler derived the Maclaurin Series for the cosine function. Admittedly, using an example of Euler's work as typical of the eighteenth century might be a little unfair. Twenty years ago Bill Dunham was the keynote speaker at the 1999 ACMS conference, just after his book *Euler, the Master of Us All* was published. For three days he enchanted us with examples of Euler's creative and sometimes unorthodox methods. While we were delighted to see examples of the genius Euler at work, I doubt that any of us left that conference resolved to enhance our calculus classes by emulating the great master. Nevertheless, we'll begin with Grabiner's review of this example. Be warned, there are places in this example where you might cringe at the liberties Euler takes. Who knows, perhaps Euler cringed as well. This example and Euler's techniques serve to highlight the distinction between acceptable practices in his day and those of today.

Euler's quest to establish the series for the cosine function begins with the identity

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz.$$

Expand the left side of the equation using the binomial theorem, then take the real part of that expansion and equate it to $\cos(nz)$. You have

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{2!}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}(\cos z)^{n-4}(\sin z)^4 - \dots .$$

Now let z be an infinitely small positive number and let n be infinitely large. Then

$$\cos z = 1, \quad \sin z = z, \quad n(n-1) = n^2, \quad n(n-1)(n-2)(n-3) = n^4, \quad \text{and so on.}$$

The equation now becomes recognizable as

$$\cos nz = 1 - \frac{n^2 z^2}{2!} + \frac{n^4 z^4}{4!} - \dots .$$

But since z is infinitely small and n is infinitely large, Euler concludes that nz is a finite quantity, so he lets $nz = v$. The result is that

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \dots .$$

While the result is correct, his techniques are unacceptable to us. If a student gave this argument we would likely not extend mercy. However, since this is the great Euler, we all smile, shake our heads in amazement, and move on. This change in attitude exemplifies what Grabiner means when she claims that mathematical truth is time dependent.

Of course, Euler wasn't the only one who took these liberties. Eighteenth century mathematicians wanted new results, to discover the truth, and they weren't about to stop for minor issues like rigor. As Grabiner states: "It is doubtful that Euler and his contemporaries would have been able to derive their results if they had been burdened by our sense of rigor" [2, p. 356]. Nevertheless, it is reasonable to conclude that eighteenth century mathematicians also felt some need to justify their results. After all, even Euler provides supporting arguments in the example above. However, comparing his method to an appropriate argument today, it's easy to agree that a revolution has taken place. But does that represent a change in the truth of the subject itself or is it a change in the standards for discovering that truth? I believe that these are different things.

Grabiner lists, then rejects, several reasons for the lack of rigor in the eighteenth century. These include the premise that eighteenth century mathematicians were primarily interested in getting results, that they placed heavy reliance on the power of symbolism, and that there was a purposeful disregard of the role of rigor. Her last point, that they simply disregarded the role of rigor was addressed by Professor Joan Richards in a talk she gave at the 1991 ACMS conference, The Rigorous and the Natural in Eighteenth Century Mathematics. Richards concludes that there appears to have been a great deal of discussion about the importance and role of rigor in eighteenth century mathematics, although the consensus was that "Rigor was a suspect value among the philosophe of the eighteenth century" [4, p. 15]. Some, such as D'Alembert thought that excessive rigor made mathematics unintelligible, that it is more important to be simple, or natural, than to be rigorous. She also describes several ways in which authors developed a geometry by taking a very "natural approach to the subject, including one by Clairaut where results are based on practical activities such as measurement and surveying. In short, questions about the role of rigor in mathematics were discussed, however, a "severe and inflexible conformity to some law," as rigor was thought to require, was rejected and other methods were found to support results. Of course, that attitude changed in the next century.

The nineteenth century saw a number of significant and important changes in mathematics; making calculus more rigorous was one of them. Starting with Lagrange and extending through the work of Cauchy, Bolzano, Weierstrass, and others, the challenge of providing a proper and rigorous foundation for the results of calculus was met. Concepts like continuity, limits, derivatives, and integrals were formalized, and the resulting theorems had rigorous proofs. What brought about this change? Clearly, it wasn't just because nineteenth century mathematicians were superior to their earlier counterparts. Who, after all, would consider Euler inferior to any of those just listed?

Grabiner then mentions several possible responses, reasons for the change in attitude toward rigor; the need to ensure errors were not made, a desire to generalize and unify results, or to meet the protests of critics, like Bishop Berkeley. She rejects these, offering instead an intriguing, plausible, and practical explanation. During the eighteenth century important mathematicians were typically supported by royal courts or wealthy benefactors. As the century came to a close, these opportunities dried up and mathematicians needed to get jobs, so many of them turned to teaching. In fact, since the time of the French Revolution, almost all mathematicians have also been teachers, a role that presents new challenges. Although mathematicians might be fascinated by Euler's techniques, a teacher is likely to feel that an example like Euler's development of the series for cosine will not win the day in a classroom. The need to provide students with a suitable explanation for aspects of the calculus forced mathematicians to concede that the foundations of the subject were inadequate and that new approaches to convey and support results had to be found. Thus Grabiner argues that the work on foundations was to a large extent based on the need for good explanations and points out that the work on the foundations of analysis of Lagrange, Cauchy, Weierstrass, and Dedekind all originated from lecture notes.

Finally, Grabiner mentions that more than just a new attitude was needed to affect a change in rigor. For instance, Lagrange's alternative definition of the derivative as a coefficient in the Taylor series is an example of an attempt that failed. This new revolution also required both the appropriate way to formulate definitions and the techniques of proof that allowed us to pass from these definitions to theorems. Fortunately, such techniques had recently become available. In an explanation that we will not pursue here, Grabiner gives examples of how eighteenth century work on approximations and perfecting the use of inequalities, were some of the tools needed to raise the standards of rigor in analysis.

Among the conclusions Grabiner draws from her description of these events is that while mathematical truth is perhaps eternal, our knowledge of it is not, and can change over time. Given that assumption, she suggests three possible reactions. One is to adopt a form of relativism, realizing that mathematical truth is just what the experts say it is. She considers this to be an invalid response, pointing out that if Cauchy had adopted this attitude, nothing would have changed. A second approach, which she also rejects, is to set the highest possible standard, to never give an argument unless we can completely and adequately explain every detail. Euler must have known that there were difficulties in dealing with the infinitely large and infinitely small. If he had waited for all the questions to be settled, there would have been no results for Cauchy or Weierstrass to work on. Sacrificing results to rigor is not a valid approach. She suggests a third alternative. We must recognize the "existential situations" that we find ourselves in, living with the realization that mathematics not only grows incrementally but, as is true of other sciences, also has occasional revolutions. By accepting the possibility of present error, we can hope that the future will bring a fundamental improvement in our knowledge.

I find the third response to be quite reasonable, especially as it applies to our own teaching. I believe that mathematical truth is eternal, however, our understanding of it is open to revision, perhaps even drastic change, as we acknowledge the need for such change. Assuming we agree with Grabiner that mathematical truth is eternal, or at least could be eternal, but that our knowledge of it changes over time, what implications might that have for us and for our work with students?

For one thing, I believe her conclusion supports the position on the discovery/invention question that is outlined in *Mathematics Through the Eyes of Faith*. As Christians, we may certainly believe that mathematics is part of God's creation. However, even if you aren't willing to go that far, you must admit that, as Mazur implies, mathematicians generally believe there is an underlying content to what we study. Our role is to discover that truth, finding appropriate theories to describe it. That part of the process is created by us and as such is subject to change over time. This is consistent with Grabiner's view. And, as mentioned earlier, I believe that the acceptance of an underlying content and our ability to interact with it, is consistent with the delight we have in mathematics.

I also appreciate Grabiner's willingness to elevate the role of the teacher/communicator of mathematics in her proposition that the revolution in analysis was at least partly the result of the need to provide legitimate explanations that can be passed on to students. As we teach our courses we must also decide the proper role played by both rigor and intuition in the development of the content. We are responsible for ensuring that our students not only have factual content, but also the necessary skills to communicate that content, that they can be skillful mathematicians. On the other hand, students also need to be inspired, to learn to use and trust their intuition, to have a little Euler in their lives. In some ways we should heed those eighteenth century arguments and allow for the simple and natural to come through, to appeal to intuition when possible. Our students need the opportunity to use their imagination, to discover on their own and to be given explicit opportunities to appreciate the beauty of our subject. They must be directed in a way that makes mathematics a pleasurable experience for them. If we don't do that, we have failed.

References

[1] Bradley, James and Howell, Russell, Editors. *Mathematics Through the Eyes of Faith*, Harper-Collins, New York, 2011.

[2] Grabiner, Judith, "Is Mathematical Truth Time Dependent?" *The American Mathematical Monthly* Vol. 81 No. 4 (1974) 354-36

[3] Mazur, Barry. "Mathematical Platonism and Its Opposites." *European Mathematical Society Newsletter* 68 (June 2008) 19-21.

[4] Richards, Joan. "Rigor and Progress in Eighteenth-Century Mathematics." Keynote paper for the Eighth Biennial Conference of the Association of Christians in the Mathematical Sciences, Wheaton College, May 29–June 1, 1991. (Published in the Proceedings).

[5] Yancey, Philip. *Soul Survivor: How Thirteen Unlikely Mentors Helped My Faith Survive the Church*, New York, Doubleday/Waterbrook Press, 2001.

Charles Babbage and Mathematical Aspects of the Miraculous

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1 Introduction

Our story begins in February of 1829 with the death of the Right Honourable and Reverend Francis Henry Edgerton, eighth Earl of Bridgewater. This unconventional man, with eccentricities that included dressing his dogs in custom-made clothing, had a deep interest in natural theology. This passion ran so deep that the Earl of Bridgewater's will made provision for eight thousand pounds to be invested and placed at the disposal of the President of the Royal Society of London. The person or persons selected by the president should be appointed to write, print, and publish one thousand copies of a work:

> On the Power, Wisdom, and Goodness of God as manifested in the Creation; illustrating such work by all reasonable arguments, as for instance the variety and formation of God's creatures in the animal, vegetable, and mineral kingdoms, the effect of digestion, and thereby of conversion; the construction of the hand of man; and an infinite variety of other arguments. [1]

Eight men were chosen by the President of the Royal Society to carry out the wishes of the late Earl of Bridgewater. The Reverend William Whewell, fellow of Trinity College Cambridge, authored the first published Bridgewater treatise entitled *Astronomy and General Physics Considered with Reference to Natural Theology*. In it he made the following comment:

> We may thus, with the greatest propriety, deny to the mechanical philosophers and mathematicians of recent times any authority with regard to their views of the administration of the universe; we have no reason whatever to expect from their speculations any help, when we ascend to the first cause and supreme rule of the universe. But we might perhaps go farther, and assert that they are in some respects less likely than men employed in other pursuits, to make any clear advance toward such a subject of speculation. [7]

Upon reading this, Charles Babbage decided to respond. At the time, the 45 year old was well known throughout Great Britain as a mathematician and inventor. He was Lucasian Professor of Mathematics at Cambridge, a post that Newton once held. Babbage's analytical difference engine, a precursor to our computer, had been funded by an act of Parliament to aid in navigational calculations.

Provoked by Whewell, Babbage replied to him, and the Bridgewater Treatises in general, by publishing a what he referred to as "a fragment" of his own. He titled it *The Ninth Bridgewater Treatise*. Although he was quick to note in the preface that this work was not part of the original set of treatises, he justified his appropriation of the title because he was furthering the intentions of the late Earl of Bridgewater. To make clear his intentions, on its title page he included the quotation from Whewell, and stated in the preface that he wrote his treatise due to the prejudice he encountered in Whewell's Bridgewater Treatise.

Babbage's motivation was not merely due to Whewell's provocation. Later in the preface Babbage stated, "One of the chief defects of the Treatises above referred to appear to me to arise from their not pursuing the argument to a sufficient extent." He went on to assert that some of his abstract mathematical inquiries, "most removed from any practical application" have led to new perspectives and analogies concerning natural theology. This is illustrated most fully in Chapters VIII through X of *The Ninth Bridgewater Treatise*.

2 Miracles in *The Ninth Bridgewater*

In chapter VIII of *The Ninth Bridgewater*, Babbage examined the nature of miracles. The theological question of how God interacts with his creation had received renewed interest in pre-Victorian England. Much of this was due to a variety of scientific discoveries, particularly in geology. Miracles in particular drew the attention of several writers of natural theology. [3] *The Ninth Bridgewater* can be examined as one of several works during this time concerning the miraculous. According to Babbage

> . . . it is more consistent with the attributes of the Deity to look upon miracles not as deviations from the laws assigned by the Almighty for the government of matter and of mind; but as the exact fulfillment of much more extensive laws than those we suppose to exist. [1]

Babbage remarked that such a view of the miraculous assigns greater power and knowledge to God than a God who is constantly intervening or even interfering in creation. He then proceeded to illustrate his views with two extended mathematical examples.

The first of these concerns a mechanical calculator known as a difference engine. Babbage described God as a master programmer of the universe. He is one who programmed the apparent exceptions - the miracles - to follow a uniform natural law. Babbage envisioned an observer of a difference engine who witnesses a sequence of numbers as outputs of the machine. Without fail, every one of these numbers is a perfect square. After millions of observations, one of the numbers is not a perfect square. The pattern of square numbers then returns exactly where it had left off, and continues on for every other observation.

This scenario is analogous to the way that a natural law would be inferred from numerous observations. In this case, the observer might say, "This machine always produces square numbers." The one supposed exception to the sequence of square numbers is likened by Babbage as a miracle. This anomaly could have resulted from several causes, but only two were considered. The first explanation was that a secret lever, hidden from view, was activated by the machine's creator to produce the intended effect. Another explanation of the aberration is that the supposed exception

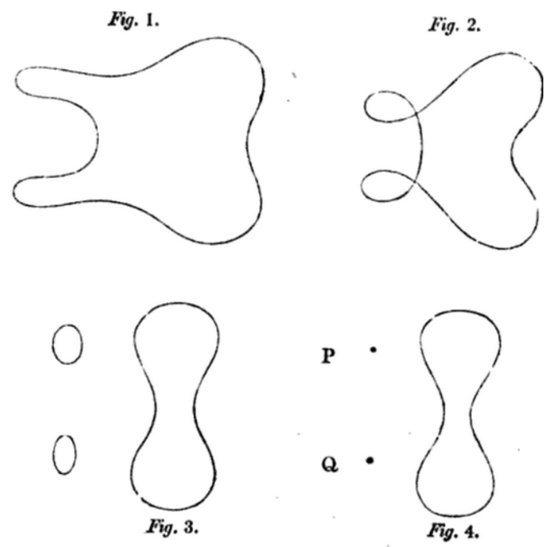

Figure 1: Figure from page 101 of *The Ninth Bridgewater*

of the rule had been programmed into the machine.

Which of these two cases shows the greater power of workmanship for one who built the difference engine? For Babbage, the second case demonstrates that a greater mind and power was at work. In a similar way, miracles, which are supposed exceptions to natural law, are really part of a more complex pattern working.

Babbage's second illustration shared the same perspective of the miraculous as his difference engine thought experiment. However, the second illustration utilized an entirely different area of mathematics. At first glance, the various curves on page 101 of *The Ninth Bridgewater*, which are shown in Figure 1, appear different from one another. The first two are connected, the third contains two portions that are separate from a larger closed curve, and the fourth contains two singular points.

For the fourth curve in particular, sight alone would lead to the conclusion that points P and Q are exceptions to the infinitely many observations that comprise the connected portion. However, as Babbage explained in a note, these singular points as well as points on the curve can be described by the same equation. Moreover, all four the figures on page 101 can be generated from the following equation, where $a, b, c, d,$ and e are all constants that can be varied:

$$(y^2 - 2)^2 = -ax^4 + bx^3 + cx^2 + dx + e. \tag{1}$$

Thus in Figure 1, the points P and Q satisfy the same equation as every point on the curve. The apparent deviations from the closed curve are really manifestations of a higher law, imperceptible to sight, but detectable by mathematics.

Babbage did not give the specific constants used to produce each of the curves in Figure 1. Experimentation, using a graphing utility such as Desmos, achieves a reasonably close approximation. An interesting exercise in curve sketching reveals how the roots of the quartic on the right side of Equation (1) affect the behavior of the curve.

We begin with a quartic polynomial possessing a negative leading coefficient. The other coefficients are given values so that there are only two real roots of the quartic, r_1 and r_2, and the three critical points of the graph of the polynomial are in the interval (r_1, r_2). The left side of Figure 2 shows one example of this. We then vary only the constant term and observe that the shapes similar to those displayed in Figure 1 begin to emerge as the roots of the quartic change. On the right side of Figure 2 there are four real roots, one of which is repeated, a critical point, and between the other two roots. On the left side of Figure 3 there are four distinct real roots. Finally, on the right side of Figure 3 there are four real roots, one of which is repeated, and is not between the other two roots. For these particular values of the coefficient the two singular points appear.

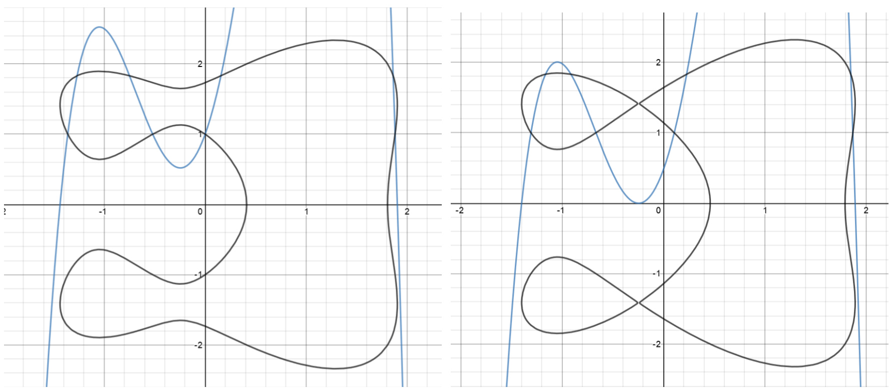

Figure 2: Graphs of $y = -3x^4 + 8.5x^2 + 4x + e$ in blue and $(y^2 - 2)^2 = -3x^4 + 8.5x^2 + 4x + e$ in black for $e = 1$ and 0.48

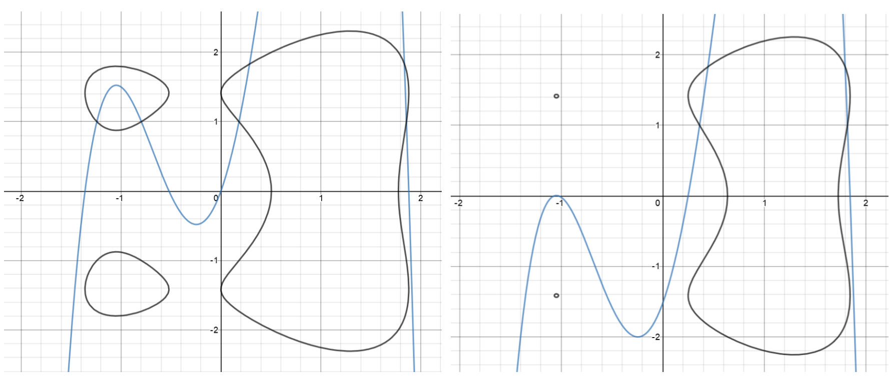

Figure 3: Graphs of $y = -3x^4 + 8.5x^2 + 4x + e$ in blue and $(y^2 - 2)^2 = -3x^4 + 8.5x^2 + 4x + e$ in black for $e = 0$ and -1.52

Babbage had Whewell's quotation in mind as he concluded chapter IX of *The Ninth Bridgewater Treatise*. According to Babbage, this example opened "views of the grandeur of creation perhaps more extensive than any which the sciences of observation or of physics have yet supplied." [1] Not everyone agreed. For instance, the Roman Catholic priest D.W. Cahill commented on the flaws of Babbage's analogy. The miraculous had been reduced to a more extended natural machinery. As Cahill wrote to the bishops of England, "Was there ever published such a monstrous conceit as to reduce miracles to a formula in algebra - to a curve of four dimensions!" [2]

Cahill's comment does spark the interesting question of why Babbage chose this particular curve for his example. Other curves of lower degree will exhibit similar behavior. For instance, the graph of $y^2 = -x^3 + x + c$ changes its number of components and even has a singular point depending on the value of c.

3 Babbage and Hume

Babbage's examination of the miraculous needed to confront a deeper issue that had arisen nearly a century earlier. In 1748 the Scottish philosopher David Hume published *An Enquiry Concerning Human Understanding*. In section X, known as "Of Miracles," Hume contended that miracles, which he considered violation of the laws of nature, were so unlikely as to not exist. There are several elements to "Of Miracles," and the one that drew Babbage attention relates to testimony of miraculous events.

In Hume's words

> The plain consequence is (and it is a general maxim worthy of our attention), that no testimony is sufficient to establish a miracle, unless the testimony be of such a kind, that its falsehood would be more miraculous than the fact which it endeavors to establish. [4]

The use of the word "miraculous" concerning testimony has obscured some of Hume's meaning, but reference to another key passage clarifies this usage. Elsewhere in section X Hume remarked

> When any one tells me, that he saw a dead man restored to life, I immediately consider with myself, whether it be more probable, that this person should either deceive or be deceived, or that the fact, which he relates, should really have happened. I weigh the one miracle against the other; and according to the superiority, which I discover, I pronounce my decision, and always reject the greater miracle. [4]

To establish that a miracle has occurred, following Hume, "the falsehood of testimony of the miracle must be *more miraculous* than the miracle." By meeting Hume on his own terms, Babbage recast Hume's criterion to read, "The falsehood of testimony of the miracle must be *more improbable* than the miracle."

This restatement was not original to Babbage. A similar argument can be traced as far back as George Campbell's *A Dissertation on Miracles* in 1762. What was novel in Babbage's approach was to use mathematical formalism to further restate Hume's criterion. A miracle has occurred if

$$P(\text{ Falsehood of Testimony of Miracle }) < P(\text{ Miracle }). \tag{2}$$

Babbage realized that Hume had simply stated an equivalent version of this criterion, and then asserted that it could never be met. Hume had made no effort to weigh the probabilities in the above inequality. Chapter X and Appendix E of *The Ninth Bridgewater* concern the calculation of these probabilities. We will follow and expand on Babbage's line of argument.

We begin with the calculation of the probability of a miracle. In order to assign a numerical value to this probability, Babbage uses Laplace's rule of succession. Laplace derived this result in order to confront the so-called sunrise problem [5]. Given that the sun has risen a specified number of times, and there is no prior knowledge on what it will do tomorrow, what is the probability that the sun rises again? More generally, suppose that there are N observations, which can either be classified as a success or a failure, and that s successes were observed. The probability that the $N+1$ observation is a success is given by the following formula

$$P(N+1 \text{ observation is a success } | s \text{ successes }) = \frac{s+1}{N+2}. \tag{3}$$

In order to use Equation 3 regarding something as miraculous as a dead person being restored to life, Babbage made several rough estimates and provided us with the details. He first supposed that the world is 6000 years old, and that there have consistently been 30 years per generation. Thus there have been 200 generations of people since the beginning of the world. He further estimates that at every point in history the earth's human population has numbered 1 billion, which was the estimated world population in the mid-nineteenth century. This gave Babbage the estimate that 200 billion people have died, which is used as the value for N in Equation 3. For the sake of argument, Babbage supposed that following the death of every one of these 200 billion, none of them were restored to life. With no successes, the value of $s = 0$. By Laplace's rule of succession, the probability that the next person who dies will be restored to life is

$$\frac{s+1}{N+2} = \frac{0+1}{200{,}000{,}000{,}000+2} = \frac{1}{200{,}000{,}000{,}002} \approx 5 \cdot 10^{-12}.$$

This is an astonishingly small probability, but it is not zero.

We now consider the probability of the falsehood of testimony concerning a miracle. Babbage supposed mutually independent witnesses who are reliable 99% of the time. Given two such witnesses, due to the crucial assumption of independence, the probability that they both agree on a falsehood is $(1/100) \cdot (1/100) = 1/10,000$. Each additional independent witness will further reduce the overall probability of mistaken or false testimony. With n such witnesses

$$P(\text{ Falsehood of Testimony }) = (1/100)^n. \tag{4}$$

To return to Hume's criterion, in order to establish by testimony that something as seemingly impossible as a dead person being restored to life had occurred, we would need to combine Equation 4 with our probability of a miracle $2 \cdot 10^{-11}$ in the inequality 2. A miracle has occurred when

$$(1/100)^n < 5 \cdot 10^{-12}.$$

The question then becomes how large must n be in order to satisfy the inequality.

With only $n = 6$ witnesses that match Babbage's description, the probability that they all give false testimony (either intentionally or unintentionally) concerning the restoration of a dead person to life is $(1/100)^6 = 10^{-12}$. This probability is less than the probability of a dead person being restored to life that was calculated above to be approximately $5 \cdot 10^{-12}$. By Hume's own criterion, such a miracle could be established by testimony.

In appendix E of *The Ninth Bridgewater Treatise* by use of several mathematical proofs, each making subtle distinctions in the nature of the testimony of a purported miracle, Babbage considered other situations. For example, what if the witnesses are less reliable than 99% of the time? Variations on the original question were summarized by Babbage (with emphasis in original):

> ... if independent witnesses can be found, who speak truth more frequently than falsehood, it is ALWAYS possible to assign a number of independent witnesses, the improbability of the falsehood of whose concurring testimony shall be greater than the improbability of the miracle itself. [1]

4 Conclusion

Although many of us, including myself, would disagree with Babbage's definition of a miracle, *The Ninth Bridgewater Treatise* provides a fascinating historical example of the integration of faith and mathematics. Babbage's refutation of Hume as well as his dispute with Whewell demonstrated a firm conviction and willingness to engage with his opponent on his opponent's terms, modelling a type of civility in disagreement.

There are several places in the undergraduate mathematics curriculum where the ideas from *The Ninth Bridgewater* could be discussed. The specific calculations performed by Babbage could be demonstrated as in-class examples of probability to illustrate concepts such as independence. An upper-division seminar course could evaluate Babbage's argument and discuss the role of mathematics in Christian apologetics.

A series of questions below would help start a seminar discussion concerning some of the issues underlying Babbage's examination of the miraculous.

- What was some more of the historical context underlying the discussion of miracles? See, for instance, [3] and [6].
- What are some flaws in how Babbage defined a miracle?
- What is your definition of a miracle? How does your definition compare or contrast with Babbage's interpretation? How does your definition compare or contrast with Hume's interpretation?
- Was Babbage's interpretation of a miracle in a naturalistic setting consistent with a Biblical understanding of the miraculous?
- How effective was Babbage's use of mathematical analogies to illustrate his conception of a miracle? What are the advantages and limitations to using mathematical analogies to illustrate theological truths?
- How effective do you think Babbage's argument concerning the miraculous would be to an unbeliever?
- What are some other arguments, besides Babbage's, that refute Hume's "Of Miracles"?
- What is an appropriate response today to those who would challenge the existence of miracles and our faith in them?

References

[1] Charles Babbage. *The Ninth Bridgewater Treatise, Second Edition.* John Murray, London, 1838.

[2] *The Bengal Catholic Herald, V. IX.* P.S. D'Rozario and Co., Calcutta, 1845.

[3] Walter Cannon. "The Problem of Miracles in the 1830's." *Victorian Studies*, vol. 4, no. 1, 1960, pp. 5–32.

[4] David Hume. "On Miracles" *Enquiry Concerning Human Understanding.* 1748

[5] Pierre Simon Laplace. *Essai philosophique sur les probabilités.* 1814.

[6] Joan L. Richards. "The Probable and the Possible in Early Victorian England" In B. Lightman (Ed.) *Victorian Science in Context*, University of Chicago Press, 2008.

[7] William Whewell. *Astronomy and General Physics Considered with Reference to Natural Theology.* William Pickering, London, 1834.

Numerical Range of Toeplitz Matrices over Finite Fields[1]

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Abstract

This paper characterizes the k-th numerical range of all $n \times n$ Toeplitz matrices with a constant main diagonal and another single, non-zero diagonal, where the matrices are over the field $\mathbb{Z}_p[i]$, with p a prime congruent to $3 \mod 4$. For $k \in \mathbb{Z}_p^*$, the k-th numerical range is always equal to $\mathbb{Z}_p[i]$ with the exception of the scaled identity. Similar techniques are used to discover a general connection between the 0-th numerical range and the k-th numerical range. Lastly, a conjecture is given regarding the general numerical range of all triangular Toeplitz matrices.

1 Introduction

Let p be a prime congruent to $3 \mod 4$. Let $M \in M_n(\mathbb{Z}_p[i])$ where $\mathbb{Z}_p[i]$ is a Galois Field of order p^2 in the form $\{a + bi : a, b \in \mathbb{Z}_p\}$, $M_n(\mathbb{Z}_p[i])$ denotes the set of $n \times n$ matrices with entries from the field in the argument, and M is a Toeplitz matrix. This paper classifies the numerical range of Toeplitz matrices, $W(M)$, over the finite field $\mathbb{Z}_p[i]$. Numerical range of matrices over $\mathbb{C}$ has been of high interest in the mathematical community with substantial advances made in the area by Hausdorff, Toeplitz, and Kippenhahn [2]. However, in this paper we work with the field extension $\mathbb{Z}_p[i]$ of $\mathbb{Z}_p$ where p is prime and $p \equiv 3 \mod 4$, which is based off of a recent publication on numerical ranges over finite fields [2]. Restricting to $p \equiv 3 \mod 4$ ensures that the element -1 is not a quadratic residue in $\mathbb{Z}_p$, reflecting the same property of -1 in $\mathbb{R}$. Note that in this analogue we have preserved the extension field having degree 2.

2 Preliminaries

Before describing the numerical range of Toeplitz matrices in a finite field, we must establish some preliminary definitions and lemmas. We have built the foundation of our research off of a paper by Coons et al ([2]), which contains the specific proofs for this section. We begin by presenting the definition of numerical range over a finite field with p prime and $p \equiv 3 \mod 4$.

Definition 1. [2, Definition 1.1] Let p denote a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. We define $W(M)$, the **finite field numerical range** of M, to be

$$W(M) = \{x^* M x : x \in \mathbb{Z}_p[i]^n, x^* x = 1\}.$$

[1]Thanks to the following organizations for funding this project: the Taylor University Women's Giving Circle, the Bedi Center for Teaching & Learning Excellence, and the Faculty-Mentored Undergraduate Scholarship program.

The reader may note that x^*x is an indefinite inner product since there may be several x that map to 0 under this operation. Therefore, the function x^*x can not be defined as a norm. However, this analogue does maintain several standard properties of the numerical range. The numerical range remains unchanged under scaling and translating, while also remaining unitarily invariant.

Definition 2. [2, Definition 2.2] Let p denote a prime congruent to $3 \mod 4$ and let $||x||^2 := x^*x$ and for any $k \in \mathbb{Z}_p$ let C_n^k denote the set of all vectors $x \in \mathbb{Z}_p[i]^n$ for which $||x||^2 := k$.

The ***k*-th numerical range** of a matrix $M \in M_n(\mathbb{Z}_p[i])$ is the set $W_k(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, x^*x = k\}$.

Lemma 1. [2, Lemma 2.3] *For all primes p congruent to $3 \mod 4$, $k \in \mathbb{Z}_p^*$, and $M \in M_n(\mathbb{Z}_p[i])$, we have $W_k = kW_1(M)$.*

Lemma 2. [2, Lemma 2.7] *Let p be a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. For any $a, b \in \mathbb{Z}_p[i]$ we have $W(aM + bI) = aW(M) + b$.*

Lemma 3. [2, Lemma 2.6] *Let $M, U \in M_n(\mathbb{Z}_p[i])$ with U unitary and p a prime congruent to $3 \mod 4$. Then, $W(M) = W(U^*MU)$.*

For simplicity, the first numerical range will now be referred to as $W(M)$ as described in Definition 1. It is important to note that Lemma 1 does not apply to $W_0(M)$. Following the work of [2], Ballico classified $W_0(M)$ for every 2×2 matrix in [1]. The work that follows also requires some number-theoretic tools; the following again comes from [2].

Lemma 4. [2, Lemma 2.1] *For all primes p congruent to $3 \mod 4$, and for all $k \in \mathbb{Z}_p$, there exists $t, s \in \mathbb{Z}_p$ for which $t^2 + s^2 = k$.*

There is a nice connection between this theorem and the kind of expressions that constrain the numerical range:

Lemma 5. *Let p be a prime congruent to $3 \mod 4$. For all $k \in \mathbb{Z}_p$ and all $x \in \mathbb{Z}_p[i]$, there exists a $y \in \mathbb{Z}_p[i]$ for which $|x|^2 + |y|^2 \equiv k \mod p$.*

Proof. First, let $x = 0$. Since $0 \in \mathbb{Z}_p[i]$, let $y = 0$. Then, $0^2 + 0^2 \equiv 0 \mod p$, and we are done.

Now, let $x \in \mathbb{Z}_p[i]^*$. It follows that the element $x\overline{x} \in \mathbb{Z}_p^*$. Since the field $\mathbb{Z}_p[i]$ has additive inverses, there exists an $m \in \mathbb{Z}_p^*$ such that $x\overline{x} + m \equiv 0 \mod p$. Therefore, $x\overline{x} + m + k \equiv k \mod p$ when $k \in \mathbb{Z}_p$. It follows that $m + k \in \mathbb{Z}_p$. By Lemma 4, we know there exist $a, b \in \mathbb{Z}_p$ for which $(m+k) = a^2 + b^2$. Letting $y = a + bi$, we have $(m+k) = y\overline{y}$. Therefore, $|x|^2 + |y|^2 \equiv k \mod p$. □

We are now prepared to investigate the numerical range of Toeplitz matrices.

3 Toeplitz Matrices

In this section, we will prove the numerical range of a specific class of Toeplitz matrices. Toeplitz matrices have constant, descending diagonals. The form of a general $n \times n$ Toeplitz matrix M is given below.

$$M = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & \vdots \\ a_2 & a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_0 & a_{-1} \\ a_{n-1} & \dots & \dots & a_1 & a_0 \end{pmatrix}$$

Our Toeplitz matrices have a main diagonal a_0 and a single, non-zero lower or upper diagonal a_r. (If only the main diagonal is non-zero, we already have $W(M) = \mathbb{Z}_p$ by [2, Corollary 3.2].) Here we state our theorem regarding $W(M)$ for such matrices, but its proof will be written as a sequence of several lemmas.

Theorem 3. *Let $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ where M is a Toeplitz matrix with a main diagonal a_0 and a single, non-zero lower or upper diagonal a_r. Then for $k \in \mathbb{Z}_p^*$, $W_k(M) = \mathbb{Z}_p[i]$.*

We now transition into proving the full numerical ranges described in Theorem 3. To begin, we will examine the numerical range of strictly lower triangular, $n \times n$ Toeplitz matrices with a single, non-zero, lower diagonal.

Lemma 6. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. Define $x^* = (c_1\ c_2\ c_3)$, and let $m := x^*Mx = c_1\overline{c_3}$. We will consider a subset of the numerical range by setting $c_3 = 1$. (We will show this subset of $W(M)$ still attains all of $\mathbb{Z}_p[i]$ as outputs.) For all $c_1 \in \mathbb{Z}_p[i]$, there exists an x such that $|c_1|^2 + |x|^2 \equiv 0 \mod p$ by Lemma 5. Let $c_2 = x$. With c_1 varying over all of $\mathbb{Z}_p[i]$ and c_2 always chosen so that $|c_1|^2 + |c_2|^2 \equiv 0$, we have $W(M) = \mathbb{Z}_p[i]$. $\square$

Lemma 7. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. Consider $m := x^*Mx = c_1\overline{c_2} + c_2\overline{c_3}$. Our goal is to show that m can become any element of $\mathbb{Z}_p[i]$. We will again show this on a subset of the numerical range by stipulating that $c_2 = 1$.

First, we show that there is a non-zero element in this set. Letting $c_1 = 1$, we have that $|c_3|^2 \equiv -1$. By the Lemma 4, there exists $a, b \in \mathbb{Z}_p$ such that $a^2 + b^2 \equiv -1$, so we will let $c_3 = a + bi$. It is important to note that since -1 is not a quadratic residue when $p \equiv 3 \mod 4$, neither a nor b can be 0. Therefore, since c_1 is chosen to be real and c_3 is guaranteed to be complex, we know that $c_1 + \overline{c_3}$ is non-zero, and that the set of all such elements is different from $\{0\}$.

Now, let $c_1 + \overline{c_3}$ be a given fixed non-zero quantity with the constraint that $|c_1|^2 + |c_3|^2 \equiv 0$. Let us now consider inputting the elements kc_1 and $\overline{k}c_3$ where k is an arbitrary element of $\mathbb{Z}_p[i]$. Note that $|kc_1|^2 + |\overline{k}c_3|^2 = |k|^2|c_1|^2 + |k|^2|c_3|^2 = |k|^2(|c_1|^2 + |c_3|^2) = |k|^2(0) = 0$, which satisfies the constraint. Then, the output becomes $kc_1 + k\overline{c_3} = k(c_1 + \overline{c_3})$. Since $c_1 + \overline{c_3}$ is fixed and k varies over all of $\mathbb{Z}_p[i]$, we have that $k(c_1 + \overline{c_3})$ maps to every element of $\mathbb{Z}_p[i]$. This is because $k \to \alpha k$ is an automorphism of $\mathbb{Z}_p[i]$ where $\alpha = c_1 + \overline{c_3} \in \mathbb{Z}_p[i]^*$. Therefore, $W(M) = \mathbb{Z}_p[i]$. □

We now complete the proof to Theorem 3 for the lower diagonal cases by proving the following lemma.

Lemma 8. *For all primes $p \equiv 3 \mod 4$, $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$ and where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is a Toeplitz matrix with a constant diagonal $a_0 \in \mathbb{Z}_p[i]$ and lower diagonal $a_r \in \mathbb{Z}_p[i]^*$ with $0 < r < n$.*

Proof. We first consider $W(M) = \mathbb{Z}_p[i]$ where M is a Toeplitz matrix with a constant diagonal $a_0 = 0$ and lower diagonal $a_r = 1$ with $0 < r < n$.

Suppose $n = 3$. Then the only two cases are already proven to be $\mathbb{Z}_p[i]$ in Lemmas 6 and 7.

Now, suppose $n \geq 4$. Let $x = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}^T$ where $c_1, c_2, ..., c_n \in \mathbb{Z}_p[i]$ and $|c_1|^2 + |c_2|^2 + ... + |c_n|^2 \equiv 1$. Consider m where $m := x^*Mx = 0\sum_{i=1}^{n} c_i\overline{c_i} + 1\sum_{m=1}^{n-r} c_m\overline{c_{m+r}} = c_1\overline{c_{1+r}} + c_2\overline{c_{2+r}} + ... + c_{n-r}\overline{c_n}$.

We begin by restricting to a subset of the numerical range by requiring $c_{1+r} = 1$. For all $c_1 \in \mathbb{Z}_p[i]$, there exists an x such that $|c_1|^2 + |x|^2 \equiv 0 \mod p$ by Lemma 5. Therefore, let $c_2 = x$ and let all other $c_i \equiv 0$ where $i \neq 1, 2, 1+r$. Therefore, we now have $|c_1|^2 + |c_2|^2 \equiv 0$ and $m = c_1(1) + c_2(0) = c_1$. Since we let c_1 vary over all of $\mathbb{Z}_p[i]$, adjusting c_2 appropriately, we have $W(M) = \mathbb{Z}_p[i]$. The only concern is the possibility that $r = 1$, so that $c_{1+r} = c_2$. In that case, since $n \geq 4$, we can let c_3 take on the role of c_2 in the work above, and the proof follows similarly.

By Lemma 2, we know that $W(a_rM + a_0I) = a_rW(M) + a_0 = \mathbb{Z}_p[i]$ since $k \to \alpha k + \beta$ is an automorphism of $\mathbb{Z}_p[i]$. Therefore, $W(M) = \mathbb{Z}_p[i]$ when M is a Toeplitz matrix with a main diagonal $a_0 \in \mathbb{Z}_p[i]$ and a lower diagonal $a_r \in \mathbb{Z}_p[i]^*$ with $0 < r < n$.

Therefore, for the matrix M described, we conclude that $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$ by Lemma 1. □

We have proven Theorem 3 for all Toeplitz matrices with a single, non-zero lower diagonal. We will now show that the result holds for matrices with an upper diagonal.

Lemma 9. *When $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is a Toeplitz matrix with a main diagonal a_0 and a single, non-zero, upper diagonal a_r with $0 < r < n$, $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$.*

Proof. In this setting, we still have $W(M^*) = \overline{W(M)}$. Every Toeplitz matrix M with a single, constant, non-zero upper diagonal can be described by a matrix with a single non-zero lower diagonal by considering M^*. Thus, here again we have $W(M) = \mathbb{Z}_p[i]$. By Lemma 1, one can conclude the same for the k-th numerical range: $W_k(M) = \mathbb{Z}_p[i]$. □

This concludes our proof for Theorem 3.

4 Connecting W_0 to W_k

While W_0 is generally distinct from W_k for $k \neq 0$, our work in this section illustrates an important, general connection between the two, for a much broader class of matrices than simply Toeplitz matrices, inspired by the proofs of the previous section. The main proof depends on direct sums, which work very differently in this setting. The following definition and proposition explain how direct sums now work with numerical range, and are lifted directly from [2].

Definition 4. [2, Definition 2.8] For any two elements s and t of $\mathbb{Z}_p[i]$, we define $L_{s,t}$, the **open line segment** connecting s and t, to be the set $\{sj + t(1-j) : j \in \mathbb{Z}_p, j \neq 0, 1\}$. Furthermore, for any two subsets S and T of $\mathbb{Z}_p[i]$ we define $\mathrm{Conv}(S,T)$, the **open convex hull** of S and T, to be the union of all open line segments connecting an element of S and an element of T, i.e.

$$\mathrm{Conv}(S,T) := \bigcup_{\substack{s \in S \\ t \in T}} L_{s,t}.$$

Finally, we define $\mathrm{Conv}(S,T;S_0,T_0)$, the **oddly-closed convex hull** of S and T with respect to sets S_0 and T_0, to be the union of $\mathrm{Conv}(S,T)$ with sets $S + T_0$ and $S_0 + T$.

Proposition 5. [2, Proposition 3.1] *Let p be a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. Assume further that M is reducible, i.e. $U^*MU = A \oplus B$ for some unitary matrix $U \in M_n(\mathbb{Z}_p[i])$ and for some lower dimensional matrices A and B with entries in $\mathbb{Z}_p[i]$. Then $W(M)$ is the oddly-closed convex hull of $W(A)$ and $W(B)$ with respect to $W_0(A)$ and $W_0(B)$.*

We are now ready to progress towards our main theorem.

Lemma 10. *Let $A \in M_n(\mathbb{Z}_p[i])$ and let B be the block matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $W(B) = \bigcup_{k \in \mathbb{Z}_p} W_k(A)$.*

Proof. Here $S = W(A), T = W(0) = \{0\}, S_0 = W_0(A)$, and $T_0 = W_0(0) = \{0\}$. Therefore $S + T_0 = W(A)$ and $T + S_0 = W_0(A)$. Additionally, we have $\mathrm{Conv}(S,T) = \bigcup W_k(A)$ where $k \in \mathbb{Z}_p \neq 0, 1$. Taking the union of all these sets as specified in Proposition 5 gives the result. ☐

Lemma 11. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Then the block matrix $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ has $W(B) = \mathbb{Z}_p[i]$.*

Proof. Consider a subset of the numerical range by requiring inputs x such that $c_k = 1$, and $c_i = 0$a for $i \neq j, k, n+1$. Then, the expression x^*Ax becomes $a_{jk}c_j\overline{c_k} + a_{kj}c_k\overline{c_j} = a_{jk}c_j$, with the constraint that $|c_j|^2 + |c_{n+1}|^2 = 0$. Letting c_j be any element of $\mathbb{Z}_p[i]$, and adjusting c_{n+1} appropriately as in the proof of Lemma 6, we have $W(B) = \mathbb{Z}_p[i]$. ☐

Theorem 6. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Then*

$$\bigcup_{k \in \mathbb{Z}_p} W_k(A) = \mathbb{Z}_p[i].$$

Proof. The theorem is a direct consequence of the previous two lemmas. $\square$

This theorem shows that for a large class of matrices, if the numerical range is not all of $\mathbb{Z}_p[i]$, then the missing elements will be found in *some* variation of the numerical range. Our constraint on the matrix - a non-symmetric 0 somewhere - may seem odd, and it is not clear how sharp this constraint is. However, the theorem certainly fails for symmetric matrices. For example, computing x^*Mx where $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $x = (c_1c_2c_3)$ gives $c_1\overline{c_2} + \overline{c_1}c_2 + c_2\overline{c_3} + c_3\overline{c_2}$ which is clearly an element of $\mathbb{Z}_p$, regardless of the constraint on $\|x\|^2$.

The theorem also gives an immediate corollary for when W_0 might "compensate" for missing elements of $W(M)$.

Corollary 7. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Suppose further that $W(A)$ is either $\mathbb{Z}_p$ or $\{x + xi : x \in \mathbb{Z}_p\}$. Then $W_0(A)$ contains the complement of $W(A)$, and $W_0(A) \cup W(A) = \mathbb{Z}_p[i]$.*

Proof. In either of these situations, $W_k(A) = W(A)$ for all $k \in \mathbb{Z}_p^*$. $\square$

5 Future Work

Throughout our research, we have found that the standard numerical range of lower or upper triangular Toeplitz matrices is always full. For some matrices, this is easily proven with the techniques we have used thus far. For example, consider

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

and $x = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}^T$ where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{Z}_p[i]$ and $|c_1|^2+|c_2|^2+|c_3|^2+|c_4|^2+|c_5|^2 = 1$. The elements of the numerical range are in the form $m = x^*Mx = c_1\overline{c_2} + c_2\overline{c_3} + c_1\overline{c_4} + c_3\overline{c_4} + c_1\overline{c_5} + c_2\overline{c_5} + c_4\overline{c_5}$. If we restrict x such that c_1 and $c_5, = 0$ and $c_3 = 1$, we are left with $m = c_2(1) + (0)\overline{c_4}$ when $|c_3|^2 + |c_4|^2 = 0$. This was proven to be $\mathbb{Z}_p[i]$ in Lemma 6.

It seems as if we can classify every upper or lower triangular Toeplitz matrix with a full numerical range if there is at least one off-diagonal of zeroes. The problem arises from a matrix where we have no such diagonal of zeroes. For example, consider the following matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

An arbitrary element of $W(M)$ is in the form $m = c_1\overline{c_2} + c_2\overline{c_3} + c_1\overline{c_3}$. There is no simple trick we know of to verify this element gives us all of $\mathbb{Z}_p[i]$. But our testing has shown that for small $p \equiv 3 \mod 4$, we know this does give us all of $\mathbb{Z}_p[i]$.

From this example, and several like it, we make the following conjecture.

Conjecture. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is either a lower or upper triangular Toeplitz matrix.*

Additionally, as mentioned earlier, W_0 is distinct from W_k. However, in all of our work, we have found $W_0(M) = W(M)$ when $W(M) = \mathbb{Z}_p[i]$. A proof of the following conjecture would also be of interest.

Conjecture. *For all primes $p \equiv 3 \mod 4$, $W_0(M) = \mathbb{Z}_p[i]$ where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is either a lower or upper triangular Toeplitz matrix.*

We have also tested non-triangular matrices and have seen, but are unable to show, that in many cases (but not all, e.g. if the matrix is symmetric), $W(M) = \mathbb{Z}_p[i]$. In general, for higher dimensional matrices, it seems that finding a numerical range other than $\mathbb{Z}_p[i]$ is rare, even when a Toeplitz form is not assumed.

We conclude with a list of further questions.

1. Is it true that every triangular Toeplitz matrix which is not a multiple of the identity has $W(M) = \mathbb{Z}_p[i]$?
2. If a Toeplitz matrix is symmetric, then $W(M)$ is a line. Is it true that for every Toeplitz matrix, $W(M)$ is either a line or $\mathbb{Z}_p[i]$? If not, what other regions are possible?
3. If T is a multiple of the identity, then $W(M) \neq W_0(M)$. Is this ever true otherwise for Toeplitz matrices?
4. How different is the situation when $p \not\equiv 3 \mod 4$?

References

[1] Ballico, E. (2018). "The Hermitian Null-range of a Matrix over a Finite Field", *Electronic Journal of Linear Algebra*, Volume 34, pp.205-216. DOI: https://doi.org/10.13001/1081-3810.3416

[2] Coons, J.I., Jenkins, J., Knowles, D., Luke, R.A., Rault, P.X.(2016). Numerical Ranges Over Finite Fields. *Elsevier: Linear Algebra and its Applications, 501.*
Retrieved From https://www.sciencedirect.com/science/article/pii/S0024379516300106

[3] Gallier, J. (2011). *Geometric Methods and Applications For Computer Science and Engineering.* New York, NY. Springer Science+ Business Media.

[4] Martińez-Avendanõ, R.A., Rosenthal, P. (2007). *An Introduction to Operators on the Hardy-Hilbert Space.* New York, NY. Springer Science+ Business Media.

[5] Zachlin, P.F., Hochstenback, M.E. (2007). *On the numerical range of a matrix.* (CASA-report; Vol.0702). Eindhoven: Technische Universiteit Eidehoven.

Computer Science: Creating in a Fallen World

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1 Introduction

God, the creator of the universe and everything in it, created every person in his image. One of the many amazing gifts this implies is the gift of sub-creation: our ability to make things in and from God's creation. Just as God repeatedly declared the goodness of his creation, we can take joy in creating things with the abilities God has given us. However, as we exercise this gift, we need to be careful of both what and how we create. The responsibility to create for good, and not for evil, is obvious, although how to do that is sometimes not. In particular, it took decades to appreciate the full impact of our imperfect, sinful nature on how we create software.

The question of how best to exercise the gift of creating is particularly important for Christians involved in Computer Science, whether as teachers, learners, or practitioners, because our field offers so many tools and opportunities for creating things. This is such a rich, broad, and deep question that a single paper cannot possibly answer it. So the present paper aspires only to elucidate and highlight some parts of the answer that I find particularly interesting and compelling in my own career and in teaching my students.

This paper has three main sections. The first section, "Sub-Creation," explores the intersection of sub-creation and computer science, not in the narrow sense of creating virtual worlds, but in the broader sense of making things within God's created world. Our creations can have both obvious and more subtle effects, and all of these are part of creating. The second section, "Creating for Christ," considers what Christians should seek to create, and how we should decide. And the third section, "Creating in a Fallen World," considers how the creative process, and the resulting creations, reflect and respond to our sinful nature and fallen world. Agile software development is an important example, because it embraces and works with our imperfect knowledge and understanding, in contrast with the discredited waterfall (or rational) model which long strove to document complete and detailed requirements and design before most coding begins.

2 Sub-Creation

Sub-creation is the God-given gift of creating, or making things, within God's created universe. People have been making things since Adam and Eve sewed fig leaves together (Genesis 3:7) after their original sin, but Dorothy Sayers argues persuasively for an earlier origin:

> [W]hen we turn back to see what [the author of Genesis] says about the original upon which the "image" of God was modelled, we find only the single assertion, "God created". The characteristic common to God and man is apparently that: the desire and the ability to make things. [22]

J.R.R. Tolkien famously used the term "sub-creation" in a more limited sense to describe the creation of vivid and self-consistent imaginary, or virtual, worlds like "middle earth" in his Lord of the Rings trilogy [27]. I follow Fred Brooks [5], my dissertation advisor, in using Sayers' more general usage. In both cases, the "sub" part recognizes we are limited to creating within, and from the materials available in, the world God created us in. As the joke goes, when an engineer challenged God by saying he could create a man from dirt, God responded, "Not so fast. You get your own dirt." [28]

There are many motives and metrics for our creative acts. According to David Downing, Tolkien viewed "sub-creation as a form of worship, a way for creatures to express the divine image in them by becoming creators." [7] C.S. Lewis takes this a step further by seeking to simultaneously serve a higher purpose, as in the deep truths and Christian metaphors conveyed in The Chronicles of Narnia. [16] Downing notes a similar difference between composers J.S. Bach, who fulfilled his sense of Christian vocation simply by writing music, and Isaac Watts, who wrote hymns. [7] Brooks [5] emphasizes a different distinction, between natural sciences, which "take the discovery of facts and laws as a proper end in itself", and disciplines of design, including computer science, which more properly measure creations by their usefulness and cost.

Interestingly, Lewis, in the context of literature, also disagreed with the tendency to value originality and innovation for its own sake. Instead, "an author should never conceive himself as bringing into existence beauty or wisdom that did not exist before, but simply and solely some reflection of Eternal Beauty and Wisdom." [17] Downing quotes Lewis as concluding that "of every idea and of every method the Christian writer will ask not 'Is it mine?' but 'Is it good?' " [7] This question of "is it good, by God's metrics?" is indeed critical, and reflects our role as creators within God's creation.

Tools and Their Effects

Much of computer science is concerned not just with sub-creation, but more specifically with creating tools [5]: software and computer systems that serve people and solve problems. So it is important to understand the full impact of those tools. A tool's first kind of impact is what it helps the user do, and the second is the effect it has on the user. John Dyer illustrates this with a shovel: it helps the user dig holes wider, deeper, and faster. And if it is used enough, the user is likely to develop calluses and stronger muscles [8] (chapter 2). In fact, whole classes of tools, like exercise machines and language learning apps, are designed primarily for their effect on the user. These effects are clearly linked to the tool and its use.

The third kind of impact is more subtle, like a "nudge." The decisions people make are predictably influenced by the way the choices are presented, sometimes even to their own detriment. For example, people save more for retirement if they are enrolled by default (and even more if the amount saved is automatically increased). Richard Thaler described this in Nudge [26], and won the 2017 Nobel Prize in Economics "for his contributions to behavioral economics." [20] This effect in web pages and other computerized tools is so important that designers can reasonably be called "digital choice architects." The middle-option bias, the scarcity effect, the decoy effect, and the status quo bias are just some of the mechanisms for influencing choices, and there is a well-documented process for designing and testing the nudge [23]. The goals, motivations, and decisions of myriad "digital choice architects" are clearly significant, particularly since users making choices are commonly unaware of these influences.

These first three kinds of impacts can be combined to create or change habits, and to cause addiction. Nir Eyal documents how companies like Google, Amazon, and the YouVersion Bible app encourage user habits around their products, and why various techniques work. He cautions against creating and knowingly sustaining addictive behavior, pointing out that software knows how it is being used and should at some reasonable point switch from encouraging use to discouraging it. And he proposes an ethical framework based on two questions: do the creators use their own product, and do they believe it can materially improve people's lives? [10]

The fourth kind of impact is yet more subtle, and is sometimes not even fully understood or predicted by a tool's creator. It is suggested by the old saying, "if the only tool you have is a hammer, everything looks like a nail," but it is really the tendencies and values built into the tool. By choosing what to make easy, by the sum of all the nudges, by the choice of feedback (since measurements are often influential), and even by omission, the tool creator helps shape the ecosystem in which the tool is used, including its users. Sometimes, the resulting impact is more apparent in aggregate, in a community, either because the effect on an individual might be small and obscured by individual differences, or because the effects are on interactions [8] (chapter 11).

Marshall McLuhan proposed in 1977 that all human creations, including language, ideas, tools, and clothing, influence humans and society in four ways. Each one enhances something, obsolesces something, retrieves something previously obsolesced, and "flips" into something when pushed to the extreme [19]. Dyer adapts and rephrases this tetrad into a Biblical context for evaluating technology. He proposes evaluating technology by asking four kinds of questions:

- Reflection - how does it reflect God's nature and help people obey God's commands?
- Rebellion - how could it help or tempt people to disobey or rebel against God? [1]
- Redemption - how does it help overcome effects of the fall?
- Restoration - what unintended problems does it bring, and how can they be avoided?

He provides a larger set of questions for each area [8] (Appendix: Technology Tetrad).

In short, software tools have obvious, subtle, and sometimes hidden and even unexpected impacts. All of these effects must be considered in evaluating our work.

3 Creating for Christ

Let us consider what guidance the Bible gives about what tools to build. While "all Scripture is God-breathed and is useful for teaching, rebuking, correcting and training in righteousness" (2 Tim 3:16, NIV), there are a few passages that are particularly relevant for guiding sub-creation in computer science.

[1] Jesus particularly cautions against this in Luke 17:1, Matt 18:6-7, and Mark 9:42.

Start with what Jesus said were the most important commandments: "'Hear, O Israel: The Lord our God, the Lord is one. Love the Lord your God with all your heart and with all your soul and with all your mind and with all your strength.' The second is this: 'Love your neighbor as yourself.' There is no commandment greater than these." (Mark 12:29-31, NIV) If we are loving God with all our being, we certainly will not want to create anything displeasing to him. Rather, we will seek both to live lives pleasing to God, and by our tool building to help others do the same. Paul and Timothy's advice, "Finally, brothers and sisters, whatever is true, whatever is noble, whatever is right, whatever is pure, whatever is lovely, whatever is admirable–if anything is excellent or praiseworthy–think about such things," (Philippians 4:8, NIV) applies both to ourselves and our tools, since they often influence our users' thoughts and attention.

Jesus' second command, "love your neighbor as yourself," particularly as he restated it in Matthew 7:12, "do to others what you would have them do to you," (NIV) provides very practical guidance, including a Biblical foundation for Eyal's two questions (do the creators use their own product, and do they believe it can materially improve people's lives?).[2]

It is notable that God gave work to Adam before sin entered the world: "The Lord God took the man and put him in the Garden of Eden to work it and take care of it." (Genesis 2:15, NIV) It was a blessing to have purposeful, productive activity. It is only after Adam and Eve sinned that God made their work painful and exhausting. (Genesis 3:16-19) So it is reasonable to interpret the joy and satisfaction of creating software as part of God's gift of work, and the frustrations of bugs and other failures as part of the curse that came from sin.

Much later, Paul and Timothy were inspired to write, "whatever you do, work at it with all your heart, as working for the Lord, not for human masters, since you know that you will receive an inheritance from the Lord as a reward. It is the Lord Christ you are serving." (Colossians 3:23-24, NIV) This has long guided believers in our attitude toward work. But it can also provide more specific guidance about creating for Christ. Consider three examples: influencing speech ("The Tongue"), being stewards of users' time ("Stewardship"), and seeking to do avoid and correct bias ("Justice").

3.1 The Tongue

One potential example is the challenging and important area of tools for communication, such as social media. James 3 has very strong warnings about the dangers of the tongue. While these certainly apply to what users say through these tools, tool builders should consider how to help users speak responsibly, or at least not harm them by encouraging the opposite. Since conflict and outrage can be strong drivers of social media "engagement" (short-term usage), simple usage-driven metrics can easily lead developers to make changes that encourage harmful speech, rather than helping forestall it. In fact, there may be a strong financial incentive, at least in the short to medium term, to encourage harmful speech.

But a Christian software engineer or product manager should consider how to help avoid, rather than encourage, harmful speech, in light of Jesus's warning: "Things that cause people to stumble are bound to come, but woe to anyone through whom they come. It would be better for them to be thrown into the sea with a millstone tied around their neck than to cause one of these little ones to stumble." (Luke 17:1-2, NIV) The ideal would be to help users think about "whatever is true,

[2]Though there's no mention in Eyal's work that he was influenced by scripture.

whatever is noble, whatever is right, whatever is pure, whatever is lovely, whatever is admirable" (from Philippians 4:8, NIV). It is not always clear how to provide nudges in this direction. But it is a worthy area for thought and investigation.

3.2 Stewardship

Stewardship is another important Biblical mandate, with multiple implications for computer scientists. Stewardship stems from the fact that "The earth is the Lord's, and everything in it," (Psalm 24:1) and "The Lord God took the man and put him in the Garden of Eden to work it and take care of it." (Genesis 2:15, NIV) In Matthew 25, Jesus defines good stewardship in the parable of the talents (bags of gold), where the master praised servants who invested the money well and punished those who ignored that responsibility. In short, everything we have, including the ability to create software, is really God's. So we must use these gifts to please God – a broad mandate requiring prayer and careful thought to apply in each situation.

An important consequence, and another application of stewardship, is that Christians have a responsibility to write software that helps the user be a good steward of their time. Because mobile devices are becoming ubiquitous in our lives, and are entrusted by their users with considerable time and attention, they have the potential to powerfully shape our habits. There is a strong potential for conflict between the system creators' interests, since they make more money when their systems are used more, and the users' interests. Encouraging, or even just enabling, unhealthy or addictive behavior is clearly bad stewardship of the user's time and attention. It also fails the Luke 17 standard of not causing someone to stumble. In this light, as stewards of God-given software abilities and remembering that Jesus commanded both "love your neighbor as yourself" (Mark 12:31, NIV) and "in everything, do to others what you would have them do to you," (Matt. 7:12, NIV) it is clear that stewardship of users' time should take priority. The opposite, choosing selfish monetary gain at the expense of others' well being, is greed.[3]

Interestingly, at least one large, profitable, secular company believes this is also good business in the long term. Google's "Ten Things We Know to be True" begins with "1. Focus on the user and all else will follow," and goes on to express the unselfish corollary, "we take great care to ensure that [our products] will ultimately serve you, rather than our own internal goal or bottom line" [13]. Nir Ayal agrees, and explains why, summarizing "With very few exceptions, when a product harms people, they use it less or look for alternatives." [9] [4]

3.3 Justice

Righteousness and justice are fundamental to God's character, and this is reflected throughout the Bible. Looking back on the whole Old Testament, Micah 6:8 summarizes "He has shown you, O mortal, what is good. And what does the Lord require of you? To act justly and to love mercy and to walk humbly with your God." (NIV) Currently, machine learning is one of the most important, interesting, and challenging areas of Computer Science in which to apply this

[3] Mark 7:20-23 makes it clear that greed is evil.

[4] Google may be an example of this in another way. One could easily argue that selling porn ads is profiting by helping to harm its users, yet for years Google considered porn "not evil" and sold ads for porn searches. But it stopped in June 2014, in "an effort to continually improve users' experiences." [12]

admonition. Machine learning systems process large amounts of past “training” data in order to discover mathematical “rules” that enable software to imitate past decisions on new data (though it is commonly described as predicting classifications). A significant danger of machine learning is that it can observe discriminatory behavior and learn to do it automatically. It would be a horrible injustice to automate decisions and actions that unfairly harm people or discriminate against them, and to make these actions even more prevalent and hard to prevent. It could camouflage evil with a veneer of impartiality, a quality which people often attribute to computer systems. In addition, it is often impossible to understand the reasons for a machine learning system’s actions, making it harder to root out and fix such problems. Christians must work to create systems that help us do justice, and never the opposite.

While I am not a machine learning expert, I wonder if a system can be trained to help find examples of discrimination in training data, allowing a more just system to be trained on data with fewer unjust examples to learn from. While justice is getting increasing attention in the design and use of machine learning systems, both from observers [21] and researchers [2], much more work is still needed.

4 Creating in a Fallen World

Although computer scientists exercise the gift of sub-creation, we do so as imperfect people in a fallen world. This affects not only what we build but, more fundamentally, how we build it. Four aspects of this problem will be explored, in successive subsections.

- Bugs, Tools, and Testing: We are inherently imperfect and mistake-prone, which means our software inevitably contains bugs (mistakes). So we have to test software to find the bugs, then figure out how to fix them.
- Agile Development: More fundamentally and less obviously, our knowledge and understanding are imperfect, so we do not even know exactly what to build. Agile development addresses this by making iterative discovery and refinement of requirements a central part of the development process.
- Diversity: Human bias and discrimination too easily corrupt our process and products, making it important and necessary to increase the diversity of teams and groups throughout our field.
- Robustness: Because we build in an imperfect world where devices fail and bad things happen, we design systems with redundancy and fault-tolerance, so they will (mostly) keep working anyway.

4.1 Bugs, Tools, and Testing

One of the first experiences of anyone learning to write software is that their code sometimes does not work. It does something wrong and unexpected, and it takes careful and often frustrating and time-consuming “debugging” to find and fix a mistake (or multiple mistakes) that caused the problem. Sadly, this is not just the experience of beginners. Every programmer, no matter how experienced, makes mistakes and introduces errors (“bugs”) into their programs. It is part of the human condition: we are imperfect, fallible, mistake-prone. It is a practical example of both the curse of Genesis and the spiritual struggle with sin that Paul describes in Romans:

> 15 I do not understand what I do. For what I want to do I do not do, but what I hate I
> do. 16 And if I do what I do not want to do, I agree that the law is good. 17 As it is, it
> is no longer I myself who do it, but it is sin living in me. 18 For I know that good itself
> does not dwell in me, that is, in my sinful nature. For I have the desire to do what is
> good, but I cannot carry it out. 19 For I do not do the good I want to do, but the evil
> I do not want to do -—- this I keep on doing. (Romans 7:15-19, NIV).

Just as we are unable to live perfectly, we are unable to write software perfectly.

Because we inevitably make mistakes writing software, new programming languages are periodically designed to help prevent, or at least detect, more and more types of errors. The fact that, over time, these languages become widely used, is an indication of the seriousness of the problem, because changing languages involves a major cost in rewriting or replacing code written in the previous language.

It is also necessary to test software to know if it works as expected. At first, testing was done manually, often by a dedicated group of test engineers. But this can easily become very slow and expensive, because a large software system may need extensive retesting even if only a small fraction of its code has been changed. The work of testing is roughly proportional to the size of the program times the number of releases. This prevents, or at least discourages, frequent releases (which have benefits discussed further on), and the resulting long times between tests makes it harder to fix problems. There are several reasons for this:

- Programmers have had more time to forget details of what they were trying to do;
- More bugs accumulate before testing, and they sometimes interact in ways that make them harder to detect and harder to fix;
- This approach tends to focus on system-level tests, which are often many layers removed from the errors that need fixing.

As a result, many developers and software organizations invest the time to write additional “unit test” code to test what they just wrote. This requires extra work, and extra debugging since the tests can have bugs. But because the tests are programs, they can be re-run quickly and easily (with a testing framework, also developed out of necessity). This makes it practical to retest a whole program regularly as new features are added. In fact, it makes the human test work much more nearly proportional to the work of creating the program.

In summary, the inevitability of human mistakes has led computer scientists to develop tools, techniques, and practices to prevent, detect, and correct bugs (mistakes in writing software). This is an important point to make clear to beginning programmers, who are often surprised and discouraged by their mistakes, and incorrectly assume that there is something wrong with them.

4.2 Agile Development

While bugs and the need for testing are a reality at every scale of software development, another kind of problem becomes apparent in the effort to build large systems: we do not know what to build. This is an illustration of Paul's statement, "For now we see only a reflection as in a mirror; then we shall see face to face. Now I know in part; then I shall know fully, even as I am fully known." (1 Corinthians 13:12, NIV) His point, that our understanding and knowledge this side of heaven are inevitably flawed and incomplete, is profound. And it has many more direct applications than to software development methods. And yet, as God's word frequently does, it explains human nature and difficulties in many contexts, including this one.

Still, the implications for software development have been slowly discovered at great cost. So let us start with some background. When writing a very small program, it is natural just to think about what to do, quickly write the code, and then test and correct it. And it is natural to try to follow the same general process for a larger project, recognizing that there is more work at each step. This pattern of doing each phase in sequence is called the "rational model" or, more vividly, the "waterfall model" (because it is hard to go back upstream past a waterfall). Many large projects following the waterfall model got bogged down and failed. Some failed completely, but most simply took much more time, work, and money than expected. Some of the problems with large projects were attributable to poor planning and poor communication. So more formal processes were put in place to carefully and precisely document the needs, thoroughly work out and document the design of the desired system, and communicate details and changes between the people working to implement and test it. This helped some, but major problems persisted.

Various models were proposed and attempted as improvements on the waterfall model, and gradually an iterative model called "agile development" has been widely adopted as a best practice [4]. While agile development is practiced in a variety of ways, its core is incremental development with frequent input from the users (or "customer"). The team focuses on getting a useful prototype working as soon as possible, and then refines it in short (often 2-week) iterations. In each iteration, some improvements are made and immediate feedback is obtained from users. This process helps refine both the developers' and the users' understanding of the requirements and possibilities [18].

When developing something new, it is often impossible to anticipate exactly how it will be used, and how that will impact related (often human) processes and systems. In other words, we often do not know quite what to build, or how to build it, until we have built and tried it. The iterative process at the heart of agile development seeks to learn quickly and immediately apply that learning to continued development. This is an effective solution for our inherently limited knowledge.

Agile's iterative development works particularly well with unit testing [1]. By writing sets of small, simple tests for each new piece of code, as it is written, the process of testing is included in each iterative step. Those tests also help ensure existing code is not broken in subsequent iterations.

Other engineering disciplines also deal with uncertainty by building and testing prototypes. The main difference is that software is much more flexible than physical building materials, so an early prototype can often be more or less continuously refined into a useful production system. Unlike most physical creations, a good software system is rarely "final": if it is used successfully, opportunities for further improvements are almost always found, and these are often implemented in further iterations as priorities and resources allow.

The point of this discussion of agile programming methodology is that it enables software development organizations to cope with not only the imperfection of humans which leads to bugs, but also our inherently limited knowledge of our users and how to help solve their problems. Since we do not know exactly what to build, it helps to build iteratively so we can test as early and frequently as possible how our creation fits (and does not fit) the need, and refine our plans appropriately.

While God could use the "waterfall" method to speak the world into creation according to his perfect plan, humans do not have the perfect knowledge required for a perfect plan. Since we are doomed to trial and error, it works best to embrace that and do it as efficiently as possible.

4.3 Diversity

Throughout history, human societies have shown a strong tendency to mistreat and discriminate against "others" unlike the locally dominant group. This seems to be part of our sinful nature, since scripture teaches:

- "The foreigner residing among you must be treated as your native-born. Love them as yourself, for you were foreigners in Egypt." (Leviticus 19:33, NIV)
- "Cursed is anyone who withholds justice from the foreigner, the fatherless or the widow." (Deuteronomy 27:19, NIV)
- "For we were all baptized by one Spirit so as to form one body – whether Jews or Gentiles, slave or free – and we were all given the one Spirit to drink." (1 Corinthians 12:13, NIV)
- "Therefore go and make disciples of all nations, baptizing them in the name of the Father and of the Son and of the Holy Spirit, ..." (Matthew 28:19, NIV)

The United States has a particularly bad history of racial discrimination [29], which contributes to and exacerbates the problem within computer science. Even when we disavow discrimination, we find that implicit bias remains and causes harm [15]. Past discrimination, implicit bias, and other factors combine to create systemic discrimination, which is very difficult to overcome [14] [25].

This is very serious, but what does it have to do with computer science? Computer science is a predominantly white male field. Women and every ethnic group other than white and Asian men are under-represented. Since computer science jobs are generally very well paid and have desirable working conditions, it is unfair that others are left out. This can be a self-reinforcing problem, since members of those groups who try computer science may easily be dissuaded by the discomfort of being such a small minority (and the implicit bias that often happens in this situation). In addition, products designed primarily by one group are often biased toward users in that group and do not work as well for others [30]. Finally, software is often a part of the discriminatory system. Sometimes this is because it was designed by those in the majority and contains their biases (whether explicit or implicit). It can also result from machine learning, which is prone to learning and repeating biases present in the past data it is learning from [21]. All of these effects combine to multiply the difficulties for people who are members of more than one disadvantaged group, like black women [6].

In short, the predominance of white males in computer science, combined with human bias and discrimination, causes a wide variety of serious problems. It is therefore important for Christians in computer science to work to diversify the field and mitigate the damage. We should work hard to recruit, welcome, support, and encourage women and members of underrepresented minorities into the field.[5] In the meantime, we should also strive to reduce and eliminate bias in the systems we create. Both of these are very important, difficult challenges.

4.4 Robustness

Human nature is not the only source of problems in our post-fall world. Physical things also fail. A lot of work and creativity are required to create a reliable system. I experienced this extensively as the first manager of Gmail's Site Reliability Engineering (SRE) team. Every level of the system had to be designed to work despite hardware, software, and human failures. At the infrastructure level, this included the Google File System (GFS), which accommodated both disk and computer failures, the job scheduling system, which would restart servers (programs) on different computers when a computer running them (or its network connection) failed, and the network, which routed packets around failures in the datacenter network as well as the links between datacenters. It also included code we wrote to take broken machines out of service for repair, notify technicians (who used their own software to schedule the repair work efficiently) and put computers back in service after they were fixed.

At a higher level, the Gmail server software and Google's authentication servers were written to tolerate hardware failure without losing data and with as little disruption to service as possible. When a server failed for any reason, another server would quickly take over its work. The Gmail storage servers were written with special care not to lose user data, even if a bug was introduced. All new data, both email received and sent and other actions like "archive" and "mark spam", was immediately logged in multiple places. Then, if either software bugs or hardware failure caused corruption or loss, the database could be reconstructed from a copy of the log.

In order to know if the service was working, and to be able to investigate and fix problems, we had monitoring systems observing and recording behavior of the system at multiple levels. We wrote rules so the system could alert us immediately to problems that needed human attention, and had an on-call rotation schedule so there was always someone to respond to these alerts quickly (and someone else to help them if needed). We generated graphs of various errors, and studied them regularly to find problems, prioritize them by user impact and risk, and consider how to improve the system to eliminate or reduce them.

When there was a major user-visible outage, the engineer most familiar with it wrote a post-mortem describing in precise detail the sequence of events leading up to the problem, how it was discovered, how it was fixed, and the impact on users. This was followed by the most important part, a prioritized list of changes needed to keep it from happening again. Since some outages were caused or exacerbated or extended by human error, it was critical for everyone on the team, as well as higher management, to understand that the post-mortem's purpose was to prevent future problems, not to assign blame. It was equally important to recognize that human errors are inevitable, so a conclusion that we should "be more careful" or add an "are you sure?" prompt was generally useless. Instead, the system needed to be changed so it was not so easy to make mistakes that

[5] The Grace Hopper Celebration of Women in Technology and the ACM Richard Tapia Celebration of Diversity in Computing seek to encourage and support people in these groups. In addition, the sponsors, AnitaB.org and the Center for Minorities and People with Disabilities in IT (www.cmd-it.org), respectively, seek to identify and publicize best practices for recruiting, welcoming, and helping them thrive.

caused an outage. Often, this meant working to automate a task (by writing a program to do it). Since this took time, we often did it in stages, first automating a safety check or a few steps in the process, until the whole thing was done. When that was not practical, another solution was to redesign part of the system so a mistake did not have such bad consequences. This helped the system tolerate more types of human errors, in addition to various types of hardware and software failure.

This extended discussion of Gmail and its SRE team is just one example of how systems are routinely created and operated to work reliably despite the myriad failures endemic to our fallen world. Google SREs later wrote a whole book about the principles and practices involved [3].

5 Conclusions

There is great joy and frustration in creating and improving computerized systems, reflecting God's gift of sub-creation and the reality of failure in our sin-stained world. This gift, like life itself, comes with God-given instructions and responsibilities. For example, our systems should help our users tame their tongues, not entice them to harmful speech, and our systems should be good stewards of their users' time, never exploiting them for greedy or self-serving ends. Justice should be a prime focus, particularly when using machine learning.

Understanding our sinful nature and fallen world leads to a richer and deeper understanding of many important software development techniques. Debugging, testing, agile development, and redundancy are all productive and appropriate responses to our human nature and fallen world. Instead of asking "why does this work?", we can wonder "why did it take so long to figure this out?" and "what else should we try?" Similarly, this understanding helps explain why we struggle with bias and need a strong focus on increasing diversity in computer science.

Finally, as a personal extension of the theme of stewardship, I am keenly (yet, sadly, only intermittently) aware that nothing I have is really mine. My abilities, my knowledge, my possessions, my time, and my family are all gifts from God. So my true call is to be a faithful steward.

References

[1] Kent Beck. *Extreme Programming Explained: Embrace Change*. Addison-Wesley, second edition, 2004. The first edition was published in 1999.

[2] Rachel K.E. Bellamy, Kuntal Dey, Michael Hind, Samuel C. Hoffman, Stephanie Houde, Kalapriya Kannan, Pranay Lohia, Sameep Mehta, Aleksandra Mojsilovic, Seema Nagar, Karthikeyan Natesan Ramamurthy, John Richards, Diptikalyan Saha, Prasanna Sattigeri, Moninder Singh, Kush R. Varshney, and Yunfeng Zhang. Think your artificial intelligence software is fair? think again. *IEEE Software*, 36:76–80, July-Aug 2019.

[3] Betsy Beyer, Chris Jones, Jennifer Petoff, and Nial Richard Murphy, editors. *Site Reliability Engineering: How Google Runs Production Systems*. O'Reilly Media, 2016.

[4] Frederick P. Brooks, Jr. *The Mythical Man-Month: Essays on Software Engineering*, pages 254–273. Addison-Wesley, anniversary edition edition, 1995.

[5] Frederick P. Brooks, Jr. The computer scientist as toolsmith II. *Communications of the ACM*, 39:61–68, March 1996. http://www.cs.unc.edu/~brooks/Toolsmith-CACM.pdf. This wise and perceptive paper, summarizing decades of thought by a fervent Christian and respected Computer Scientist, has stood the test of time. It's worth reading now, before finishing the present paper.

[6] Kimberly Crenshaw. Demarginalizing the intersection of race and sex: a black feminist critique of antidiscrimination doctrine, feminist theory and antiracist politics. *University of Chicago Legal Forum*, 1, 1989. Article 8, https://archive.org/details/DemarginalizingTheIntersectionOfRaceAndSexABlackFeminis.

[7] David C. Downing. Sub-creation or smuggled theology: Tolkien contra Lewis on christian fantasy, undated. www.cslewisinstitute.org/node/1207.

[8] John Dyer. *From the Garden to the City: The Redeeming and Corrupting Power of Technology.* Kregel Publications, 2011.

[9] Nir Eyal. The real reason Apple and Google want you to use your phone less, June 2018. https://www.nirandfar.com/2018/06/google-apple-less-phone-use.html#more-3348.

[10] Nir Eyal and Ryan Hoover. *Hooked: How to Build Habit-Forming Products.* Portfolio Penguin, 2014. Eyal's work was influenced by Stanford Pofessor BJ Fogg; see [24]. BJ Fogg summarizes his theory of behavior at http://www.behaviormodel.org/index.html.

[11] Verilyn Flieger. On fairy-stories, undated. www.tolkienestate.com/en/writing/translations-essays/on-fairy-stories.html.

[12] Google Inc. Adult content. In Advertising Policies Change log, https://support.google.com/adspolicy/answer/4271759?hl=en&ref_topic=29265.

[13] Google Inc. Ten things we know to be true, 2004. https://www.google.com/about/philosophy.html, accessed Aug. 6, 2018. (Google has certainly been criticized for not always living up to this lofty goal.) An earlier version, from 2004, https://web.archive.org/web/20040603020634/http://www.google.com/corporate/tenthings.html, says "by always placing the interests of the user first, Google has built the most loyal audience on the web." A version in similar form existed before Sept. 2002.

[14] Daniel Hill. *White Awake: An honest look at what it means to be white.* Intervarsity Press, 2017.

[15] Kirwin Institute for the Study of Race and Ethnicity, The Ohio State University. Understanding implicit bias, undated. http://kirwaninstitute.osu.edu/research/understanding-implicit-bias/.

[16] C.S. Lewis. Sometimes fairy stories may say best what's to be said. *New York Times*, November 1956. www.nytimes.com/1956/11/18/archives/sometimes-fairy-stories-may-say-best-whats-to-be-said, also found at apilgriminnarnia.com/2014/01/27/sometimes-fairy-stories/.

[17] C.S. Lewis. *Christian Reflections*, chapter Christianity and Literature, page 7. Wm. B. Eerdmans Publishing Co., 1967.

[18] Robert C. Martin. *Agile Software Development: Principles, Patterns, and Practices.* Prentice Hall, Pearson Education, 2003.

[19] Marshall McLuhan. Laws of the media. *ETC: A Review of General Semantics*, 34(2), June 1977. https://www.jstor.org/stable/42575246.

[20] Nobel Media. The prize in economic sciences 2017, July 2018. www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2017/.

[21] Cathy O'Neil. *Weapons of Math Destruction*. Crown Publishing Group, Penguin Random House, 2016.

[22] Dorothy L. Sayers. *The Mind of the Maker*. Harcourt, Brace, 1941.

[23] Christoph Schneider, Markus Weinmann, and Jan Vom Brocke. Online user choices through interface design. *Communications of the ACM*, pages 67 – 73, July 2018.

[24] Simone Stolzoff. The formula for phone addiction might double as a cure. *Wired*, February 2018. https://www.wired.com/story/phone-addiction-formula/.

[25] Beverly Daniel Tatum. *Why Are All the Black Kids Sitting Together In the Cafeteria: And other conversations about race*. Basic Books, Hachette Book Group, 1997, 2017.

[26] Richard Thaler and Cass Sunstein. *Nudge: Improving Decisions about Health, Wealth, and Happiness*. Yale University Press, 2008.

[27] J.R.R. Tokien. On fairy stories, 1947. brainstorm-services.com/wcu-2004/fairystories-tolkien.pdf. According to [11], this paper started as a lecture in 1939, was published in *Essays Presented to Charles Williams* (1947), was included in *The Tolkien Reader* (1968), and was eventually published stand-alone (2008).

[28] Reddit.com user 'Jonzell'. Get your own dirt. https://www.reddit.com/r/Jokes/comments/b0faut/get_your_own_dirt/.

[29] Jim Wallis. *America's Original Sin: Racism, White Privilege, and the Bridge to a New America*. Brazos Press, 2016. I found this book to be particularly enlightening and convicting.

[30] Gayna Williams. Are you sure your software is gender-neutral? *Interactions*, page 36ff, January-February 2014. http://interactions.acm.org/archive/view/january-february-2014/are-you-sure-your-software-is-gender-neutral.

Thinking Beautifully about Mathematics
A View of Mathematics as the Science of Measurable Orders

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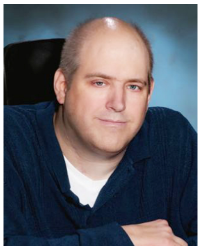

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Abstract

We initiate a view of how beauty is seen in mathematics as one that highlights the human endeavor to explore and articulate the orderly features of creation. To that end, we focus on what order is , how order is understood in our physical universe as the created order, and how mathematicians study order through the means of measuring through quantities (generally understood). It is in terms of this last way that we will define the notion of measurable order and further define mathematics as the science of them. We will describe how mathematics arises from and relates to the natural sciences by viewing the latter as aimed at an empirical understanding of the created order and the former as an intellectual endeavor that is initiated by our encounter with the created order and then discovered through abstraction. Mathematics will then be seen as pertaining to a natural science when its subject-matter can be understood with respect to a measurable order by way of abstraction. In this light, measurable orders can be viewed, in general, as abstractions from orders that (potentially) pertain to the created order.

From such a vantage point, we articulate a perspective of beauty as one that arises from the human endeavor to explore and discover the wonders found in the created order. Here we broaden an understanding of exploration and discovery to pertain to the created order in either an empirical mode, as with the natural sciences, or in a speculative mode, as with mathematics.

We close with a theological perspective of order that forms a basis for our view of mathematics as the science of measurable orders.

1 Introduction

The goal of this paper is to initiate a viewpoint of mathematics in which the subject-matter is seen as a synthesis of both being originated in human understanding and as central in the achievement of scientific advancement. To that end we will be guided in our discussion by addressing two important questions pertaining to the nature of mathematics:

1. Wigner's Problem[1]: Why is mathematics unreasonably effective to the natural sciences?
2. Ontology/Epistemology Problem [2] : Are mathematical objects invented or discovered?

[1] Eugene Wigner, "Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications in Pure and Applied Mathematics*, Volume 13, No. I 1960

[2] For recent discussions on this problem, see: Reuben Hersh, "On Platonism." European Mathematical Society Newsletter 68 (June 2008): 17; Barry Mazur, "Mathematical Platonism and Its Opposites." European Mathematical Society Newsletter 68 (June 2008): 19-21; *Mathematics through the Eyes of Faith,* eds. James Bradley and Russel Howell, 2011: Ch. 10.

Our approach will be to address both of these questions together by articulating a broadly understood notion of **order** and seeing the natural sciences and mathematics as human endeavors to explore order in their proper contexts. In particular, we will understand the type of orders that the natural sciences explore, the orderings found in our physical world, as uniformly part of the **created order**. As such, we shall think of the physical world as ordered in a particular way, due to it being an **existing order** arising from an **act of creation** which determines both its existence and its intelligibility. Uncovering aspects of that order is a process that is initiated through the senses, in order to disclose its particular features. Understanding, though, arises as an act of intelligence which theorizes the relevant properties and relations that pertain to that order. An act of senses then, through appropriated tools, conditions, and experimentations, seek to verify, modify, or reject the hypothesis. Within this process, only certain parts and properties of the physical world are paid attention to in order to discover the pertinent objects and the laws that pertain to them within that ordering. We will want to be explicit about this mode of selective attention, which we refer to as **abstraction**, and which places the act of understanding completely within the cognitive realm. It is within this realm that order may be speculated on as it pertains to a potential created order, which we will refer to as a **cosmic order**. It is this last concept that we seek to find an explicit description of that pertains to orders that are either potential or created.

An important aspect of both theorizing and then verifying through experimentation is the way of associating quantities to properties and relations and then making measurements. In terms of orderings and the things that are ordered, quantifying them pertains to only certain aspects of their overall nature. These aspects are arrived at by abstraction and we will seek to characterize precisely those things and the ways they may be ordered which can be understood by quantities and quantifying. We will refer to such orders as **measurable orders** and define **mathematics as the science of measurable orders**. Moreover, we will hold, with Thomas Aquinas, that, to the extent a measurable order pertains to material things, the matter of the measurable order describes is strictly intelligible in nature. As such, measurable orders are not restricted to orders within creation alone, but, again, to any order that can be potentially created. In order to make that idea coherent, we need an understanding of what underlies our knowing of creation *per se* that is either potential or actual. Here we will need a sufficient account of **being** which underlies all that which does or can exist. We will see that the condition of being is not only the minimum condition for an order to be part of a creation, potential or actual, but provides the basis for understanding the minimal conditions for things to be understood as participating in an order.

With these perspectives of order in place, we may view material orders as occurring within a **hierarchy of orders**, beginning with the most particular, the created order, to the most universal, measurable orders. The levels within this hierarchy can be understood through the sciences, related by ascending via greater degrees of abstraction. For example, biology → chemistry → physics → mathematics. In terms of how this hierarchy is understood to pertain to a creation, either actually or potentially, we will identify two modes of thinking: creational thinking and thinking beautifully [3].

Creational thinking is a way of thinking of orders in two related modes. The first is the **empirical mode**, typically engaged within the natural sciences, in which the focus is on determining properties and relations within orders valid for the created order and verified via proper experimentation. The second mode is the **speculative mode** which aims to ascertain what is valid pertaining to properties of and relations among things within the order based purely upon that order's intrinsic nature. As such, there is no presumed expectation that such validations pertain to the created

[3] My thanks to Dr. Laura Smit for suggesting this last phrase.

order, but to orders that may be potentially created. Thus, for example, in chemistry, elements may be speculated to exist (beyond the current table of elements) and then determine possible ways molecules may be formed from them. Or contemplate, in physics, a physical universe classically behaving according to an inverse cube law. Ultimately, discovery in a natural science occurs when the two modes are intertwined: First, by posing theories, in the speculative mode, that serve as hypotheses for what is true of the created order. Then, in the empirical mode, such theories are verified or rejected by experimentation.

Finally, **thinking beautifully** is an enhancement upon creational thinking in which the seeking and revealing of truths pertaining to an order is made intentionally as part of our conscious awareness. Here, again, we distinguish two modes. The first, we call **(the mode of)** ***invenire*** is one which aims at making or enriching discoveries of objects by the selection and/or invention of tools which seek out ever richer and ever deeper properties and relations within the order the objects belong to. The second mode we will refer to as the **reflective mode** which seeks to disclose the very order by which a discovery is arrived at and made. This seeks to make clear the ways in which the human investigator, as a participant of the creator order, comes to make discoveries pertaining an order from the harmonious interplay of the acts of human senses and human understanding.

It is the aim of this paper to be the first part of two in which the concept of thinking beautifully is unveiled and made explicit in the context of mathematics. To that end, we first clarify our notion of order and distinguish between the created order and a more broader notion of cosmic order. This will then lead us to the concepts of abstraction and of creational thinking and its two modes. We will then focus on the particular type of orders relevant to the subject-matter of mathematics. Here we first clarify what we take to mean by quantity and how orders are quantified. This leads to our notion of measurable orders and of mathematics as the science of them. We then analyze how we understand thinking beautifully about mathematics in terms of the mode on *invenire*. We will reserve our analysis of thinking beautifully about mathematics in the mode of reflection to part two. Finally, we will analyze how any cosmic order, potential or actual, can pertain intelligently to a potential creation when we properly understand what it is for there to be a Creator which is the source of the existence and intelligence of that cosmos were it to be created. We will base this analysis on the theology of Thomas Aquinas and suggest how scripture supports it.

We close this introduction by noting that the ideas formulated in this paper resulted from a long engagement with and meditation upon the thought of Thomas Aquinas. In particular, the work by Bernard Lonergan in his magnum opus *Insight* [4] has been instrumental. In the sequel to this paper, we will take a closer look at how Lonergan's notion of insight provides a medium for understanding how a mathematician discovers and verifies mathematical truths.

2 Understanding creation through order

In order to get a sense of the subject-matter of mathematics as a science, we will first describe how we understand those features common to things. These are features that are perceived by our senses. It is our aim to identify those features of things generally conceived sufficient for enabling a scientific investigation, broadly understood.

To begin, we need to be specific about what it is to understand a thing as a concrete individual in

[4]Bernard Lonergan, *Insight,* University of Toronto Press; 5th edition (1992) (Hereafter *Insight*)

the world. Following the **common sense** perspective of Lonergan[5] in which we initially understand a **thing in itself** as *a concrete, whole, identity, unity which is, in relationship to ourselves, what is immediately received and understood through our senses.* In this initial awareness of such a thing, there is no effort to divorce that thing from its environment, nor sequester from it particular features from what it is in its holistic unity. Rather, the incipient understanding of that thing is as a **being**, that is, *as the objective of the pure desire to know.*[6] As such, a true understanding of that thing must conform to two laws. First, the **law of non-contradiction:** *a thing cannot simultaneously both be and not-be.* As such, any scientific claim about something in our universe that violates this law fails in some way to say something true about that thing. As Tuomas Tahko frames it: "[T]he law of non-contradiction is a general principle derived from how things are in the world. For example, there are certain constraints as to what kind of properties an object can have, and especially: some of these properties are mutually exclusive."[7] Second, **the law of excluded middle:** *a thing must either be or not-be.* This law preserves the connection between the speculative and empirical understanding: the success of an act of judgement that is made in the speculative mode ensures the success that a corresponding act of judgement can be made in the empirical mode (see next section). In other words, this law serves to bridge hypothesis and verification.

A further consequence of understanding a thing as a being, is its conformity to the **law of sufficient reason:** *everything that comes into being has a cause for its existence.* Thus scientific reasoning is movement from cause to effect, and so truths are either self-evident truths, first principles, or obtainable in an orderly way from self-evident truths, first principles and/or established truths. [8]

In this common sense first approach to understanding, we take as primary those things that inhabit the real world. As such, it takes the position that a true understanding of what a thing is is one which must be non-reductionistic, as it must first and foremost be taken as a whole unity. Thus a common sense view of a thing is to be distinguished from a scientific view, in that the latter seeks to understand that thing in terms of a particular viewpoint that reduces what it is to be to how it is to be constituted by those elements which that science takes as fundamental. From this incipient viewpoint, we understand a thing in itself to be **intrinsically ordered** as it is part of the order that is integral to creation. We will then identify this initial act to understand a thing in itself, received via common sense, as an **act of contemplation** of the intrinsic order of that thing.

A science will then be understood to traffic in a mode of understanding which views a thing as a whole reducible to certain fundamental parts (henceforth **elements**). We therefore distinguish between a thing in itself, as first understood by common sense, and the same thing as understood scientifically. In terms of the former, we will, henceforth, refer to that thing as a **created thing** and, then, use the term **thing** more generally as the subject of any mode of understanding. It is then the question of when a scientific approach to a created thing has achieved a degree of knowledge of that created thing by the way that thing is investigated by that science. The standard approach is by the method of **abduction**: the interplay of hypothesis and verification. In the end, though, the knowledge gained by that science is limited by the material assumptions and relations that that science understands how that thing is to be so constituted.

[5] *Insight* §IV.1

[6] *Insight*, §XII.1

[7] Tuomas E. Tahko, "The Law of Non-Contradiction as a Metaphysical Principle", *Australian Journal of Logic* (7) 2009, 32

[8] For more on these two laws, see W. Norris Clarke, *The One and the Many: A Contemporary Thomistic Metaphysics*, University of Notre Dame Press, 2001, pp. 19-23

To frame how we take a science to understand a thing, we define what it is to be an order. To that end, we need to be able to distinguish between what an order is within a thing and what is an order among things. To do so, we first consider Lonergan's understanding of **genus** [9]. This is a way to broadly portray how a range of things are considered in a unified way by a specific science in terms of the elements that science takes as fundamental. So sub-atomic physics considers the genus of atomic things; chemistry considers the genus of molecular things; biology considers the genus of living cellular things. In every case, a genus of things unifies those things as constituted by the pertinent fundamental elements and relations between them. In terms of the latter, these **correlations** are how the science of that genus understands and relates those things within it.

What is important in Lonergan's account is that while genera can be viewed in a hierarchy (genus of cellular things can be studied in terms of the genus of molecular things which, in turn, can be studied in terms of the genus of atomic things) the correlations of a higher genus are neither independent of nor reducible to that of a lower genus. Thus while things in a higher genus can be in certain ways described by things of a lower genus, those things in the lower genus will not be found as things in the higher genus.

Now, we will define an **order** to be an organization of things in accord with the relevant elements and correlations that constitute the formation of a genus. An **order within** a given thing will refer to how it can be in its particular genus in accord with things as constituted by the elements and their correlations coming from lower genera, in the hierarchy, for which that particular genus is at the apex. From that perspective, that thing is understood to **participate** in that genus or order. An **order among** given things is the totality of correlations that pertain to those given things that are understood as participants of that genus.

Finally, we use the term **created order** to refer to our own physical universe with its hierarchy of orders and define a **cosmic order** to be the view of a cosmos together with its particular hierarchy of orders that arises from speculation about how our universe could be.

3 Abstraction and creational thinking

In order to come to how we will understand the subject-matter and science of mathematics, we will describe the way humans understand and reason about cosmic orders. What is foremost about delineating such a description is the recognition that any disclosure of truths pertaining to an order is initiated in the senses, but is only finally understood in the mind. This transition is accomplished through the process of abstraction to which we need to give careful attention.

We begin by following Thomas Aquinas [10] to define **abstraction** as the "operation ... by which in understanding what a thing is, it distinguishes one thing from another by knowing what one is without the other, either that it is united to it or separated from it". In our particular context, it is by way of abstraction from things of our experience that orderings of those things (that is, their genera and its elements and correlations) are discovered and determined. Furthermore, in arriving at an understanding of things in terms of a particular genus, we need to only understand first how each are ordered within, with respect to that genus. As such, those aspects that are not pertinent to understanding a thing within its particular genus (for example, particular time, particular place)

[9] *Insight* pp. 280ff

[10] Armand Mauer, *Thomas Aquinas: The Division and Methods of the Sciences*, (Mediaeval Sources in Translation) Pontifical Institute of Mediaeval Studies; 3rd ed. (1963), pp. 28-31

are abstracted away. Further, those aspects that remain and are pertinent to the genus it inhabits, along with the relevant ordering, are universal with respect to that genus.

Scientific investigation within the created order begins by understanding that order as pertaining to a cosmos. From such a standpoint, investigation of things is initially conducted in abstraction. As such, classical laws pertaining to things are first understood as pertaining to some cosmos. Discoveries then arise and are understood as perceived within that cosmos and with respect to its particular order. To ascertain whether the created order conforms to that cosmic order, investigations are conducted, through particularly crafted experiments, in which the required abstract conditions are sharply fulfilled, to obtain a verification. We will take this perspective on the practice of science as a special case of **creational thinking**. In particular, we will take the activity of the scientist as a creational thinker to be one in which hypotheses are proffered about the created order, but initially understood to pertain to some cosmos. In this way, investigation is conducted within that science's particular theoretical framework. The scientist may then endeavor to determine whether the resulting consequences obtained are valid to the created order by empirical means via, for example, experiments within a laboratory.

Thus, in general, creational thinking aims to understand ways our universe may be organized as a cosmic order. The initial stance is to engage in creational thinking in the **empirical mode**. This particular type of creational thinking requires the common sense elements of experimenting, testing and verifying theories conducted within the physical realm. The development of those theories, though, involve engaging in speculations by posing such questions as "Is it the case ...?" or "What if ...?". This is the **speculative mode** of creational thinking and is involved in any effort to understand any ordering that can take possibly place in our cosmos. For example, such theoretical models in physics, such as superstring theory, or, more generally M-theory, is well developed speculatively, but still seeks experimental verification. Such a theory may be viewed as creational thinking pertaining to a cosmic order which physicists, in the empirical mode, are seeking to establish whether that order is (relevant to) the created order.

By ordering creational thinking so that the empirical mode is first, knowledge begins with our sense reception of the world and our common sense understanding that results. Subsequently, understanding the world more deeply requires the speculative mode by abstracting in things what it takes to theorize within the particular scientific mode and testing the theories produced within (for example) a laboratory to verify, modify, or reject. The speculative mode can also be engaged to postulate theoretical frameworks for orders and produce theories that pertain to such orders without any immediate requirement that they be found relevant to the created order via the empirical mode. Here, things may simply be understood as being potentially created things by simply presuming them to possess being and test their potential existence through the scientific theories themselves. Ultimately, though, it is only in the empirical mode that speculation may begin and it is only a return to that mode that the pertinence of those theories to the created order may be tested for their validation.

As we continue to set out how we perceive the concept of order and what constitutes a science of order , we note again that orders can be viewed in a hierarchical fashion in which one order found on a higher level will be common to all the orders found on lower levels. For example, we have the following hierarchy

$$\text{living things} \to \text{molecular things} \to \text{atomic things}\ .$$

Each level has their orders within and among, but also may be understood via the higher levels. In each level, there is a corresponding science: biology, chemistry, and physics. A science of each level may be conducted independent of the lower levels and each science may be conducted either speculatively or empirically. Finally, we note that any and all such hierarchy of orders are understood level-wise and together as bound by the condition of potentially or actually having being.

We now aim to concentrate on the type of orders that are potentially common to all possible orders, namely orders of things understood through quantity alone. Such an order is called a measurable order and mathematics as its science.

4 Quantity and the quantified

We now look at a specific type of order, one that we claim every other order participates in, namely order that is understood through **quantity**. Thus, to frame quantity in a way to speak to order in the broadest sense, we first identify two characteristics of quantity that we will take as fundamental at the basic level:

I. That which can be increased or diminished (L. Euler [11])

II. That by which things are inclined, distinguished, and limited. (Bonaventure [12])

The first of these is straightforward and expected in how we initially conceive quantity. The second brings to quantity the way equality works by, for example, the requirement that quantities associated to equal things must be equal. We further codify this utilizing Euclid's Common Notions, which we refer to as **basic rules of calculation**:

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole is greater than the part.

Taken together, these propositions provide not only the first principles for being a quantity, but also for a thing to be **quantified**. Furthermore, to be clearer on this last point, we shall stipulate the conditions under which quantifying occurs. To obtain such conditions, enough to be sufficiently universal, we follow James Franklin [13] by requiring:

A. Parts unified into a whole.

B. Intrinsic notions of same and difference.

[11] Leonard Euler, *Elements of Algebra*, Springer; 1972 edition
[12] Saint Bonaventure, *Journey to the Mind of God*, Translated by Oleg Bychkov, I.11
[13] James Franklin's *An Aristotelean Realist Philosophy of Mathematics*, Palgrave Macmillan; 2014 edition

Together these two conditions determine a **structure**. In keeping with our current theme, we will define a **measurable order** to be a structure together with the parts quantified by the **rule of invariance:** *two parts that are the same have equal quantities.*

Before we examine how to define mathematics in terms of measurable orders, we first need to identify what constitutes the range of things that can be counted as quantity. Here we primary follow Lonergan by taking an **ordinal** perspective of quantity and note that a **cardinal** perspective is also equally important for framing quantity, but we will forgo an analysis of that viewpoint.

We begin with the **natural number system**:

$$0 \quad < \quad 1 \quad < \quad 2 = 1 + 1 \quad < \quad 3 = 2 + 1 \quad < \quad 4 = 3 + 1 \quad < \quad \dots$$

Implicit in this description is the sufficiency of 0, 1, $+$, $=$, and $<$ to generate this number system along with the "$\dots$" to indicate the **well-ordering principle** at work to provide this system with it's potentially infinite completeness. Additional operations of $\times$ and exponentiation can be further defined.

Next, inverse versions of the operations (difference, division, roots, logarithms) can be constructed and properties determined, but are limited in when they can be defined. These limitations can be overcome by defining new numbers from these operations (rational numbers, irrational numbers, imaginary numbers) limited only by the rules of non-contradiction (e.g. no division by 0) and sufficient reason. This in turn leads to new number systems (rationals, reals, complex numbers) each having operations and rules of calculation under which they operate. Furthermore, the rules enable the operations to act on numbers in an indeterminate fashion, allowing the introduction of variables, and leading to polynomials, rational functions, and, eventually, functions in general.

This process of developing new systems from old by the introduction of new operations and constructions that require, in turn, the introduction of additional new objects and the modification of rules for the operations and constructions is the process of producing (after Lonergan) **higher viewpoints** [14]. From this concept, we will define a **general quantity** to be any term found in a system that is constructed as a higher viewpoint of the natural number system. We note that general quantities, either of specific or abstract type, are themselves measurable orders. Moreover, we further modify our definition of measurable order to allow for quantifying via general quantities. Thus, we may finally we make the following definition: **mathematics is the science of measurable orders.**

5 Measurable orders from cosmic orders

We now compare our concept of measurable orders with the notion of cosmic order. In particular, we show that any cosmic order has the capacity to be understood at a certain level as a measurable order in a way consistent with that particular cosmic order. To begin, we first observe that any thing viewed as participating in a particular cosmic ordering must, in a primary way, participate in being. Following Aristotle, as delineated by Thomas Aquinas, we can understand how things participate in being in terms of a proper order:

- First as being.

[14] *Insight* pp. 40ff

- Second in terms of quantity.
- Third in terms of sensible qualities.
- Fourth in terms of particular place and time.
- Fifth in terms of being subject to change.

Now, understanding a thing in terms of this hierarchy at a particular level depends on understanding that thing in terms of earlier levels. From the perspective of the matter that constitutes the thing, the last three levels are understood in terms of **sensible matter**. This is not so for the first two levels which are understood independent of the subsequent levels and so are, together, understood in terms of **intelligible matter**. From this, we draw two conclusions. First, quantity and the quantifiable (and so, measurable orders) are materially constituted in terms of intelligible matter. Moreover, understanding a thing in terms of intelligible matter is arrived at by abstracting from the sensible matter in the thing. We will refer to this manner of abstracting intelligible matter in a thing as **formal abstraction**. Second, any thing viewed as participating in a particular cosmic order will do so in terms of some aspect of its sensible matter. Scientifically, such aspects must belong to the sensible qualities in order to locate classical laws that are empirically verified. Furthermore, the elements and correlations of that order will also be understood in terms of those sensible qualities. By formal abstraction, the thing in terms of those elements and correlations can be quantified. From that, a measurable order is determined by abstracting the intelligible matter in those elements and their correlations. Thus, measurable orders serve cosmic orders by:

- interpreting correlations in terms of identities,
- identifying when correlations hold, and
- giving limits to when correlates and/or correlations may obtain.

We therefore conclude that a scientific perspective of a cosmic order is one that can be understood through measurable orders that potentially arise from abstractions. In turn, those cosmic orders that can be understood and analyzed by a particular measurable order will be ones that fit a **model** of that measurable order through abstraction. Thus, the science of measurable orders provides a context to study any cosmic order. Finally, Wigner's Problem is addressed to the extent that that science discloses an understanding of the created order by speculation of the physical universe as a cosmic order. Then the particular order that is the purview of that science can be framed within a measurable order once things within that scientific genus, in terms of elements and correlations, are properly quantified and modeled.

6 Thinking beautifully about mathematics

We now initiate a look at the shape of creational thinking in regards to the science of mathematics. To that end, we will describe a perspective on the structure of mathematical knowledge as it pertains to the search for and classification of mathematical objects.

The aim to understand mathematical objects and provide a classification of them raises the Ontology/Epistemology Problem. From that perspective, we will consider the elements of what is part of invention. In particular, we need to clarify the ways the human will plays a role in navigating a path toward making a discovery. In particular, forming questions, selecting tools, reviewing prior research, making calculations, performing experiments, forming and proving propositions, etc. We will then examine how objects are defined, analyzed, and understood as existing. This will lead us to addressing in what ways the role of discovery is at play. In the end, we will understand these two objectives as unified within the **mode of *invenire*** when thinking beautifully about mathematics.

We begin by describing how objects are defined. A **mathematical object** will be understood to be *a thing that is constituted simply by intelligible matter alone.* Such objects can then be understood within mathematics through a **field of mathematics**: a framework which involves prescribing either

- a type of general quantity through formally specifying definitions and rules of combination and relations, or
- a type of structure by formally defining how basic parts are understood, how such parts are understood to be formally the same or different, and by formally identifying postulates that stipulate how parts are to be constructed or related within the whole.

We will say that a field of mathematics is **developed** through the introduction of operations and constructions, to produce further objects, as well as the use of those processes that lead to higher viewpoints. Furthermore, development also involves the production of modes of quantification that connect (systems of) general numbers to structures to form measurable orders.

Now, mathematical objects need not be understood solely with respect to a single field of mathematics. In fact, distinct fields of mathematics may have a non-trivial overlap, or may be viewed in a hierarchical fashion. Thus objects may be studied from the vantage point of different fields of mathematics if one or more scenarios like this hold. Furthermore, an object may be framed in two different ways so as to be studied and understood in terms of two different fields of mathematics. In fact, true mathematical objects are not determined a priori by any field of mathematics, but rather by framing an object within a particular field with properties and relations disclosed and understood in terms of the framework and language of that field. Thus it is here that the tension between invention and discovery can be seen. The erection and development of a field of mathematics requires forming questions, the articulation of definitions, the selection of postulates, and the choices of symbols and language to frame objects to enable the discourse and exploration of those objects within that particular field (making calculations, stating and proving propositions, . . .). These may be viewed as **acts of invention** by the mathematician. Additionally, these acts include the posing of questions to be explored, the gathering of evidence, the formation of hypotheses, and, of great importance, the ***design*** *of frameworks and methodology.* **Discovery of objects** then can occur by understanding them sufficiently as framed within a field, articulating or specializing properties which characterize a potential (type of) object within that field, testing and refining the hypotheses as regard to them, and, finally, drawing the conclusions that give a deeper understanding.

But what of mathematical objects themselves? First, we will say, like things, that an **object** is a being which is a unity, identity, whole. We take such an object as midway between a created thing and a mathematical object. In this way, a mathematical object is an object that can be understood through the lens of (a field of) mathematics by abstraction. And so, we will say that our understanding of mathematical objects occur by **acts of discovery** understood as the revealing of the properties and relations those objects are seen to possess through a framing within mathematics. Furthermore, once these objects are understood, further acts of invention seek to speculate on ways to deepen the understanding of such objects and the relationships between them, opening up to new acts of discovery.

But what is the state-of-being of objects? We will take them to *arise from abstractions of created things by speculation*, but only to be considered as beings-to-be-understood by mathematics without committing them to be the subjects of any particular mathematical field for analysis. From this perspective, the object is understood by a mode of abstraction that is prior to a formal abstraction. In that manner, such an object may be considered as being participated by things as subject to a cosmic order in general, or participated possibly by created things. In the latter situation, it is by experimentation that created things are verified to be an object by participation. If such is the case, we say that such a created thing is a **similitude** of the object. For example, a chemist can understand water in a lake as the similitude of molecular objects ordered within by the molecule H_2O. Or speculate that a created thing understood as ordered within by atomic point particles be considered, instead, as ordered within by string particles, i.e. closed loops curled up in tightly curved space existing in higher hidden dimensions. Here, we have string objects which are increasingly well understood in the cosmic order governed by string theory, but there has yet been any verification that created things are similitudes of string objects.

Thus the status of objects are on par with the status of things as potentially created. These are dependent on the status of cosmic orders in relationship to the created order. Hence, the first test that properties pertaining to a mathematical object potentially reflect properties of an object is their persistent conformity to the law of non-contradiction. That is, an understanding of a mathematical object reflects an understanding of a thing within a cosmos insofar as that object arises from abstracting that potential thing. Hence any knowledge of a mathematical object provides knowledge of a thing in a cosmos which can be understood as a similitude of that object. In the next section, we indicate in what way this last way of understanding is grounded in the divine activity of the Creator.

We close this section by describing ways acts of invention and discovery can play out in the research of a mathematician. Here we follow the approach of David Mumford [15] to identify such categories. [16] Before outlining them, we summarize the overall aims of these modes of research in terms of two objectives: the **construction and analysis of tools** (acts of invention) for the purposes of **identification and classification of objects** (acts of discovery). Within this context, mathematicians can be divided into the following modes of researchers:

- Explorers: These are seekers of objects with particular or special properties. These need not be initially understood within a particular field of mathematics, but make use of particular fields in order to frame the motivating questions and build and/or utilize the tools to better understand those objects. These are often sought in terms of their properties and the relations between them and to other types of objects, and, ultimately, classify them. In the end, some seek special types of objects with unusual properties (gem collectors) or give an organization to objects of a particular type into a novel ordering (mappers). Examples of the mathematics

[15] David Mumford, **Math & Beauty & Brain Areas**, http://www.dam.brown.edu/people/mumford/blog/2015/MathBeautyBrain.html

[16] A similar analysis of Mumford's categories as understood in terms of St. Bonaventure's theology is given by the author in "Seeing Beauty in Mathematics: Reflections on Bonaventure's "Reduction" of Mathematics to Theology", Chithara 57 (1) pp. 38 – 57

discovered by explorers include:

  - Pythagoras's Theorem arising from the study of many tables of data produced from surveying efforts in ancient Mesopotamia.
  - Theaetetus and Schlafli's determination of regular polytopes in all dimensions.
  - Cantor's exploration of higher infinities in set theory.
  - William Thurston's program to classify 3-manifolds.
  - The classification of finite simple groups.
  - Michael Artin's adaptation of the theory of algebraic geometry to construct and study varieties associated to non-commutative algebras.

- Alchemists: These are mathematicians who find connections between two areas (not necessarily fields) of mathematics that have not been previously connected. Here, areas of mathematics can be less precise than fields of mathematics and may be prescribed in terms of conditions or properties on objects that can be interpreted in more than field. This can lead to a synthesis of two existing fields or a founding of a brand new field. Examples of the mathematics discovered by alchemists include:

  - Descartes's efforts to connect geometry to algebraic equations to form analytic geometry.
  - DeMoive's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Oscar Zariski's initiating the use of the tools of commutative algebra to address difficult questions in algebraic geometry.

- Wrestlers: These are mathematicians who are focused on the quantifying of objects within a measurable order and "[focus] on relative sizes and strengths of this or that object; [they] thrive not on equalities between numbers but on inequalities, what quantity can be estimated or bounded by what other quantity, and on asymptotic estimates of size or rate of growth". Examples of the mathematics discovered by wrestlers include:

  - Hardy and Ramanujan's formula for numbering the partitions of an integer.
  - Riemann and Hademard's work on the Prime Number Theorem
  - Zhang's work in estimating the sizes of gaps between prime numbers.

- Detectives: These are researchers who tackle questions that have shown to be deep and difficult to answer, but who, nonetheless, seek clues, follow trails, and attempt alternative perspectives, confident that a solution can be found. Some (Strip miners) seek to "[uncover] a hidden layer underneath a visible superficial layer in order to solve a problem. The hidden layer is often more abstract." Others (Baptizers) invent "[a name for] something new, making explicit a key object that has often been implicit earlier but whose significance is clearly seen only when formally defined and given a name." Examples of the mathematics discovered by detectives include:

  - Eudoxus and Archimedes's explorations in the nature of number in understanding and estimating their occurrences in geometry and the physical world giving the beginnings of the theory of the real number system.
  - Grothendieck's program revolutionizing algebraic geometry.
  - Andrew Wiles's work to prove the Taniyama-Shimura Conjecture.
  - Egorov and Luzin's work on the Continuum Hypothesis by naming and investigating sets that may violate it (thereby launching the field of descriptive set theory).

- Naming of π as the ratio of circumference and diameter of the circle
- Naming the number e so that $\dfrac{d}{dx}\, e^x = e^x$.
- Naming of i as the "number" that satisfies $i^2 = -1$.

Such approaches to engaging in mathematical research see the search for mathematical objects as being prior to any framing of them via fields of mathematics. Thus the aim to identify objects and disclose their properties is initiated by identifying enough of their structural properties (possibly within a particular field of mathematics) to quantify the parts and, hence, view those objects within the context of a measurable order. This is a move from being to quantification that enables it to be understood in a formal way and so be understood within a field (or fields) of mathematics. This is a way of thinking beautifully about mathematics which can be seen as a marriage of invention and discovery. In fact, following David Bentley Hart, we will define this marriage as an **act of *invenire***: "The Latin *invenire* means principally 'to find', 'to encounter', or (literally) 'to come upon'. Only secondarily does it mean 'to create', or 'to originate'" and so "every genuine act of human creativity is simultaneously an innovation and a discovery, a marriage of poetic craft and contemplative vision that captures traces of eternity's radiance in fugitive splendors here below by translating our tacit knowledge of the eternal forms into finite objects of reflection, at once strange and strangely familiar." [17]

7 Cosmic orders as divine ideas in the Creator

We end this paper by giving a theological view on cosmic orders as grounded in the divine activity of God. Our primary source is the medieval theologian Thomas Aquinas.

First, Aquinas observes (Summa Contra Gentiles 3.97-8) that the primacy of a thing's existence over what a thing is is in its being. Here we take first the act of being (*esse*) that brings a thing into existence (*ens*) as determining that thing's essence. The *esse* is then the act in accord with the intended form that brings the pure potency of matter from non-being to being as a thing with an essential nature. Thus *esse* is the ground of all forms and all things that come to be come as a proportion of *esse* and essence. As such, the source of *esse* is one in which *esse* and essence are at once one and the same and thus is referred to as *ipsum esse subsistens* (subsistent act of being itself) which we may take to be God, who is his own existence pure and simple. And so by having form, a thing exists and by existing it resembles God, so that form is nothing else than God's resemblance in things. Now things can resemble something absolutely simple by closeness or remoteness, things more closely resembling God being the more perfect, so forms differ by degrees of perfection. Thus variety in forms requires different levels of perfection. Variety in things requires an order in levels in things and mere equality.

Next, Aquinas shows that God's providence orders everything to a goal - his own goodness - not as if what happens can expand that goodness, for things are made to reflect and express that goodness as much as possible. Created things must all fall short of the full goodness of God. So in order that things may reflect that goodness more perfectly there is a variety of things for which what one thing couldn't express perfectly could be more perfectly expressed in a variety of ways by a variety of things. In other words, any form (e.g. owl-ness, cat-ness, rock-ness, sphere-ness) that God intends

[17] David Bentley Hart, "A Perfect Game", reprinted in *A Splendid Wickedness and Other Essays* Eerdmans Publishing 2016 p. 44

to be part of creation has an unlimited way of expressing itself as that form. As a created thing is a finite material expression of a form, that form is expressed in that thing in a limited way and, hence, has only a finite portion of the power to act in terms of that form as possessed by that form. On the other hand, the full expression of that form can be seen by the variety of ways and by the variety of things that form is the expression of materially, in creation, and throughout time. All those things that do express that form can be seen as being similar by that expression, but no two express that form identically.

Thus, the perfect goodness that exists one and unbroken in God can exist in creatures only in a multitude of fragmented ways. Variety in things comes from different forms determining their species. Because of their goals things differ in forms. Hence there is an order among things. Thus in the hierarchy of reasons behind God's providence, we have:

- God's own goodness: the ultimate goal which first starts activity off.
- The many-ness of things: determined by different degrees of forms and matters, agents and patients, activities and properties.

Aquinas notes that while God must love his own goodness, it doesn't follow necessarily that creatures must exist to express it, since God's goodness is perfect without it. The coming to be of creatures finds its first reasons in God's goodness and depends on a single act of God's will. The reason for the variety of creatures is because God does will to share his goodness as far as that is possible by way of resemblance to God. This does not necessitate this or that measure of perfection or this or that number of things. Thus a reason for a thing to have this form or that matter is by God through an act of will, deciding the number of things and the measures of perfections. Furthermore, as acutely analyzed by Fran O'Rourke: [18] "The beauty of the universe consists in the harmony, proportion, order, and mutual solidarity of beings which are infused with a single desire for their unique and universal end" [19] for

> ... divine beauty is the cause and goal of creation. Out of love for his beauty God wishes to multiply it through the communication of his likeness. He makes all things, that they may imitate divine beauty. Aquinas is thus able to declare: 'The beauty of the creature is nothing other than the likeness of divine beauty participated in things ... Created being itself (*ipsum esse creatum*) is a certain participation and likeness of God.' The beauty of the creature is its very being ... Each being is a participation in the divine beauty, an irradiation of the divine brilliance. [20]

God's providential ordering of the universe speaks to the nature of mathematics via a perspective of created things as essences formed by an act of existence which is something divine in things. [21] In terms of the ordering of things, we take the *mathematical objects as intelligible species that are ordered in a measurable way*[22]. These are arrived at by abstracting from the order experienced in the physical world. In particular, we may take the items of the world as the existing things for which their variety in unity is undergirded by the analogy of existence (*esse*). This holds for all things, especially necessary truths, such as found in logic and mathematics. For here, these truths,

[18] Fran O'Rourke: *Pseudo-Dionysius and the Metaphysics of Aquinas*, University of Notre Dame Press, 2005
[19] Cf. 273
[20] Cf. 273-74
[21] Here we follow David Burrell's *Knowing the Unknowable God* (University of Notre Dame Press (2001)) pp. 66-70
[22] emphasis mine

as pertaining to created things, are, in God, “divine ideas" (*logoi*): not by which God understands, but that which God understands.

> “God’s knowing is not propositional and God’s creation is not to be considered literally after a craftsman who must look at blueprints, or be thought to work from an idea. Rather, God in knowing his essence as imitable in this particular way by this particular creature, knows his essence as the nature and idea proper to that creature." (Summa Theologica I.15.2).

The “ideas" can be likened in God to the “form in the mind of the builder" so that for a creation to be structured in such a way is for that created nature to imitate the divine nature. Thus God does not create these ideas, but in creating “they express the structure creation will assume as God’s creation - ‘imitating the divine essence’". The ideas then depend upon God’s nature and not on God’s will and are in the mind of God in an exemplary way by understanding what must be the case if the divine essence is to be initiated by such a nature. Thus the ideas act as “rules of inference, or constraints of matter, purposes and agents in building something" in which the world can be taken to be structured in such a way as to be open to scientific inquiry, not by being made of such formal features, but by its constituent parts yielding to intellectual inquiry through those features.

Thus the nature of mathematics, as pertaining to the abstract structures reflecting the formal features in creation, is grounded in God’s intentional act in ordering the universe: for “in every effect, the ultimate end is the proper intention of the principle agent . . . the highest good is the good of the order of the universe . . . therefore the order of the universe is the proper intention of God" (Summa Theologica I.15.2). The “divine ideas" are thus instrumental to God’s making, being distinct from God as he is utterly first, creating by giving *esse* and intending the “good order of the universe," so that all emanates from God. Thus the truths of logic and mathematics become instrumental to creation as the necessary condition of the divine nature being imitated in such a way. So in order for there to be such a creation, it must issue in such a matter that these truths share in as part of the hypothetical necessity proper to contingent being. Thus, as pertaining to the divine act of creation, mathematics understands not only quantifiable created things, but also quantifiable things in any cosmic ordering of a cosmos.

We close by giving a scriptural perspective of this theological account of cosmic orders, and our understanding of them, in a Trinitarian key.

> In the beginning God created the heavens and the earth. The earth was without form, and void; and darkness was on the face of the deep. And the Spirit of God was hovering over the face of the waters. Then God said, “Let there be light"; and there was light. And God saw the light, that it was good; and God divided the light from the darkness. God called the light Day, and the darkness He called Night. So the evening and the morning were the first day. – Genesis 1:1-5 (NKJV)

> The Lord possessed [Wisdom] at the beginning of His way, before His works of old. I have been established from everlasting, from the beginning, before there was ever an earth. When there were no depths I was brought forth, when there were no fountains abounding with water. Before the mountains were settled, before the hills, I was brought forth; While as yet He had not made the earth or the fields, or the primal dust of the

world. When He prepared the heavens, I was there, when He drew a circle on the face of the deep, when He established the clouds above, when He strengthened the fountains of the deep, when He assigned to the sea its limit, so that the waters would not transgress His command, when He marked out the foundations of the earth, then I was beside Him as a master craftsman; And I was daily His delight, rejoicing always before Him, rejoicing in His inhabited world, and my delight was with the sons of men. – Proverbs 8:22-31 (NKJV)

Wisdom reacheth from one end to another mightily: and sweetly doth she order all things ... O God of my fathers, and Lord of mercy, who hast made all things with thy word, and ordained man through thy wisdom, that he should have dominion over the creatures which thou hast made, and order the world according to equity and righteousness, and execute judgment with an upright heart ... For thy Almighty hand, that made the world of matter without form ... thou hast ordered all things in measure and number and weight. – Wisdom of Solomon 8:1; 9:1-3; 11:17a, 20b (Apocrypha, KJV)

In the origin there was the Logos, and the Logos was present with God, and the Logos was god; This one was present with God in the origin. All things came to be through him, and without him came to be not a single thing that has come to be. In him was life, and this life was the light of men. And the light shines in the darkness, and the darkness did not conquer it ... It was the true light, which illuminates everyone, that was coming into the cosmos. – John 1:1-5, 9 [23]

The four living creatures, each having six wings, were full of eyes around and within. And they do not rest day or night, saying:

“Holy, holy, holy,
Lord God Almighty,
Who was and is and is to come!"

Whenever the living creatures give glory and honor and thanks to Him who sits on the throne, who lives forever and ever, the twenty-four elders fall down before Him who sits on the throne and worship Him who lives forever and ever, and cast their crowns before the throne, saying:

“You are worthy, O Lord,
To receive glory and honor and power;
For You created all things,
And by Your will they exist and were created."

– Revelation 4:4-11 (NKJV)

God, in creating, creates by first understanding fully and in unity the universe that He will create. This understanding is, in God, an act of artistry that the Father gives to the Son, the Logos, to design by giving creation its creative meaning and form (the divine ideas). This is the light that

[23]David Bentley Hart, *The New Testament: A Translation*, Yale University Press (2017) 168

gives creation its order and the means for humans to see that order and thereby come to understand it in its possibilities and actualities. It is the Spirit that proceeds from the Father and the Son as the act of being, carrying out God's will, by which creation is *ex nihilo* in accord with the meaning and form the Son has designed and the Father wills to be. It is also the Spirit that reveals the purposes of creation to mankind that are in the Father and the Son.[24] In part 2 to this paper, we will examine the reflective mode of thinking beautifully by further exploring the mathematician's aim to know truths in mathematics and the means to fulfill that aim framed within a theological conception of illumination and Lonergan's notion of insight.

[24] 1 Corinthians 2:6-16 (NKJV)

Replacing Remedial Mathematics with Corequisites in General Education Mathematics Courses

Alana Unfried (California State University, Monterey Bay)

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1 Introduction

Every year, millions of students in the United States begin their college education. Most of these students will be required to complete a college-level mathematics course in order to graduate. However, consider what happens to students pursuing a college degree who are underprepared for college-level mathematics. Sometimes with hard work, these students find success at college-level mathematics; however, there are three other common scenarios: 1) They are simply not be admitted at certain universities; 2) They attempt a college-level mathematics course, but can not pass the course; and 3) They are required to take an Elementary Algebra or Mathematics Remediation course before being able to take a general education (GE) mathematics course. In fact, across the nation, more than one million students begin in Mathematics Remediation each year [7].

With so many students needing mathematics remediation, one would hope that remediation works. However, at four-year colleges, only 36 percent of students who begin in remedial mathematics successfully complete a GE mathematics course after completing remediation. At two-year colleges, only 20 percent of students are successful [7].

With the failure of remedial mathematics programs becoming more evident, several states are beginning to offer corequisite support courses in place of traditional remediation. In a corequisite model, students can enter directly into a college-level mathematics course instead of requiring remediation first. Since they might still need additional support to successfully complete the course, they can enroll in a corequisite support course alongside the college-level mathematics course to give them extra practice. This can improve time-to-graduation since students need to complete less units overall.

Research also shows that students in a corequisite model have improved pass rates compared to a corequisite model. A randomized experiment showed that students assigned directly to a college-level statistics course with a corequisite had a 16% higher pass rate than students placed in remedial algebra, and they earned college credit at the same time [13]. Burns Childers et al. [1] also found that many students get lost in the "pipeline towards earning college mathematics credit" when placed in a remedial mathematics structure, and recommend placing as many students as possible into a corequisite course rather than a remedial course. Results from the Tennessee community college system in 2015 showed that 51 percent of students enrolled in a corequisite mathematics course successfully passed the college-level course, whereas only 12.3 percent of students who began in mathematics remediation and attempted a college-level course were able to pass the college-level course [15]. Due to findings such as these, many states are moving to widespread implementation of corequisite models. This includes Georgia, West Virginia, Tennessee, Indiana, Colorado, and Texas [7, 9, 15].

Switching to a corequisite model, however, can be challenging. Daugherty et al. [9] identify issues such as scheduling and advising, buy-in from stakeholders such as students and the institution, and the cost of professional development for faculty to implement the new instructional model.

This paper discusses the switch from a mathematics remediation model to a corequisite support model at one mid-sized university in California. Section 2 gives university context and the motivation for the change. Section 3 describes the structure of the corequisite courses that were developed, and Section 4 similarly describes how GE mathematics courses were adapted to support this change. Section 5 provides results from the first year of implementation, and Section 6 gives recommendations for readers interested in moving to a corequisite model.

2 Context

As of 2017, California was added to the list of states practicing widespread implementation of corequisite models. In August of 2017, the Chancellor's Office of the California State University (CSU; 23 campuses serving 428,000 undergraduate students as of Fall 2018 [2]) issued Executive Order 1110, mandating the end of both mathematics and written communication remediation by Fall of 2018 across all CSU campuses [19]. In place of remediation, each campus may require only one unit of non-college-credit-bearing developmental math, either in the form of a corequisite course or a stretch course. Also in 2017, the California government passed AB705, a bill which requires community colleges to "maximize the probability that the students will enter and complete transfer-level coursework in English and Mathematics within a one-year time frame" [18]. This bill aims to substantially reduce the number of students placed into remedial courses, which in effect is increasing the use of corequisite courses in community colleges across California.

The university that is the subject of this article, California State University Monterey Bay (CSUMB), is a part of the CSU system and therefore subject to Executive Order 1110. CSUMB is classified as a Hispanic-Serving Institution, with 42% of students identifying as Latino in Fall 2018. Fall 2018 enrollment was just over 7,500 students, with 63% female, 95% coming from California, and 72% of students age 24 or younger [4]. In Fall 2018, CSUMB complied with the Executive Order by ending mathematics remediation and moving fully to a corequisite model across all GE mathematics courses, of which there are four (Quantitative Literacy, Finite Mathematics, Precalculus, and Introductory Statistics). The choice of GE mathematics course is determined by major; Precalculus is required by STEM majors, Finite Mathematics serves Business majors, Quantitative Reasoning serves Liberal Studies students, and Introductory Statistics is required for most other majors such as Psychology, Kinesiology, and Collaborative Health and Human Services.

Historically, close to 40 percent of students at CSUMB have begun in mathematics remediation. These students were required to complete either a one- or two-course remediation sequence of four-unit non-credit-bearing courses; the majority required two-course remediation. Our remediation program was recognized with a $3 million grant due to its success moving students through remediation efficiently; 90 percent of students completed mathematics remediation in their first attempt, when national rates hovered around 50 percent [3]. However, similar to national trends, successful completion of mathematics remediation did not guarantee successful completion of the subsequent GE mathematics course. Table 1 shows pass rates for GE mathematics courses in Fall 2016. Pass rates for students who never required remediation ranged from about 72 to 90 percent, whereas students who had successfully completed remediation generally had much lower pass rates, even though they were considered fully remediated. The equity gap for Quantitative Literacy is notably lower than for the other courses; this is likely due to smaller class sizes and being the GE mathematics course that requires the lowest level of mathematical skills.

Table 1: General Education Pass Rates in Fall 2016 as a function of Remediation. Parentheses display the total number of students in each category.

Course	No Remediation	2-Course Remediation	Equity Gap
Introductory Statistics	79.9% (189)	48.2% (112)	31.7%
Quantitative Literacy	90.5% (42)	88.7% (30)	1.8%
Finite Math	78.0% (82)	29.4% (17)	48.6%
Precalculus	72.1% (287)	39.0% (41)	33.1%

Although CSUMB was considered successful at mathematics remediation, it did not naturally follow that students were successful in their general education courses. Therefore, the 90% remediation completion rate should not be indicative that CSUMB is somehow different from other universities in terms of GE mathematics success. The demographics of our university are very similar to those of other California State Universities and public universities more generally. The high remediation completion rate is, however, evidence that CSUMB faculty strive to be innovative and effective educators. The efforts that faculty put into the remedial mathematics program were transferred to the development of the corequisite model instead.

Upon abandoning the mathematics remediation model, CSUMB faculty chose to create a one-unit corequisite course tied to each GE mathematics course to offer additional support to underprepared students. At the same time, faculty recognized that this change would result in a new mix of student preparedness in our GE mathematics courses, requiring careful consideration of pedagogy in those course courses as well. The following two sections detail the structures of the corequisite and GE courses, respectively.

3 Corequisite Support Course

On campus, we often refer to the corequisite course as a "support course" in order to remove any negative stigma from the course; it is not meant to be a punishment, but rather a resource for students. Therefore the phrases "corequisite course" and "support course" will be used interchangeably.

3.1 Who Enrolls in a Corequisite?

Under the mathematics remediation model, math placement exam scores were used to determine which students were placed into mathematics remediation. Under Executive Order 1110, however, the CSU implemented a Multiple Measures placement in which factors such as high school GPA, high school mathematics courses and grades, and ACT/SAT score, are used to determine if students are required, recommended, not recommended to enroll in a corequisite course.

Instead, CSUMB piloted Directed Self-Placement (DSP) in which students complete a short online reflective experience prior to enrolling in their courses. Directed self-placement is commonly used for placement in writing courses [16]. Some questions require mathematical reasoning, but other questions ask about a student's attitude toward mathematics and their previous experiences with mathematics. The DSP module offers them a recommendation to either enroll in the corequisite course or not, but students are given agency to decide for themselves if they will enroll. Our investigation of the validity of our DSP instrument is ongoing. In total, 21.6 percent of GE math students chose to enroll in a corresponding support course during the 2018-19 academic year.

3.2 Corequisite Structure

The corequisite course is offered as a one-unit, two-hour activity period. The corequisite course is comprised of students from any section of the corresponding GE mathematics course. Corequisites are capped at 25 students, and an embedded peer mentor from the university tutoring center assists in each class period so that students can receive more individualized attention and learn strategies from their peers. The corequisite course is always taught by an instructor who is also teaching a section of the GE course so that the instructor is up-to-date on current materials and challenges in the GE course.

3.3 Corequisite Class Components

The corequisite class has three components: 1) support for corequisite mathematics knowledge, 2) support for GE course content, and 3) study skill development.

3.3.1 Corequisite Mathematics Knowledge

Corequisite mathematics knowledge is developed through the use of an online adaptive learning system, EdReady. Each instructor creates a skill set of mathematical knowledge needed in a GE course prior to each exam, for a total of about three skill sets per semester. Each student completes a diagnostic quiz online to assess what mathematical skills need work, and then EdReady creates a custom study path for each student. Students can then learn and self-assess through the online platform, and work towards 90% proficiency on the skill set (Figure 1).

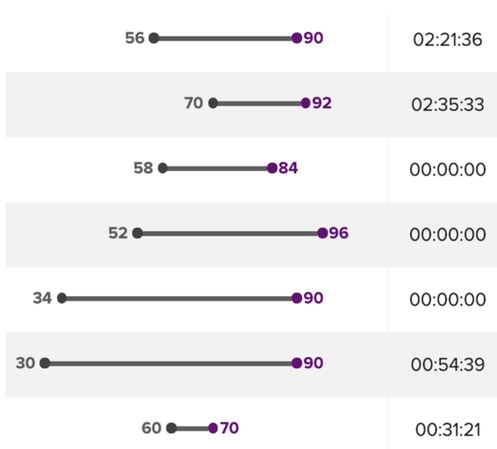


Figure 1: An example of starting and ending scores for students in EdReady, along with the amount of time they worked on learning additional skills. (A time of zero indicates students improved their scores on the assessment without needing to complete additional learning modules.)

Students can spend as much time as they would like working toward proficiency, rather than having a limited number of attempts. Some time is given in class for EdReady practice, but students are mostly expected to complete EdReady outside of the activity period.

3.3.2 Support for Course Content

Each support course is geared towards active learning, with the use of group activities and worksheets that give extra practice on concepts learned in the GE mathematics course (see Figure 2 for an example). Instructors also give mini-lectures on particularly difficult concepts for students, and sometimes open work time is provided for students to work on homework assignments, exam preparation, or projects. Some support courses also devote time to working on test corrections after exams. Mathematical knowledge development is also integrated into practice of GE course materials. For example, prior to practicing how to interpret regression coefficients, students begin by practicing plotting a line from a basic equation and interpreting the meaning of the slope and intercept (Figure 3).

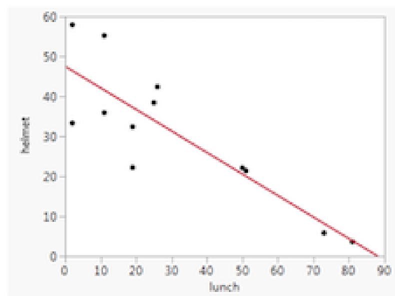
Use the following to answer questions 8 through 26.

The scatterplot shows the relationship between socioeconomic status measured as the percentage of children in a neighborhood receiving reduced-fee lunches at school (lunch) and the percentage of bike riders in the neighborhood wearing helmets (helmet). The average percentage of children receiving reduced-fee lunches is 30.8% with a standard deviation of 26.7% and the average percentage of bike riders wearing helmets is 38.8% with a standard deviation of 16.9%. The JMP output below shows correlation and regression results for this data.

8. Which variable (lunch or helmet) has more variability? How do you know?

9. In this scenario, which is the response variable and which is the explanatory variable?

10. What is a case in this dataset?



Linear Fit

helmet = 47.490446 - 0.5386091*lunch

Summary of Fit

RSquare	0.721253
RSquare Adj	0.693378
Root Mean Square Error	9.384827
Mean of Response	30.88333
Observations (or Sum Wgts)	12

Lack Of Fit

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	1	2278.9268	2278.93	25.8748
Error	10	880.7499	88.07	Prob > F
C. Total	11	3159.6767		0.0005*

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	47.490446	4.242456	11.19	<.0001*
lunch	-0.538609	0.105885	-5.09	0.0005*

Bivariate Normal Ellipse P=0.950

Variable	Mean	Std Dev	Correlation	Signif. Prob	Number
lunch	30.83333	26.72361	-0.84927	0.0005*	12
helmet	30.88333	16.94825			

Figure 2: An example of a worksheet assigned during the Introductory Statistics corequisite for regression practice.

3.3.3 Study Skills Development

Most students enrolled in the corequisite course are new to college, so time is spent developing general skills that will help them in the GE mathematics course and beyond. Our Center for Student Success provides a time management workshop, and our counseling center provides a workshop on anxiety. Time is also dedicated to discussing test anxiety and test-taking strategies. Additionally, CSUMB mathematics faculty continue to develop this component of the course by referencing strategies from [14].

Use the following equation to answer questions 1 through 6. $y = 3x - 2$

1. Fill out the following chart to calculate y for given values of x.

x	y
-1.0	
0.0	
1.0	
1.5	
2.0	

2. On the graph to the right, plot the equation $y = 3x - 2$. Use the chart form question 10 to help you.

3. On the graph, label which axis is the y axis and which is the x axis.

4. Using the equation and the graph to guide you, interpret the meaning of slope.

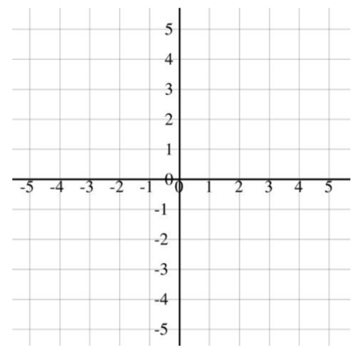


Figure 3: An example of a problem assigned during the Introductory Statistics corequisite prior to completing the regression exercise in Figure 2.

3.4 Corequisite Grading Structure

The course is non-college-credit bearing, but students receive a letter grade to help them assess their progress during the semester (this grade does not affect their college GPA). The corequisite grade and GE grade are independent from one another; it is possible to pass both courses, fail both courses, or pass one but fail the other. Students who fail their GE mathematics course are required to take the corequisite course when they retake the GE course (although this is difficult to enforce). The ultimate goal is for students to pass their GE course. Therefore, if a student passes the GE course but fails the corequisite course there are no negative consequences since the corequisite course does not count for college credit or affect GPA. The corequisite course grade is comprised of points for completing EdReady assignments and participation/attendance points.

3.5 Corequisite Student Experiences

At the end of the Fall 2018 and Spring 2019 semesters, students enrolled in the corequisite courses for any of the four GE mathematics courses (Quantitative Literacy, Finite Mathematics, Precalculus, and Introductory Statistics) were asked to complete a survey describing their experiences in the support course. The response rate for students who consented to participate in the research study and answered the survey was 33 percent (107 students). Results are displayed in Figure 4.

Experiences in the support course are highly positive. Only two items show more mixed reviews. Regarding learning better study habits, we expect this experience to increase as more activities are integrated from [14]. Regarding EdReady, we find that students have mixed opinions about the online mathematical skills development. Focus groups revealed that some students found the EdReady content did not seem to align well with what they needed to know in their GE mathemat-

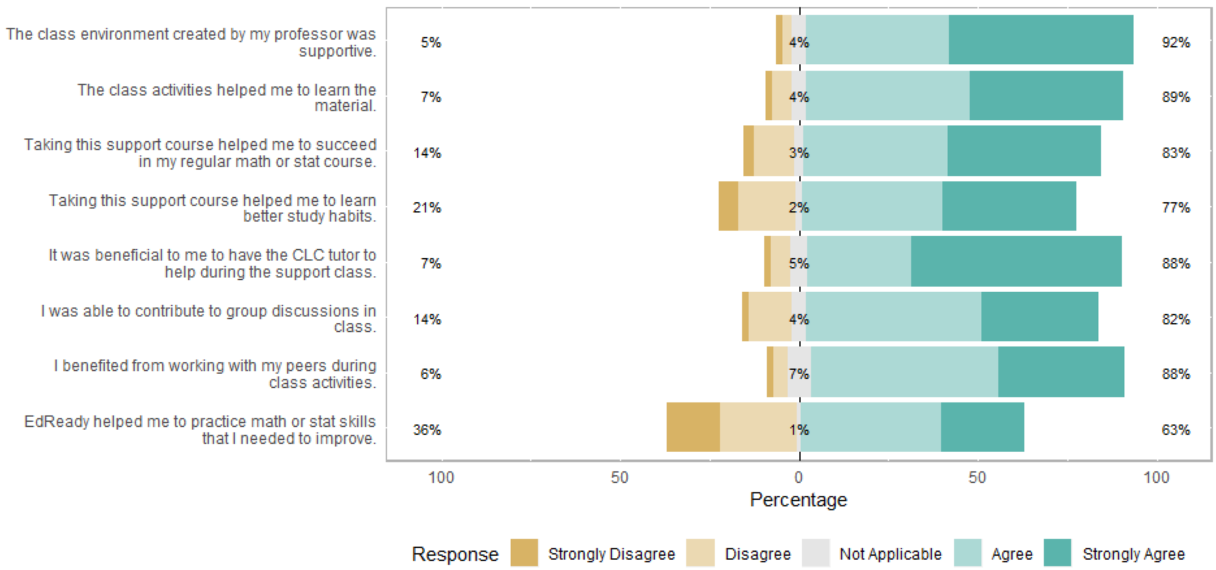


Figure 4: Student experiences in a corequisite course during the 2018-19 academic year.

ics course. Finite Mathematics and Precalculus students found EdReady most useful (80 percent and 72 percent respectively), where only 57 percent of Introductory Statistics and 62 percent of Quantitative Literacy students found the platform useful. In addition to the experiences displayed in Figure 4, 98 percent of corequisite students who responded to the survey said they would recommend the support course to a friend. Based on these results, we conclude that students are generally having positive experiences in the corequisite courses.

Note that GE course pass rates were found to differ between students who completed the corequisite experiences survey and who did not complete the survey. About 90 percent of students who completed the corequisite experiences survey passed their GE mathematics course, where only about 63 percent of students who did not complete the survey passed their GE mathematics course. This difference is not surprising since the survey was administered during class at the end of the semester, and many students had stopped attending the corequisite course. The pass rate difference might suggest that reported experiences are biased in the positive direction since responses were from students more likely to be satisfied with their grade in the GE course.

4 General Education Math Course

The general education mathematics course structure at CSUMB is considered co-mingled, in corequisite language [8]. This means that college-ready and underprepared students are mixed together in the GE course, and the underprepared students enroll in their corequisite course separately. The alternative is "cohorting," in which sections of the GE course are reserved for underprepared students.

4.1 Why Change the GE Course?

Due to the co-mingled nature of the GE courses, students in each section are largely varied in terms of their mathematical preparation for the course. But, our goal is to help all students succeed, not just some. A common concern with moving to a corequisite model is that the "rigor" of the college-level course will be lost when under-prepared students are included. Therefore, CSUMB faculty were careful to utilize pedagogy that caters to the success of all students without losing the strength of the content or creating bimodal outcomes. This pedagogy is described in Section 4.3.

4.2 GE Course Structure

General education mathematics courses at CSUMB are capped at 36 students. Each course is highly coordinated by a tenured or tenure-track faculty member overseeing all sections. This ensures that all sections implement common pedagogical practices, lesson plans, an exams for a consistent student experience. This coordination was also crucial due to the co-mingled structure, in which each corequisite contains students from many different sections of the GE course, so that each student was receiving the same course content each week and was ready for common material in the corequisite. Each GE course also uses open educational resources to keep costs at a minimum for our students. For example, Precalculus instructors designed their own course activity pack that serves that the textbook and class materials. Statistics instructors use *Introductory Statistics with Randomization and Simulation* [10], which costs less than 10 dollars per copy and is available for free online.

4.3 GE Course Pedagogy

The goal with our chosen pedagogy was to develop each student's sense of belonging in the course, no matter their mathematical background, and also to empower our students to be successful learners. In a traditional lecture-based learning environment, students engage with mathematics solely through the instructor, rather than directly themselves (Figure 5). There is much evidence showing that active learning, in which students interact directly with course content, benefits all students and closes equity gaps [11]. In an environment geared towards equitable student learning (Figure 6), the teacher develops systems for students to interact directly with mathematical content, and serves to manage the interaction so that it is fruitful [12, p.17].

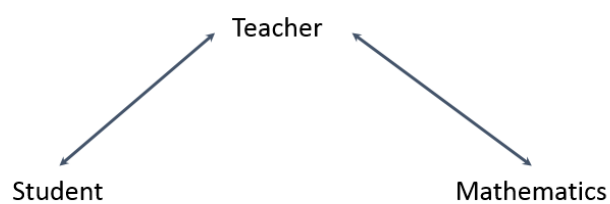


Figure 5: Instructional Triangle in a Lecture-Based Model.

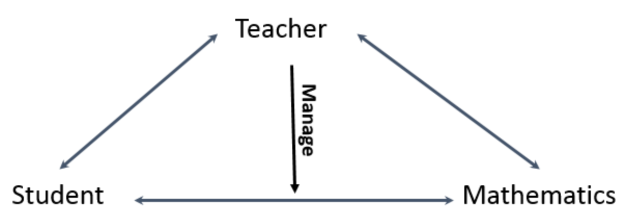


Figure 6: Instructional Triangle in an Active-Learning Environment (Horn, 2012).

The questions remain, though, of how to create productive student interactions with the content, and how to manage the interactions effectively. These issues are addressed through the use of *complex instruction* and *reading apprenticeship* frameworks.

4.3.1 Complex Instruction

Complex instruction, first developed by [5], is "a combination of pedagogical strategies used to create a classroom 'social system' that directly attends to problems of social inequality, which undermine academic access and achievement if left unexamined" [Lisa Jilk, personal communication, 2018]. Generally speaking, in a complex instruction classroom, students work in groups to complete tasks and learn mathematical content directly. However, specific structures are put into place in order to make this an equitable learning environment (Figure 7).

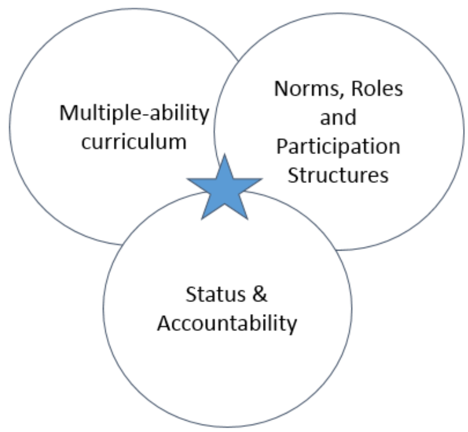


Figure 7: Components of Complex Instruction [Lisa Jilk, Personal Communication, 2018].

The groupworthy tasks that students complete in groups are open-ended and require multiple abilities to solve; that way each student has an entry point into the problem even if each student does not possess the same skill. Students must rely on one another to solve the problem (*multiple-ability curriculum*).

In order for these groups to function well, *norms, roles and participation structures* are introduced. Groups of three to four students are assigned randomly, and re-randomized every two to three weeks. Norms describe how students learn together (see Table 2). Each student in a group is given a role (Facilitator, Resource Manager, Recorder/Reporter, or Team Captain) in order for group members to hold each other accountable to learning [12, p. 50].

Table 2: Norms for group learning. Posters with these norms are posted in mathematics classrooms at CSUMB.

Norms for Group Learning
No one is done until everyone in your group is done.
You have the right to ask anyone in your group for help.
You have the duty to assist anyone in your group who asks for help.
Helping peers means explaining your thinking, not giving answers or doing work for others in the group.
Provide justification (say why!) when you make a statement.
Only ask the instructor a question when it's a team question.
Think and work together. Don't divide up the work.
Work only within your group — no crosstalk with other groups.
No one is as smart as all of us together!

Lastly, the instructor is responsible for managing the status of students in the classroom and holding groups accountable for learning (*status and accountability*). Status characteristics refer to criteria that students use to deterine whether or not they are smart (past mathematical experiences, gender, speed of problem solving, race/ethnicity, reading comprehension, social class, etc.). It is the instructor's job to manage student status, or students will exhibit unequal participation. Much of this is accomplished by what has already been mentioned. One way to manage status is by randomizing groups to demonstrate that all students and groups are equally capable of solving a problem. Another way is by assigning roles so that students do not define roles based on expectations of competence; e.g. "You do the coding, you will be better at it than me." Creating problems that require multiple abilities to solve allows equitable access to a problem. Lastly, instructors should assign competence to their students; identify the various intellectual abilities and skills that students posess, and tell them out loud [6].

4.3.2 Reading Apprenticeship

With so much groupwork happening in class, it is difficult to move through content at the same pace that one would during lecture. Further, to be successful in mathematics courses, students need to develop the skills needed to read mathematical texts, which is very different from other types of reading. We therefore implement Reading Apprenticeship strategies in our GE mathematics courses, a framework in which instructors apprentice students into reading within the discipline [17]. Reading Apprenticeship centers on having metacognitive conversations with students about how we learn and process information. CSUMB faculty model their reading processes out loud in the classroom, showing students how they annotate text.

As for saving time in class, with little room for lecturing in the classroom, many of our GE mathematics courses assign daily reading assignments so that students are engaging with concepts outside of class prior to working on the topics in class, allowing groups to move directly into more complex tasks rather than starting with basic definitions and surface-level understanding. Most courses have students complete daily reading logs, which are incorporated into course grades (Figure 8).

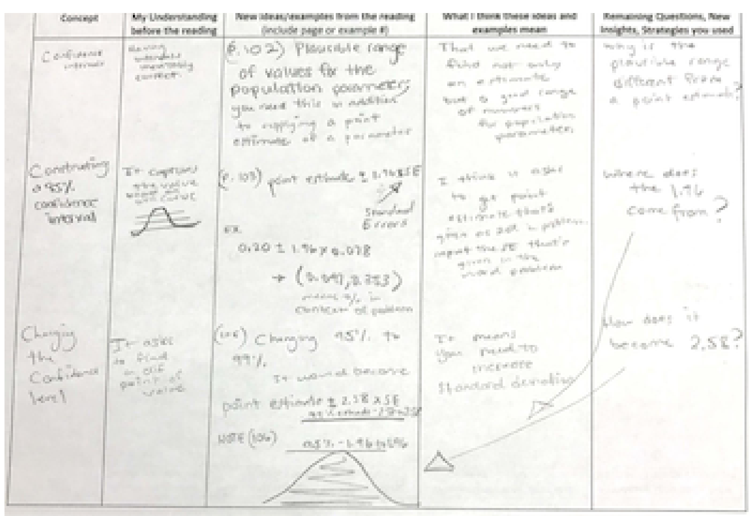


Figure 8: Reading log from an Introductory Statistics student.

4.4 GE Student Experiences

At the end of the Fall 2018 and Spring 2019 semesters, students enrolled in three out of four of our GE mathematics courses (Quantitative Literacy, Finite Mathematics, and Statistics) were asked to complete a survey describing their experiences in the GE mathematics course. The response rate for students who consented to participate in the research study and answered the survey was 44 percent (405 students). Results are displayed in Figure 9. These results demonstrate that the vast majority of students are finding reading apprenticeship and complex instruction to be valuable frameworks for their learning.

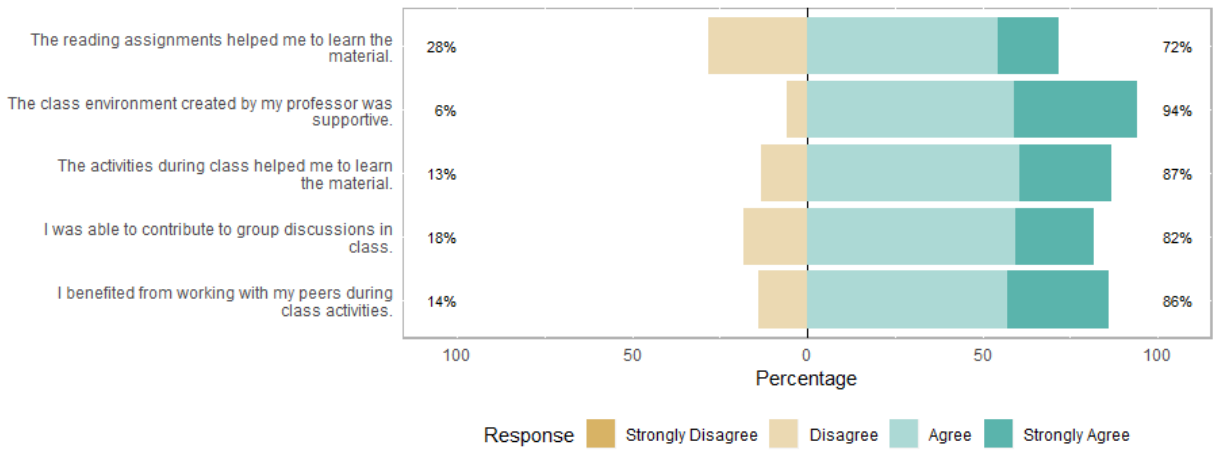


Figure 9: Student experiences in a corequisite course during the 2018-19 academic year. (Does not include Precalculus students.)

What about maintaining the "rigor" of the course? Although only one metric for the "rigor" of the course, 74 percent of students found their GE mathematics course somewhat or very challenging, which we consider a good thing. (Please note that this number does not include Precalculus students.)

Similar to the corequisite student experiences survey, some charactersticis were found to differ between students who completed the GE course survey and who did not complete the survey. About 88 percent of students who completed the corequisite experiences survey passed their GE mathematics course, where only about 72 percent of students who did not complete the survey passed their GE mathematics course. Additionally, 71 percent of survey-completers were female whereas only 57 percent of non-survey-completers were female. No other meaningful differences were found. The pass rate difference might suggest that reported experiences are biased in the positive direction since responses were from students more likely to be satisfied with their grade in the course.

5 Results

Recall that under a remedial model, large equity gaps existed in CSUMB GE course pass rates between students who had completed remediation and those who did not require remediation. Table 3 displays the corresponding data under the corequisite model.

Table 3: General Education pass rates in Fall 2018 as a function of corequisite enrollment. Parentheses display the total number of students in each category. A negative equity gap implies that the pass rate was higher among students who completed the corequisite course. The remediation equity gap refers to Fall 2016 data from Table 1.

Course	No Corequisite	Corequisite	Equity Gap	Remediation Equity Gap
Introductory Statistics	79.9% (204)	83.3% (78)	-3.4%	31.7%
Quantitative Literacy	90.7% (54)	93.8% (16)	-3.1%	1.8%
Finite Math	73.1% (93)	53.8% (13)	19.3%	48.6%
Precalculus	83.0% (336)	66.7% (99)	16.3%	33.1%

Compared to the remediation model Fall 2016 data in Table 1, the equity gaps between between corequisite students and those not enrolled in the corequisite have either shrunk substantially or actually reversed.

One point to consider is that the cohort of students who enrolled in a corequisite course is not analagous to the cohort of students who were required to complete remediation. Students self-selected into the corequisite course, and it was a smaller group. Therefore it might seem unfair to compare the pass rates for remediation versus corequsite students. However, it is important to note that since fewer students opted into the corequisite course, many students completed GE mathematics courses without additional support who in past years would have been required to complete remediation. One might expect that this would cause the pass rates for the "No Corequisite" group to drop significantly compared to previous years. However, it is evident from the data that this is not what happened. Among students who did not enroll in a support course, pass rates compared to Fall 2016 have either been maintained or even been surpassed (in the case of Introductory Statistics and Precalculus).

These results demonstrate that when looking at students in aggregate, moving to a corequisite model allowed just as many students, and in some cases more students, to successfully complete their GE mathematics course without requiring the burden of four to eight units of remedial mathematics. Further, under the new model, underprepared students have a much reduced equity gap compared to their peers, which we attribute to the pedagogical changes addressing classroom dynamics and student learning. Together, this evidence suggests that with a careful course design and well-structured corequisite courses, remedial mathematics courses are not necessary for student success in general education mathemaitics courses. This aligns with the literature discussed in Section 1 showing that corequisite models can lead to improved pass rates, in contrast to the failures of many remedial mathematics programs to push students through completion of GE mathematics courses.

6 Limitations and Future Directions

We made changes to our course structure and pedagogy all at the same time, so it is impossible to determine which individual components had the greatest impact on students. We know that students are succeeding in our GE mathematics courses without mathematics remediation, sometimes at even higher rates than during the remediation model. However, we cannot say definitively if these changes are due to the reduced class size, the implementation of complex instruction and reading apprenticeship frameworks, better course coordination, etc. It is the CSUMB mathematics and statistics faculty's belief that these changes are synergistic and together create the positive results that we are seeing. Further, the data presented here is only from one academic year of implementation. It remains to be seen how students perform in subsequent mathematics and statistics courses, such as a Research Methods course or Calculus.

Future work will focus on additional comparisons of characterstics between those students required to complete remediation and those who enrolled in corequisite courses. Additionally, we will further examine other types of equity gaps based on demographic characteristics and use more advanced modeling methods to understand, aside from corequisite course enrollment, what covariates play a role in GE mathematics course success.

7 Recommendations

Readers of this article might be involved with many differing types of institutions. If you work at an institution which offers elementary algebra and/or mathematics remediation, consider checking the pass rates of remedial students in subsequent courses. Do they succeed at the same rates as their non-remedial peers? If not, consider switching to a corequisite model. Better yet, even if the equity gaps are minimal, try switching to a corequisite model to reduce the burden of extra units that these students must complete. If you work at an institution in which students must enter directly into college-level mathematics, consider if you might be excluding students from your university who might be able to succeed just as well as their peers if given the opportunity. Further, consider adding corequisites to other courses in which students may struggle, such as Calculus. Lastly, for all readers, I encourage you to believe that any student can succeed in mathematics, no matter how underprepared they may be at the start of college. Moving beyond belief, move into action by enacting equitable learning environments in your courses so that students themselves can develop the belief that they can succeed in mathematics.

8 Acknowledgments

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References

[1] Annie Burns Childers, Lianfang Lu, Joshua Hairston, and Timothy Squires. Impact and Effects of Co-Requisite Mathematics Remediation. *Primus*, 0(0):1–28, 2019.

[2] California State University. Total Enrollment by Sex and Student Level, Fall 2018, 2018.

[3] California State University Monterey Bay. CSUMB Awarded $8 Million Innovation Grants, 2015.

[4] California State University Monterey Bay. Institutional Assessment and Research: Enrollment Overview, 2018.

[5] Elizabeth G. Cohen and Rachel A. Lotan. *Working for Equity in Heterogeneous Classrooms: Sociological Theory in Practice.* Teachers College Press, 1997.

[6] Elizabeth G. Cohen and Rachel A. Lotan. *Designing Groupwork: Strategies for the Heterogeneous Classroom.* Teachers College Press, New York, 3rd edition, 2014.

[7] Complete College America. Corequisite Remediation: Spanning the Completion Divide. Technical report, Complete College America, 2016.

[8] Dana Center Mathematics Pathways. Co-requisite Courses: Narrowing the Gap betwee Instruction and Supports. Technical report, University of Texas at Austin Charles A. Dana Center, Austin, TX, 2017.

[9] Lindsay Daugherty, Celia J. Gomen, Diana Gehlhaus Carew, Alexandra Mendoza-Graf, and Trey Miller. Designing and implementing corequisite models of developmental education. *RAND Corporation*, 2018.

[10] David M. Diez, Christopher D. Barr, and Mine Çetinkaya-Rundel. *Introductory Statistics with Randomization and Simulation.* 1st edition, 2014.

[11] Scott Freeman, Sarah L. Eddy, Miles McDonough, Michelle K. Smith, Nnadozie Okoroafor, Hannah Jordt, and Mary Pat Wenderoth. Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, 111(23):8410–8415, 2014.

[12] Ilana Seidel Horn. *Stength in Numbers: Collaborative Learning in Secondary Mathematics.* The National Council of Teachers of Mathematics, Inc., Reston, VA, 2012.

[13] A. W. Logue, Mari Watanabe-Rose, and Daniel Douglas. Should Students Assessed as Needing Remedial Mathematics Take College-Level Quantitative Courses Instead? A Randomized Controlled Trial. *Educational Evaluation and Policy Analysis*, 38(3):578–598, 2016.

[14] Lisa M. Nunn. *33 Simple Strategies for Faculty: A Week-By-Week Resource for Teaching First-Generation Students.* Rutgers University Press, New Brunswick, 2018.

[15] Office of the Vice Chancellor for Academic Affairs. Co-requisite Remediation Pilot Study - Fall 2014 and Spring 2015 and Full Implementation Fall 2015. Technical report, Tennessee Board of Regents, 2016.

[16] Anne Ruggles Gere, Laura Aull, Timothy Green, and Anne Porter. Assessing the validity of directed self-placement at a large university. *Assessing Writing*, 15(3):154–176, 2010.

[17] Ruth Schoenbach, Cynthia Greenleaf, and Lynn Murphy. *Reading for Understanding: How Reading Apprenticeship Improves Disciplinary Learning in Secondary and College Classrooms.* Jossey-Bass, San Francisco, second edition, 2012.

[18] State of California. Assembly Bill No. 705, 2017.

[19] Timothy P. White. Executive Order 1110: Assessment of Academic Preparation and Placement in First-Year General Education Written Communication and Mathematics/Quantitative Reasoning Courses, 2017.

Models, Values, and Disasters

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Abstract

Decision-support models have values embedded in them and are subjective to varying degrees. Philosophical and ethical perspectives on operations research models are used to describe this subjectivity. Approaches to model building are then suggested that take into account subjectivity and values. For the decisions to reflect the right values, the model must align with the decision-maker's values. I argue that it is appropriate and important for Christians applying mathematical models to be keenly aware of decision-maker's values and seek to reflect them in the model. Disaster response planning is presented as an example where incorporating values is challenging. The responding organizations have multifaceted goals. How is equity balanced with efficiency? How is cost and donor interest considered? I report on a study of how Christian relief organizations differ from non-faith based organizations in ways that can be reflected in their logistics procedures and in these models.

1 Introduction

This paper came about because, after 37 years of using mathematical models and reflection on modeling, I am still struck by how challenging it is to explain and justify the assumptions in a model, especially *prescriptive* models. My interest resurfaced during a recent study of how Christian relief organizations make operational decisions. Modeling in this context is particularly challenging and subjective. The study has raised more questions than it has provided answers. This paper is an attempt to lay out these questions as they relate to modeling more generally and suggest some pragmatic approaches to dealing with subjectivity in modeling. It is my hope that acknowledging and thinking through the subjective side of modeling will be useful for those trained in the mathematical sciences, where models are generally not viewed as subjective.

The unsuspecting undergraduate who starts modeling things beyond "hard" scientific phenomena steps into a long debate about the nature and quality of models. At Gordon College, a core science course includes thinking about the limitations and possible biases in science. When using models to support business or organizational decisions, many other issues about the use of models present themselves. The questions center around how well a model of a specific situation can describe it and the implications for the decision maker or the stakeholders in a decision.

Good models require data and an understanding of the dynamics or process being modeled. However, decision-support models also have values embedded in them. For the decisions to reflect the right values, the model must align with the decision-maker's values. This rather obvious statement raises several questions for the model builder, or analyst. How can the analyst know the values of the decision maker? What if the analyst's values differ notably from the decision-maker's? When is it appropriate for the Christian (or other) faith of the decision-maker, which frames her overall values, to influence the model used? The author's position is that Christians applying mathematical models to support decisions should be keenly aware of decision-maker's values and seek to reflect them in the model. This principle is usually germane even when the decision-maker does not share the analyst's Christianpresuppositions.

This question about the decision-maker's and the analyst's values is explored in the specific context of decision-support models for humanitarian response after a disaster. The decisions that will be considered are about logistics: the transportation and distribution of goods. The responding organizations have multifaceted goals rather than an overarching profit objective. Because each disaster is unique and requires a rapid response, there are limitations on the information available to drive the model. Christian disaster response organizations (DROs) play a large and increasing role in global disaster response. Thus, it is natural to ask how their faith values influence their decisions and whether these values can be incorporated in mathematical models.

Rather than dip into the broad philosophical literature about subjectivity in science and social science, I will focus on the small, but noteworthy, literature on decision-support models in the operations research community. First, some ethical guidelines in modeling are reviewed and the role of values is considered. Section 3 describes the philosophical problem of justifying operations research models and Section 4 suggests some factors that affect the ability to construct useful models. Section 5 describes ways to incorporate values in the models used in disaster response. The author's study of Christian relief organizations and their values is described in Section 6. The final section suggests how students could be introduced to values in models.

2 Trusting modelers: ethics and values

The ethical pitfalls of applying operations research has received some attention in the field's journals. A lively earlier discussion of ethics in modeling is the edited volume [20]. Another early voice in this area was Saul Gass [7]. A survey of the literature on ethics is given in [19] and a report on the operations research community's stand on ethics is in [4]. Much of the literature and dialogue on this topic has been centered in Europe, including [6] and [14]. However, ethics is not emphasized as much in operations research as in professions such as engineering or business. Graduate students in operations research, business analytics, or related majors generally do not take courses in ethics. Business ethics would be the closest course offering, and is more likely to be taken by MBA students. There is no broadly endorsed code of ethics in the United States, suggesting a lack of agreement about what such a code would contain. There was a voluntary code proposed in Europe starting in 2000 [3], focusing on respect for the people affected by business decisions and environmental concerns.

When analysts develop a model, they certainly share the ethical responsibilities that come with the application area and decisions that are being made, and scientific standards of honesty and integrity. They also have a unique role because they are proposing a model, which they have chosen or developed, to a decision-maker or to stakeholders whose values may differ from the analyst's. Others may not understand the model, either because of the mathematics or not having access to the information that the analyst does. As discussed in the next section, there are limitations to the objectivity of models, leaving more room for the analyst to shape the results. The issue of trusting the model builder looms larger when the model is more subjective or opaque. Is the analyst simply being trusted to be honest in creating an objective, scientific model or is the process permeated with the knowledge, experience, and biases of the analyst?

One common pitfall discussed in this literature is that the analyst may place too much trust in a model. What we can expect from an operations research model depends greatly on the system being modeled and available data. The gold standard of validating a model against exogenous data is not possible in many settings. Another dilemma is pressure from a client to produce a model or a reach a certain conclusion, biasing the results. To what degree should the analyst represent the interests of the client? Put another way, is it appropriate to use a model in a certain situation for advocacy? Concerns about subjectivity and bias presuppose that there is an objective reality that the analyst should seek to capture in the model and the data that drives it.

3 Trusting models: epistemology in operations research

John Mingers explains the various philosophical views of operations research models in [16] and [15]. The founders of operations research, with backgrounds in science and engineering, embraced realism and empiricism. They saw operations research models as subject to testing and refinement, broadly applicable, and capturing objective shared experience. These views appear in early textbooks, such as [1]. However, the systems being modeled include or are designed by humans, so the field has aspects of a social science. The positivist view of social science asserts that knowledge in these fields is empirical and can be objective, or theory-free. Realism leads to an emphasis on established models or problems, and a tendency to teach models, and how to justify them, rather than teaching a discovery process of modeling [17]. A course might present the Verhulst logistics growth model for a population, or a linear programming approach to choosing an investment portfolio, then ask students to apply this type of model to a situation. Empiricism emphasizes that models are based on data and observation. Statistical and simulation models exemplify this approach.

In contrast to empiricism, a pragmatic or instrumentalist philosophy views models as useful for prediction, but not as containing truth or explanatory power. The "system" is in the model, not the real world. Proponents of this view see operation research as technology, not science–which is still viewed as realist.

A third view, prevalent in the philosophy of science, is that theories are socially constructed. In the social sciences, theories–or paradigms–tend to coexist. Rather than one theory making predictions that could falsify the other, they may be incommensurable, essentially talking past each other. Mingers sees social construction as explaining the alternative methods that have flourished in operations research. He lists

- traditional optimization models
- cognitive mapping of qualitative models in decision making
- soft systems methodology [5], which is strongly relativist and constructs models based on the decision-maker's views
- critical operations research, advocated by Mingers based on a philosophy of critical realism
- decision analysis [12], which recognizes multiple criteria and assesses a decision-makers values or utility function.

Observing the models used by individuals with different training within operations research, it is difficult to avoid the view that the models employed in a specific situation are partly chosen because of the person's background. To a degree, models are chosen by modelers, not derived from the reality being modeled.

The categories of models described above, and the focus of this paper, excludes the many "models" that are essentially statistical. While all of these models use data, a statistical model makes few structural assumptions and has no dynamics. Rather, its goal is to generalize, or draw inferences, from data. The most common example is a regression model, which might assume linear relationships between certain variables, but even these assumptions are usually tested. The model fits the data and seeks to predict outputs based on inputs, but offers little or no additional structure. The use of statistical models is growing rapidly. This is largely due to the availability of data and better algorithms from machine learning to draw inferences from large amounts of data. Machine learning and data science blur the lines somewhat between statistical and more structural models. Techniques like clustering and regularized regression (ridge or LASSO regression) can be considered "just" statistical, but they also identify patterns out of so many possibilities that one could view them as finding structure. Either way, data-driven projects that involve machine learning often include a more structural or dynamic model, to which the comments in this paper would apply.

Another growing use of models is when organizations classify and make decisions about individuals. The models tend to be of the machine learning type: large and data-intensive. For example, the credit scores used by companies to decide what loans and mortgages to offer to individuals are based on a complex model [18]. These data-driven systems are receiving increasing scrutiny because of their opaqueness, potential for bias or discrimination, and unintended consequences. While these issues are very important, this paper focuses on models used for organizational-level decisions and control systems.

4 Accuracy of models

Operations research models have continually been used in new domains and on more complex systems. However, for an application to be successful, some conditions must be met. Here is a partial list.

- The system must obey mathematical, or at least statistical, laws (law-like).
- The laws must be discernible. Encrypted messages, for example, contain information and conform to the syntactical laws of language. But this information cannot be extracted or the laws observed because the encryption precludes—as a matter of computing limitations but not in principle—detecting the regularities in the data.
- Data for the particular instance must be available.
- The phenomenon should be repeatable, either in a deterministic or statistical sense. This is a standard criterion in science. For knowledge to be scientific it must be public. In experimental science, this standard is maintained by requiring that other scientists be able to reproduce a result. In other sciences, the methodology (including models) and the existing data are subjected to review.

For models of systems that include people in certain roles, intentionality– actions that cannot seem to be described and predicted by scientific laws—creates limitations to modeling. These limits are widely recognized in the social sciences, where different theories of human agency place different limitations on the ability, in principle, to predict human behavior and its impact on models of human institutions and systems.

Some systems cannot be modeled without aggregating because of their sheer size. However, it is often possible to model aggregate behavior in these systems. One very successful example of modeling aggregate behavior is economic theory, where individual people's actions are not predicted but the relationships between market variables can be accurately modeled. Similarly, demographic and epidemiological models aggregate individuals into large classes, or cohorts, and apply a single rate parameter to the cohort to predict future population or the spread of a disease.

The ability to construct a predictive model of a system depends critically on its stability and related properties. Consider three examples.

1. An aircraft carrier is preparing to launch its full complement of aircraft. How long will it take until all (or almost all, since aircraft are not as reliable as the carrier) are launched?
2. A factory assembles many different, but closely related complex pieces of electronic equipment. They have plenty of business, so their goal is to produce as much as possible. How many items of each type will they produce in the next month?
3. A large group of people is gathered on the grounds of a rural home when a fire breaks out in the house. No help is nearby, but there is a well on the property and they attempt to put out the fire. How long does it take them to put out the fire?

These three examples have roughly comparable size, or complexity, as measured by the number of interacting people and objects that come to mind as we think about them. The fire brigade lacks complex equipment and so might be considered somewhat smaller. Yet even if we possessed all the data and knowledge about these systems that we can imagine, our ability to construct models of them would differ dramatically.

Carrier operations are a remarkable example of a very tightly designed and centrally controlled human/machine system. Although its individual elements are fraught with randomness and even some creativity, the overall behavior—the launching of aircraft—is very predictable, even though the system operates very nearly at capacity. Even the uncertainty in the model's prediction could be well captured by probability models of equipment reliability and human performance.

The fire brigade is at the opposite extreme. It has no predefined plan or control; it lacks design. Even given detailed knowledge of the house, the surroundings, the people present, how the fire started, and the weather, we would be at a loss as to how to construct a meaningful model. Given a large database of past fire brigades (the existence of which is problematic because the point is that the situation is unusual), we might find some correlations. Or we might establish bounds on water volume delivery and its efficacy on certain sizes of fires. But we would still be left with a huge uncertainty: Will the fire be brought under control while it is small or will it consume the house? This example differs not so much in the randomness exhibited by its components—both systems have that characteristic—but by its lack of an operational plan. We have no idea how the people will organize and how the system will operate.

The factory is in some sense in the middle. It is designed, with a fair degree of control over the actions of people, but it is controlled in a less centralized fashion than the carrier. Various people in the factory make decisions that affect scheduling, routing, and tempo. External circumstances and the response to them can easily alter the schedule. For example, a customer with a "hot" order may have their job expedited, resulting in other jobs (and total production) being delayed. A sole-source supplier may fail to deliver a part on time, making the production of certain items impossible. Models of the factory will be very helpful in understanding it and exploring sensitivities, but they are likely to miss some of the innovative behavior it exhibits. Predictions will contain randomness, but have the potential to be rather accurate in a statistical sense.

In summary, human agency without a clear purpose to constrain individual actions limits the ability to model. Such constraints occur in a system with a high degree of design and control. Repeatable situations are most easily modeled. Since systems with humans may evolve and learn, they can exhibit more change, so that a model no longer applies.

If accurate models are possible in a certain application, then as work continues on the models and more data is collected to drive them, they should perform better, converging on an accurate description of the system. Is this a realistic expectation? There are cases where accuracy has greatly improved over time and others where it has not. One problem that was the subject of intense study, particularly from about 1990 to 2010, is how to staff an inbound call center. Models in this area try to predict the proportion of calls made to customer service that will have to wait more than a specified time. The firm operating the call center may have contracted to provide this level of service to the company whose customers they are, or may be concerned about losing their own customers. Once the service level can be predicted from the staffing level, an optimal staffing level can be found. Early models were very inaccurate at predicting service level.

The first major improvement was to replace naive assumptions about the shape of the service time distribution with a shape fit to large amounts of data, which has a "heavier" tail. Then the same was done for customer impatience (how long they will wait on hold). A third major achievement was to develop good approximate models of the system dynamics, called many-server queues. The models study the limit as the number of servers (and calls) approaches infinity. The approximation is quite accurate and the resulting model can be studied and optimized more easily than simulation models. Forecasting call volume also improved. Today, these models are being used for staffing, resulting in steadier service levels and huge cost savings. This example has many of the characteristics described above: known system dynamics (laws), observability, voluminous data, and enough size to aggregate unpredictable human behavior.

5 Values in models

Next we describe how values appear in decision support models. These models are prescriptive or normative; they compare options, leading to a recommended decision or a ranking of alternatives. When there are many possible decisions, the general framework of constrained optimization is used. In this framework, there is a set of decision variables, constraints limiting the possible choices, and an objective function. Values may shape the choice of decision variables through the assumption of what things are subject to change and what is already fixed. Some constraints reflect logical requirements, operating procedures, or physical capacities; these are not likely to be influenced by values, once the choice of decision variables is made. Other constraints are goals set by the decision-maker, based on their values or need to accommodate other stakeholders. For example, an airline scheduling flight crews may want the constraint that no one is away from home more than two weekends in a month. See [2] for a history of optimization models in the airline industry.

Objective functions are most clearly value-laden. Some objectives are singular, such as maximizing profit. In other situations, there are many competing objectives. Although multi-criteria decision making and goal programming can be used to find non-dominated decisions, conceptually we can assign weights to each criteria, creating a single utility function. Utility theory, expert systems, and more recently supervised learning all have methods to elicit the preferences from decision-makers. Still, often the desired objective cannot be measured or modeled, so surrogate objective functions are used.

The analyst needs to consider how to learn the decision-maker's values and the extent to which they will include them in the model. The client can also shape the analysis in many ways, some to be expected and some perhaps surprising. I offer an anecdote. In the 1990s I developed an inspection model for a large manufacturer of printers. Some batches of parts from suppliers were inspected by sampling a few parts. If defective parts were not detected until they were in subassemblies, or until the finished printer was tested, the cost of the defect was much larger. They wanted to better choose which parts to inspect before assembly to reduce their quality-related costs. They provided detailed data about their inspection procedures, costs involved, and the defects found over a two year period. This was precisely the data needed to measure their cost of quality, but contained very little information about how to improve it. The issue was that, if they do not use screening (100% inspection) of any items, the reduction in downstream defects due to sample inspection depends on common-cause defects. If defects occur independently in each part, then rejecting batches has very little effect (the uninspected items all have the same probability of a defect) and taking corrective action with the supplier has no effect. If common-cause defects affect whole batches or all future batches from the supplier, then sampling from batches and corrective action can be very effective. Their sampling procedures presumed common-cause defects, but they had very little relevant data: the distribution of defect rates across batches of the same item is needed.

The client didn't see the need for this data. When I persisted, they referred me to another department that designed their sampling plans. A statistician in this group fully understood the issue. No more data were available, but he guided me toward empirical Bayesian estimation procedures in the quality literature and what could be done with limited data. The client wanted results, so I developed a model using the dubious approach of pooling the data for all items, giving enough data to estimate the distribution. As is often the case in a large organization, the client had other needs and agendas. They wanted to use the model to justify doing less inspection, which had high labor costs. Reducing inspection would eliminate some jobs in that department. The model was used once to identify a list of items where inspecting batches was most cost effective, but ultimately had little effect on their decisions. This project illustrates that the client's view of the problem can drive which aspects are modeled, what data they make available, what activities they fund, and ultimately what models or results they use.

Analysts, trained in mathematical models, will tend to bring different motivations and values to their work than a typical client. The analyst may also differ from the client on broader values, such as social responsibility, environmental impact, or the importance of different stakeholders in the decisions being made. I would suggest that Christians have additional motivation to understand a decision-maker's values and priorities, as well as the larger values that underpin them. Within ethical boundaries and personal conscience, they should seek to reflect these values in the model, even if they do not hold the same values. The additional role, or calling, of the Christian in this situation may be to think more deeply about the values implied by the modeling choices and how they impact other stakeholders, societies, the environment, and the broader notion of shalom.

6 Disaster response planning

Academics in operations research and logistics have been intensely studying how to better respond to humanitarian disasters, such as hurricanes and earthquakes [13]. This field of humanitarian logistics considers strategies for propositioning, transportation, warehousing, last mile distribution, debris removal, etc. Some of the methods apply to rapid onset man-made disasters as well.

Although many models have been developed, using them is challenging because of the unique features of disaster response. A large disaster requires an intense, rapid response, often needing supplies and personnel from far away. Infrastructure is usually damaged, vulnerable populations tend to be in remote areas, and control of the response is decentralized, with many organizations responding. Each disaster is different, and the needs in a disaster are difficult to assess and predict when it first occurs. On the spectrum in Section 4, disaster response is a hard situation to model accurately. Its unique features are discussed in [10].

Values are very important in these models. In the commercial sector, the objective is to minimize logistics cost, while in the military sector operating capability is maximized. In humanitarian logistics, the overall objective is to reduce loss of life and suffering due to lack of basic needs: water, sanitation, food, shelter, and emergency health care. One approach is to model deprivation costs, which are convex in the time until aid is delivered, and add them to logistics costs [11]. The secondary, more measurable objectives include the quantity and speed of aid, sending priority items to priority locations, cost, the impact of aid on local suppliers and the recovery phase, and media coverage to stimulate donations. Disaster relief organizations (DROs) need to consider all of the secondary objectives; most models combine several of them as in [8], or use constraints such as equity of distribution to different communities. More issues involved in these models are discussed in [9].

7 Christian disaster response organizations

Along with Paul Isihara and Danilo Diedrichs at Wheaton College (IL), I have been studying Christian DROs. Although several faiths are active in disaster response, we chose to only study medium to large, U.S. based, Christian organizations involved in disaster response. The primary research question is how their Christian mission and values influence the planning and implementation of disaster response. We have focused on their decision of whether to respond to a disaster and the operational decisions they make in distributing aid.

A small comparison group of non-faith-based DROs was included in the study; however, this group was not a random sample and is not very representative of all non-faith based DROs. Results from the surveys are not yet published, but here are some distinctives voiced by representatives of Christian organizations that we interviewed.

- A willingness to make more personal sacrifices when helping those in need.

	Speed	Cost	% of request	Priority items	Priority locations
Christian ($n = 21$)	3.0	3.8	4.7	1.9	1.7
Non-faith based ($n = 6$)	3.0	4.5	3.2	2.5	1.8

Table 1: Average priority given to attributes of a disaster response (1 is highest, 5 is lowest).

- A preference for meeting more needs, including emotional, psychological, for fewer people rather than meeting immediate physical needs for more people.
- Combining disaster relief with other activities, such as supporting local churches or witnessing to individuals.
- Staying longer in a disaster area, as relief transitions to recovery.
- A preference for a relational approach, where staff get to know beneficiaries.
- A desire to respond to disasters according to the needs, even when donors are not interested in a particular disaster.
- Some Christian DROs are very selective about who they partner with and do not work with governments or partner with organizations that do not share their Christian values.

One Christian organization we worked with has a system for classifying goods that are donated to the organization according to four levels of mission priority, depending on how well they align with their ongoing development programs. For example, because of their emphasis on children, new children's clothing is usually classified in the highest mission priority. A simple model supporting the decision to accept the donation takes into account the mission priority.

In our survey, one question asked logistics managers to rank five factors of a typical first phase response to a large disaster, where requests for aid exceed the combined capacity of all respondents to deliver. The factors are taken from a study that assessed how much weight managers at large non-faith based DROs give to each factor [8]. Table 1 shows the average priority in our survey, on a five point scale where 1 is the highest priority. The Christian DROs give little importance to the total volume of aid, measured by the percent of the request met, and more importance to delivering high priority items to the locations with the greatest need. Cost is rated fairly low in importance; the difference between the two groups may be due to sampling or to the context of the question, rather than the overall concern for cost efficiency in the two groups. These results can be used as inputs to a model that optimizes the distribution of aid according to these values, e.g., truck or helicopter scheduling and routing, as done in [8].

8 Teaching modeling with values

In my course on optimization models for undergraduate mathematics majors, I don't say much about values, ethics, or the ability to model some systems better than others. But maybe I should. Another course, Mathematical Models for Industry, brings up questions about the client's values and what sort of model is possible. Students wrestle with a semester-long project from industry. Many students have said this course was very helpful. Other pedagogical changes could address values in models more directly.

One specific suggestion, which has been made repeatedly in forums about teaching operations research, is to teach fewer models and more modeling. When a model is taught, we are in a sense giving students the answer instead of the question. The quantity to be measured or optimized is usually identified for the student, as well as the general model structure. In contrast, the process of modeling includes choosing the type of model to use and the quantities to predict or optimize. This skill is often needed for real-world projects.

A second suggestion is to use messy, unstructured modeling projects that directly raise questions about values. A good starting place might be modeling in the public and nonprofit sectors, where there are often stakeholders with different values. Given the many ways that values influence models, and the desirability of getting the right values in the model to support the decision-maker, anything we can do to prepare students in this regard seems valuable.

References

[1] Russell L. Ackoff. *Scientific method: Optimizing applied research decisions.* Wiley, 1962.

[2] C. Barnhart, P. Belobaba, and A.R. Odoni. Applications of operations research in the air transport industry. *Transportation science*, 37(4):368–391, 2003.

[3] J Pierre Brans. OR, ethics and decisions: The oath of Prometheus. *European Journal of Operational Research*, 140(2):191–196, 2002.

[4] Jean-Pierre Brans and Giorgio Gallo. Ethics in OR/MS: Past, present and future. *Annals of Operations Research*, 153(1):165–178, 2007.

[5] C West Churchman. *The Design of Inquiring Systems: Basic Concepts of Systems and Organization.* Basic Books, 1971.

[6] Giorgio Gallo. Operations research and ethics: Responsibility, sharing and cooperation. *European Journal of Operational Research*, 153(2):468–476, 2004.

[7] Saul I Gass. Model world: Models at the OK corral. *Interfaces*, 21(6):80–86, 1991.

[8] Erica Gralla, Jarrod Goentzel, and Charles Fine. Assessing trade-offs among multiple objectives for humanitarian aid delivery using expert preferences. *Production and Operations Management*, 23(6):978–989, 2014.

[9] José Holguín-Veras, Miguel Jaller, Felipe Aros-Vera, Johanna Amaya, Trilce Encarnación, and Tricia Wachtendorf. Disaster response logistics: Chief findings of fieldwork research. In *Advances in managing humanitarian operations*, pages 33–57. Springer, 2016.

[10] José Holguín-Veras, Miguel Jaller, Luk N Van Wassenhove, Noel Pérez, and Tricia Wachtendorf. On the unique features of post-disaster humanitarian logistics. *Journal of Operations Management*, 30(7-8):494–506, 2012.

[11] José Holguín-Veras, Noel Pérez, Miguel Jaller, Luk N Van Wassenhove, and Felipe Aros-Vera. On the appropriate objective function for post-disaster humanitarian logistics models. *Journal of Operations Management*, 31(5):262–280, 2013.

[12] Ralph L Keeney and Howard Raiffa. *Decisions with Multiple Objectives: Preferences and Value Trade-offs*. Cambridge University Press, 1993.

[13] Nathan Kunz and Gerald Reiner. A meta-analysis of humanitarian logistics research. *Journal of Humanitarian Logistics and Supply Chain Management*, 2(2):116–147, 2012.

[14] Marc Le Menestrel and Luk N Van Wassenhove. Ethics in operations research and management sciences: A never-ending effort to combine rigor and passion. *Omega*, 37(6):1039–1043, 2009.

[15] John Mingers. *Systems Thinking, Critical Realism and Philosophy: A Confluence of Ideas*. Routledge, 2014.

[16] John Mingers et al. The contribution of critical realism as an underpinning philosophy for OR/MS and systems. *Journal of the Operational Research Society*, 51(11):1256–1270, 2000.

[17] William T Morris. On the art of modeling. *Management Science*, 13(12):B–707, 1967.

[18] Cathy O'Neil. *Weapons of Math Destruction: How Big Data Increases Inequality and Threatens Democracy*. Broadway Books, 2016.

[19] Richard J Ormerod and Werner Ulrich. Operational research and ethics: A literature review. *European Journal of Operational Research*, 228(2):291–307, 2013.

[20] William A Wallace. *Ethics in Modeling*. Pergamon, 1994.

Maximum Elements of Ordered Sets and Anselm's Ontological Argument

Doug Ward (Miami University, Ohio)

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Abstract

I present a theorem about a set containing a maximum element with respect to some asymmetric ordering. This theorem aims to elucidate Anselm's ontological argument, a classical proof of the existence of God.

1 A Theorem on Ordered Sets

In this note I will state and prove a simple mathematical theorem about a set endowed with an ordering. This theorem is very easy to prove, but it has a significant application.

I will begin by establishing some basic notation. Let T be a set. I will employ the usual notation $x \in T$ to indicate that x is an element or member of T. If x is not an element of T, I will write $x \notin T$. For a set E, I will say that E is a subset of T, denoted $E \subset T$, if every element of E is also an element of T. I will denote by $T \setminus E$ the set of elements of T that are not elements of E.

I will denote by $T \times T$ the Cartesian product of T with itself; that is, $T \times T$ consists of all ordered pairs (x, y), where x and y are elements of T. I will refer to any set of ordered pairs R, where $R \subset T \times T$, as a *relation on* T.

Now let R be a relation on T. We can think of R as an *ordering on* T by writing $x \succ y$ ("x is better than y") whenever $(x, y) \in R$. I will say that R is asymmetric if whenever $(x, y) \in R$, then $(y, x) \notin R$. In the notation of orderings, the ordering $\succ$ is asymmetric if whenever we have $x \succ y$, then we do not have $y \succ x$.

I will say that $m \in T$ is a *maximum element of* T *with respect to* $\succ$ if $m \succ y$ for all $y \in T$ with $y \neq m$. Notice that if $\succ$ is asymmetric, then maximum elements are unique. (If m_1 and m_2 are maximum elements with $m_1 \neq m_2$, then we have both $m_1 \succ m_2$ and $m_2 \succ m_1$, violating the assumption that $\succ$ is asymmetric.)

To illustrate these definitions, suppose that $T = \mathbb{R}$, the set of all real numbers, and let R be the set of ordered pairs of real numbers (x, y) such that $x > y$. Then R is an asymmetric relation, and $x \succ y$ simply means $x > y$. So, for example, $(5, 3) \in R$, since $5 > 3$, but $(3, 5) \notin R$.

In this example the set T has no maximum element, since there is no largest real number. On the other hand, suppose that we let T be the set of all real numbers along with an additional element ∞ (infinity) having the property that $\infty > y$ for all real numbers y. This new enhanced set, which I will call the set of extended real numbers, does have a maximum element: ∞

Here is another example. For real numbers a and b with $a < b$, Let $[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$, and let $T = [a, b] \times [a, b]$. For elements (x_1, y_1), (x_2, y_2) of T, define $(x_1, y_1) \succ (x_2, y_2)$ to mean that $x_1 > x_2$ and $y_1 > y_2$. Then T contains the maximum element (b, b).

The ordering in this second example is known as the *coordinate-wise ordering.* Notice that there are many pairs of elements of $\mathbb{R} \times \mathbb{R}$ that cannot be compared using the coordinate-wise ordering. For instance, when $x > 0$ and $y < 0$ or when $x < 0$ and $y > 0$, then neither $(x, y) \succ (0.0)$ nor $(0, 0) \succ (x, y)$ is true.

I can now state a theorem about sets that have a maximum element with respect to some asymmetric ordering.

Theorem: Let T and E be sets such that $E \subset T$. Let $\succ$ be an asymmetric ordering on T such that

1. T has a maximum element m with respect to $\succ$;
2. for each $y \in T \setminus E$, there exists $x \in E$ such that $x \succ y$.

Then $m \in E$.

The theorem is easy to prove. Suppose that $m \notin E$. Then $m \in T \setminus E$, and by assumption 2, there exists $x \in E$ such that $x \succ m$. Now by assumption 1, we also have $m \succ x$, contradicting our assumption that $\succ$ is an asymmetric ordering. Since the assumption that $m \notin E$ leads to a contradiction, it must be that $m \in E$.

As initial illustrations of the theorem, consider the aforementioned examples. In the case where T is the set of extended real numbers, the set E can be any subset of T that includes the maximum element ∞. In our second example, where $T = [a, b] \times [a, b]$, E can be any subset of T that contains (b, b).

2 An Application in Philosophy

My motivation in formulating the theorem is a particular application in philosophy. In the corollary below, let T be the set of all things that can possibly be imagined, and suppose that T is endowed with an asymmetric ordering $\succ$. Let E be the set of all things that exist in reality, where we assume that E is a subset of T. Then we have the following result:

Corollary: Suppose that T has a maximum element M with respect to $\succ$. Suppose that for any thing imaginable that does not exist in reality, something exists in reality that is better than that thing. Then M exists in reality.

This corollary is a simple form of the ontological argument, a classical proof of the existence of God first proposed by Anselm of Canterbury (1033-1109 AD) in his *Proslogion* (1078 AD). If God can be defined as the best thing imaginable, and if every nonexistent thing is surpassed by something that exists in reality, then God (the maximum element M in the corollary) exists in reality.

Anselm was a Benedictine monk who later served as the Archbishop of Canterbury. He wrote *Proslogion* as an exercise in "faith seeking understanding." His goal was to present an argument for God's existence that would lead to a greater understanding of, and appreciation for, God's attributes.

Anselm defined God as "that than which nothing greater can be conceived," a definition consistent with his Christian beliefs. In both Christian and Jewish theology God is unique as the Creator of everything, and he is greater than all of creation (see for example Isa 44:6,24; 45:22-23).[1] God's status as Creator and Ruler of all makes him a "maximum element" among all the things that can be conceived.

Anselm's argument was controversial from the start. The argument was first challenged by a contemporary, fellow Benedictine monk Guanilo of Marmoutiers. Guanilo posed the following question: Suppose we think of a kind of "fantasy island," the greatest, richest, most beautiful island imaginable. By the reasoning used in the ontological argument, would not such an island have to actually exist? I would answer Guanilo's question by suggesting that the set of all fantasy islands, like the set of real numbers, will have no maximum element. For any such island that one can imagine, it is always possible to imagine an island that is a bit greater.

The mathematical formulation in the corollary also raises questions. One such question involves the composition of the set E of things that exist in reality: Does E include, for example, mathematical objects like numbers, sets, and theorems? One model for the ontological status of mathematical objects, proposed in chapter 3 of [3], says that mathematical objects do indeed exist as thoughts in the mind of God. In this model, mathematical objects are part of creation, and their existence is continually sustained by God because God continually thinks them.

The existence of mathematical objects suggests a second question on the difficulty of drawing comparisons between pairs of elements of the set T. How might we consider one mathematical object to be "better" than another, for example? In response, I will point out that there is no requirement that the ordering $\succ$ be able to compare every pair of elements of T. (We saw this in the example of the coordinate-wise ordering on $\mathbb{R} \times \mathbb{R}$.) The Theorem only requires that each element of T can be compared with the maximum element M, and that each element of $T \setminus E$ can be compared with some element of E.

A third question involves the plausibility of assumption 2 in the context of the corollary. This assumption has an intuitive appeal. Having a real friend, for example, should be better than having an imaginary one. For a person stranded in the desert, a real oasis is better than a mirage. A correct proof of a mathematical theorem is certainly better than an argument containing a flaw. But does adding "existence" to the description of something always improve that thing? Assumption 2 has been a controversial part of the ontological argument.

This note merely scratches the surface of a fascinating topic. For further discussion of the ontological

[1] Analogous statements made about Jesus in the New Testament (Rev 22:13; Phil 2:9-11) are thus affirmations of the deity of Jesus; see for example [1].

argument, see for example the fourth chapter of [2], the third chapter of [5], and [4]. Anselm went on to argue that not only does God exist, but God must in fact exist necessarily—that is, God exists in every conceivable world. Modern discussions of this form of Anselm's argument often use *modal logics* that distinguish between possible and necessary existence.

I formulated the theorem in this note as a means of better understanding Anselm's original argument. This theorem also might an interesting addition to a mathematics class that discusses relations and their properties.

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References

[1] Richard J. Bauckham, *God Crucified: Monotheism and Christology in the New Testament*, Eerdmans, Grand Rapids, Michigan, 1999.

[2] William Lane Craig, *Reasonable Faith: Christian Truth and Apologetics*, Crossway Books, Wheaton, Illinois, 2008.

[3] Russell W. Howell and W. James Bradley, Editors, *Mathematics in a Postmodern Age: A Christian Perspective*, Eeerdmans, Grand Rapids, Michigan, 2001.

[4] Alvin Plantinga, Editor, *The Ontological Argument: From Anselm to Contemporary Philosophers*, Anchor Books, Doubleday & Company, Garden City, New York, 1965.

[5] Nathan Schneider, *God in Proof: The Story of a Search, from the Ancients to the Internet*, University of California Press, Berkeley, 2013.

Closing Banquet Eulogies


David Lay


John Roe

Tribute to David Lay
(Prepared by Russell Howell)

Being asked to eulogize our good friend David Lay is a high honor for me. Because Indiana Wesleyan University has roots in the Wesleyan tradition, it seems appropriate to begin with a well-known quote from John Wesley. Those who knew David would agree that he lived by the following dictum:

Do all the good you can,
By all the means you can,
In all the ways you can,
In all the places you can,
At all the times you can,
To all the people you can,
As long as ever you can.

Focusing on those phrases makes their application to David easy to see:

Do all the good you can by all the means you can
Like many of you I first met David at an ACMS meeting, and was instantly drawn to him because of his affable nature and welcoming spirit. Sometime in 1998 he phoned me out of the blue stating that he was one of the lottery winners for a free room at the Joint Mathematics Meetings in San Antonio. Many people, I suppose, would have taken that opportunity to enjoy a room by themselves, but David invited me to share in his reward. Indeed, whenever he had the means and opportunity David was always thinking of the other.

If you teach at a Christian institution you may have received a note similar to the one Westmont got. It said, “The Lord Jesus has blessed the sales of my Linear Algebra book. Please use the enclosed check in whatever means will best further your program.” The check was very generous, and receiving such a gift by an author is an extraordinary event. But of course, David was an extraordinary person.

In all the ways you can
One way of doing good is to do your job well. David was an excellent teacher, having won Maryland’s Distinguished Scholar-Teacher award in 1997. We all know him as the author of a leading Linear Algebra book, but we might not know of some of the stories behind it. Have you noticed his dedication? It reads, “To my wife, Lillian, and our children, Christina, Deborah, and Melissa, whose support, encouragement, and faithful prayers made this book possible.” More than most any

one I knew, David sought to bear witness to his Lord and savior. His good work earned the respect of everyone who knew him, which resulted in his having a special platform in promoting Christian perspectives. In an earlier edition of his book David wanted a picture of a family walking in a park. The page proofs came back with two women and a young boy. He objected. His publisher gave a lame reason for making the change. David insisted that he wanted to depict a family, and that he wouldn't agree to publishing if the picture wasn't changed back. The publisher relented of course. After all, David's good work had translated into healthy profits for them.

In all the places you can

I was fortunate enough to spend a semester teaching a course in complex analysis at the University of Maryland beginning in January of 2000. David went out of his way to roll out the welcome mat. Sometime early on he and Lillian invited Kay and me to celebrate their anniversary at a dinner theater. I suggested we split the bill, but he insisted on treating us—to *their* anniversary dinner—saying that this was a practice in which they regularly engaged.

And I observed firsthand the extraordinary commitment he had to his church, his teaching, and his family. He was a leader in organizing a weekly lunch group with other Christian faculty, and took delight in Christians doing scholarship in whatever field they were engaged.

As a faculty member David sought to serve rather than to be served, and was not at all concerned about status. I remember a discussion we had in which he expressed disappointment that the Maryland faculty were continually comparing themselves with Harvard, with nuanced language of plans for getting to that level. He then made a rhetorical comment that stuck with me: "Why do we always have to strive to be something we're not? Why can't we just be as good as possible given the situation in which God has placed us?"

At all the times you can to all the people you can

Students no doubt saw Christ's love through David. If, for whatever reason, I happened to leave the mathematics building late, David would often still be there. On one occasion he had deliberately waited until most faculty had gone home. He wanted to enter more helpful problems in Maryland's "Blackboard" web system. But during the day the system was too slow, so waiting until after normal business hours for that task made his time more efficient. This commitment to teaching left an impact on young people that cannot be overstated. Listen to an acolade given to him by Chris Beach, one of his former students. I found it on web page setup by his family in conjunction with the Schoedinger Funeral Parlor: "He made Math exciting for me and when I asked for help, he was always there for me with a smile and encouraging voice ... I just wanted you to know that 26 years after he taught me, I still consider [him to be] the finest teacher I ever had."

I'm sure that comment represents just a tiny tip of the iceberg of all tributes that could be collected from former students. And the fact that we are having this time of reflection speaks well of the impact he had on many of us.

As long as ever you can

Until his final breath, David exhibited the fruit of the Spirit. I continue to be inspired by his work ethic, his brilliance, his humility, his love for his family, and his dedication to God. His life reinforces for me the truth of Proverbs 22:29. "Do you see a man skilled in his work? He will stand before kings; He will not stand before obscure men." Let's take the skill that David exhibited in all his work as a model for us to emulate. But not for the purpose of standing before kings. Rather, for the hope of hearing from the King of kings words with which David has no doubt already been blessed: "Well done, thou good and faithful servant."

Tribute to David Lay

(Prepared by C. Ray Rosentrater)

It is my honor to be given this opportunity to say a few words in honor of our friend David Lay.

My association with David has two different threads: through the ACMS, and through his work in revitalizing Linear Algebra and his associated, best-selling Linear Algebra text.

A couple of decades ago, David, along with Gilbert Strang, David Strong, and others began a conversation on the way that Linear Algebra was being taught. This conversation continues to the present time. At the most recent joint mathematics meetings there were several large sessions on Linear Algebra pedagogy.

As part of this reform movement, David authored a Linear Algebra text. In David's view, most introductory texts spent about half of the semester on computational ideas and matrix manipulations. Then midway through the semester, more theoretical ideas and proofs were suddenly introduced. This left students disoriented and somewhat in shock. David wrote a text that gently introduces theory and proofs along with the computational ideas from the very beginning. Gradually, the number and level of the proofs increases as students develop the ability to handle more sophisticated ideas. This student-centric orientation is typical of David. (More on his concern for students anon.)

The text was a rousing success. David's contact at Pearson told me that he felt very fortunate to be associated with David's text. Only a few people get to be part of publishing such a successful text. David, himself, mused that his book was probably a more significant contribution to the discipline than all of his other professional papers put together.

But David did not view the book's success as his own accomplishment. He frequently expressed gratitude to his wife, Lillian, for her support as he worked on the text. And David told me that as he was writing the text and submitting it for publication, he often prayed and offered the work to God. During the process of negotiation for publication in particular, David felt lead to ask for an unusual contract: a lower royalty rate for typical sales, but if the book exceeded expectations, the royalty rate would be significantly higher than usual. Due to the book's success, David tapped into the higher rate earlier on. And these funds helped provide for David's care as dementia gradually overtook his mind and proceeds from the book continue to support his wife Lillian after his passing.

David not only offered the book to God as it was being written. He maintained the attitude that he was a steward of something that belonged to God. I offer three examples of the way that David put this attitude into practice.

First, David donated some of the royalty funds to Christian endeavors. The checks were accompanied by a letter acknowledging God's goodness in granting success to the book and conveying David's desire to use the proceeds to further God's kingdom.

Second, as one of the latter editions of the text was being written, the publisher wanted to include an image that David felt did not honor God. David fought with the publisher until the picture was replaced.

Finally, as dementia began to overtake David, he knew that responsibility for the text would pass to someone else. David's strong desire was that responsibility for the text would be handed to another Christian. Unfortunately, David lost this battle as his contract specified that the publisher would select subsequent authors in the event that David stopped producing new editions.

My other association with David was through this organization, the ACMS. I particularly remember a Joint Mathematics Meeting in San Diego when we met at a nearby restaurant. (This was in the era when we met at the message boards and then walked as a group to a nearby restaurant.) David was our post-dinner speaker. In his talk, David spoke of his interactions with students, his concern for them, and his regular prayer for them. I was impressed by David's student orientation at an institution that did not place significant value on such things and where the number of students involved was much greater than many of us experience.

David served the ACMS as a board member and, even after he had served out his terms, he was instrumental in advancing ACMS. I remember David coming to a board meeting with a proposal that we change our meeting protocol at the Joint Meetings. He proposed that instead of meeting by the message board and walking to a restaurant, we should host an official banquet held in the meeting facilities. This would allow our ACMS gathering to appear in the conference schedule. David's proposal was approved. While, due to cost considerations, we have we subsequently switched from holding banquets to holding a reception with a speaker and multiple, smaller dinner gatherings at local restaurants, the exposure we have gained from being in the JMM program has resulted in greatly increased attendance at our JMM events. There is no possible way that we could meet at a local restaurant anymore. Thank you, David. You have provided a significant avenue for us to be salt and light in our mathematical community.

I strongly suspect that additional thanks are in order. Holding a banquet at the official venue is not cheap and would have been prohibitively expensive for a significant portion of our membership had it not been for an anonymous donor (or donors) who subsidized the banquets. I do not know, but I strongly suspect that David provided significant financial support for the banquets. This conclusion is strongly in alignment with David's proposal to initiate a JMM banquet, his other donations to support mathematics in a Christian setting, and the humble way in which he conducted his affairs. So, assuming that you contributed financially toward making the banquets possible, thank you, David.

To shift gears a bit, one of my earliest recollections of David is of him leading the conference choir. I don't recall him leading the choir many times, but he sang in subsequent conference choirs. He loved to sing God's praises. So it is fitting that at the end of his life when his mind was failing, one of the last points of connection he had with this world was a session of hymns. David, we will miss your voice. Farwell dear brother in Christ until we meet again around God's throne.

Tribute to John Roe
(Prepared by James Sellers)

I am humbled at the opportunity to share some thoughts about my very good friend, John Roe, and I count it a privilege to do so. I am thankful to Liane and the extended family for this opportunity.

I first met John when I interviewed for my job here at Penn State in the summer of 2001. Just days after the interview, I received an email message from John. In it, he welcomed me to the department, and he did something that I didn't really expect as I returned to my alma mater—he openly and graciously invited me and my family to join him and his family for church services once we arrived in State College. At that moment, I sensed that John and I shared a special bond, as believers in Christ, and our journey as friends began.

John and I were very different from one another in many ways. He was raised in England, completing studies at Cambridge and Oxford, I grew up in the Deep South of the United States, and my academic career wasn't as impressive. He loved rock climbing while I am terrified of heights. He was an extremely healthy eater, I am most certainly not! He held to many liberal views, socially, politically, theologically, while I was much more conservative (at least at the time). He once told me that he was a zero-point Calvinist, a statement which I still don't completely understand (as someone who has followed a Calvinistic theology for most of my life). And yet, John single-handedly helped me to grow in so many ways, especially as he modeled for me what it looked like to be a caring husband and father, a mathematician, and a man of faith!

As many of you are aware, John was a mathematician of the highest caliber. He published numerous books and papers during his career. He was a prolific writer. I learned from John a great deal about completing research in mathematics - the drive to learn something new, the desire to do work that was worthwhile and to write in a clear, readable manner, the willingness to put in countless hours to complete a task.

But John was much more than a research mathematician. His worldview, which was heavily informed by his faith, demanded that he influence the department in broader ways. So while at PSU, John served in various administrative roles, first as associate chair of the department, and later as the chair. In the early years of my relationship with him, he worked diligently to develop a much-needed, semester-long graduate teaching assistant training program. This was essential for the professional development of our graduate students, many of whom spend five years in our department teaching a wide variety of courses which serve students throughout the university. John felt that it was essential that these new graduate students, whom he viewed as colleagues, learn how to teach, how to manage a classroom, how to give priority to teaching, and how to navigate the policies and procedures within a university in the US. He strove to pass on to them his passion and excitement for teaching. The program was a huge success, and was really a forerunner for similar programs which are now popping up around the country.

John was also very sensitive to the educational needs of all students at Penn State, not just the mathematics majors. So he spent significant time developing my department's "Math for Money" course some years ago, and in more recent years, he developed "Math for Sustainability" as well as a corresponding textbook for the course. (After all, there really wasn't a text suitable for such a course - so the right thing to do, in his mind, was to write one!). This project was extremely near and dear to John's heart; he was significantly burdened in the waning years of his life that this textbook come to fruition. Thankfully, within the last few months, this text was completed and is now published by Springer. It was, arguably, John's professional "last hurrah." He was sincerely concerned about climate change, and convicted that we should be treating the creation with dignity and respect, and to serve as excellent stewards of the earth that we have inherited.

To be honest, John's "behind the scenes" work impressed me even more. I will never forget the day, during John's chairship, when he came to me shaken, broken, after he had to compassionately share with a fixed-term instructor that the department might not be able to renew her contract because of the significant financial struggles which the university was facing. He shed tears over his deliberations prior to this meeting; he didn't simply see such actions as "departmental administration;" he always understood that his decisions, his actions, his words, impacted *people*, people about whom he cared a great deal.

And in typically thorough fashion, John also took on the truly "base," non-glamorous task of developing a new system for course scheduling in my department. Motivated by his desire to see the department handle all of these details in a much more effective manner, and his love for programming and electronic gadgetry of all sorts, he single-handedly took it upon himself to develop (from scratch) a database app which kept track of every imaginable detail related to teaching in our department as well as providing a means to assign each member of our teaching corps to their courses each semester. And just for "fun," John wrote a user's guide and documentation for the program - its more than 40 pages long! All as part of his work as department chair.

John completed all of the above, and more, with seemingly unbounded energy and tireless drive. And much of this was being done as he was pouring his heart and soul into the department, while his own personal life contained a great deal of heartache and grief. As John and I would go to lunch together, we would talk about home life, church life, and what we were learning in our faith walks. He spoke many times about his deep concerns for his youngest child Eli. He was often consumed with this concern. And, of course, our conversations often centered around his physical health which, eventually, became the focus of much of his energy. And yet, I spent time with a man who cared deeply about what was going on in *my* life, even while his life seemed to be turned upside down. I enjoyed the presence of a friend who wanted to know more about my kids, my marriage (which was crumbling during the same time), my aspirations, my faith journey. I will always be grateful for his strength and his listening ear. (As a personal aside, the hearing in John's right ear was minimal due, in part, to the cancer that affected that side of his neck and the treatments that followed. In contrast, the hearing in my *left* ear is highly impaired, so even in this way, John and I were truly complementary. In time, we learned that I should always walk on his left side so that we could actually hear one another!)

I am grateful to have had the opportunity to serve as John's caregiver at Johns Hopkins for two weeks during the summer after his cancer surgery. Words cannot express the deep impact those days had on me as I accompanied John to his chemo and radiation treatments and doctors visits (dutifully carrying my pen and pad in hand to take down notes that he and I would reference later if needed). He showed such grace, openness, and thankfulness to others during this time. These shared experiences brought us together in a way that could not have been replicated.

I have been truly blessed to know John Roe as my friend. He impacted me in many ways, from doing everything that he could to nurture my teaching and research career, to challenging me to "eat something green" every once in a while! I am proud to say that I had the opportunity to work side by side with him, as an administrator, mathematical colleague, and educator over the better part of the last two decades. But I am even more thankful, and proud, to have been counted as one of John's friends. His impact on me, and many of us, will be felt for years to come.

Appendix 1: Conference Schedule

Tuesday, May 28	
6:00–7:30 pm	Off-campus dinner for pre-conference attendees
Wednesday, May 29	
7:30–8:30 am	Breakfast (Baldwin)
9:00–12:00 am	Pre-Conference Workshops I (R – Ott Hall 155; Prof Dev – Ott Hall 157)
12:00–1:00 pm	Lunch (Baldwin)
1:00–4:00 pm	Pre-Conference Workshops II (R – Ott Hall 155; Prof Dev – Ott Hall 157)
5:30–6:30 pm	Dinner (Baldwin)
7:00–7:15 pm	Welcome and announcements (Globe)
7:15–8:15 pm	Plenary Talk I: Ken Ono (Globe)
8:30–9:30 pm	Reception / Program (West)
Thursday, May 30	
7:00–7:30 am	Morning Prayer (Bedford)
7:30–8:30 am	Breakfast (Baldwin)
8:30–9:00 am	Devotions and announcements (Globe)
9:00–10:00 am	Plenary Talk II: Ken Ono (Globe)
10:00–10:30 am	Break / Snacks (Piazza)
10:30–11:55 am	Parallel Sessions I (Jones, Leedy, Bedford, Globe)
12:00–1:00 pm	Lunch (Baldwin)
1:15–2:30 pm	Parallel Sessions II (Jones, Leedy, Bedford, Globe)
2:45–3:30 pm	Prayer walk / Testimonials (Piazza)
3:30–4:00 pm	Break / Snacks (Piazza)
4:00–5:25 pm	Parallel Sessions III (Jones, Leedy, Bedford, Globe)
5:30–6:30 pm	Dinner (Baldwin)
6:45–7:45 pm	Plenary Talk III: Joan Richards (Globe)
8:00–9:00 pm	Topics Discussion I (Leedy, Bedford)
Friday, May 31	
7:00–7:30 am	Morning Prayer (Bedford)
7:30–8:30 am	Breakfast (Baldwin)
8:30–9:00 am	Devotions and announcements (Globe)
9:00–10:30 am	Plenary Talk IV: Michael Alford (Globe)
10:30–11:00 am	Group Photo and Break / Snacks (Commons, Piazza)
11:00–11:55 am	Parallel Sessions IV (Jones, Leedy, Bedford, Globe)
12:00–1:00 pm	Lunch (Baldwin)
1:15–3:00 pm	Parallel Sessions V (Jones, Leedy, Bedford, Globe)
3:15–4:15 pm	Topics Discussion II (Jones, Leedy, Bedford)
4:15–6:00 pm	Break
6:00–7:15 pm	Banquet (West)
7:30–8:00 pm	ACMS Members Memorial Session (West)
Saturday, June 1	
7:30–8:30 am	Breakfast (Baldwin)
8:45–9:00 am	Announcements (West)
9:00–10:00 am	Plenary Talk V: Joan Richards (West)
10:15–10:45 am	ACMS Business Meeting (West)
10:45–11:45 am	Worship Service (West)
12:15 pm	Vans depart for Indianapolis Airport
12:15 pm	ACMS Executive Board Meeting (Bedford)

Parallel Session Schedule

Thursday, May 30				
Session 1	**Jones (Computer Science)**	**Leedy (Mathematics Ed.)**	**Bedford (Mathematics)**	**Globe (Faith)**
10:30–10:45	Ryan Yates *Image Data: A Project-Based Exploration of Computer Vision*	Mary Vanderschoot *A Modeling-driven Approach in Teaching Differential Equations*	Brian Beasley *From Perfect Shuffles to Landau's Function*	James Turner *Ways of Thinking Beautifully about Mathematics*
10:50–11:05	Michael Janzen *EventFinder: A Program for Screening Remotely Captured Images*	Lauren Sager *Specifications Grading in a Math for Liberal Arts Course*	Lisa Hernandez *Mosaic Number of Torus Knots*	Josh Wilkerson *Math is ________*
11:10–11:25	Seth Hamman *Character and Cybersecurity Education*	Daniel Showalter *Small is Beautiful: Leveraging the Small Size of Departments to Make Rapid Cultural Changes*	Maddison Guillaume Baker, Amish Mishra, Derek Thompson *Numerical Range of Toeplitz Matrices Over Finite Fields*	Richard Stout *Is Mathematical Truth Time Dependent? Some Thoughts Related to a Paper from Judith Grabiner*
11:30–11:55	Lori Carter, Catherine Crockett *Including Ethics in Computer Science and Mathematics Education*	Jayleen Wangle *An APOS Analysis of Calculus Student Comprehension of Continuity and Related Topics*	Karl-Dieter Krisman *Marin Mersenne: Minim Monk and Modern Messenger of Monotheism, Mathematics, and Music*	Bob Mallison *Faith, Mathematics and Science: The Priority of Scripture in the Pursuit and Acquisition of Truth*

Session 2	**Jones (Mathematics)**	**Leedy (Statistics)**	**Bedford (Mathematics Ed.)**	**Globe (Faith)**
1:15–1:30	Matthew Bone, James Turner *The Boundless Confusion Between Mathematics and Metaphysics*	Judith Canner *Supporting Underrepresented and First-Generation Students in Data Science*	Brandon Bate *Flowcharts in Introduction to Proofs*	Jeremy Case *Almost 20 Years of "Mathematics in a Postmodern Age": A Personal Reflection*
1:35–1:50	Doug Ward *Maximal Elements of Ordered Sets and the Ontological Argument*	Stacey DeRuiter *Statistical Consultancy as Service Learning in Undergraduate Statistics Courses*	Robert Brabenec *A Different Way to Teach Infinite Series*	Thomas Clark *Doing Mathematics the Wright Way*
1:55–2:10	Crow, Zack *Practical Examples of Bin Packing and Critical Path Scheduling*	Rachel Grotheer *Paradigm Shift: One College's Transition from Math to Data Analytics*	Kristin Camenga *Effective Practice and Feedback Methods in Calculus*	Mark Colgan *25 Bible Verses to Connect Faith and Mathematics Using Weekly Devotionals, Written Reflections, and Memory Verses*
2:25–2:30	Andrew Mosteller *Abracadabra: Math, Magic, and More!*	Alana Unfried *Replacing Remedial Mathematics with Corequisites in General Education Mathematics Courses*	Deborah Thomas *Analyzing Retention and Graduation Rates at Bethel University*	Bryant Mathews *25 Designing a New Sequence of Three Seminars on Math and Faith at Azusa Pacific University*

Parallel Session Schedule (Continued)

Thursday, May 30				
Session 3	**Jones (Computer Science)**	**Leedy (Mathematics Ed.)**	**Bedford (Mathematics)**	**Globe (Faith)**
4:00–4:15	Hyunju Kim *Making the Connection Between Biblical Concepts and Programming Basics*	Sarah Nelson *Team Work and Evaluation: Finding the Missing Link*	Michael Pilla *Random Walk on Three "Half-Cubes"*	Kathy Lewis *Teaching at a University in a Developing Country — My Experience*
4:20–4:35	Michael Leih *Using Agile Project Management Techniques as a Pedagogical Framework*	Rebekah Yates *Making Stuff Up: A Model for Undergraduate Research in Mathematics*	Adam Hammett *The n-Children Problem*	Mandi Maxwell *Number Patterns and Insights for the Mathematically Apprehensive*
4:40–4:55	Patrice Conrath *The Joys & Pains of Managing Real World Course Projects*	Andrew Simoson *A Unifying Project for a TEX/CAS course*	Kevin Vander Muelen *Approaching History with Graph Theory – A Review*	Doug Phillippy *Glimpses of God Through a Mathematician's Eyes*
5:00–5:25	David Schweitzer *Addressing Challenges in Creating Math Presentations*	Valorie Zonnefeld *Outreach Activities to Attract Majors*	Michael Veatch *TENZI: A Fun Introduction to Markov Chains and Decisions*	Owen Byer, Deirde Smeltzer *Our "Top Ten" Problem*
Discussion 8:00–9:00		Ryan Botts, Lori Carter, Catherine Crockett, Mike Leih *Mentoring Students with Extra Challenges*	Bob Brabenec *What New Collegiate Faith/Integration Resources Can ACMS Provide?*	

Friday, May 31				
Session 4	**Jones (Mathematics Ed.)**	**Leedy (Statistics)**	**Bedford (Mathematics)**	**Globe (Faith)**
11:00–11:15	Melissa Lindsey *Calculus: A Part of God's Story*	Randall Pruim, Stacy DeRuiter, Matthew Bone *Clean Water for Liberia*	Russell Howell *When the Fundamental Theorem of Algebra Goes Awry*	Patrick Eggleton *Developing Mathematicians: The Benefits of Weaving Spiritual and Disciplinary Discipleship*
11:20–11:35	Bryan Dawson *Ongoing Adventures in Writing a Calculus Textbook*	Ricardo Cordero-Soto *The Applicability of Abstract Mathematics and the Naturalist Die*	Jesús Jiménez *Lagrange's Interpolation, the Chinese Remainder Theorem, and Linear Equations*	Cory Krause *Viewing Mathematics as an Opportunity to Practice Intellectual Virtue*
11:40–11:55	Jim Freemyer, Dave Klanderman, Lauren Sager *Teaching Mathematics Conceptually: Promoting Change in K-12 Math Classrooms*	Randall Pruim *Less Volume, More Creativity: R for Busy Humans*	Sarah Klanderman *Computations in Topological CoHochschild Homology*	Matt Lunsford *A Christian Mathematician's Response to* A Mathematician's Apology

Parallel Session Schedule (Continued)

Friday, May 31				
Session 5	**Jones (Computer Science)**	**Leedy (Statistics)**	**Bedford (Mathematics Ed.)**	**Globe (Faith)**
1:05–1:30	Benjamin Mood *Towards Non-Violent and Christian Video Games*	Katie Fitzgerald *What are We 95% Confident About Anyway? A Software-Embedded Curriculum for Learning Statistical Inference*	Jessie Hamm *Overcoming Stereotypes Through a Liberal Arts Math Course*	Courtney Taylor *Charles Babbage and Mathematical Aspects of the Miraculous*
1:35–1:50	Russ Tuck *Computer Science: Sub-Creation in a Fallen World*	Ray Rosentrater *Celebrity Politicians*	Jill Jordan *Speak for Yourself: Self-Assessment as a Tool for Measuring Participation*	
1:55–2:10	Stefan Brandle *A Modest Conjecture Based on Eternity in Their Hearts*	Ryan Botts, Greg Crow *Adventures in Introductory Statistics: Hybrid, Traditional, and then Hybrid Again*	Bill Kinney *Mathematical YouTubing and Blogging*	Chris Micklewright *Mathematics as Sub-Creation*
2:15–2:30	James Vanderhyde *A Metaphor from Competitive Gaming: the Crown that will Last Forever*	Michael Stob *Does Harvard Discriminate Against Asian Americans in Admission Decisions?*	Amanda Harsy, Marie Meyer, Michael Smith, Brittany Stephenson *Analyzing the Impact of Active Learning in General Education Mathematics Courses*	Emily Pardee *A Conversation About Health Insurance*
2:35–3:00	Derek Schuurman *Transhumanism, Faith, and the Human Future*	Linn Carothers *Mom and the JABEZ Principle – Getting from Vulnerable to Resilient*	Dave Klanderman, Benjamin Gliesmann, Josh Wilkerson, Sarah Klanderman, Patrick Eggleton *Factors that Motivate Students to Learn Mathematics*	Samuel Alexander *Theological Implications of the Identical Ancestors Point*
Discussion 3:15–4:45	Derek Schuurman *New Evangelical Statement of Principles on AI*	Stephen McCarty *Advising Students for Government and Industry Job Searches*	Josh Wilkerson *ACMS Resources for K-12 Christian Educators*	

Appendix 2: Parallel Session Abstracts

Theological Implications of the Identical Ancestors Point

Samuel Allen Alexander

Let t be the most recent time (called the identical ancestors point) such that every human alive at time t was either an ancestor of all humans alive today, or of no humans alive today. Candidates for t include the time of Adam or Noah, but mathematical models suggest t might be more recent (see Rohde et al, 2004, https://doi.org/10.1038/nature02842). We discuss mathematics of the identical ancestors point and explore theological implications that would follow from various values of t within Biblical times.

Flowcharts in Introduction to Proofs

Brandon Bate

For most students, the transition from concrete, computationally intensive content to abstract, proof-based course work can be daunting. The task of developing logical arguments that connect hypothesis to conclusion and then expressing these arguments in succinct, accurate writing can be deeply discouraging for students. At the high school level, flowchart proofs are frequently used to help with this transition as they provide a mechanism both for proof discovery and expression. The same can also be done at the college-level, although some adaptations are required due to the increased complexity that college-level work often entails. In this talk, I will share how I have used flowcharts in my Introduction to Proofs and also discuss the strengths and weaknesses of this approach.

From Perfect Shuffles to Landau's Function

Brian D. Beasley

If we view a given shuffle of a deck of cards as a permutation, then repeatedly applying this same shuffle will eventually return the deck to its original order. In general, how many steps will that take? What happens in the case of so-called perfect shuffles? What type of shuffle will require the greatest number of applications before restoring the original deck? This talk will address those questions and provide a brief history of the work of Edmund Landau on the maximal order of a permutation in the symmetric group on n objects. It will also note some recent progress in refining his results.

The Boundless Confusion Between Mathematics and Metaphysics

Matthew Bone, James Turner

Many metaphysicians have attempted to co-opt topology in the pursuit of understanding real space. It's no surprise why topology is an incredibly useful tool in solving spatial problems and has the benefit of being a mathematically precise framework. However, adopting topology into a metaphysical framework is a precarious endeavor and often leads to undesirable or inconsistent metaphysical conclusions. One such conclusion is that boundaries are real spatial objects. Since boundaries are an integral part of understanding the topology of a space, when people use topology to describe real space, they take boundaries to be real objects. Using topology to describe the essence of real space is untenable because topology is an abstraction from objects in real space and, therefore, cannot capture its whole essence. In this talk, we hope to show why the idea of a real boundary is untenable and how topology is limited in its ability to describe real space. In doing so, we hope to make a more general point: just because people have a good mathematical model for something does not mean they have a good understanding of its essence.

Adventures in Introductory Statistics: Hybrid, Traditional, and then Hybrid Again

Ryan Botts, Greg Crow

Introductory statistics is a high-enrollment service course with a diverse student population posing many interesting teaching challenges. At PLNU we have approached this using a blended flipped pedagogy. However, when our content provider priced us out of their software, we returned to the traditional modality for one semester allowing us to test different pedagogical approaches. We have been gathering student performance and student satisfaction data before, during and after the traditional semester. The student performance data includes nearly identical mid-semester and final exams. So how did they do, and how did they like the hybrid, traditional and then hybrid again?

A Different Way to Teach Infinite Series

Robert Brabenec

The chapter on infinite series in most calculus texts follows a standard rigorous approach which was developed by the end of the nineteenth century. It begins with the definition of a convergent series in terms of a sequence of partial sums, followed by the statement and proof of many convergence tests, and concludes with the Maclaurin series representation of functions and their applications. This approach does not take into consideration the fact that for hundreds of years, individuals worked on specific examples, often for decades before finding a solution. During this time, there was no thought of starting with an attempt to prove that a given series converged. The only concern was to find a value for the given series by any possible method.

Calculus students today experience little of this exciting challenge of looking for patterns and techniques to find a value for a given series. In the fall 2018 semester, I will teach this unit from my own notes. I will use examples of specific series that were solved prior to the development of the axiom system approach described above, which is non-intuitive and confusing to our calculus students. The emphasis will be on group work and student explanation of their ideas about these specific examples. I have rather high hopes and great expectations of a positive response from my students, but my talk will provide an honest account of what really happened.

A Modest Conjecture Based on *Eternity in Their Hearts*

Stefan Brandle

Don Richardson was a missionary to a tribal group called the Sawi. He was starting to despair of communicating the gospel until he learned about their cultural "peace child" tradition. When Don told them that Jesus was God's peace child, the Sawi hearts changed essentially instantly. Based on his experience as related in *Peace Child* and using a title inspired by Ecclesiastes 3:11 ("[God] has planted Eternity in their heart"), Don conjectured that every people group has been prepared by God to receive the Gospel and has "redemptive analogies" that can cross the communication divide.

A parallel conjecture is that within every discipline there are faith and learning integration opportunities waiting to be used. As an example of a redemptive analogy, the expression $3N^2+N\log_2 N+20$ describes the complexity of a particular algorithm. Identifying the most significant factor allows categorization into general performance groupings (constant time, logarithmic, linear, quadratic, etc.). As N gets larger, the dominant term is N^2; everything else can be discarded when classifying the algorithm. The point is to isolate the most significant factor.

The application to faith and learning is: isolate the most significant factor and act based on that factor. For instance, in stimulating personal growth, don't get sidetracked by the less significant factors when dealing with sin, habits, etc. (or use them for avoidance purposes). There is little point in taking some extra vitamins when major aspects of your eating and exercise lifestyle are in a mess. Don't bother with little acts of piety if you do not love God and your brother. Order of magnitude, you must first remove the beam from your own eye, before removing the splinter from someone else's eye. A set of other redemptive analogies will be presented.

Top Ten

Owen Byer and Deirdre L. Smeltzer

We will present our "Top Ten" problems, choosing mostly from those that appear in our recently published textbook *Journey into Discrete Mathematics* (MAA/AMS Press, November 2018). The problems vary in difficulty, but a common feature is that each of the solutions is elegant or clever. Here is one of them: How many positive integers have their digits arranged in strictly decreasing order?

Effective Practice and Feedback Methods in Calculus I

Kristin A. Camenga

How do students learn most effectively in Calculus? Research suggests that the time students spend practicing problems and the feedback that they receive have a positive effect on student learning. However, there are many ways to accomplish this practice and give feedback, so what is most effective? In the context of a small private liberal arts college with Calculus 1 sections of about 30 students, we are studying the effects of three types of practice and feedback on student achievement on exams. The methods of practice and feedback include WeBWorK, textbook homework with students self-checking, and teacher-designed homework with a student grader. We will share preliminary results from two semesters of the study.

Supporting Underrepresented and First-Generation Students in Data Science

Judith E. Canner

In 2015, through the support of the NIH BD2K dR25 Enhancing Diversity in Biomedical Data Science Grant, California State University, Monterey Bay (CSUMB), a federally classified Hispanic Serving Institution, started to develop several programs to support undergraduate training in biomedical data science. The programs include new majors, concentrations, and minors relevant to biomedical data science and professional and research training for undergraduate students and faculty at CSUMB. We will present the evaluation of the first three years of the program and discuss plans for the upcoming years. We will provide case studies of the impact of academic, professional, and personal supports developed specifically for our program to support first-generation and underrepresented students to pursue graduate studies in biomedical data science. Specifically, we will discuss the program specific structures we developed to support students "late" in their undergraduate careers that want to pursue graduate education in biomedical data science and our plans to institutionalize the new structures through programs and coursework. The institutionalization of structures beyond our funded program will allow us to recruit more first-generation and underrepresented students, especially transfer students, into the data science graduate programs and careers.

Mom and the JABEZ Principle – Getting from Vulnerable to Resilient

Linn Carothers

Jabez was more honorable than his brothers, . . . his mother named him Jabez saying, “. . . I bore him with pain.” 1 Chronicles 4:9 NASB95

The pain of family dysfunction is part of 21st Century life. As parents and parental surrogates, what are key factors to communicate faith, hope and love to the next generation? Using the longitudinal perspective of a Danish cohort ($N = 9125$), variables measuring the family’s stability and structure, parental characteristics, household characteristics, and the child’s social and academic competencies were examined. Biostatistical data mining using discriminant analysis, coupled with loglinear modeling, reveal additive and interactive contributions of instability of adult constellations, parent crime, poor economic conditions, and the work organization and athletic skills of the child; however, critical elements of mothers and their response or reaction played pivotal roles. For parents and parental surrogates, the key to resilience centers on teaching self-monitoring skills, valuing work, and engagement during latency, providing tools for faith-integrated change.

Including Ethics in Computer Science and Mathematics Education

Lori Carter, Catherine Crockett

Several speakers at recent ACMS conferences have pointed out that the discussion of computing ethics extends beyond a security course looking at computer viruses, identity theft, and hacking. Ethical discussions now must involve issues of transparency, mental and physical health, accessibility, dignity, the environment, and the list is growing. Furthermore, with the surging popularity of data science, a blend of computer science and mathematics, computing ethics must extend into the mathematics realm as well. Math-related topics may include ethical data visualization, data cleaning, and transparent predictive algorithms. Leaving the discussion of ethics to a class in the philosophy department sometimes leaves students unable to make the connection between theory and practice, especially discipline-specific practice. Even housing a dedicated ethics course in the mathematics or computer science department runs the risk of making it seem separate from daily practice. We believe that the best way to help students think about ethical issues related to computer science and mathematics is to integrate these discussions into core courses. Spreading this integration over the four years, will, we believe, make it more naturally come to mind in the future. This presentation describes an ethics curriculum consisting of modules to be embedded into computer science and mathematics courses so students can practice recognizing and making judgements on ethical dilemmas in context. The contents of the modules go from general concepts in early courses to more area-specific concepts in later courses. In addition to the overview, we will present several modules in both computer science and mathematics.

Almost 20 Years of *Mathematics in a PostModern Age* A Personal Reflection

Jeremy Case

Mathematics in a Postmodern Age was published in 2001 and has been used in our Mathematics Capstone since 2002. Initiated by members of the ACMS, the book examined how Postmodernism might apply to the field of mathematics in its truth claims, its apparent universality, and its cultural influence. While some material is not as hotly debated today, the rejection of mathematical imperialism and other points continue to remain relevant and are almost prescient. The benefits of MIAPA as a course text include its examination of mathematical history, mathematical philosophy, and mathematical education. There are plenty of challenging perspectives to prompt students to evaluate and articulate their own beliefs and values. This presentation will report on how students have viewed the book and its ideas. Many students find the reading difficult, and pedagogical strategies will be provided on how to alleviate their struggle. Finally, I will assess the book's influence on my own spiritual journey and teaching practice.

Doing Mathematics the Wright Way

Thomas J. Clark

The British theologian N.T. Wright describes how the Scriptures, specifically the books of John and Colossians, reveal that the creation was made in and through Christ. It reasonably follows that aspects of the character of Jesus should arise naturally in the creation. Therefore, mathematics, the study of the numerical and spatial aspects of creation, should be rife with Christ's fingerprints. We explore what it means to do mathematics with this in mind, juxtaposing the character of Christ as shown in the gospels with the experience of actually doing mathematics and find a veritable bouquet of connections. A look into various ways in which students can participate in this endeavor will conclude the session.

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25 Bible Verses to Connect Faith and Mathematics Using Weekly Devotionals, Written Reflections, and Memory Verses

Mark Colgan

One of the goals for all my students is to help them make connections between issues of faith and mathematics. As a former Bible quizzer myself, I sought to find Bible verses that illustrated some of these connections. I present a math memory verse as a devotional each Monday in all my classes that connects in some way to the content we are studying that week. I ask students to write short reflections that relate to the verses, and they have the option to memorize the verses for bonus points. I have categorized the verses into mathematical topics that connect with issues of faith such as beauty, fourth dimension, infinity, saving and exponential growth, debt and credit, reasoning and proof, gambling, group work, maximizing and minimizing, why study mathematics, etc. I will share my list of verses and include a video, a short article, and a few discussion questions that go with each verse.

The Joys & Pains of Managing Real World Course Projects

Patrice Conrath

Leading students through real world projects is an exciting exercise that can inspire students and show them how to use their math and computer science skills to make a difference in the world. However, the management challenges of these projects can deter even the most eager professor from providing these opportunities. Finding clients, managing teams, and pushing students towards significant solutions are some of the trials.

At the 2013 ACMS conference, I presented, "Simulation Projects in an Operations Research Course," highlighting techniques I have used for project management in my Operations Research course. Since then, I have continued to manage semester long projects, such as our last project related to a proposed Feed My Starving Children packing plant in Nepal (student portfolio). I recognize the value of these projects, but management continues to be challenging. Therefore, I am currently taking a project management course and will have a sabbatical in spring 2020, where I will examine project management techniques. In this session, I will share some of my challenges and current materials, and will also gather pain points and ideas from attendees to guide my future study. Through the gathering, I hope to start a small group to share resources and encourage one another in this worthy activity, as we inspire students to serve God and others with their talents.

The Applicability of Abstract Mathematics and the Naturalist Die

Ricardo J. Cordero-Soto

Philosopher and Christian apologist William Lane Craig has proposed a valid deductive argument for God's existence that is rooted in the applicability of mathematics to the physical universe. This argument was presented by Craig during a debate with philosopher and atheist Alex Rosenberg. During the debate, Rosenberg presented a rebuttal to the soundness of this argument by appealing to chance as an explanation to the applicability of mathematics to the physical universe. In this talk, the presenter will discuss how the naturalist die is unable to produce "chance application" of mathematics while defending the soundness of the argument in light of the ontology of mathematics. In passing, the problem of abstract objects and God will be addressed.

Marin Mersenne: Minim Monk and Modern Messenger of Monotheism, Mathematics, and Music

Karl-Dieter Crisman

If you have taught a number theory course or even watched the mathematical news, you know that occasionally a new (and enormous) "Mersenne prime" is discovered. Those who have introduced students to the prehistory of calculus may know of a certain Marin Mersenne as the interlocutor who drew Fermat and Descartes (and others) out to discuss their methods of tangents (and more).

But who was Mersenne, and what did he actually do? This presentation will give an overview of his times, his role in the history of science, and his own writings. We'll especially look into why a monk from an order devoted to being the least of all delved so deeply into (among other things) exploratory mathematics, practical acoustics, and defeating freethinkers.

Ongoing Adventures in Writing a Calculus Textbook

Bryan Dawson

Issues, challenges, joys, and benefits are just a few of the words that describe the textbook writing process. This talk will describe the speaker's thoughts on intentionality, textbook design, the writing process, and what motivated the journey in the first place.

Statistical Consultancy as Service Learning in Undergraduate Statistics Courses

Stacy DeRuiter

Best practices and recommendations for undergraduate statistics courses increasingly emphasize student analysis of real data, effective use of statistical computing software, and meaningful problem-solving and decision-making. Academic service learning allows students to put their academic and technical skills to use for community good. Integration of faith and statistical learning, at a Christian institution, often means considering the moral or religious motivations or consequences of drawing conclusions from data. Combining real data with community service is one way to bring all these ideas together. At Calvin College we recently introduced a second course in statistics (Advanced Data Analysis) in which students learn modern regression techniques. As a term project for this course, groups of students act as statistical consultants for area non-profit organizations or academic researchers. These statistical consultancy service-learning projects combine real-world data wrangling and analysis with community service, and provide an opportunity for meaningful integration of faith and learning. In this presentation, I will consider case studies from the first two years of the Calvin course, offering comparisons with similar projects at other institutions and highlighting areas that show promise and others that need revision.

Developing Mathematicians: The Benefits of Weaving Spiritual and Disciplinary Discipleship

Patrick Eggleton

Part of the goal of discipleship at the Christian university is for faith development to seep into the hearts of the students. Similarly, the goal of the development of future mathematicians is for the mathematical proficiencies, the practices like problem solving and analytical reasoning that permeate each of the courses, to seep into the hearts of our majors. This presentation shares how the weaving of our spiritual and disciplinary discipleship efforts benefits the faith development of our students while also helping them to think like a mathematician.

What are We 95% Confident About Anyway? A Software-Embedded Curriculum for Learning Statistical Inference

Katie Fitzgerald

Much of scientific inquiry relies on statistical inference, and nearly every person has consumed "evidence" that arose from statistical inference, whether they are aware of it or not. Despite being ubiquitous in our society, the mechanisms behind statistical inference are notoriously abstract and confound students and users of statistics at all levels. The widespread and persistent lack of sound statistical reasoning is well-documented and researched. Introductory statistics courses often focus on formulas and procedures but fail to develop intuition for why statistical methods work. This talk discusses a software-embedded curriculum I designed to make the mechanisms behind statistical inference more salient. The software and supporting curriculum help learners explore—and even construct for themselves—properties of the normal distribution that play a critical role in determining what constitutes valid and convincing statistical evidence. The design is informed by both a constructionist learning philosophy as well as my own experience working with undergraduate students and social science researchers who use statistical methods but lack sound statistical reasoning. The software harnesses the power of simulation-based methods and an agent-based modeling environment, NetLogo, to allow students to see stable aggregate patterns and complex properties emerge from individual observations, without relying on knowledge of calculus or probability theory.

Teaching Mathematics Conceptually: Promoting Change in K-12 Mathematics Classrooms

Jim Freemyer, Lauren Sager, Dave Klanderman

Recent measures of mathematical achievement by students in the United States document limited mastery problem solving and more conceptual, as compared to procedural, understanding of mathematics, as reported by the National Assessment of Educational Progress (NAEP, 2018). Comparisons to developed countries around the world have shown the US to be average or worse (cf. TIMSS, 2015). In response to these data, the mathematics education community has called for a greater focus on conceptual learning, as seen in *Principles and Standards for School Mathematics* (2000) and *Principles to Actions: Ensuring Mathematical Success for All* (2014). This session provides an overview of a current book project written for school principals, superintendents, and other school decision makers. The thesis of the book is that a collaborative transformation of mathematics teachers is required to prepare K-12 students for a competitive workplace in the current and coming decades. Attention is given to providing teachers with resources, time for both professional development and lesson design, and appropriate recognition for improved student learning outcomes.

Paradigm Shift: One College's Transition from Math to Data Analytics

Rachel Grotheer

Goucher College has recently gone through a campus-wide curriculum change to rethink the meaning of a liberal arts education and how to best prepare students for life after college. As a result of this reflection, our "Mathematical Reasoning" general education requirement has changed into a Data Analytics requirement that requires students to take one semester-long course learning the foundations of data analytics and then another semester-long course learning data analytics techniques in the context of another discipline, usually their major. Further, a program prioritization process in the last year led to the discontinuation of the traditional mathematics major. In response to both stimuli, Goucher math and computer science faculty have designed a new Integrative Data Analytics major. This talk will discuss the process and challenges of creating such a major at the undergraduate level, as well as the challenges and successes of incorporating data analytics across the college curriculum.

Numerical Range of Toeplitz Matrices Over Finite Fields

Maddi Guillaume Baker, Amish Mishra, Derek Thompson

We characterize the kth numerical range of all $n \times n$ Toeplitz matrices with a constant main diagonal and another single, non-zero diagonal, where the matrices are over the field $\mathbb{Z}_p[i]$, with p a prime congruent to 3 mod 4. We find that, for $k \in \mathbb{Z}_p^*$, the kth numerical range is always equal to $\mathbb{Z}_p[i]$ with the exception of the scaled identity. We also use similar techniques to discover a general connection between the 0th numerical range and the kth numerical range. Lastly, we conclude with a conjecture regarding the general numerical range of all triangular Toeplitz matrices.

Overcoming Stereotypes Through a Liberal Arts Math Course

Jessie A. Hamm

"I'm just not a math person." We've heard this comment countless times from our students. It is a mentality that both paralyzes and strangely comforts them. In this talk I will describe how I use my liberal arts Joy of Mathematics course to help students address and overcome stereotypes. In particular, I will discuss a specific assignment as well as share some student comments and perspectives on how this course helped change their viewpoint on more than just math.

Character and Cybersecurity Education

Seth Hamman

It has been said that a person's character is who they are when nobody is looking; today it may be better said that a person's character is who they are in cyberspace. This is because the medium of cyberspace embodies a troublesome combination of temptation, opportunity, and plausible deniability. But cyberspace is also the enabler of extraordinary modern conveniences, and its impact on daily life will only continue to expand. These two facts, cyberspace's bias towards malicious activity and society's increasing dependence upon it, make cybersecurity, which seeks to protect the rights of individuals and organizations in cyberspace, a worthy challenge. Colleges around the country are creating cybersecurity programs to help meet this challenge. In a nutshell, these programs teach the good guys what the bad guys already know. Cyber students are taught technical knowledge and skills, adversarial thinking, and the tricks of the hacking trade. This is essential for helping to level the cyber battlefield. But this type of education, especially when it is combined with the privileged access grads will require, is also a cause for concern. Will the good guys not also face moral ambiguity and temptation in this perversely-bent world? Can they be trusted to recognize right from wrong and to make the right choice every time even when nobody may ever know the difference? The best cyber defenders will have the moral clarity and integrity to match their technical expertise. For this reason, cyber programs that deliberately cultivate character will enrich the cybersecurity education landscape.

The n-Children Problem

Adam Hammett

In 1959, Martin Gardner posed the Two-Children Problem and provided its solution. In it, he asked two questions:

1. Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
2. Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?

Gardner stated that the answer to the first question is 1/2 (as expected), but then caused quite a ruckus by claiming that the answer to the second is 1/3. However, he later corrected his solution due to the ambiguity involved in the procedure for obtaining the given information. In 2010, Gary Foshee posed a generalization of Gardner's problem by considering an additional condition beyond gender, namely the birth day of the week. In this talk we will seek to unify and extend both these famous problems, but not before taking in the historical, and sometimes controversial, landscape surrounding them. At the conclusion of our exploration, we will be able to state and prove a robust generalization to any number of children and any number of conditions we might require of them, under one explicit procedure for obtaining the given information.

Analyzing the Impact of Active Learning in General Education Mathematics Courses

Amanda Harsy, Marie Meyer, Michael Smith, Brittany Stephenson

This talk shares the preliminary results of a study that explores the general perceptions and attitudes of students in general education mathematics courses taught using primarily active learning-based methods (like group work, projects, and discovery learning), and compares them with those enrolled in a general education mathematics course taught in a more traditional and lecture-based method. We present an analysis of survey data collected throughout the semester, which explores the disposition and mindset of students, their mathematical confidence and anxiety, and perceptions of pedagogical methods used for the teaching of mathematics. We also explored how these perceptions and dispositions changed throughout the course by comparing pre, mid, and post surveys.

Mosaic Number of Torus Knots

Lisa Hernández

A torus knot is a knot that can be embedded into the surface of a torus. The Mosaic Number of a knot is the minimum number n such that a knot can be fit onto an $n \times n$ grid in such a way that each tile contains at most one arc or crossing. We investigate the relationship between torus knots and their mosaic numbers.

When the Fundamental Theorem of Algebra Goes Awry

Russell W. Howell

One form of the fundamental theorem of algebra states that a polynomial of degree n has n roots (counting multiplicities). This talk will show that, as stated, the theorem is false. To illustrate, we exhibit a polynomial of degree 3 (with real coefficients) that has 9 (distinct!) zeros. We then explore why such an anomaly occurs, and give initial results and conjectures regarding the number of zeros that certain types of polynomials might attain.

EventFinder: A Program for Screening Remotely Captured Images

Michael Janzen

Camera traps are becoming ubiquitous tools for ecologists managing and making decisions for wildlife. While easily deployed they require human time to organize, review, and classify images including sequences of images of the same individual, and non-target images triggered by environmental conditions. For such cases we developed an automated computer program, EventFinder, to reduce operator time by pre-processing and classifying images using background subtraction techniques and colour histogram comparisons. We tested the accuracy of the program against images previously classified by a human operator. The automated classification, on average, reduced the data requiring human input by 90.8% with an accuracy of 96.1%, and produced a false positive rate of only 3.4%. Thus, EventFinder provides an efficient method for reducing the time for human operators to review and classify images making camera trap projects, which compile a large number of images, less costly to process.

Lagrange's Interpolation, the Chinese Remainder Theorem, and Linear Equations

Jesús Jiménez

Consider a finite set of points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ in $\mathbb{R}^2$. The Lagrange's interpolation problem is to find a polynomial $p(x)$ of degree $k-1$ satisfying $p(x_i) = y_i$ for $1 \leq i \leq k$. We will recall the solution to Lagrange's interpolation problems as an instance of the Chinese Remainder Theorem. Next, we will show that a similar approach can be used to construct solutions to a system of linear equations.

Speak for Yourself: Self-Assessment as a Tool for Measuring Participation

Jill Jordan

In each of my classes, I give my students responsibility for grading themselves on the attendance and participation component of their final grade, following rubrics that I provide. In this talk I will explain both my reasoning for incorporating participation into the final grade calculation and my rationale for having students be the ones who determine these grades. In addition, I will give specific examples of participation grading for various types of classes, including a general education math course, calculus, and an upper-level math major course.

Making the Connection Between Biblical Concepts and Programming Basics

Hyunju Kim

With the exception of ethical issues, it is not easy to integrate Christian faith into Computer Science (CS) and relate the principles and teachings from the Bible to what we teach and learn in the classroom. This talk will share my effort to identify biblical concepts that can be associated with some of the basics of programming. Since joining the college some three years ago, I have taught Programming I, the first programming course for CS and non-CS students, and wanted to show students, as well as myself, that what they learn in the programming course is not separate from what the Bible teaches and reveals. This includes framework and modeling, naming and identity, languages and communications, causality, inheritance, recursion, and diagrams. As part of class devotions, these programming concepts have been discussed along with the scriptures, Genesis in particular, which are associated with the creation, the cultural mandate, the Fall, the dispersion at Babel, and so on. Through this talk, I hope to gain a better insight into faith integration and expand this list through feedback from colleagues.

Mathematical YouTubing and Blogging

Bill Kinney

Interested in posting mathematical content on YouTube and/or a blog? Interested in collaborating on such platforms? I'll share my experiences with my YouTube channel and my blog, as well as some of my content. I'll also get into the details of how I post and promote content, as well as working with some of the details in Wordpress. If you are interested and have the time, I welcome collaborating on and/or developing synergy between our content.

Factors that Motivate Students to Learn Mathematics

Dave Klanderman, Benjamin Gliesmann, Josh Wilkerson,
Sarah Klanderman, Patrick Eggleton

What motivates some students to want to learn mathematics while others do not share similar motivation? Are these factors intrinsic, extrinsic, or a combination of both? To answer these questions, we adapted a survey originally developed by Tapia (1996) and later shortened by Lim and Chapman (2015). We administered the survey in multiple middle schools, a high school, and multiple colleges and universities. We obtained over 100 completed surveys for each of these educational levels. This presentation offers an analysis of these data, including descriptive statistics and confidence intervals for each educational level. For the college and university sample, we also provide comparisons among students majoring in mathematics or mathematics education, those majoring in elementary education, and those with a variety of other majors. In addition to the Likert scale items from the original survey, we explore qualitative data from a free response item. Join us to learn more about why students enjoy learning mathematics and later choose undergraduate majors in the discipline.

Computations in Topological CoHochschild Homology

Sarah Klanderman

Hochschild homology (HH) is a classic invariant of rings that is defined in the realm of algebra and that can be extended as an invariant in topology, called topological Hochschild Homology (THH), to apply to ring spectra. Dually, Doi gave a way to extend Hochschild's work when he defined coHochschild Homology (coHH), which applies to coalgebras, and then Shipley and Hess defined topological coHochschild homology (coTHH) to be the topological extension of these ideas to coalgebra spectra. In this talk we will discuss coTHH calculations and the tools needed to do them.

If the last few sentences were complete jargon to you, come hear about what they really mean and how they exemplify approaches found in the field of algebraic topology in general. Sarah is graduating in May 2020, and so this presentation is designed to give an overview of her dissertation work.

Viewing Mathematics as an Opportunity to Practice Intellectual Virtue

Cory Allen Krause

The most important thing you learn in any course is a way of thinking. When students take a course in mathematics, we wish them to finish having gained the ability to think, in some small way, more like a mathematician. But are such habits of mind also good for us as human persons with a moral dimension? The answer may depend on whether we choose to view mathematics as an opportunity to practice intellectual virtue. Virtue is what the ancients called the practiced conformity of one's thinking and action toward a positive standard. In this talk, I will discuss how I view the study of mathematics as providing a unique way to practice the virtues of honesty, humility, patience, diligence, and even kindness.

Using Agile Project Management Techniques as a Pedagogical Framework

Michael Leih

In the fall of 2015, a Faculty Lead Academic Research Experience (FLARE) course was offered to students to design and build a mobile app for a new start-up company. Each student in the course was responsible for a different aspect of app development, including mobile programming, UX design, graphic design, project management, testing, documentation, cloud technology and database design. Students in the course consisted of juniors and seniors and majored in business, information technology, graphics design, and other disciplines. The challenge of the course was developing a pedagogical framework that would support activities and assessments where each student had unique tasks and responsibilities. In this talk, I will describe how agile project management techniques along with an action research approach were used as a pedagogical framework to support a course were each student had a unique responsibility and the primary deliverable of the course was unknown at the beginning of the semester.

Teaching at a University in a Developing Country – My Experience

Kathy Lewis

For eight years (2010-2018), I was the head of the math department at The University of The Gambia, a small, new (started in 1999) university in a very small, mostly Muslim, West African country that most Americans have never heard of. During that time, a number of other Americans and Europeans came to teach for shorter periods of time.

I will talk about what this experience was like for us, both the good and the bad. I will describe the possibilities for others to spend either a sabbatical or an extended period of time at such a university and suggest some questions to ask of a university before going there. I think teaching overseas can be a real ministry opportunity.

Calculus: A Part of God's Story

Melissa Lindsey

The Educational Framework of Dordt College sets forth a structure for the overall education program at Dordt College. The content of the curriculum and the curricular goals are organized under four broad headings: religious orientation, creational structure, creational development, and contemporary response. To better tell the story of Calculus—and how it fits into God's story, I had students read the Educational Framework and complete a series of four assignments throughout the semester that were geared towards those four curricular-coordinates. This talk will cover what those four assignments were, how they tied to the curricular coordinates, what their strengths were, where they can be improved, and student responses to them.

A Christian Mathematician's Response to *A Mathematician's Apology*

Matt D. Lunsford

G. H. Hardy was a prominent British mathematician during the first half of the twentieth century. In 1940, Hardy wrote an essay defending his career choice of becoming a mathematician. Hardy's essay, *A Mathematician's Apology*, has become a definitive piece in the history of mathematics. Since its publication, several authors have attempted to revise and expound upon Hardy's essay. Possibly the best known of these is the British mathematician Ian Stewart's book *Letters to a Young Mathematician*. In Letters, Stewart seeks to bring Hardy's message into the 21st century, with a particular goal of gender inclusivity. Christian mathematicians James Bradley and Russell Howell, co-editors of *Mathematics Through the Eyes of Faith*, also reference Hardy's essay in their text. In the concluding chapter "An Apology," Howell restates two key questions posed originally by Hardy, hoping to convey Hardy's questions to a broader audience of readers. This talk will provide an overview of the conversation initiated by Hardy, and expounded on by Stewart and Howell, and will incorporate the speaker's endeavor to examine Hardy's essay from a Christian viewpoint.

Faith, Mathematics and Science: The Priority of Scripture in the Pursuit and Acquisition of Truth

Bob Mallison

Sanctify them by the truth; your word is truth. (John 17:17)

This research will examine some approaches for identifying truth as well as some issues involved in recognizing reliable sources of information. We will proceed from a decidedly Christian perspective including the conviction that God created an orderly universe (and that studying nature provides valuable information about Him) and that His Word, the Bible, even more clearly expresses information about Him. We will discuss some of the essential tools used by mathematicians and scientists for the discovery of truth – namely, models. We will examine some valuable models from history, and briefly discuss that as additional scientific information became available, the models required refinement, and sometimes replacement. The Bible, on the other hand, is perfect and needs no corrections. We will also consider the following items:

1. The nature of higher dimensions and possible relationships with certain Bible passages;
2. The role of hermeneutics in Biblical interpretation (and scientific interpretation) regarding reconciliation of science with the Bible;
3. Finally, we will speculate about possible implications for the frequent use of the phrase "[God] stretched out the heavens."

We conclude by summarizing the results and recognizing that as we study mathematics and science, along with Gods Word, we can know the truth. In fact, God's plan for each of us is to know Him Who is the Truth.

Designing a New Sequence of Three Seminars on Math and Faith at Azusa Pacific University

Bryant Matthews

Three years ago, the Azusa Pacific University math faculty began collaborating on a new sequence of seminars intended to help our majors to integrate their mathematical studies with their faith development. The first seminar considers the influence of Christian belief on the historical development of math and physics as well as the strengths and limitations of mathematical and scientific reasoning in our attempts to learn about the world around us. The second seminar looks at the purpose of life, of work, of cultural work, and of mathematical work while supporting students in developing their own sense of vocation. The final seminar helps students to design pathways for missional application of their mathematical training and to plan for intentional growth in character along the way. In this talk, I will share about our design process as well as lessons learned thus far in the implementation phase.

Number Patterns and Insights for the Mathematically Apprehensive

Mandi Maxwell

In the general public, math tends to get a bad reputation; it's misunderstood. It is all too often characterized as just a bunch of rules and procedures, but we know that this a limited and stale view. Mathematics can be beautiful. Fundamentally, it is a study of patterns with intriguing and unexpected connections. It describes and reveals the structure of God's creation and its imagery can deepen our appreciation of our Lord and Savior. My purpose in this talk is to share how some of the connections and beauty that I see in the realm of numerical sequences might enable those outside the mathematical community to gain a deeper appreciation of mathematics.

Reflections Upon the Relationship Between Mathematical and Biblical Truth

Dale McIntyre

And he who talked with me had a gold reed to measure the city [Jerusalem], its gates, and its wall. The city is laid out as a square; its length is as great as its breadth. And he measured the city with the reed: twelve thousand furlongs. Its length, breadth, and height are equal.

(Rev. 21:15-16)

This paper seeks to explore the relationship between mathematical and biblical truth in a number of ways. Pursuit of mathematical truth parallels theology; mathematics reflects the divine nature; mathematical concepts pervade Scripture; and biblical arguments follow the same rules of logic that are foundational to mathematics. It is hoped that reflection upon these connections will encourage the Christian mathematician to serve the Lord through his/her vocation with increased vitality and devotion.

Mathematics as Sub-Creation

Chris Micklewright

As with so many debatable topics today, discussions of mathematical philosophy too often focus on the polarized alternatives of Platonism and nominalism. In these discussions, mathematicians are either discovering eternal truths that could not be otherwise (Platonism), or they are playing meaningless games according to arbitrary rules (nominalism). Faced with these alternatives, Christians have naturally favored Platonism, and there is a well-known joke that all mathematicians are Platonists except on Sundays. However, both perspectives tend toward abstraction, neglecting the created world around us. I will explore a middle way between these alternatives, seeing mathematics as a form of sub-creation (to borrow the term from Tolkien). Like Adam, mathematicians study the created world, identifying patterns and structures, naming and classifying them. However, we are also created in God's image and given the freedom to go beyond what we see in creation, to create new concepts and to forge new connections. I will also explore the implications of this perspective for our understanding of mathematical beauty, and for our work as educators. These ideas been developed with the support of a grant given by Bridging the Two Cultures of Science and the Humanities II, a project run by Scholarship and Christianity in Oxford, the UK subsidiary of the Council for Christian Colleges and Universities, with funding by Templeton Religion Trust and The Blankemeyer Foundation.

Towards Non-Violent and Christian Video Games

Benjamin Mood

Video games have become more prevalent in our society since their spread during the last quarter of the 20th century. Unfortunately, most of the games that are successful are violent and non-Christian. For example, in the popular version of Fortnite, the rules are essentially be the last one left standing by killing all your opponents. The overall goal of this project is to explore how to create alternative games, which are non-violent and Christian. This talk will focus on the progress of the first part this project, which will examine why humans perceive some aspects of games as violent and others as not.

Math, Magic, and More!

Andrew Mosteller

In this talk, we will briefly discus the history of math, magic, and how they interact. We will also explore two very powerful, self-working (relying fully on mathematics, no sleight of hand required) card tricks. In addition, we will reveal the underlying mathematics and method behind these tricks which use key concepts from Group Theory and Number Theory. We will also go over how generalizations of this method can be used to create new possibilities of self-working card tricks.

Team Work and Evaluation: Finding the Missing Link

Sarah A. Nelson

As I learn more about inquiry based learning, I find myself increasingly drawn to finding the most effective ways to facilitate team work during class time. Working together helps level the playing field for all of my students while encouraging them to all learn the material in a deeper way. There are so many benefits to working together. Historically, though, assessments come in the form of exams during which students are completely isolated. As a result, these traditional exams undermine my teaching practices and send mixed messages to my students. There should be a link between the learning process and how we assess students. On the other hand, having students complete all their assessments together presents too many opportunities for individual students to slip between the cracks. In this talk, we will share the process of finding common ground between our teaching and assessment practices. We will share our observations and what we have learned along the way.

A Conversation About Health Insurance

Emily Sprague Pardee

One evening I found myself explaining Medicare to a new recipient about to undergo a much-needed hip replacement. My Christian friend, long an opponent of the Affordable Care Act, thought of this access to his necessary treatment as a miracle. I pulled out Galatians 6:2, "Bear ye one another's burdens and so fulfill the law of Christ," as a gateway to introduce health insurance simply, perhaps even to regard insurance premiums as tangible expressions of charity. This talk builds on those simple beginnings to provide a quick overview of the Affordable Care Act with a spotlight on the mathematics behind its provisions. We continue by honing in on California's particular implementation of the ACA and attempt to sort out the mathematical choices behind differences of opinion about costs and benefits. We end with a few remarks about how this Christian bridges the gap between density curves and spreadsheets to speak to the hearts of concerned citizens.

Glimpses of God Through a Mathematician's Eyes

Doug Phillippy

The ability of mathematics to describe the world around us has long been of interest to mathematicians. I share this interest and my training as an applied mathematician has allowed me to gain insight into the world through the modeling process. Nevertheless, my understanding of the modeling process was limited by a secular education. This talk will describe my journey from secular education to Christian educator and how that journey has expanded my understanding of the modeling process. In particular, I will highlight several mathematical models that have strengthened my understanding of God and my faith.

Random Walk on Three "Half-Cubes"

Michael R. Pilla

In this talk, we will study random walks on some interesting graphs including a clock, a cube, and three "half-cubes." Random walks are ubiquitous in applications, appearing everywhere from animal movements to Google search algorithms. In our case, take a cube of cheese and slice it with a knife through its center of gravity. It turns out there are three topologically distinct way of doing this, leaving a "half-cube." Starting from a designated origin, we will place a cookie at each vertex of the half-cube. Each step will be taken from one vertex to an adjacent vertex with equal probability, independently of all previous steps. Every time we reach a vertex for the first time, we will eat the cookie and move on (the cookies are not replaced). We will find the expected time to eat a given cookie, the probability a given cookie is eaten last, and the time it takes to eat all of the cookies.

Less Volume, More Creativity: R for Busy Humans

Randall Pruim

R is a general purpose programming language. R is also a data analysis tool with a vast array of statistical procedures available to its users. Although a core team develops R, it is also community-supported software with thousands of package authors. While these features make R powerful and useful, they can also make it overwhelming. We will present an introduction to a "Less Volume, More Creativity" approach to using R that makes R more hospitable to its human users and helps optimize our scarcest resource: analyst (including faculty and student) time.

Clean Water for Liberia

Randall Pruim, Stacy DeRuiter, Matthew Bone

A team of Calvin faculty and students from statistics, sociology, and GIS are partnering with Sawyer (a manufacturer of water filters) and NGOs in Liberia in a project to bring clean water to everyone in that country by 2022. This summer will mark the mid-point of this five-year effort. We will present some of the results so far and discuss the benefits and challenges of working in such a large and varied team.

Celebrity Politicians

Ray Rosentrater

We all know of celebrities who have gone on to hold high political office. Do celebrities have a significant advantage over seasoned and amateur (those having not held office previously) politicians? Initial results seem to give a resounding "yes," but a closer analysis indicates that things might not be as simple as they first appear.

Specifications Grading in a Math for Liberal Arts Course

Lauren Sager

Concepts of Mathematics is a quantitative reasoning core course with a dual audience: non-mathematics majors looking for core credit, and elementary education majors. After watching students, who are often self-proclaimed "not math people," panic before exams, I turned to specifications grading to try to ease some mathematics anxiety in the classroom. In this talk, I will give a brief overview of the course and of specifications grading. I will also talk a bit about both student and grading outcomes.

Transhumanism, Faith, and the Human Future

Derek Schuurman

The adage that "we shape our tools, and thereafter our tools shape us" takes on a new meaning with transhumanism. Transhumanism is a movement that seeks to enhance humans using technology far beyond the limits of their current physical and intellectual capacities to evolve into something better.

Technology like glasses, pacemakers and artificial limbs already augment human capabilities, but the goal of these technologies is to restore normal human capacities that have been lost or damaged due to disease or accidents. In contrast, the goal of transhumanism is for humanity to take control of its evolutionary destiny and move towards a "posthuman" future.

The longings of transhumanists point, in part, to a recognition of our fallen condition, and that things are not the way they are supposed to be with disease, suffering, and death. The issue is that they look to technology as savior of the human condition instead of God (or in addition to God).

The notion of disembodied existence in certain transhumanist ideals reflects some themes in Gnosticism. We need to recall that the incarnation reveals the value God places on our physicality and humanity. We need to remember how Christ, "the Word who became flesh" (1 John 3:2), models what it means to be truly human. Furthermore, the importance of valuing bodies should inform our current technology use and design by encouraging embodied experiences and practices.

Addressing Challenges in Creating Math Presentations

David Schweitzer

When it comes to composing presentation slides with extensive mathematical content, each of the slide creation platforms has at least one significant drawback. Whether it is Beamer and its steep learning curve, PowerPoint and its relative inefficiency with math, Google Slides and its complete lack of math capabilities, or some other platform, no one tool single-handedly offers an ideal solution. Additionally, if users desire creative flexibility, such as the ability to easily change fonts or colors, the platforms' respective limitations can become even more pronounced.

In a project that has been well suited for undergraduate research, the presenter and his team are actively developing new tools and enhancing existing ones to address these issues. This talk will discuss two such tools for PowerPoint. By using a new VBA macro and a modified version of a pre-existing, open source VBA add-in, users can have math creation and rendering capabilities largely on par with TEX's, all while maintaining PowerPoint's simplicity for all other tasks. These tools consequently greatly reduce the amount of time necessary to create slides with mathematical content. Time-permitting, the talk will also include brief discussion of long-term project goals, such as the team's early inroads for web platforms and efforts to improve educators' ability to meet the accessibility needs of students with visual impairments.

Small is Beautiful: Leveraging the Small Size of Departments to Make Rapid Cultural Changes

Daniel Showalter

Several faculty in our mathematical sciences and biochemistry departments received an NSF grant to transform our culture and make it a more welcoming home for underrepresented minority and first generation college students. Within two years, over 90% of our full-time math and science faculty had gone through a one-year Diversity Response Training course, and many continued on to conduct related studies in their classrooms. This talk will present some of the challenges, failures, and successes of our attempt to impact some of the more invisible parts of our journey as Christian professors to see beyond how students are performing by conventional academic standards.

A Unifying Project for a TEX/CAS course

Andrew Simoson

We describe a CAS and TEX usage course for mathematics majors. As a unifying project, each student selects two primes p and q with $pq < 100$, explores mathematical pq ideas, and generates associated graphs, figures, tables for a final TEX paper. We summarize several pq explorations: students render page pq from Schwartz's picture book about primes, *You Can Count on Monsters*, via Mathematica and TEX's picture environment; students generate fractal images of pq; and students discover the primes of the ring $\mathbb{Z}\left[\sqrt{pq}\right]$.

Does Harvard Discriminate Against Asian Americans in Admission Decisions?

Michael Stob

The organization Students for Fair Admissions sued Harvard University alleging that Harvard discriminates against Asian-American applicants in its undergraduate admissions decisions. At the trial, each side featured an expert witness presenting statistical analyses supporting their respective cases. In this talk, we look at these analyses not to resolve the title question but to highlight some of the difficulties in using such statistical models to resolve discrimination cases.

Is Mathematical Truth Time Dependent? Some Thoughts Related to a Paper from Judith Grabiner

Richard Stout

Judith Grabiner, a renowned historian of mathematics, has written many papers related to significant changes in the content and nature of analysis, from the 17th through the 19th century. In her paper "Is Mathematical Truth Time Dependent?" Professor Grabiner gives several reasons for the changing nature and requirements in rigor that occurred over this period of two hundred years. In this talk I will briefly summarize her conclusions, particularly in light of how they might influence a Christian perspective on mathematics.

Charles Babbage and Mathematical Aspects of the Miraculous

Courtney K. Taylor

Charles Babbage is widely known as the father of the computer, but he is lesser known for his contributions to natural theology and apologetics. In 1837 Babbage wrote the Ninth Bridgewater Treatise in response to a series of writings concerning faith and science that had been commissioned by the Royal Society. Among the remarkable features of the Ninth Bridgewater are mathematical analogies concerning the miraculous. We will explore these ideas, which range from the difference engine to a family of fourth degree curves, illustrating that for Babbage, miracles are not exceptions to natural law, but rather instances of a larger pattern. In addition we will see how Babbage employed probability to refute Hume's argument against miracles.

Analyzing Retention and Graduation Rates at Bethel University

Deborah Thomas

Academic institutions such as ours are always looking for ways to increase retention rates. Being able to describe successful students and learn how to better cater to all students is a key goal of universities. Two such markers are the percentage of returning students between the freshman and sophomore year as well as the graduation rate for a particular class after four years. Two years ago, I presented initial results. However, at that stage, I had mainly focused on data cleaning. Here, will present more robust results, looking at the effect of participation in sports, dorm choice and first generation college student status on the retention of students between their first and second year and also at graduation rates. We will also make recommendations based on our results to increase retention rates.

Computer Science: Sub-Creation in a Fallen World

Russ Tuck

When God created people in his image, he gave us the gift of sub-creation. One of the great joys of Computer Science is exercising that gift to create tools : software and computer systems that serve people and solve problems. Like all God's gifts, he charges us to exercise the gift of sub-creation wisely and for good. While there are many obvious implications and challenges, being good stewardship of users' time and reducing discrimination are particularly relevant and perhaps less obvious examples.

Although computer scientists exercise the gift of sub-creation, we do so as fallen people in a fallen world. This affects not only what we build but, more fundamentally, how we build it. First, we are inherently imperfect and mistake-prone, which means our software inevitably contains bugs (mistakes). So we have to test software to find the bugs, then figure out how to fix them.

More fundamentally, but less obviously, our knowledge and understanding are imperfect, so we don't even know exactly what to build. The modern practice of "agile" software development can be understood as addressing this problem. Its focus on incremental development and immediate testing seeks to explore and better understand the requirements and to refine the design. It also helps find mistakes quickly in order to fix them more easily. This is a productive response to human imperfection in both knowledge and action.

Ways of Thinking Beautifully about Mathematics

James M. Turner

At Calvin College, First Year Students are required to take a section of our Discovering the Christian Mind course during our January term. For the past 3 years, I have taught a section of this course titled *Thinking Beautifully about Mathematics*. In it, I explored, with the students, various areas of mathematics, as well as how mathematicians have explored them, while addressing such questions as "Is mathematics invented or discovered?" and "Why is mathematics unreasonably effective?" Ultimately, we look to identify ways and characterizations for how beauty displays itself in mathematics and how and it what ways beauty is seen by mathematicians. In this talk, I will report on some of the questions, observations, and speculations that arose from teaching this course.

Replacing Remedial Mathematics with Corequisites in General Education Mathematics Courses

Alana Unfried

Many colleges and universities offer courses, such as Remedial Mathematics or Elementary Algebra, that underprepared students must complete before they can take a college-level mathematics course. However, nationally there is a push to replace remedial mathematics courses with corequisite courses instead. This design allows students to enter directly into their general education mathematics course instead of first overcoming the barrier of a remedial course. Corequisite mathematics courses were implemented across the 23-campus California State University system during the 2018-19 academic year. I will discuss the design and implementation of a corequisite structure at California State University, Monterey Bay, in particular focusing on the redesign of our Introductory Statistics course and corresponding corequisite course. I will discuss the rationale for moving to this structure, the logistics of making this change, the design and pedagogy used for Introductory Statistics and its corequisite, as well as preliminary findings about student success in our first year implementation.

A Metaphor from Competitive Gaming: the Crown that will Last Forever

James Vanderhyde

David Sirlin, competitive video game tournament champion, wrote a book (2005) that explains how to win at games. Sirlin begins the book by describing a "mountain" of competitive gaming: a few gamers are already "on the journey" to the mountain peak, but most only think they are. "They got stuck at a chasm at the mountain's base," and "they are imprisoned in their own mental constructs of made-up game rules. . . . 'Playing to win' is largely the process of shedding the mental constructs that trap players in the chasm who would be happier at the mountain peak."

This is how I see the Christian life, or more broadly the spiritual or moral life. When a person has not come to terms with the sinful nature and accepted the grace of God and the saving work of Jesus, "they are imprisoned in their own mental constructs." All they have to work with are the moral codes they either come up with on their own or inherit from their parents and peers. Paul (Col. 2:21 NIV) writes, "Such regulations indeed have an appearance of wisdom, with their self-imposed worship, their false humility and their harsh treatment of the body, but they lack any value in restraining sensual indulgence." This way of living sounds just like what Sirlin calls the "scrub" mentality. Life is not a game, but let's see how far the analogy takes us, and what we need to do to win.

Approaching History with Graph Theory – A Review

Kevin Vander Meulen

The book *The Square and the Tower: Networks and Power, from the Freemasons to Facebook* by Niall Ferguson is a 2017 popular history book that has a leading role for mathematics. The book notes that the subject of history tends to focus on institutions and leaders of institutions. Ferguson proposes that the significant impact of social networks has been overlooked. He explores many historical events, including the Enlightenment and the Reformation, as well as more current events. Ferguson uses mathematical tools, especially graph theory, to make his point. I will review and critique what the book offers.

A Modeling-Driven Approach to Teaching Differential Equations

Mary Vanderschoot

Wheaton College uses a modeling-driven approach in teaching Differential Equations by incorporating several modeling projects throughout the course. I'll share several creative variations on canonical models (solution flowing into a tank, mass on a spring, predator-prey) that cater to students majoring in economics, physics, environmental science, and social science. These projects not only promote understanding, but also increase students' ability to work in groups and to communicate assumptions and results in written form with clarity and professionalism as a scientific report.

TENZI: A Fun Introduction to Markov Chains and Decisions

Michael Veatch

TENZI, "The world's fastest game," sounds simple: roll 10 dice until they all match, setting aside the dice you want to keep. On average how many rolls does it take to get 10 matches? Answering this question, for both the optimal and a naive policy, requires combinatorics, order statistics, and Markov chains. The proof that the strategy is optimal is more advanced, using dynamic programming. The mathematical framework of Markov chains often gets little or no treatment in the undergraduate curriculum, but is now essential in machine learning and dynamic pricing, for example. General dynamic programming approximation schemes make it possible to handle larger models with decisions. The TENZI problem could be used to introduce probability students to these important ideas.

An APOS Analysis of Calculus Student Comprehension of Continuity and Related Topics

Jayleen Wangle

This study concerns Calculus I students' comprehension of the concepts of function, limit, and continuity. Items were designed to inquire about participant depth of understanding of function, limit, and continuity. Participant responses were viewed through the lens of the constructs depicted by Dubinsky's (1991) Action-Process-Object-Schema (APOS) theory. APOS theory was developed by Dubinsky and his colleagues as a means of measuring perceived student depth of understanding. The theory purports that one's schema consists of the constructs of actions, processes, and objects (Asiala et al., 1996). For example, one demonstrates a strong schema of the concept of function when one shows that one is able to think about a function as a process or an object as appropriate to the task. Even though quantitative and qualitative methods were used in this study, this talk will center on results from participant interviews. A prominent finding was that participants who demonstrated a stronger conception of the notion of function displayed a more in-depth understanding of continuity.

Maximal Elements of Ordered Sets and the Ontological Argument

Doug Ward

I will present a simple theorem concerning maximal elements of a set T endowed with an ordering "$>$" that is antisymmetric, i.e., if A and B are elements of T, we cannot have both $A > B$ and $B > A$. A special case of this theorem is a simple version of the ontological argument, one of the classical proofs for the existence of God.

Math is ______

Josh Wilkerson

Many discussions of mathematics from a Christian perspective focus on presenting math as true, good, and beautiful. While this is undoubtedly an integral conversation to bring into the math classroom, if the conversation stops there, at true, good, and beautiful, then we are painting an incomplete picture of mathematics. If the conversation stops there then we are analyzing an abstract discipline with abstract language and many students leave our doors feeling as if they have had an intellectual exchange but they remain ultimately unchanged in how they practice and understand mathematics.

This presentation will challenge educators on how to complete the sentence "Math is ______" with language that remains faithful to the true, good, and beautiful but also considers the practical experience students are having of mathematics. How do we understand not only the philosophy of mathematics but the practice of mathematics from a Christian perspective? How do the practices and liturgies of the math classroom impact the mathematical affections of students?

This presentation will end by offering some practical examples that math teachers can implement in their own classroom.

Making Stuff Up: A Model for Undergraduate Research in Mathematics

Rebekah Yates

In order to introduce our students to mathematical research, several years ago we began offering a 1-credit Math Research Seminar each spring for any students who have completed our Introduction to Proofs course. In this talk, I will describe the structure of the course, research problems we have explored, and our experiences with the course, including how the course broadens students' perspectives on mathematics as a creative, active discipline.

Image Data: A Project-Based Exploration of Computer Vision

Ryan Yates

Computer vision systems appear in many well-known cutting-edge domains such as self-driving cars, augmented reality, and facial recognition. There are also many lesser-known applications throughout industry and scientific domains. Our Image Data class is a special topics course in Data Science at Houghton College where we attempt to demystify tools for building computer vision though hands-on projects including building practical vision systems out of low cost components and using standard open source software tools for computer scientists and data scientists.

Practical Examples of Bin Packing and Critical Path Scheduling

Maria Zack, Greg Crow

In the process of designing and implementing the gut remodel of a post-Sputnik science building, we created a host of examples for quantitative literacy classes that are rooted in real world problems. How many cubicles can you fit in a construction trailer? What are the prerequisite constraints on furniture installation in the building? Why are the painters still here? How can you number rooms so that it is intuitive to professors and high school student visitors and yet allows sufficient flexibility that the rooms could be sub-divided later?

Outreach Activities to Attract Majors

Valorie L. Zonnefeld

The demand for graduates with mathematical and statistical skills remains high. In education, the National Science Foundation (2014) reported that 27% of high school math teachers in the US did not possess a degree in mathematics. The need for mathematics and statistics graduates is evident, but the students are not always available. This presentation will share outreach activities that have and have not worked for recruiting more majors at Dordt University.

How do we add more students to the quantitative sciences pipeline? One strategy is to reach out to students prior to college. Dordt has used multiple approaches to share the joys and opportunities in mathematics with Kindergarten through 12th grade students. These strategies include math competitions, a March Madness Data Analytics Battle, classroom visits, postcards, game nights, and week-long camps. Outreach for current students has included a mini-internship, a booth at the campus fair, game nights, club activities, and the addition of new programs.

This presentation and many of the outreach opportunities are funded by a NSF Noyce grant [DUE 1660632]

Discussion Abstracts

Discussion on Student Mentoring

Ryan Botts, Lori Carter, Catherine Crockett, Mike Leih

College professors wear many hats in addition to teaching and research. One of the most daunting tasks can be mentoring. Whether it be about academic trajectory, career options, life issues, or research, most of us have no shortage of students lining up at our doors for advice. More recently we've found that those we are advising have even more specialized needs. They may have disabilities about which we have little information, may be a first generation student finding it difficult to navigate current and future requirements, or be someone of an ethnicity or gender who feels different from the rest of the students in the class. At Point Loma Nazarene University we have been working together to find good ways to support these sometimes challenging students. We believe that as Christians, this is a part of our calling. We'd like to share what we have learned, and hear about how others have helped their most challenging students.

What New Collegiate Faith/Integration Resources Can ACMS Provide?

Bob Brabenec

Over the years, the ACMS organization has supported the integration of mathematics and the Christian faith in a variety of ways. These include many talks at the biennial conferences which are in the conference Proceedings, along with the writing and publication of two books: *Mathematics in the Postmodern Age: A Christian Perspective* in 2001 and *Mathematics Through the Eyes of Faith* in 2010. The purpose of this informal discussion session will be to ascertain what kind of additional materials are desired, and some ways in which these can be provided. Content will be shared from my *Mathematics and its Foundations* manual used for my math capstone course at Wheaton.

Advising Students for Government and Industry Job Searches

Stephen McCarty

I am an operations research analyst and work for the US Army. This topic discussion will consist of a short presentation on working for the federal government and applying for jobs on the US government job website https://www.usajobs.gov/ followed by Q&A. The session will then be open for discussion and presentations from other audience members with tips to share on helping students with job searches.

New Evangelical Statement of Principles on AI

Derek Schuurman

The Southern Baptist Convention issued a statement titled "Artificial Intelligence: An Evangelical Statement of Principles" (click here). Christianity Today has a report on this as well (click here). I am curious to hear from others: what do you think of the statement? I am glad to see the church responding to this issue with a voice that is "biblical and relevant" – but much more remains to be done in this rapidly evolving field!

ACMS Resources for K-12 Christian Educators

Josh Wilkerson

What role might ACMS play in K-12 Christian education? How can ACMS help to develop resources for math educators in Christian schools—or at least engage educators in critiquing available resources for integrating faith and mathematics?

Appendix 3: Participant Information

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