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Formal Differential Variables and an Abstract Chain Rule
Big-Oh Notations, Elections, and Hyperreal Numbers:
A Socratic Dialogue

AI and Writing:
Do We Speak the Words of Stochastic Parrots?

The Great Integration

The Noodle Problem

Intentional Community:
Fostering Faith ... Developing Mathematicians

Faith-Learning Integration in Mathematics:
A Beginner's Guide

Correlation, Causation, Statistics, and Soteriology

Restoring Faith-Based Mathematics

Discovery, Invention, and the Mathematical Logic Loop

The Intersection of Mathematics and Creative Habits

Surviving a Pandemic Using Alternate Assessments

Using Graphic Novels in the Teaching and Learning
of Mathematics and Physics

A Mathematical Analysis of Large Numbers
in the Book of Numbers

Innovative Assessments via Creative Assignments

Mathematical Minds:
A Senior Seminar for the Mathematically-Minded

Incorporating Perspectival Elements
in a Discrete Mathematics Course

The Nature of Reasoning
in Theology, Philosophy, and Mathematics

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Can Both Exist Simultaneously?

An Interdisciplinary Approach to Integrating Faith
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A Homily on Hardy's *Apology*

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Editor's Introduction

The twenty-third biennial conference of the [Association of Christians in the Mathematical Sciences](#) was held at [Azusa Pacific University](#) from May 31 until June 4, 2022. Thanks go to [Bryant Mathews](#) and his colleagues for all their efforts that went into hosting it.

Many thanks also to the three invited speakers:

- [Matt DeLong](#) ([Marian University](#))

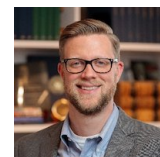
- *Saving Beauty: On Myshkin, Misery, Meaning, and Mathematics*
- *Child's Play: Mathematical Exploration and Exposition at MathPath*



Matt DeLong (B.A., Northwestern University; Ph.D., University of Michigan) chairs the Department of Mathematics at Marian University in Indianapolis. He was awarded the Alder and Haimo awards for distinguished teaching, and has served the MAA in various capacities: on the leadership team of Project NExT, as chair of the Indiana Section, and as chair of the Council on Meetings. Matt is also the Academic Director of MathPath, an advanced summer program for middle-school students. He has published articles in knot theory, number theory, mathematics education, and the scholarship of teaching and learning.

- [Jason Thacker](#) ([The Ethics and Religious Liberty Commission](#))

- *Ethics in the Age of AI: Navigating Emerging Technologies with Biblical Wisdom*
- *Always Known, but Rarely Loved: Facial Recognition Technology and the Nature of Privacy*



Jason Thacker serves as chair of research in technology ethics, and director of the research institute at The Ethics and Religious Liberty Commission of the Southern Baptist Convention. He also serves as an adjunct instructor of philosophy, ethics, and worldview at Boyce College in Louisville, KY. He is the author of the forthcoming book, *Following Jesus in the Digital Age* (B&H Publishing), as well as *The Age of AI: Artificial Intelligence and the Future of Humanity* (Zondervan). He is the project leader and lead drafter of [“Artificial Intelligence: An Evangelical Statement of Principles.”](#)

- [Talithia Williams](#) ([Harvey Mudd College](#))

- *Power in Numbers: The Rebel Women of Mathematics*
- *Scam or Scriptural: Could Plant-Based Diets be God's Healing Mechanism?*



Talithia Williams (B.A., Spelman College; M.S., Howard University; Ph.D., Rice University) develops statistical models that emphasize the spatial and temporal structure of data, and applies them to problems in the environment. She's worked at NASA, NSA and the JPL, and has partnered with the World Health Organization on research regarding cataract surgical rates in African countries. Faith and family round out a busy life that she shares with her husband and three amazing boys. Through her research and work in the community at large, she is helping change the mindset regarding STEM in general and mathematics in particular.

Two pre-conference workshops were offered on May 31–June 1.

1. *Mathematics in Context: Its History, Philosophy, and Connections with the Christian Faith* was organized by [Bob Brabenec](#), [Russell Howell](#), [Dick Stout](#), and [Dusty Wilson](#).
2. *Integrating Ethics into your Statistics and Computer Science Courses: Integrated Ethics Labs* was organized by [Lori Carter](#), [Catherine Crockett](#), and [Stacy De Ruiter](#).

The conference schedule is presented in [Appendix 1](#), [Appendix 2](#) gives the abstracts for the various sessions, [Appendix 3](#) lists information for the individual participants by name, and [Appendix 4](#) organizes them by institution.

The contributed paper sessions had a total of 72 presentations from the 156 conference attendees. Bryant Mathews has placed many of the corresponding files, in addition to some from other talks, on Google Drive. Click [here](#) to access them. Also on Google Drive are photos taken by a variety of people throughout the conference. They can be accessed by clicking [here](#).

Not every paper presented at the conference was submitted to this *Journal and Proceedings*, and starting with this issue the *Journal and Proceedings* is accepting papers for consideration that were not presented at the corresponding conference. The following pages contain all submissions that made their way through the single-blind review process, each having been scrutinized by a minimum of two reviewers. Thanks go to the authors for their good work. Too numerous to mention are all the referees that were involved, but heartfelt thanks go to them for their diligence.

ACMS is now a 501(c)(3) organization, so any monetary gifts to it are tax deductible. The conference at [Azusa Pacific University](#) featured in this *Journal and Proceedings* was originally scheduled for 2021, but had to be postponed until 2022 because of COVID protocols. We are grateful to God that we were able to gather in 2022, despite some COVID infections that occurred during the conference. Because of the postponement, the ACMS membership voted to change its by-laws to stipulate that conferences will henceforth be held on evenly-numbered years, D.V. Thus, the twenty-fourth biennial conference is slated for May 29–June 1, 2024, with [Dordt University](#) as the host institution. Further details can be found at the ACMS website: <https://acmsonline.org>.

[Russell W. Howell](#) (Westmont College)
ACMS *Journal and Proceedings* Editor

Formal Differential Variables and an Abstract Chain Rule

Samuel Alexander (The U.S. Securities and Exchange Commission)



Samuel Alexander completed his Ph.D. in mathematical logic in 2013 at The Ohio State University, with a dissertation about ordinal notation systems applied to the epistemology of machine knowledge. He did his undergraduate studies at the University of Arizona. His current position is in finance at The United States Securities and Exchange Commission. His research interests include mathematical epistemology, theoretical biology, and reinforcement learning.

Abstract

One shortcoming of the chain rule is that it does not iterate: it gives the derivative of $f(g(x))$, but not (directly) the second or higher-order derivatives. We present iterated differentials and a version of the multivariable chain rule which iterates to any desired level of derivative. We first present this material informally, and later discuss how to make it rigorous (a discussion which touches on formal foundations of calculus). We also suggest a finite calculus chain rule (contrary to Graham, Knuth and Patashnik's claim that "there's no corresponding chain rule of finite calculus").

1 Introduction

Consider the following statement, uncontroversial in an elementary calculus context (*): "For all variables u and v , $d(uv) = v du + u dv$." In his popular calculus textbook [11], Stewart says:

... the **differential** dx is an independent variable ...

So if $*$ really does hold for *all* variables u and v , and if x is a variable, and if (as Stewart says) dx is also a variable, then, by letting $u = x$ and $v = dx$, we get $d(x dx) = dx dx + x ddx$. We do not know whether Stewart intended us to make such an unfamiliar-looking conclusion from his innocent-looking statement, but let's continue along these lines and see where it leads us. We will formalize this kind of computation using machinery from first-order logic, and show that it leads to an elegant higher-order multivariable chain rule.

A weakness of the familiar chain rule is that it does not iterate: it tells us how to find the *first* derivative of $f(g(x))$, but it does not tell us how to find second- or higher-order derivatives of the same (at least not directly). Our abstract chain rule will iterate: the exact same rule which tells us $df(g(x))$ will also tell us $d^k f(g(x))$ for any integer $k > 1$.

Our d operator has some similarities with the Δ operator of Huang et al [7]. Our work improves on theirs in that we explicitly distinguish differential variables from others, so that the operator we develop better reveals the connection to differentials. For example, in Huang et al, one has $\Delta_1 e^{x_0} = e^{x_0} x_1$ and $\Delta_2 e^{x_0} = e^{x_0} (x_1^2 + x_2)$, which is equivalent to our $de^{x_0} = e^{x_0} dx_0$ and $d^2 e^{x_0} = e^{x_0} (dx_0 dx_0 + ddx_0)$. Besides better emphasizing the connection to differentials, the latter version should also be more familiar, since we already routinely write things like $de^x = e^x dx$ in elementary calculus classes.

2 Computing Iterated Partial Derivatives: Informal Examples

In this section, we will informally describe a way to compute iterated partial derivatives of a multivariable function. We will make the method formal in subsequent sections.

Example 1. Compute the differential $dd x^2 = d(dx^2)$, treating differential variables just like ordinary variables.

Solution. The differential $dx^2 = 2x dx$ involves two variables: x and dx . Thus, $d(dx^2)$ will have two terms, one where we differentiate with respect to x and multiply the result by dx , and one where we differentiate with respect to dx and multiply the result by ddx :

$$\begin{aligned} ddx^2 &= d(dx^2) \\ &= d(2x dx) \\ &= \frac{\partial(2x dx)}{\partial x} dx + \frac{\partial(2x dx)}{\partial dx} ddx \\ &= 2 dx dx + 2x ddx. \end{aligned}$$

Note that when we compute $\frac{\partial(2x dx)}{\partial x}$, we treat dx as a variable independent from x , so it can be treated as a constant. Likewise when we compute $\frac{\partial(2x dx)}{\partial dx}$, x is treated as a constant. \square

Example 2. Compute the differential $dd e^x$, treating differential variables just like ordinary variables.

Solution. As in Example 1, since $de^x = e^x dx$,

$$\begin{aligned} dde^x &= d(e^x dx) \\ &= \frac{\partial(e^x dx)}{\partial x} dx + \frac{\partial(e^x dx)}{\partial dx} ddx \\ &= e^x dx dx + e^x ddx. \end{aligned}$$

\square

Example 3. Compute $dd f(x)$, treating differential variables just like ordinary variables.

Solution. Just as above,

$$\begin{aligned} dd f(x) &= d(f'(x) dx) \\ &= \frac{\partial(f'(x) dx)}{\partial x} dx + \frac{\partial(f'(x) dx)}{\partial dx} ddx \\ &= f''(x) dx dx + f'(x) ddx. \end{aligned}$$

\square

In a later section, we will formalize and prove a formal chain rule (Corollary 20). For now, we will state it informally:

Remark 1. (Abstract Chain Rule, stated informally) Let T and U be expressions and let x be a non-differential variable. Assume T , U , and all of their sub-expressions are everywhere infinitely differentiable. Then

$$d(T[x|U]) = (dT)[x|U],$$

where the operator $[x|U]$ works by simultaneously replacing all occurrences of x by U , all occurrences of dx by dU , all occurrences of d^2x by d^2U , and so on.

The Abstract Chain Rule can be stated in English: “substituting first and then applying d gives the same result as applying d first and then substituting, provided that when one substitutes U for x , one also substitutes dU for dx and so on.”

Example 4. Compute $(e^{x^2})''$.

Solution. By Example 3, $(e^{x^2})''$ is the $dx\,dx$ -coefficient of $dd\,e^{x^2}$. We compute:

$$\begin{aligned} dd\,e^{x^2} &= dd\,(e^x[x|x^2]) \\ &= (dd\,e^x)[x|x^2] && \text{(Abstract Chain Rule)} \\ &= (e^x\,dx\,dx + e^x\,ddx)[x|x^2] && \text{(Example 2)} \\ &= e^{x^2}\,d(x^2)\,d(x^2) + e^{x^2}\,dd(x^2) && \text{(Substituting)} \\ &= e^{x^2}\,(2x\,dx)^2 + e^{x^2}\,(2\,dx\,dx + 2x\,ddx) && \text{(Example 1)} \\ &= (4x^2 + 2)e^{x^2}\,dx\,dx + 2xe^{x^2}\,ddx. \end{aligned}$$

The answer is the above $dx\,dx$ -coefficient:

$$(e^{x^2})'' = (4x^2 + 2)e^{x^2}.$$

□

Our Abstract Chain Rule works for multivariable and higher-order derivatives, too.

Example 5. The iterated total derivative

$$d^3 \sin xy = d^3(\sin x [x|xy]) = (d^3 \sin x)[x|xy]$$

encodes:

- $\partial^3 \sin xy / \partial x^3$ as its $dx\,dx\,dx$ -coefficient.
- $\partial^3 \sin xy / \partial y^3$ as its $dy\,dy\,dy$ -coefficient.
- $\frac{\partial^3 \sin xy}{\partial x \partial y \partial y} = \frac{\partial^3 \sin xy}{\partial y \partial x \partial y} = \frac{\partial^3 \sin xy}{\partial y \partial y \partial x}$ times 3 as its $dx\,dy\,dy = dy\,dx\,dy = dy\,dy\,dx$ -coefficient (the fact that there are three ways to write this coefficient is why we write “times 3”).

In Sections 5–7 we will formalize and prove the Abstract Chain Rule. But first, we will connect these higher-order differentials to a more concrete higher-order chain rule known as Faà di Bruno’s formula, and also show how the same ideas lead to a finite calculus chain rule.

3 Faà di Bruno's Formula

Faà di Bruno's formula, named after the 19th century Italian priest Francesco Faà di Bruno, is a formula for the higher derivatives of $f(g(x))$. See [8] and [3] for the history of Faà di Bruno's formula (see also [10] for related work in category theory, of all places, by another ACMS presenter). The formula can be stated combinatorially:

$$f(g(x))^{(n)} = \sum_{\pi \in \Pi_n} f^{(|\pi|)}(g(x)) \prod_{B \in \pi} g^{(|B|)}(x)$$

where π ranges over the set Π_n of all partitions of $\{1, \dots, n\}$ (so for each such partition π , B ranges over the blocks in π).

The ideas of Section 2 offer an intuitive way to understand the above formula¹. For any partition $\pi = \{B_1, \dots, B_k\}$ of $\{1, \dots, n\}$, let $I(\pi)$ be the expression

$$I(\pi) = f^{(k)}(x) d^{|B_1|}x d^{|B_2|}x \dots d^{|B_k|}x$$

involving iterated differentials as in Section 2. By an inductive argument, it is easy to see that

$$d^n f(x) = \sum_{\pi \in \Pi_n} I(\pi)$$

(for the inductive step, consider the different ways of obtaining a partition $\pi' \in \Pi_{n+1}$ from a partition $\pi \in \Pi_n$: one can either add $\{n+1\}$ as a new block, which corresponds to changing $f^{(k)}(x)$ to $f^{(k+1)}(x)dx$ when using the product rule to calculate $dI(\pi)$; or one can add $n+1$ to existing block B_i of π , which corresponds to changing $d^{|B_i|}x$ to $d^{|B_i|+1}x$ when using the product rule to calculate $dI(\pi)$).

By similar reasoning as in Examples 3 and 4, $f(g(x))^{(n)}$ is the $(dx)^n$ -coefficient of $d^n f(g(x)) = d^n(f(x)[x|g(x)]) = (d^n f(x))[x|g(x)]$. Thus $f(g(x))^{(n)}$ is the $(dx)^n$ -coefficient of

$$\sum_{\pi \in \Pi_n} I(\pi)[x|g(x)] = \sum_{\pi \in \Pi_n} f^{(|\pi|)}(g(x)) \prod_{B \in \pi} d^{|B|}g(x).$$

It is easy to see that $d^{|B|}g(x) = g^{(|B|)}(x)d^{|B|}x + o$ where o is a sum of terms involving higher-order differentials (which can be ignored because they contribute nothing to the $(dx)^n$ -coefficient we seek). Faà di Bruno's formula follows.

4 Application to Finite Calculus

The ideas in this paper also lead to a chain rule for the so-called finite calculus. The finite calculus is described in Section 2.6 of Graham, Knuth and Patashnik's *Concrete Mathematics* [4]. In finite calculus, one defines an operator Δ on functions by $\Delta f(x) = f(x+1) - f(x)$. This operator has many surprising analogies with differentiation, but Graham et al claim: "there's no corresponding

¹Shortly after presenting this argument at ACMS, we realized that the argument can actually be applied directly, without using iterated differentials at all, yielding a shockingly short elementary proof of Faà di Bruno's formula. Examining the literature, we found that the basic idea is already known [9] [6], but both published proofs which we found are actually proofs of more complicated multivariable generalizations of Faà di Bruno's formula. For the single-variable special case, the idea (essentially the same idea which we presented using iterated differentials at ACMS) is so simple that it can be written with a single sentence [2].

chain rule of finite calculus, because there's no nice form for $\Delta f(g(x))$." To the contrary, since $\Delta x = (x + 1) - x = 1$, an equivalent way to write $\Delta f(x)$ is

$$\Delta f(x) = f(x + \Delta x) - f(x).$$

One can then easily check that

$$\Delta f(g(x)) = f(g(x + \Delta x)) - f(g(x)) = f(g(x) + \Delta g(x)) - f(g(x)),$$

which can be expressed as a chain rule

$$\Delta(f(x)[x|g(x)]) = (\Delta f(x))[x|g(x)],$$

where $[x|g(x)]$ operates by replacing x by $g(x)$ and Δx by $\Delta g(x)$.

Of course, to make this rigorous, it would be necessary to work in a formal language so as to carefully track which "1"s are " Δx "s. For example, if $f(x) = 1/(1 + x^2)$, we want $f(x)[x|g(x)]$ to be $1/(1 + g(x)^2)$, not $\Delta g(x)/(\Delta g(x) + g(x)^2)$, even though $\Delta x = 1$. We will not go through the necessary formalism in this paper, but it would be very similar to the formalism required for the d operator, which we devote the whole rest of the paper to.

5 Formalizing Terms

In this section, we will formalize the terms (or expressions) of differential calculus. We attempt to make this formalization self-contained. The machinery we develop here is very similar to the machinery used to define terms in first-order logic, except that we assume more structure on the set of variables than is assumed in first-order logic.

Note that one could strongly argue that elementary calculus already implicitly operates on terms, abusing language to call terms "functions". For example, $x \mapsto x^2$ and $y \mapsto y^2$ are two names for the exact same function. Yet, nevertheless, in elementary calculus, the expressions x^2 and y^2 are *not* interchangeable [5]. Evidently, such discrepancies point to the fact that elementary calculus really is done using formal terms, implicitly. In the following, we make it explicit.

Definition 1. (Variables) We fix a set of *variables* defined inductively as follows.

1. For the base step, we fix a countably infinite set $\{x_0, x_1, \dots\}$ of distinct elements called *precalculus variables*, and we declare them to be variables.
2. Inductively, for every variable v , we fix a new variable dv , which we call a *differential variable*; we do this in such a way as to satisfy the following requirement (we write $d^n v$ for $ddd \cdots dv$ where d occurs n times):
 - (Unique Readability) For all $n, m \in \mathbb{N}$, for all variables v and w , if $d^n v$ is the same variable as $d^m w$, then $n = m$ and $v = w$.

We write \mathcal{V} for the set of variables.

Examples of variables include x_1 , x_{50} , dx_0 , ddx_3 , $d^4 x_{50}$ (shorthand for $ddddx_{50}$), and so on. The unique readability property guarantees that, e.g., dx_1 is not the same variable as dx_2 or ddx_3 or

$ddd x_1$, etc. We allow n or m to be 0 in the unique readability requirement, so, e.g., x_1 and dx_1 are not the same variable (since $d^0 x_1$ denotes x_1). Every variable is either a precalculus variable (in which case it is x_n for some $n \in \mathbb{N}$) or a differential variable (in which case it is $d^m x_n$ for some $n, m \in \mathbb{N}$ with $m > 0$).

Definition 2. (Constant symbols and function symbols)

1. We fix a distinct set $\{\bar{r}\}_{r \in \mathbb{R}}$ of *constant symbols* for the real numbers. For any $r \in \mathbb{R}$, \bar{r} is the *constant symbol* for r .
2. For every $n \in \mathbb{N}$ with $n > 0$, we fix a distinct set $\{\bar{f}\}_f$ of *n -ary function symbols*, where f ranges over the set of all functions from \mathbb{R}^n to \mathbb{R} . For any such f , \bar{f} is the *n -ary function symbol* for f .

We make these choices in such a way that no variable is a constant symbol, no variable is an n -ary function symbol (for any n), and no constant symbol is an n -ary function symbol (for any n).

For example, the exponential function \exp gives rise to a 1-ary (or *unary*) function symbol $\overline{\exp}$. The addition function $+$ gives rise to a 2-ary (or *binary*) function symbol $\bar{+}$.

Definition 3. (Terms) We define the *terms of differential calculus* (or simply *terms*) inductively as follows.

1. Every variable v is a term.
2. Every constant symbol is a term.
3. For all $n \in \mathbb{N}$ ($n > 0$), for every $f : \mathbb{R}^n \rightarrow \mathbb{R}$, for all terms U_1, \dots, U_n , $\bar{f}(U_1, \dots, U_n)$ is a term.

Examples of terms include $\bar{5}$, $\bar{\pi}$, x_1 , dx_2 , $\overline{\sin}(x_1)$, $\bar{+}(x_0, x_1)$, and so on. We often abuse notation and suppress the overlines and possibly parentheses when writing terms. For example, we might write $\sin x_0$ instead of $\overline{\sin}(x_0)$, $\cos \pi$ instead of $\overline{\cos}(\bar{\pi})$, and so on. For certain well-known functions, we sometimes abuse notation further, for example, writing:

- $x_0 + x_1$ instead of $\bar{+}(x_0, x_1)$;
- $2x_0$ instead of $\bar{\cdot}(\bar{2}, x_0)$;
- $x_0 dx_1$ instead of $\bar{\cdot}(x_0, dx_1)$;
- x_0^2 instead of $\overline{x \mapsto x^2}(x_0)$;
- e^{x_1} instead of $\overline{\exp}(x_1)$;
- $x_0 dx_1 + x_1 dx_0$ instead of $\bar{+}(\bar{\cdot}(x_0, dx_1), \bar{\cdot}(x_1, dx_0))$;
- and so on.

This should cause no confusion in practice.

Definition 4. (Term interpretation)

- By an *assignment*, we mean a function $s : \mathcal{V} \rightarrow \mathbb{R}$ (recall that \mathcal{V} is the set of variables).
- Let s be an assignment. For every term T , we define the *interpretation* $T^s \in \mathbb{R}$ of T (according to s) by induction on term complexity as follows.
 1. If T is a constant symbol \bar{r} , then $T^s = r$.
 2. If T is a variable v , then $T^s = s(v)$.
 3. If T is $\bar{f}(U_1, \dots, U_n)$ for some $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and terms U_1, \dots, U_n , then $T^s = f(U_1^s, \dots, U_n^s)$.

For example, if $s(x_0) = 5$, then $\overline{\exp}(x_0)^s = e^5$. If $s(x_0) = 9$ and $s(dx_0) = 0.1$, then $(x dx)^s = 9 \cdot 0.1 = 0.9$.

Definition 5. (Free variables) We define the *free variables* $\text{FV}(T)$ of a term T as follows.

1. If T is a constant symbol, then $\text{FV}(T) = \emptyset$ (the empty set).
2. If T is a variable v , then $\text{FV}(T) = \{v\}$.
3. If T is $\bar{f}(U_1, \dots, U_n)$ for some $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and terms U_1, \dots, U_n , then

$$\text{FV}(T) = \text{FV}(U_1) \cup \dots \cup \text{FV}(U_n).$$

For example, $\text{FV}(\bar{5}) = \emptyset$, $\text{FV}(x_6) = \{x_6\}$, $\text{FV}(dx_2) = \{dx_2\}$ (note that x_2 is *not* a free variable of dx_2), $\text{FV}(e^{x_0+x_1}) = \{x_0, x_1\}$, $\text{FV}(x_1 dx_2) = \{x_1, dx_2\}$.

Lemma 1. Suppose T is a term, v is a variable, and s is an assignment. If $v \notin \text{FV}(T)$, then T^s does not depend on $s(v)$.

Proof. By induction. □

Definition 6. (Semantic equivalence) If T and U are terms, we declare $T \equiv U$ (and say that T and U are *semantically equivalent*) if for every assignment s , $T^s = U^s$.

For example, $\sin(x_0 + 2\pi) \equiv \sin x_0$, by which we mean $\overline{\sin}(\overline{+}(x_0, \overline{2\pi})) \equiv \overline{\sin}(x_0)$.

6 Formal Derivatives

Definition 7. (Ordered free variables) If T is a term, we define the *ordered free variables* $\text{OFV}(T)$ to be the finite sequence whose elements are the free variables $\text{FV}(T)$ of T (each appearing exactly one time in the sequence), ordered such that:

- Whenever $0 < n < m$ then $d^n x_i$ precedes $d^m x_j$.
- Whenever $0 < i < j$ then $d^n x_i$ precedes $d^n x_j$.

For example,

$$\text{OFV}(e^{x_1+x_3+x_2+x_2+x_{99}} dx_1 d^3 x_1 dx_2 d^{50} x_0) = (x_1, x_2, x_3, x_{99}, dx_1, dx_2, d^3 x_1, d^{50} x_0).$$

Definition 8. If s is an assignment, w is a variable, and $r \in \mathbb{R}$, we write $s(w|r)$ for the assignment defined by

$$s(w|r)(v) = \begin{cases} r & \text{if } v \text{ is } w \\ s(v) & \text{otherwise.} \end{cases}$$

In other words, $s(w|r)$ is the assignment which is identical to s except that it overrides s 's output on w , mapping w to r instead.

Lemma 2. For any assignment s and variable v , $s(v|s(v)) = s$.

Proof. Trivial. □

Definition 9. (Everywhere-differentiability) Let T be a term, w a variable. We say that T is *everywhere-differentiable with respect to w* if for every assignment s , the limit

$$\lim_{h \rightarrow 0} \frac{T^{s(w|s(w)+h)} - T^s}{h}$$

converges to a finite real number.

Lemma 3. Let T be a term with $\text{OFV}(T) = (v_1, \dots, v_n) \neq \emptyset$, and let w be a variable. Assume T is everywhere-differentiable with respect to w . For all r_1, \dots, r_n , let

$$f(r_1, \dots, r_n) = \lim_{h \rightarrow 0} \frac{T^{s(w|s(w)+h)} - T^s}{h}$$

where s is some assignment such that each $s(v_i) = r_i$. Then $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is well-defined.

Proof. In other words, for any $r_1, \dots, r_n \in \mathbb{R}$, $f(r_1, \dots, r_n)$ does not depend on the choice of s , as long as each $s(v_i) = r_i$. This follows from Lemma 1 since T has no free variables other than v_1, \dots, v_n . □

Definition 10. If T is a term with $\text{OFV}(T) = (v_1, \dots, v_n)$, w is a variable, and T is everywhere-differentiable with respect to w , then we define the *derivative of T with respect to w* , a term, written $\frac{\partial T}{\partial w}$, as

$$\frac{\partial T}{\partial w} = \bar{f}(v_1, \dots, v_n)$$

where f is as in Lemma 3. We define $\frac{\partial T}{\partial w}$ to be the term $\bar{0}$ if $\text{FV}(T) = \emptyset$.

Example 6. (Some example term derivatives)

1. $\partial x_0 / \partial x_0 \equiv 1$.
2. $\partial x_0 / \partial x_1 \equiv 0$.
3. $\partial x_0 / \partial dx_0 \equiv 0$.
4. $\partial(e^{x_1 x_2} dx_1) / \partial x_1 \equiv x_2 e^{x_1 x_2} dx_1$.

Proof. (1) The function f of Lemma 3 is

$$f(r) = \lim_{h \rightarrow 0} \frac{x_0^{s(x_0|s(x_0)+h)} - x_0^s}{h}$$

(for any assignment s with $s(x_0) = r$). By Definitions 4 and 8 this simplifies to $f(r) = \lim_{h \rightarrow 0} \frac{s(x_0)+h-s(x_0)}{h} = 1$. The claim follows.

(2) The function f of Lemma 3 is

$$f(r) = \lim_{h \rightarrow 0} \frac{x_0^{s(x_1|s(x_1)+h)} - x_0^s}{h}$$

(whenever $s(x_0) = r$). This simplifies to $f(r) = \lim_{h \rightarrow 0} \frac{s(x_0)-s(x_0)}{h} = 0$. The claim follows.

(3) Similar to (2).

(4) By Definition 7, $\text{OFV}(e^{x_1 x_2} dx_1) = (x_1, x_2, dx_1)$. So, letting $v_1 = x_1$, $v_2 = x_2$, $v_3 = dx_1$, the function f of Definition 3 is

$$f(r_1, r_2, r_3) = \lim_{h \rightarrow 0} \frac{(e^{x_1 x_2} dx_1)^{s(v_1|s(v_1)+h)} - (e^{x_1 x_2} dx_1)^s}{h}$$

(where each $s(v_i) = r_i$). By Definitions 4 and 8 this simplifies to

$$f(r_1, r_2, r_3) = \lim_{h \rightarrow 0} \frac{e^{(r_1+h)r_2} r_3 - e^{r_1 r_2} r_3}{h},$$

which is $r_2 e^{r_1 r_2} r_3$ by calculus. The claim follows. \square

Another way to prove Example 6 would be to use the following lemma.

Lemma 4. For each term T , variable w , and assignment t , if T is everywhere-differentiable with respect to w , then

$$\left(\frac{\partial T}{\partial w}\right)^t = \lim_{h \rightarrow 0} \frac{T^{t(w|t(w)+h)} - T^t}{h}.$$

Proof. If $\text{FV}(T) = \emptyset$, the lemma is trivial. Assume not. Let $(v_1, \dots, v_n) = \text{OFV}(T)$. By definition, $\frac{\partial T}{\partial w} = \bar{f}(v_1, \dots, v_n)$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is such that for all $r_1, \dots, r_n \in \mathbb{R}$, for any assignment s with each $s(v_i) = r_i$,

$$f(r_1, \dots, r_n) = \lim_{h \rightarrow 0} \frac{T^{s(w|s(w)+h)} - T^s}{h}.$$

In particular, let each $r_i = t(v_i)$. Then:

$$\begin{aligned} \left(\frac{\partial T}{\partial w}\right)^t &= \bar{f}(v_1, \dots, v_n)^t && \text{(Definition 10)} \\ &= f(t(v_1), \dots, t(v_n)) && \text{(Definition 4)} \\ &= f(r_1, \dots, r_n) && \text{(Choice of } r_1, \dots, r_n) \\ &= \lim_{h \rightarrow 0} \frac{T^{t(w|t(w)+h)} - T^t}{h}, && \text{(Since each } t(v_i) = r_i) \end{aligned}$$

as desired. \square

Definition 11. (Term total differentials) Suppose T is a term. We say T is *everywhere totally differentiable* if T is everywhere-differentiable with respect to every variable. If so, we define the *total differential* $\mathbf{d}T$, a term, as follows. If $\text{FV}(T) = \emptyset$ then we define $\mathbf{d}T = \bar{0}$. Otherwise, let $\text{OFV}(T) = (v_1, \dots, v_n)$ and define

$$\mathbf{d}T = \frac{\partial T}{\partial v_1} dv_1 + \dots + \frac{\partial T}{\partial v_n} dv_n.$$

Furthermore, we inductively define $\mathbf{d}^1 T$ to be $\mathbf{d}T$ and, whenever $\mathbf{d}^n T$ is defined and is everywhere totally differentiable, we define $\mathbf{d}^{n+1} T = \mathbf{d}\mathbf{d}^n T$.

For example,

$$\begin{aligned} \mathbf{d}(x_1 dx_2) &= \frac{\partial(x_1 dx_2)}{\partial x_1} dx_1 + \frac{\partial(x_1 dx_2)}{\partial dx_2} ddx_2 \\ &\equiv dx_1 dx_2 + x_1 ddx_2. \end{aligned}$$

Lemma 5. If term T is everywhere totally differentiable and if v_1, \dots, v_n are distinct variables such that $\text{FV}(T) \subseteq \{v_1, \dots, v_n\}$, then

$$\mathbf{d}T \equiv \frac{\partial T}{\partial v_1} dv_1 + \dots + \frac{\partial T}{\partial v_n} dv_n.$$

Proof. Follows from the commutativity of addition and the fact that clearly $\frac{\partial T}{\partial v_i} \equiv \bar{0}$ if $v_i \notin \text{FV}(T)$. \square

In order to prove an abstract chain rule in Section 7, we will need a form of the classical multivariable chain rule, expressed for formal terms. For this purpose, we first introduce shorthand for finite summation notation².

Definition 12. If $m > 0$ is an integer and T_1, \dots, T_m are terms, we write $\sum_{i=1}^m T_i$ (or just $\sum_i T_i$ if no confusion results) as shorthand for $T_1 + \dots + T_m$.

Lemma 6. (Classic Multivariable Chain Rule for Terms) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Suppose $\vec{T} = (T_1, \dots, T_n)$ are terms with each $\text{FV}(T_i) \subseteq \{v_1, \dots, v_m\}$ (where v_1, \dots, v_m are distinct). Assume that $\bar{f}(\vec{T})$ and T_1, \dots, T_n are everywhere totally differentiable. Then for all $1 \leq i \leq m$,

$$\frac{\partial(\bar{f}(\vec{T}))}{\partial v_i} \equiv \sum_{j=1}^n \bar{f}_j(\vec{T}) \frac{\partial T_j}{\partial v_i},$$

where $f_j = D_j f$ (the partial derivative of f (in the usual sense) with respect to its j th argument).

Proof. Let s be an assignment and fix $1 \leq i \leq m$. We must show (Definition 6) that

$$\left(\frac{\partial(\bar{f}(\vec{T}))}{\partial v_i} \right)^s = \left(\sum_{j=1}^n \bar{f}_j(\vec{T}) \frac{\partial T_j}{\partial v_i} \right)^s.$$

²It is also possible to incorporate summation notation formally into Definition 3, but the details are complicated. See [1].

Define functions $F, G_j : \mathbb{R} \rightarrow \mathbb{R}$ ($1 \leq j \leq n$) by

$$\begin{aligned} F(z) &= \bar{f}(\vec{T})^{s(v_i|z)}, \\ G_j(z) &= T_j^{s(v_i|z)}. \end{aligned}$$

For all $1 \leq j \leq n$ and $z \in \mathbb{R}$,

$$\begin{aligned} F(z) &= \bar{f}(\vec{T})^{s(v_i|z)} && \text{(Definition of } F_i) \\ &= f(T_1^{s(v_i|z)}, \dots, T_n^{s(v_i|z)}) && \text{(Definition 4)} \\ &= f(G_1(z), \dots, G_n(z)), && \text{(Definition of } G_j) \\ \text{so } (*) \quad F'(z) &= \sum_j f_j(G_1(z), \dots, G_n(z)) G'_j(z) && \text{(Classic multivar. chain rule)} \end{aligned}$$

(the hypotheses of the classic multivariable chain rule are implied by the everywhere-total-differentiability of $\bar{f}(\vec{T})$ and each T_i , by Lemma 4). So armed, we compute:

$$\begin{aligned} \left(\frac{\partial(\bar{f}(\vec{T}))}{\partial v_i} \right)^s &= \lim_{h \rightarrow 0} \frac{\bar{f}(\vec{T})^{s(v_i|s(v_i)+h)} - \bar{f}(\vec{T})^s}{h} && \text{(Lemma 4)} \\ &= \lim_{h \rightarrow 0} \frac{F(s(v_i) + h) - F(s(v_i))}{h} && \text{(Def. of } F) \\ &= F'(s(v_i)) && \text{(Def. of } F') \\ &= \sum_j f_j(G_1(s(v_i)), \dots, G_n(s(v_i))) G'_j(s(v_i)) && \text{(By } (*)) \\ &= \sum_j f_j(T_1^{s(v_i|s(v_i))}, \dots, T_n^{s(v_i|s(v_i))}) G'_j(s(v_i)) && \text{(Def. of } G_j) \\ &= \sum_j f_j(T_1^s, \dots, T_n^s) G'_j(s(v_i)) && \text{(Lemma 2)} \\ &= \sum_j f_j(T_1^s, \dots, T_n^s) \lim_{h \rightarrow 0} \frac{G_j(s(v_i) + h) - G_j(s(v_i))}{h} && \text{(Def. of } G'_j) \\ &= \sum_j f_j(T_1^s, \dots, T_n^s) \lim_{h \rightarrow 0} \frac{T^{s(v_i|s(v_i)+h)} - T^{s(v_i|s(v_i))}}{h} && \text{(Def. of } G_j) \\ &= \sum_j f_j(T_1^s, \dots, T_n^s) \lim_{h \rightarrow 0} \frac{T^{s(v_i|s(v_i)+h)} - T^s}{h} && \text{(Lemma 2)} \\ &= \sum_j f_j(T_1^s, \dots, T_n^s) \left(\frac{\partial T_j}{\partial v_i} \right)^s && \text{(Lemma 4)} \\ &= \left(\sum_{j=1}^n \bar{f}_j(\vec{T}) \frac{\partial T_j}{\partial v_i} \right)^s, && \text{(Def. 4)} \end{aligned}$$

as desired. □

7 An Abstract Chain Rule

Recall that \mathcal{V} denotes the set of all variables. Let \mathcal{T} denote the set of all terms.

Definition 13. For any $\phi_0 : \mathcal{V} \rightarrow \mathcal{T}$, the *extension of ϕ_0 to all terms* is the function $\phi : \mathcal{T} \rightarrow \mathcal{T}$ defined by induction as follows:

1. If T is a constant symbol then $\phi(T) = T$.
2. If T is a variable then $\phi(T) = \phi_0(T)$.
3. If T is $\bar{f}(S_1, \dots, S_n)$ then $\phi(T) = \bar{f}(\phi(S_1), \dots, \phi(S_n))$.

Lemma 7. Let $\phi_0 : \mathcal{V} \rightarrow \mathcal{T}$ and let ϕ be the extension of ϕ_0 to all terms. Then:

1. (The Substitution Lemma) For any assignment s , if $\phi(s)$ is the assignment defined by $\phi(s)(v) = \phi(v)^s$, then for every term T , $\phi(T)^s = T^{\phi(s)}$.
2. For all terms T and U , if $T \equiv U$ then $\phi(T) \equiv \phi(U)$.

Proof. (1) By induction on T . If T is a constant symbol or variable, the claim is trivial. Otherwise, T is $\bar{f}(U_1, \dots, U_n)$. Then

$$\begin{aligned}
 \phi(T)^s &= \bar{f}(\phi(U_1), \dots, \phi(U_n))^s && \text{(Definition 13)} \\
 &= \bar{f}(\phi(U_1)^s, \dots, \phi(U_n)^s) && \text{(Definition 4)} \\
 &= \bar{f}(U_1^{\phi(s)}, \dots, U_n^{\phi(s)}) && \text{(Induction)} \\
 &= T^{\phi(s)}. && \text{(Definition 4)}
 \end{aligned}$$

(2) Assume $T \equiv U$. For any assignment s , if $\phi(s)$ is as in (1), then $T^{\phi(s)} = U^{\phi(s)}$ by Definition 6. Thus $\phi(T)^s = \phi(U)^s$ by (1). By arbitrariness of s , $\phi(T) \equiv \phi(U)$. \square

Definition 14. Say $\phi_0 : \mathcal{V} \rightarrow \mathcal{T}$ *respects* d if for each variable v , $\phi_0(dv) \equiv \mathbf{d}\phi_0(v)$.

Definition 15. (Strong differentiability)

1. We define the *subterms* of a term T by induction as follows. If T is a variable or constant symbol, then T is its own lone subterm. If T is $\bar{f}(U_1, \dots, U_n)$, then the subterms of T are T itself along with the subterms of each U_i .
2. A term T is *strongly differentiable* if every subterm of T is everywhere totally differentiable.

Thus, a term is strongly differentiable if it is built up from pieces which are everywhere totally differentiable. An example of a term which is everywhere totally differentiable but not strongly differentiable is $|x_0|^2$, which is everywhere totally differentiable despite having a subterm $|x_0|$ which is not. Note that the ordinary chain rule for $f(g(x))'$ fails when $f(x) = x^2$ and $g(x) = |x|$ (these functions fail the chain rule's hypotheses): $(|x|^2)' = 2x$, but $|x|'$ is undefined at $x = 0$. We avoid such traps in the following theorem by requiring strong differentiability.

Theorem 16. (General Abstract Chain Rule) Let $\phi_0 : \mathcal{V} \rightarrow \mathcal{T}$ and assume that $\phi_0(v)$ is strongly differentiable for every variable v . Let ϕ be the extension of ϕ_0 to all terms. If T is strongly differentiable and ϕ_0 respects d , then $\mathbf{d}\phi(T) \equiv \phi(\mathbf{d}T)$.

Proof. By induction on T . If T is a constant symbol, the theorem is trivial. If T is a variable, the theorem reduces to the statement that ϕ_0 respects d , which is one of the hypotheses. It remains to consider the case when T is $\bar{f}(\vec{T})$ where $f : \mathbb{R}^m \rightarrow \mathbb{R}$ and $\vec{T} = T_1, \dots, T_m$ are simpler terms. Then T_1, \dots, T_m are subterms of T , so, since T is strongly differentiable, it follows that T_1, \dots, T_m are strongly differentiable. By induction, each $\mathbf{d}\phi(T_i) \equiv \phi(\mathbf{d}T_i)$. Let $\{v_1, \dots, v_\ell\} = \text{FV}(\phi(T_1)) \cup \dots \cup \text{FV}(\phi(T_m))$. For the rest of the proof, whenever S is a term and v is a variable,

we will write S_v for $\frac{\partial S}{\partial v}$. Let $\overrightarrow{\phi(T)}$ denote $\phi(T_1), \dots, \phi(T_m)$. We calculate:

$$\begin{aligned}
& \mathbf{d}\phi(\overrightarrow{f(T)}) \\
& \equiv \sum_{i=1}^{\ell} \phi(\overrightarrow{f(T)})_{v_i} dv_i && \text{(Lemma 5)} \\
& = \sum_i \overrightarrow{f}(\overrightarrow{\phi(T)})_{v_i} dv_i && \text{(Definition 13)} \\
& \equiv \sum_i \sum_{j=1}^m \overrightarrow{f_j}(\overrightarrow{\phi(T)}) \phi(T_j)_{v_i} dv_i && \text{(Lemma 6)} \\
& \equiv \sum_j \overrightarrow{f_j}(\overrightarrow{\phi(T)}) \sum_i \phi(T_j)_{v_i} dv_i && \text{(Basic algebra)} \\
& \equiv \sum_j \overrightarrow{f_j}(\overrightarrow{\phi(T)}) \mathbf{d}\phi(T_j) && \text{(Lemma 5)} \\
& \equiv \sum_j \overrightarrow{f_j}(\overrightarrow{\phi(T)}) \phi(\mathbf{d}T_j) && \text{(Induction Hypothesis)} \\
& = \phi\left(\sum_j \overrightarrow{f_j}(\overrightarrow{T}) \mathbf{d}T_j\right) && \text{(Definition 13)} \\
& \equiv \phi\left(\sum_j \overrightarrow{f_j}(\overrightarrow{T}) \sum_{i=1}^{\ell} (T_j)_{v_i} dv_i\right) && \text{(Lemma 5)} \\
& \equiv \phi\left(\sum_i \sum_j \overrightarrow{f_j}(\overrightarrow{T}) (T_j)_{v_i} dv_i\right) && \text{(Basic algebra)} \\
& \equiv \phi(\sum_i \overrightarrow{f}(\overrightarrow{T})_{v_i} dv_i) && \text{(Lemma 6)} \\
& \equiv \phi(\mathbf{d}\overrightarrow{f}(\overrightarrow{T})) && \text{(Lemma 5)}
\end{aligned}$$

(in the last few lines, we use Lemma 7 part 2). □

A weakness of the familiar chain rule is that it does not iterate. The following corollary shows that the abstract chain rule does iterate.

Corollary 17. For all ϕ_0 , ϕ and T as in Theorem 16, for all $k \in \mathbb{N}$ ($k > 0$), if $\mathbf{d}^\ell T$ exists and is strongly differentiable for all $\ell < k$, then

$$\mathbf{d}^k \phi(T) \equiv \phi(\mathbf{d}^k T).$$

Proof. By repeated applications of Theorem 16. □

In Sections 2 and 3 we used a special case of Theorem 16 which we will now formalize. Recall that a precalculus variable is one that is not of the form dv for any variable v .

Definition 18. (Variable substitution respecting differentials) Let v be a precalculus variable, U a term such that $\mathbf{d}^k U$ is strongly differentiable for all k . For every term T , we will define the *result of substituting U for v in T while respecting differentials*, written $T[v|U]$, as follows. First, we define $\phi_0 : \mathcal{V} \rightarrow \mathcal{T}$ so that:

1. $\phi_0(v) = U$.
2. For every $k > 0$, $\phi_0(\mathbf{d}^k v) = \mathbf{d}^k U$.
3. For all variables w not of either of the above two forms, $\phi_0(w) = w$.

We define $T[v|U]$ to be $\phi(T)$ where ϕ is the extension of ϕ_0 to all terms (Definition 13).

Corollary 19. (Abstract Chain Rule) Let U, v be as in Definition 18. If term T is strongly differentiable, then

$$\mathbf{d}(T[v|U]) \equiv (\mathbf{d}T)[v|U].$$

Proof. If ϕ_0 is as in Definition 18 then it is easy to see that ϕ_0 satisfies the hypotheses of Theorem 16. The corollary then immediately follows from Theorem 16. \square

Corollary 20. (Iterated Abstract Chain Rule) Let v, T, U be as in Corollary 19. For all $k > 0$, if $\mathbf{d}^\ell T$ is strongly differentiable for all $\ell < k$, then

$$\mathbf{d}^k(T[v|U]) \equiv (\mathbf{d}^k T)[v|U].$$

Proof. By repeated applications of Corollary 19. \square

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Big-Oh Notations, Elections, and Hyperreal Numbers: A Socratic Dialogue

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Abstract

We provide an intuitive motivation for the hyperreal numbers via electoral axioms. We do so in the form of a Socratic dialogue, in which Protagoras suggests replacing big-oh complexity classes by real numbers, and Socrates asks some troubling questions about what would happen if one tried to do that. The dialogue is followed by an appendix containing additional commentary and a more formal proof.

1 Act One: Replacing Big-Oh Notations by Real Numbers

Protagoras: My dear Socrates, don't you agree mathematicians make everything too complicated?

Socrates: You're right, Protagoras. How would you reform mathematics?

Protagoras: I'd start by removing big-oh notations; $O(n^2)$, $O(n \log n)$, $O(n!)$, These notations are an eyesore!

Socrates: You'd get rid of big-oh notations, like $O(n^2)$ and $O(2^n)$? What would you replace them with?

Protagoras: The real numbers. Everyone loves \mathbb{R} .

Socrates: I agree \mathbb{R} is less confusing than big-oh notation. I'm slow, though. I need examples. What would you replace $O(n)$ with?

Protagoras: It's not important. Haven't you noticed that degrees and radians both work just fine for measuring angles?

Socrates: Yes. But please humor me. I know it's arbitrary, but exactly which real would you replace $O(n)$ with?

Protagoras: I'd replace $O(n)$ with 1.

Socrates: And $O(n^2)$?

Protagoras: I'd replace $O(n^2)$ with 2.

Socrates: So instead of saying, "The algorithm has complexity $O(n)$," you'd say, "The algorithm has complexity level 1"?

Protagoras: Yes.

Socrates: And instead of, "My algorithm runs in time $O(n^2)$," you'd say, "My algorithm runs in time complexity 2"?

Protagoras: Precisely!

Socrates: So $O(n)$ becomes 1 and $O(n^2)$ becomes 2. I suppose next you'll say $O(n^3)$ becomes 3?

Protagoras: Naturally.

Socrates: And $O(n^k)$ becomes k , for every natural number k ?

Protagoras: Now you've got it!

Socrates: What would you replace $O(2^n)$ by?

Protagoras: Let's make $O(2^n)$ be 1000. Any big number would work.

Socrates: So, "Complexity $O(2^n)$ " becomes "Complexity level 1000"?

Protagoras: Yes.

Socrates: And, "Complexity $O(n^k)$ " becomes "Complexity level k "?

Protagoras: Yes.

Socrates: I feel a lot smarter. See, my slow brain takes $O(2^n)$ time to think about anything. But now I realize $O(2^n)$ is polynomial time!

Protagoras: I don't follow.

Socrates: Isn't n^{1000} a polynomial? Aren't $O(2^n)$ and $O(n^{1000})$ both replaced by 1000?

Protagoras: You trickster! Very well, let's fix that. Let's replace

$$O(1), O(n), O(n^2), \dots$$

by a strictly increasing sequence

$$r_0 < r_1 < r_2 \dots$$

of reals in $[0, 500)$. We're replacing $O(2^n)$ with 1000 so we better ensure this sequence doesn't get anywhere near 1000.

Socrates: We have to, if my brain runs slower than polynomial time. So we'll replace $O(1)$, $O(n)$, $O(n^2)$, ... by $r_0 < r_1 < r_2 < \dots$ in $[0, 500)$. If the r_i converge, can we assume $r_i \rightarrow 500$?

Protagoras: They must converge, by the Monotone Convergence Theorem. Without loss of generality, we can assume $r_i \rightarrow 500$.

Socrates: I suppose r_{100} must be around 499.99 then.

Protagoras: r_i must be at least 499.99 for some i . I see no harm in letting $r_{100} = 499.99$.

Socrates: So instead of saying, "My algorithm has complexity level $O(n^{100})$," you'd say, "My algorithm has complexity level 499.99"?

Protagoras: That is what we have decided.

Socrates: Would you object if I proposed that we should next declare that $O(n^{200})$ should be replaced by 499.99001?

Protagoras: Yes, I would certainly object!

Socrates: Why?

Protagoras: Because $O(n^{200})$ is far bigger than $O(n^{100})$, Socrates. But 499.99001 is hardly any bigger than 499.99 at all.

Socrates: I see. Well then, what should we replace $O(n^{200})$ by, if 499.99001 is too close to 499.99, which is $O(n^{100})$?

Protagoras: I see where you're going with this. No matter which number I choose for $O(n^{200})$, whether it be 499.999 or even 499.9999, you'll say that it's still barely any larger than 499.99, which is $O(n^{100})$. And even if you accept that 499.9999 is big enough compared to 499.99, you'll just go right on and ask me what I'd replace $O(n^{500})$ by, and then I'll be stuck even worse. Socrates, I'm starting to think you're deliberately trying to make me look foolish.

Socrates: I am guilty as charged.

2 Act Two: The Republic

Protagoras: It seems whatever realm I choose, you'll catch me in one of your infamous traps. How would you replace big-oh notation, then?

Socrates: I doubt I'm wise enough, Protagoras. But if you like, we can try to reason it out together. Consider a Republic...

Protagoras: You and your Republics!

Socrates: Isn't a Republic a big group of people making decisions together?

Protagoras: Yes, but I don't see how that's relevant.

Socrates: Isn't big-oh notation all about comparing growth rates?

Protagoras: Yes, I suppose so...

Socrates: Do the numbers have a king ruling them?

Protagoras: Certainly not.

Socrates: Then if the natural numbers lived together in a Republic, how would they decide, given $f, g : \mathbb{N} \rightarrow \mathbb{R}$, which function grows faster?

Protagoras: They have no king, so they would have to call a vote.

Socrates: Good idea. Let the natural numbers vote whether f outgrows g , or whether g outgrows f , or whether they grow at the same rate. How does natural number n vote?

Protagoras: Hmmm... I suppose that...

- If $f(n) > g(n)$ then n votes that f outgrows g .
- If $f(n) < g(n)$ then n votes that g outgrows f .
- If $f(n) = g(n)$ then n votes that they grow at the same rate.

Socrates: So, if $f(75) > g(75)$, then 75 votes that f outgrows g ? And if $f(30) = g(30)$, then 30 votes that f and g have equal growth rate?

Protagoras: Yes.

Socrates: Now I see why Democritus called you a math genius.

Protagoras: But how can we define the outcome of infinitely many votes?

Socrates: Call $S \subseteq \mathbb{N}$ a *winning bloc* if S 's votes alone already guarantee electoral victory. What axioms can you think of for the collection of all winning blocs?

Protagoras: Well, let's see ...

- (Properness) You lose if no one votes for you: \emptyset is not a winning bloc.
- (Monotonicity) More votes can't hurt: If S is a winning bloc, then every superset of S is a winning bloc.
- (Maximality) Someone wins: for any finite partition $\mathbb{N} = S_1 \cup \dots \cup S_k$, one of the S_i must be a winning bloc.

Socrates: I don't understand that Maximality axiom. Can you explain it to me?

Protagoras: Well, isn't the point of an election to determine a winner?

Socrates: Yes.

Protagoras: Wouldn't it be a scandal, then, if the votes were collected and then there was no winner determined by them?

Socrates: Quite so.

Protagoras: So there you have it. If the natural numbers vote between k different candidates, that's a partition of \mathbb{N} into k different pieces.

Socrates: Ahh I see. Someone must win in that case, ergo, one of those k different pieces is a winning bloc.

Protagoras: Precisely.

Socrates: If voters decide that f outgrows g , and that g outgrows h , don't you think they'd better also decide that f outgrows h ?

Protagoras: Hmm... I think we could force that by requiring:

- (Closure Under Intersections) If S and T are winning blocs, then so is $S \cap T$.

Socrates: And didn't we also agree that the natural numbers have no king?

Protagoras: Oh, right!

- (Non-Dictatorialness) There is no $n \in \mathbb{N}$ such that $\{n\}$ is a winning bloc. (Such an n would be a *dictator*.)

Socrates: But I fear we're getting nowhere. Surely there's no way to satisfy all these axioms simultaneously, is there?

Protagoras: You're thinking of Arrow's Impossibility Theorem. But Socrates, Arrow's Theorem assumes there are finitely many voters; \mathbb{N} has infinitely many voters. Arrow's Theorem isn't applicable. Let's see... yes! Using Zorn's Lemma, I'm quite sure our axioms are consistent!

Socrates: Slow down. Can you state some definitions?

Protagoras: Definition:

- A set \mathcal{U} of subsets of \mathbb{N} (called *winning blocs*) is an *ultrafilter* if it satisfies Properness, Monotonicity, Maximality, and Closure Under Intersections.
- \mathcal{U} is *free* if it also satisfies Non-Dictatorialness.

Theorem: Free ultrafilters exist.

So when the natural numbers vote, we can decide the outcome. But Socrates, how does this help us replace big-oh notations?

3 Act Three: The Hyperreals

Socrates: Tell me, why are $1/2$ and $2/4$ considered the same number?

Protagoras: Because $(1, 2)$ and $(2, 4)$ are in the same equivalence class modulo a certain equivalence relation.

Socrates: So the rational numbers are equivalence classes of pairs?

Protagoras: Yes. The whole point of the rationals is to compare proportions (you could even say, “growth rates,” in some sense) between pairs.

Socrates: Could we adapt the construction of the rationals to get numbers for comparing growth rates of functions like n^3 and 2^n ?

Protagoras: Oh, I see. Using our “voters”! Okay, fix a free ultrafilter \mathcal{U} ...

- Definition: If $f, g : \mathbb{N} \rightarrow \mathbb{R}$, declare $f \sim g$ if the naturals vote that f and g have the same growth rate (as decided by \mathcal{U}).
- Lemma: The relation \sim is an equivalence relation.
- Definition: The \sim equivalence classes are called *hyperreal numbers*.

Socrates: Do these “hyperreal numbers” have any structure?

Protagoras: Yes! Let $[f]$ be f ’s equivalence class.

- Definition: For all $f, g : \mathbb{N} \rightarrow \mathbb{R}$, we define $[f] + [g] = [f + g]$, $[f][g] = [fg]$, and $[f] < [g]$ if and only if the naturals vote that g outgrows f (using \mathcal{U}).
- Theorem: This makes the hyperreals an ordered field extension of \mathbb{R} .

Socrates: Which number should replace $O(f(n))$?

Protagoras: The hyperreal number $[f]$. Or an appropriate neighborhood thereof, if we must respect that $O(f(n)) = O(C \cdot f(n))$ for any positive real number C .

Socrates: Aren’t you worried I’ll find some f and g such that $O(f(n))$ is far bigger than $O(g(n))$ and yet $[f] \approx [g]$?

Protagoras: Not any more, Socrates. I’m protected from your tricks now by a whole Republic of voters!

4 Conclusion

If we try replacing big-oh complexity classes by real numbers, we paint ourselves into a corner. But comparing growth rates by letting natural numbers vote leads to ultrafilters and hyperreal numbers via electoral axioms (this was previously observed in [2]). We can then replace big-oh complexity classes by (classes of) hyperreal numbers without painting ourselves into a corner.

This suggests the hyperreals could potentially be quite familiar to computer scientists. They’ve (almost) been using them all along!

5 Appendix

The electoral axioms used by our fictional Protagoras to define an ultrafilter translate easily into the usual axioms of an ultrafilter. Those axioms and the additional results cited by Protagoras can be found in [4]. The idea of using electoral axioms to motivate ultrafilters was suggested in [2]. It was known since the 1970s that free ultrafilters provide infinite-voter counterexamples to Arrow's Impossibility Theorem [5]. Alexander observed in [1] that the *real-life* Protagoras made a certain claim (reported by Plato) with non-Archimedean implications similar to those of our fictional dialogue:

The very day you start [as my student], you will go home a better man, and the same thing will happen the day after. Every day, day after day, you will get better and better [6].

In this case, if Protagoras and his student live forever, and if Protagoras's goodness level does not change, and if students do not excel their teachers, and if "better" means "significantly better" (ruling out diminishing returns), then Protagoras's goodness is implied to exceed all real numbers.

Critics might claim Act 1 of this dialogue is trivial because the set of big-oh complexity classes has larger cardinality than \mathbb{R} . But Socrates' argument suggests that even the *countable* subset $\{O(n), O(n^2), \dots\} \cup \{O(2^n)\}$ is already non-embeddable in \mathbb{R} in some sense.

A more formal proof that the big-oh complexity classes cannot be meaningfully embedded into \mathbb{R} can be accomplished by a diagonalization argument, as follows.

Let \mathcal{O} be the set of big-oh complexity classes. Note that each complexity class is an equivalence class of functions $f : \mathbb{N} \rightarrow \mathbb{R}$; for instance, the complexity class $O(n^2)$ contains not only $f(n) = n^2$, but also $g(n) = 14n^2 + 3n - 14$ and $k(n) = 0.0001n^2 + \log n$. Also, there is an order $<$ on the set of complexity classes.

Suppose there is a correspondence $h : \mathcal{O} \rightarrow \mathbb{R}$ that is strictly increasing. Suppose further that h is unbounded.

Construct a sequence (a_n) as follows. For $n = 1$, there exists a complexity class C_1 such that $h(C_1) \geq 1$. Choose a function $f_1 \in C_1$. Let $a_1 = f_1(1)$. For $n = 2$, there exists a complexity class C_2 such that $h(C_2) \geq 2$ and $C_2 > C_1$. Choose a function $f_2 \in C_2$ such that $f_2(n) > f_1(n)$ for all n . Let $a_2 = f_2(2)$. Continue in this manner; for each $k \geq 2$, there exists a complexity class C_k such that $C_k > C_{k-1}$ and $h(C_k) \geq k$. Choose a function $f_k \in C_k$ such that $f_k(n) > f_{k-1}(n)$ for all $n \in \mathbb{N}$. Let $a_k = f_k(k)$. Call the resulting sequence (a_n) a *complexity diagonal* sequence.

Now, consider an algorithm requiring a_n steps for n inputs. Let C be the complexity class for this algorithm. Let $r = h(C)$ and let $M = \lceil r \rceil$. Note that for all $n \geq M + 2$, $a_n = f_n(n) > f_{M+1}(n)$, hence our algorithm has complexity level L for some $L \geq C_{M+1} > C$. This is a contradiction. Hence there can be no such correspondence h .

If h is bounded, modify the argument by letting $u = \sup \{h(C) \mid C \in \mathcal{O}\}$ and replacing $h(C_k) \geq k$ with $h(C_k) \geq u - \frac{1}{k}$. There is no complexity class C such that $h(C) = u$ (since there is no largest complexity class), and with $r = h(C)$, let $M \in \mathbb{N}$ such that $\frac{1}{M} < u - r$. Then for all $n \geq M + 2$, $a_n = f_n(n) > f_{M+1}(n)$, hence our algorithm has complexity level higher than C_{M+1} . Since $h(C_{M+1}) \geq u - \frac{1}{M+1} > u - \frac{1}{M} > r$, the complexity level is greater than C , which is a contradiction. Again, there can be no such correspondence.

The above construction of (a_n) only makes use of countably many elements of \mathcal{O} , so again, the argument is not trivialized by the fact that $|\mathcal{O}| > |\mathbb{R}|$.

For a different description of an embedding of big-oh complexity classes in the hyperreals, see Section 5.10 of [3].

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AI and Writing: Do We Speak the Words of Stochastic Parrots?

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Abstract

Generative Artificial Intelligence systems have recently and publicly become increasingly comparable to the capabilities of many humans in some domains, such as text and image generation. These systems are rapidly being incorporated into human contexts, for example as phone keyboards and email apps offer us AI-generated suggestions of words and even complete phrases to enter with a single tap or swipe. But these suggestions don't just reduce physical effort; they reduce cognitive effort: we can accept a suggested phrase before even thinking about what to say. In this article, I first describe how mimicry pervades the design and implementation of generative AI systems and expound on some of the implications thereof. I then critique how these systems are designed and implemented in light of our God-given identity, a biblical perspective on mimicry, and a reflection on virtue. I finally outline several "swords-to-plowshares" approaches that re-form generative AI technologies to support human thinking in writing tasks: *structural editing tools*, *writer-directed questions* to support writers without speaking for them, and *visualizations* to enable writers to reflect on their work.

1 Introduction

AI systems for generating and suggesting content have become commonplace. Phone keyboards suggest the words and even phrases we might write next. Search engines suggest queries. Other systems generate images, music, or code based on prompts.¹ Recent systems have reached unprecedented levels of quality in their outputs, as Figures 1 (on the following page) and 2 (later in this paper) illustrate. As this article goes to press, news outlets are reporting "The End of High-School English"² in response to the surprising level of quality of text generated by a language model (in this case, ChatGPT by OpenAI).³

Despite the superficial novelty of the content generated by these systems, generative AI systems mimic their training data: they recombine elements of the examples they've seen, and even that recombination is according to tropes of how elements are typically combined. That is, mimicry is deeply woven into the design of interactive AI systems, both in the underlying models (and the associated mathematics, algorithms, and data) and in the design of people's interactions with those

¹Generative AI can be distinguished from the pervasive recommendations provided on social media sites, video sharing sites, shopping sites, etc., since those systems recommend *existing* content that is (typically) created by other people. These recommendations also have enormous influence on what we buy, who we socialize with, and even what we believe. But the *generative* technology I discuss here generates new content, rather than selecting from among content that other people have already created.

²For the reference, click [here](#). An additional reference is available by clicking [here](#).

³<https://openai.com/blog/chatgpt/>

models. Recent scholarship has identified ways in which these so-called “stochastic parrots” can be harmful [1]; in this article I extend that critique to the interactive interfaces built on those systems and how we use those systems in creating the texts we use to communicate with each other. I argue that, although mimicry is useful in some cases, we must think deeply about how to both *build* and *use* these systems in ways that align more deeply with our God-given identity as co-creators who are called to love and serve God and our neighbors with thankfulness, humility, and integrity. In the following sections, I:

- Summarize how interactive generative AI systems work, in terms of data and computation,
- Draw implications about these systems from a Christian world view, and
- Propose some ideas for repurposing the underlying technology redemptively.

Input: I was supposed to start writing the paper at 5:00 PM. But then I started playing with this cool new language model for 10 minutes. 10 minutes later, it’s suddenly 9:30 PM!

Model Output: This joke is about how time flies when you’re having fun. The person was supposed to start writing a paper at 5:00 PM, but they started playing with a language model instead. They thought they were only playing with it for 10 minutes, but it was actually 4.5 hours.

Figure 1: The PaLM language model (a type of generative AI system) explains a joke, after being given only 2 examples of joke explanations. Example from the research paper published by Google [2]; see there for more details.

2 How Interactive Generative AI Systems Work

We begin with a systems-level overview of generative AI. The two main takeaways of this section are: (1) AI systems are mathematical not magical, and (2) a bias towards mimicry is deeply embedded into the entire life cycle of these systems. Informally, the generative modeling approach has a “groove” that it is following, the bias that the approach embeds.

We limit our scope here to what is technically called *self-supervised learning*, meaning that the system is trained to perform a prediction task that is based only on unlabeled examples. We will focus here on language modeling using Transformer-based large language models (sometimes abbreviated LLMs), but most of the comments will also apply to other language tasks and to image generation tasks using Generative Adversarial Networks (GANs), diffusion models such as DALL-E and Stable Diffusion, and others. Some models deviate slightly from this recipe by tuning the models based on human feedback; this sort of tuning, called Reinforcement Learning from Human Feedback, allows selecting for desirable behaviors among the set of behaviors that the model has learned.[3]; this was the approach used to make the headline-grabbing ChatGPT.

2.1 Collect Data

Building a generative AI system starts with **collecting data**. The beginning is typically a very large collection of text. For example, a model recently described by Google was trained on hundreds of billions of words drawn from publicly available text, such as web pages, books, Wikipedia articles, news articles, source code for computer programs, and conversations on social media platforms [2].

2.2 Train a Model of the Data Distribution

The dataset is then typically used to train a statistical model of the distribution of training data. That is, the model is a function that can compute a probability for any possible text.⁴ *Intuitively*, a “good” model is one that assigns larger probabilities to texts that are more likely to occur in its training data. The model’s function has many parameters (e.g., 540 billion, for Google’s PaLM model [2]) that are used in that computation. For practical reasons, the probability computation is typically performed sequentially through a text, one token (word, punctuation symbol, or sub-word unit) at a time, thus computing the probability as a product of conditional probabilities:

$$P(\text{text}) = P(\text{word1})P(\text{word2}|\text{word1})P(\text{word3}|\text{word1}, \text{word2})\dots$$

The model can then be trained by an iterative process: give the model a text up until a certain word, ask it to compute the discrete probability distribution of all possible words that come next, then modify the parameters of the model so that it assigns a slightly greater probability to the word that did come next. In other words, the model becomes *less surprised* by the sequences of words that occur in its training documents. *Formally*, the model is trained using stochastic gradient descent to minimize the average cross-entropy loss across every token (word, subword, punctuation, etc.) in the training data, one small batch at a time; see [4] for details.

2.3 Fine-Tune a Foundation Model

However, few developers train models from scratch. Instead, they download a large pre-trained “foundation” model, then *fine-tune* that model for their specific domain or task, if necessary. Most pre-trained models are released by large organizations such as Google, Microsoft, or NVIDIA, but some have been trained by academic and independent research groups.⁵ Fine-tuning can be done either by further training the model on new data or by crafting ways that the existing model can be queried in ways that encourage certain types of behavior (such as “prefix tuning” or “prompt engineering”).⁶ For example, the joke explanation of Figure 1 was created by prepending to the given joke two examples of a joke followed by an explanation, and appending “Explanation:” to the prompt; this *prompt engineering* caused the model to generate words that would fit the pattern, i.e., an explanation of the preceding joke. Even though joke explanation was not explicitly anticipated in its data collection or training process, the corpus of text that the model was trained on presumably had sufficient examples of jokes followed by explanations that the model learned to utilize this pattern.

2.4 Query the Model Interactively

Interactive interfaces developed on top of these mathematical models tend to align with the task that the model is trained to do. Since the underlying mathematical models are based on modeling the distribution of existing examples, the interfaces built on them tend to offer samples from that distribution. For example, for systems powered by text models trained to predict the next word, the output presented by the system is typically a plausible continuation of a given text. These continuations may be presented as individual words (for smartphone keyboards), next phrases (for

⁴Encoder-only models like BERT work slightly differently; they only need to assign a probability to *parts* of a text; they perform a “fill-in-the-blanks” task.

⁵Examples include <https://bigscience.huggingface.co/> and <https://www.eleuther.ai/>.

⁶Federated Learning can be used to incorporate private data while limiting the degree to which the learned model can memorize private data.

systems like Google’s Smart Compose), or larger units like paragraphs. (Some models are instead trained as *translators* or *summarizers*, sometimes called *seq2seq* or sequence-to-sequence models; although it may seem fundamentally different, this setting can almost always be mapped to a next-word prediction task with a suitable prefix.)

One of the ways that generative systems can support people is as a creative partner, in a setting known as human-AI co-creativity [5–7]. These systems can support people in various ways. Digital artists can iteratively explore and refine creative concepts by exploring different prompts and partial images they give to systems like DALL-E or Stable Diffusion (see example in Figure 2).⁷ Creative writers can collaborate with large language models like GPT-3 to explore new ideas, such as names for characters in a story [8, 9].

Generative systems can also support people through facilitating repetitive or predictable tasks. For example, systems can generate fluent emails given a rough summary of what to communicate, generate blog posts about desired topics, or summarize long articles.⁸

3 Implications of How Generative AIs Work

3.1 Since They’re Based on Data...

Since generative AI systems are entirely based on data, the availability of training data powers the behavior of the system. Thus, the system needs no explicit instructions about any of the specific tasks it is to perform, in contrast to decades of manual effort spent towards manually engineering domain-specific features or laboriously encoding expert knowledge.

However, the data also *limits* the behavior of the system. For example, data, at best, reflects the world as it *is*, which implicitly allows systems to stereotype on the basis of categories like race, gender, or religion. And data is rarely a faithful representation of the world as it is; datasets include biases of selection, representation, aggregation, and evaluation,⁹ which lead to biases in system behavior. For example, language models may associate Islam with terrorism and Black people with crime.¹⁰ When system behavior is dictated by data, system designers have limited ability to embed in the system any notion of the way the world *should* be, such as discouraging such stereotypes.

Since current approaches require enormous datasets, organizations with more data of sufficient quality can train better models. For example, Google’s sibling DeepMind trained a high-performing model on 1.7 trillion tokens, including a large collection of “books,” presumably from Google’s large collection of scans. An open-source dataset called The Pile [10], in contrast, contains a smaller collection of books, which will necessarily limit the capabilities of systems trained on it.

3.2 Since They’re Trained to Minimize Surprise on that Data...

For the mathematically inclined, it should be refreshing and even encouraging to see that the systems that generate seemingly impressive results are, at their core, college-level calculus (the

⁷See, for example, the DALL-E Prompt Book, <https://dallery.gallery/the-dalle-2-prompt-book/>

⁸Products utilizing these approaches include Flowrite and Jarvis.

⁹Many systematic overviews are available for the interested reader, for example, the [fast.ai Data Ethics online course](#)

¹⁰O’Sullivan, Liz; Dickerson, John P. (August 7, 2020). “[Here are a few ways GPT-3 can go wrong](#)”

chain rule), linear algebra (matrix multiplication), probability (conditioning on discrete events), and numerical methods (gradient descent) applied to large amounts of data. The functions employed have complex behavior but are composed of simple primitives; for example, the so-called Self-Attention module at the core of the popular Transformer model uses little more than matrix multiplication, element-wise exponentiation, and dividing a vector by the sum of its elements.

The word-by-word (“autoregressive”) nature of language models, which may be key to their success, aligns with how God made the world: step by step. He could have spoken the world into being in a single act, but instead he constructed it in methodical sequence, one aspect at a time. Human co-creators likewise generally construct step by step, refining partial ideas.

Since the model’s performance is measured on averages, the model can perform badly in uncommon situations without its creators noticing. Thus, parameter settings that cause the model to work well in common situations may have unintentional adverse effects [11].

3.3 Since Many Systems Leverage Big retrained Models...

The availability of large foundation models has presented many new opportunities by allowing practitioners to obtain good performance without extensive training data, but researchers have urged caution in using them because any models based on a foundation model inherit its flaws, which in most cases are poorly understood [12]. The use of foundation models also contributes to *homogenization* of AI system behavior, making many different systems all prone to exhibit similar flaws and biases. It also further disconnects those who use the model from the original authors of the work (text, images, etc.) that the algorithm was trained on, making it more difficult to practice *thankfulness* to the authors of the work.

3.4 Since They Usually Show People Generated Samples...

Efficient text entry can be desirable: it takes some of the tedium out of entering text. Predictive text interfaces are one way of doing this, but contemporary interaction techniques only work for the most highly predictable text, so it tends to nudge writers towards entering such text. Decoder-based approaches such as gesture typing [13], auto correction [14] and speech recognition post-editing work for a much broader range of predictability, but fail abruptly for the least predictable text (such as names), forcing users to make awkward transitions to different interaction techniques. New interfaces for text entry are regularly presented at conferences on human-computer interaction, but creative new interaction techniques are still needed in text entry.

Since interactive systems tend to offer samples from a distribution of existing examples, they tend to encourage users of those systems to create things that are likewise comparable to existing examples. The “groove” for the technology becomes a “groove” for the users: it will be easiest to say things that have been said, to continue in the tracks forged by others.

For example, my own studies of writers found that word and phrase predictions, though typically presented as simply ways to enter text more easily, can nudge writers to write content that conforms to the expectations of the system along aspects of word choice, level of detail, and even sentiment [15–17].

In many cases it’s good for our creations to be grounded in what has already been created. The more unusual or unexpected the words of a message are, the more difficult it is for others to under-

stand. Even God’s communication with humanity has worked this way, using words, analogies, and even patterns of documents (like covenants) that were culturally familiar to the original audience. Although Jesus’s words had unique content and authority, he spoke about himself by referring to the Law and Prophets, a foundation of understanding that many of his listeners shared.

But imitation of past examples can become excessive. Predictive text suggestions can lead us to communicate platitudes or even seemingly thoughtful statements without thinking: consider a system suggesting “I hope you have a wonderful weekend” at the end of a message, or “How adorable!” in response to a friend sharing a baby picture. Our words don’t just communicate information; they communicate human relationship; acknowledging the Image of the speaking God in the other. Generative AI systems have become, alongside social media and other developments, more ways that we are able to mimic the external signs of thoughtfulness and care without having exhibited those virtues.

When aiming to maximize use, mimicry looks like success (more use of the suggestions). So although systems like Google’s Smart Compose might have stated aims to increase user satisfaction, some of the easiest metrics (such as number of suggestions accepted or number of words inserted) could be maximized by finding how to nudge people into following the system’s lead more.

3.5 Is Mimicry Good?

We have seen that mimicry is woven throughout the design and development of generative AI systems. How should we evaluate mimicry? A consideration of the Biblical narrative shows that we are encouraged to practice mimicry, but with discernment.

Imitation is not the primary thrust of the Biblical narrative; more focus is given to accurate perception of and action toward God and others. Nevertheless, the Biblical narrative presents humans as mimics from the very beginning. The first one to speak in Genesis is God the creator; when man the creature finally spoke in Genesis 2, verses 19 (in naming every living creature) and 23 (in delighting in his companion), he echoed the words of God, who had previously given names to things and called them good.

The Bible authors commend imitation of God and of wise behavior. For example, Proverbs encourages imitating the diligence of ants (Proverbs 6:6) and to walk in the paths of wisdom. Paul frequently gives himself as an example to imitate (e.g., 1 Cor 4:16, 11:1, 1 Thes 1:6, 2 Thes 3:7-12). Other epistles tell their readers to imitate those who are faithful and righteous (3 John 11, Hebrews 6:12). The ultimate goal of all of these commands is that we imitate God (Eph 5:1, 1 Cor 11:1).

However, imitation must be done with discernment. Even imitating God’s actions might not be wise, as illustrated by the Egyptian magicians in response to the signs that Moses used to show God’s authority over the oppression of that country’s leader. They are somehow able to mimic the staff-to-snake, water-to-blood, and frog signs—but their mimicry only worsened the trouble for their country (more frogs!); they were not able to *reverse* the plague. Eventually the magicians concede that the signs are “the finger of God” (Exodus 8:19). More commonly, the people of Israel are frequently tempted to imitate the ungodly practices of other people groups, such as those that inhabited the land of Canaan (e.g., Deut 18:9). We could view the entire Pentateuch as an extended reflection on what to imitate and what not to imitate, towards the goal of creating a place and community on Earth that is an image of God’s heavenly dwelling.

3.6 Since the Systems Hide all the Nuance Behind Push-button Interfaces...

Beyond text entry, generative AI models are also very frequently used for *inspiring* writers. But they do so in ways that hide and disconnect us from the *people* whose ideas and effort the system is leveraging. For example, OpenAI provides a “playground” interface for its customers to interact with GPT-3. This interface presents the system’s generation as a magic writing demon, including showing the generation happening one word at a time as if it were a “ghost in the machine”. But in fact these words are generated using patterns that the model learned from reading many examples of other people’s work. Issues of copyright have been raised about this system,¹¹ and more prominently with image generation system such as OpenAI’s DALL-E or Stable Diffusion¹² In the case of image generation, researchers have found that generative image models would even learn to add watermarks like Shutterstock’s to their generated images; current systems avoid this behavior but the creators of both academic or industrial image generation systems have not thoroughly engaged the questions of attribution that arise. Christians might articulate that we want to be able to practice *gratitude* towards the people whose work our work is based on. The task of *example-based explanation* has seen some study in classification tasks [18, 19], but has not been sufficiently explored for generative AI systems.

Educators may rightly be concerned about the availability of generative AI systems, especially for writing both natural language and programming language, as these systems could be used to generate assignment solutions of reasonable quality that do not reflect the student’s understanding or effort. For example, language models can generate acceptable and diverse solutions to common programming exercises that do not trigger suspicion from plagiarism detection tools [20]. Full discussion of this issue is beyond the scope of this article, but in short, the situation is similar to that posed by the introduction of graphing calculators and computer algebra systems in mathematics instruction.

3.7 A Brief Normative Reflection on Generative AI

One line of Christian scholarship about technology is organized around Christian philosopher Herman Dooyeweerd’s “modal aspects,” which help avoid reductionism by prompting reflection along a wide range of aspects that cannot be reduced to each other. For many aspects, Christian scholars have identified corresponding *design norms* that reflect implications of a biblical worldview in each of these aspects [21]. A comprehensive consideration of the design norms applied to generative AI is beyond the scope of this paper, but I will mention a few observations and questions that we might raise based on some of these norms.

- *cultural norm*: What effect will generative models have on textual and visual media (and so on culture) overall? As generating text becomes cheaper, will the value of text further decrease? As increasing amounts of the text we read and images we see are generated at least partially by models that are trained to mimic the status quo, will our collective thinking be broadened or narrowed? What effects could these systems have on how people with differing literacy or language proficiency interact with language?
- *clarity norm*: Language has long been used to mislead, but will large language models increase the scale or effectiveness of disinformation?¹³ On the other hand, might better language

¹¹See [GPT-3 – Copyright Law and Power - Dr Michael Guihot - YouTube](#)

¹²See [DALL-E works with the images of creators who do not receive anything in return: what copyright says about AI - Digis Mak and Is DALL-E's art borrowed or stolen? | Engadget](#).

¹³OpenAI recently published a report from a workshop they convened on the topic: <https://openai.com/blog/forecasting-misuse/>

technology give us better tools for finding clarity through summarization, better search, etc.?

- *social norm*: Text that is kind and contextually appropriate has served as a signal of care and thoughtfulness in written communication. But if those signals can be generated automatically, will they lose their meaning? If the people we’re communicating with suspect that our text is computer-generated (even if it is not), will they trust us less? The nascent field of AI-Mediated Communication studies (AI-MC) explores these and related questions.
- *stewardship norm*: With computation and storage demands of models only increasing, will generative AI technology waste energy? (Historical precedent suggests that power-efficient implementations will come.) How will we curate quality training data?
- *harmony norm* (or *beauty norm*): How will our cultural sensibilities of what is beautiful and valuable change as it becomes easier to produce artifacts that we once deemed as high-quality? How can tools work harmoniously with the human process of writing, rather than intruding on it?
- *justice norm*: Will the benefits of AI technology be distributed equitably? Will those whose jobs are transformed or replaced by AI advances be equipped to shift to different work? As these systems become used in sense-making and decision-making contexts, will they amplify, perpetuate, or even create biases towards or against some groups of people?
- *trust norm*: Of what sort of trust can the outputs of large language models be worthy? How can we assess their reliability?

4 Redeeming Artificial Parrots

Although many applications of generative AI can be positive, the general trend towards better mimicry leads to concern about the future impacts of these systems. But rather than abandon these systems, I encourage a “swords into plowshares” (Isaiah 2:3-5) approach, where we re-mold existing systems to serve new goals. In particular, we can leverage the extensive research and computational effort that has gone into large pre-trained language models such as GPT-3. Although these models were initially trained for mimicry and are often used to substitute for human thinking, they can be repurposed to empower and encourage human thinking. I discuss two potential approaches below and provide examples from my own research.



Figure 2: Example of creative use of a modern AI system that can generate or refine images based on textual descriptions. A full description of the artist’s process is given at the source: <https://andys.page/posts/how-to-draw/>.

4.1 Empowering Writers with Structural Editing and Visualization Tools

One approach to redeeming generative models is to use them to power *editing* and *visualization* tools that give creators more powerful tools to understand and reshape their work. These tools present the opportunity for writers to remold and reshape their work without ceding authorship of any part of that work to an AI system.

Low-level revision tools, typically for grammatical error correction [22] or contextual spelling correction, typically suggest edits that can be accepted or rejected, but the edits are typically at a small scale, which helpfully constrains their potential influence on the content and makes it easy for writers to critically evaluate the suggestion. Other interfaces allow exploration of alternatives to single words using a contextual thesaurus [23].

Other tools allow manipulation at a broader level, but at the cost of writers' control. For example, a system can generate paraphrases of a writer's sentence, perhaps to match a desired style [24], but writers' control over these paraphrases is typically indirect. Still others provide indirect controls over the system's edits [25, 26], but we seek direct control. Systems like OREO [27] can flexibly reword selected spans of text to better achieve a writer's goal, although the specification of that goal is difficult.

In contrast, Dang et al. consider providing summaries of each paragraph in the document margin [28]. Writers found that these summaries provided a perspective on their writing that helped them revise. Although the system generates text, it does so in the process of summarizing the writer's own work. Compared with approaches that complete partial texts, this approach reduces the risk of the system hallucinating information or making its own point. Nevertheless, writers still observed a difference between their own writing and the style of the generated text, and sometimes adapted the summaries into revisions of their own work. Dong et al. deliberately designed their interface to avoid implying that the system's outputs were suggestions to be accepted. That choice reduces the coercion for a writer to appropriate the system's words as their own.

The approach of developing AI-powered editing tools has been more common in visual domains than text. Adobe has recently released Neural Filters for Photoshop that give visual artists a new palette of brushes and sliders to shape their work, leveraging familiar interaction patterns in a new way. While the result is clearly artificial, the process utilizes generative AI models in ways that keep much of the control of the result in the hands of the artist. However, some current work in image editing does so in ways that leave substantial creative direction to the system rather than the artist. For example, GANPaint [29] gives artists "brushes" that cause the model to make localized edits to the brushed area of the image, to hallucinate the presence of some desired attribute (such as clouds or trees). Although the approach took a major step towards providing more direct control over the otherwise black-box process of image generation, further work was needed to give artists more explicit control over how the system enacts the desired edits. As text understanding and image generation both become more powerful, opportunities emerge to give more precise editing instructions, for example as demonstrated by InstructPix2Pix [30]. However, the same advances allow artists to give even higher-level instructions of what edits to make, leaving the system with considerable latitude in interpreting the instruction, which may reduce the creative agency of the artist.

To further explore interaction techniques and technical approaches for structural revision of documents, several undergraduate students and I built several prototypes for revising structure at the sentence level. Figure 4 shows a proposed direct manipulation interaction: the writer drags a

selected phrase to a new location and the system performs the necessary edits on the rest of the sentence (highlighted in green in the figure). However, in this interaction it is not obvious to the writer which manipulations are likely to be successful, so we also explored interactions in which the system presents several plausible reorderings that the writer can choose among and then refine using further selection operations (Figure 3). We are currently working to extend these ideas to help writers revise the structure and flow of wider areas of documents.

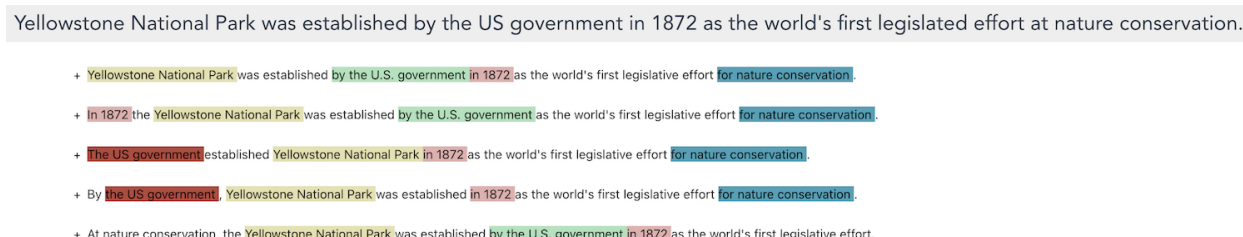


Figure 3: A selection interaction for clause reordering within a given sentence. The example (unedited output of a prototype system by Calvin students April Volzer and Noah Madrid) shows five different possibilities for which phrase opens the sentence. Phrases are given colored backgrounds to visualize the relationships between the alternatives. Each alternative can be expanded to show alternative completions of that phrase.

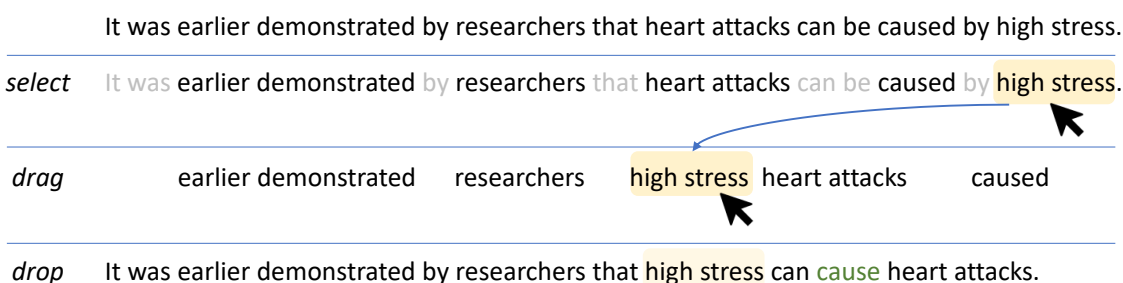


Figure 4: Mock-up of an example direct manipulation interaction for rearranging a sentence by dragging and dropping a selected phrase. During dragging, non-content words are dimmed. Dropping the phrase constrains its relative position; green highlights indicate other words that the language model needed to modify or move.

4.2 Inspiring Writers by Asking Questions

Language models have learned patterns that characterize writing; how can human writers leverage that knowledge without the model writing for them? One approach my collaborators and I have explored is having the model generate *questions* for the writer, rather than text that they might appropriate themselves.

In one exploratory study, we compared the approach of giving guidance in the form of content questions (“Questions”) with two alternatives: no guidance, and exemplar sentences (“Snippets”) from high-quality related documents. We designed a scenario for which writers would need to write isolated sentences in encyclopedia-style writing with optional guidance from a “bot.” Results of the study, summarized in Figure 5, suggested that while both types of guidance helped productivity compared to the baseline, writers found prompts shown as Questions gave them usable ideas more

often than the same semantic content shown as Snippets. Writers also expressed strong preference for Questions over Snippets presentation. More details about the study can be found in our workshop paper [31].

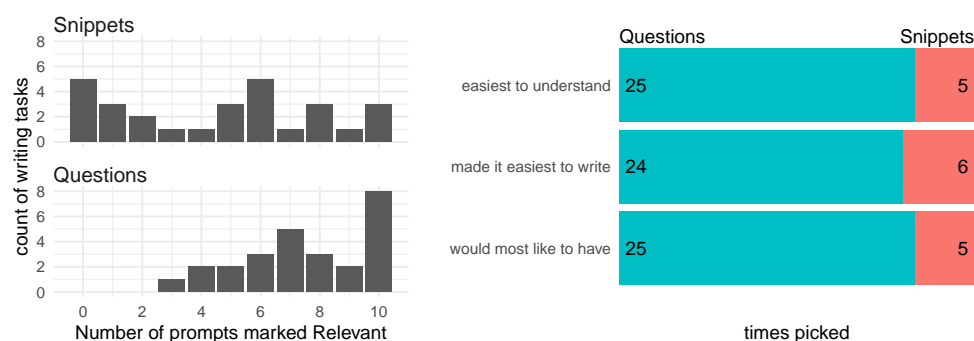


Figure 5: Left: Prompts presented as Questions were more often marked as relevant by participants. Right: Participants chose the Questions prompt as most preferable along all three measures asked.

We have also undertaken formative research (conducted Summer 2022, not yet published) to understand what sort of questions would be helpful to writers at various stages in their writing process (we contrasted drafting stage vs revision stage) and what sorts of interaction techniques might be useful to provide questions to writers. Based on a Wizard-of-Oz style study where participants wrote argumentative essays and other participants wrote questions for them, we identified categories of questions that could be helpful (typically those we classified as Open-Ended, Elaboration, and Structure), and found that useful questions were typically grounded in the existing text, without introducing many external ideas. We found that presenting these questions as word processor comments, like a human reviewer might leave, was perceived as helpful without being overly intrusive. Preliminary explorations of providing our participants’ writings to language models encouraged us that we should be able to get language models to generate these sorts of questions.

5 Conclusion

Generative AI systems such as large language models are powerful mimics. Uncritical use of them risks pressuring humans into appropriating the mimic’s words as our own. But people can choose to use them wisely, both as end users and as software developers.

As users, some people have chosen to disable predictive text on their phones and computer systems; others have chosen to leave them enabled but be thoughtful about when to look at and accept suggestions. When the system suggests a generic pleasantry, a writer could instead choose to write something more specific.

As developers of AI systems and of interactive applications that utilize them, we can consider how to mold these systems. Here are a few questions we might ask:

- How can a person’s interaction with this system encourage their creativity and individuality?
- Are there ways to help writers celebrate the thought and effort that they have put into their work, instead of just trying to reduce thought and effort?

- What sort of writing is the system encouraging? Does the system make it easier or harder for that writing to love and serve the reader?
- How can an AI system highlight the contributions of the people whose work became the system's training data? Can it help people be thankful to those people? Can it give them credit?

I have discussed two ways of repurposing generative AI systems towards a vision of empowering writers. But many other ways of repurposing these systems are also possible. For example, I have informally started to explore using language models like GPT-3 to help me come up with ideas for educational materials that are highly contextualized to the needs of my class but also leveraging the insights of educators who have tried to teach similar materials before. Also, these models are typically used to attempt to model expert-level understanding of language or code; could they be repurposed to simulate a student's developing understanding of a concept or skill, at a sufficiently high level of detail that an educator could get useful feedback on what their students individually understand and how they will respond to potential examples?

It is an exciting time to be working in interactive artificial intelligence. I hope that this article inspires some Christians to work towards shaping this developing technology.

6 Appendix

All text in italics was generated by the model `text-davinci-003`, accessed via the OpenAI API.

Title: Some tough questions at the intersection of generative AI and Christian faith. Author: GPT-3

1. How does generative AI challenge traditional Christian understandings of human creativity and free will?

Generative AI presents a challenge to traditional Christian understandings of human creativity and free will by introducing a new kind of artificial intelligence that has the ability to create original content. Generative AI can generate text, images, music, and more, and it can do so without the need for a human programmer or artist. This raises the question of whether humans can still be considered the sole creators of their work and whether they have full control over their creative processes. It also raises questions of whether the use of generative AI can be ethically justified, particularly with respect to religious beliefs.

2. How does generative AI shape our understanding of our relationship with God?

Generative AI can reshape our understanding of our relationship with God by introducing a new way of expressing our creativity. While traditional Christian faith has emphasized the importance of humans as creative agents, generative AI can enable us to create new forms of art and expression that may be seen as a form of prayer or worship. In addition, generative AI can allow us to explore our relationship with God in ways that were not possible before. By allowing us to explore the boundaries of creativity, generative AI can open up new avenues for spiritual reflection and growth.

3. Are there ethical considerations that must be taken into account when using generative AI?

Yes, there are many ethical considerations that must be taken into account when using generative AI.

These include questions of authorship and ownership, privacy, and the potential misuse of generative AI for malicious purposes. Additionally, ethical considerations must be taken into account when considering the impact of generative AI on human creativity, freedom, and the potential for the proliferation of “fake news” and other forms of misinformation.

4. How might Christians use generative AI to further their faith?

Christians can use generative AI to further their faith in a variety of ways. For example, Christians can use generative AI to create new forms of art and expression that can be used to explore and express their spiritual beliefs. Generative AI can also be used to generate new kinds of sermons and reflections that can be used to explore and deepen faith. Additionally, generative AI can be used to generate new types of music, poetry, and other forms of expression that can be used to explore and express faith.

Explain generative AI.

Generative AI is a type of AI technology which attempts to generate new data based on existing data sets. For example, generative AI could be used to create new images, music, or text based on existing data sets. Generative AI can be used to create novel and creative outputs, as well as help in the development of more complex AI systems by providing additional training data.

Translate into pop song lyrics.

*Verse 1: Generative AI, a technology so new It can generate data, create something brand new
Create images, music, and text so fine It can help create a complex AI design*

*Chorus: Generative AI, it's the way to go It can create something special, you just gotta know
Generative AI, it's the wave of the future It will help create something brand new, you'll be sure*

*Verse 2: Generative AI, it's the way to go It can help us create something we never thought we
could know Novel and creative, yeah it's the thing It'll help us create a whole new kind of thing*

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The Great Integration

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Abstract

When considering how math and faith integrate, Christians and non-Christians alike can easily view the two concepts as fairly independent of each other. What should mathematics as a whole look like for a Christian? We can use God’s revelations about His broader purposes to help us seek out godly purposes for math. As Christians who view building the kingdom of God as an ultimate concern, we can ask and hope to answer many questions about integrating our subject and our faith. Math can serve to complement our Christian walks as we obey the Great Commandment, the Great Commission, and the Great Requirement.

1 Introduction

When considering how math and faith integrate, it’s quite natural to think of the two concepts as fairly independent of each other. Indeed, the material I teach in my various classes does not change based on one’s worldview. The statement “ $4 + 3 = 7$ ” is true for Christians and non-Christians alike. A Christian’s ability to do Calculus does not rely on incorporating nor ignoring a Christian worldview to arrive at suitable solutions that agree with those of non-Christian colleagues. So what should math look like for a Christian? Do we get a free pass to do our work in the same way as every other mathematician in the world? Are we free to leave our Christian walk at the door when working in the world of math? If we bring our Christianity with us to our work, does it make a difference?

While mathematical truths do not appear to influence nor be influenced by a Christian worldview, that does not mean that math and faith are wholly independent. What are God’s purposes for mathematics? We would be hard-pressed to find direct statements in the Bible about God’s purposes for math, but He has revealed through scripture His broader purposes. We can seek out godly purposes for math based on these revelations [3]. Math is a tool that we can use for good (or not), and it can be used effectively to help us obey what we can call the *Great Commandment* (Matthew 22:37), the *Great Commission* (Matthew 28:18-20), and the *Great Requirement* (Micah 6:8) [1].

2 The Great Commandment

And He said to him, “You shall love the Lord your God with all your heart, and with all your soul, and with all your mind.” Matthew 22:37 (NASB 1995)

We are commanded to love the Lord our God with all our heart, soul, and mind. What does this look like? How do we as Christian mathematicians follow such a command?

We all have gifts and talents from God, and we all have the ability to do mathematics. As Francis Su highlights, we can all embrace the “wonder, power, and responsibility of mathematics by nourishing our affection for it.” He continues on, likening a society lacking mathematical affection to a city without concerts, parks, or museums. He makes a well-supported point that math is for human flourishing, referring to a “wholeness of being and doing, of realizing one’s potential and helping others do the same, of acting with honor and integrity even in challenging circumstances” [6]. We were designed as curious beings. When we learn any new mathematical concept, we can play with it in our minds. To learn more about God and His nature is an act of worship. Anyone with the capacity to do math has both the privilege and the obligation to do so. As we work with mathematics, we should do so with joy, gratitude, and confidence in knowing we are fulfilling God’s purposes for us. Christianity emphasizes God’s character, especially His love and value for His creation and all aspects of it, including math. When studying math, we as Christians can know we are studying something that God values [3].

We can know more about God and His nature through the beauty and truth in math. God uses his creation to reflect aspects of who He is. According to Howell and Bradley, “Mathematical truths are necessary, unchangeable, and universal. God in His wisdom made use of mathematical ideas in structuring the world at the beginning of time. Humans can study the great book of nature to appreciate the order and beauty of God’s handiwork” [3]. Many find beauty in the elegance of the myriad patterns and simple formulas that occur quite naturally in mathematics at large and specifically in the study of creation. This beauty inspires feelings of harmony, balance, and awe while spurring us on to further investigations; it sparks curiosity and a search for more insight—even to the point of experiencing something that goes beyond a specific object or idea to point to greater truth, insight, connections, or significance. Why should math be as effective as it is to explain so many things in the world? Why should there be mathematical results that connect disparate fields of study? As we investigate instances of deeper beauty and associated mathematical connections, it is very natural to think we are being pointed to some deeper truth, to some Creator and Sustainer of all things. As we seek truth in math, motivated by its beauty or inspired otherwise, we learn more about the nature of God and His truth while also developing important traits. The search for deep knowledge in math helps us to learn to think for ourselves. We can discern when answers make sense, not having to rely on blind trust in authority. We hone our skills of thinking rigorously, handling ideas well, and forming sound arguments. Thinking rigorously helps us to know the limits of our arguments, avoiding over generalization. Learning the limits of our arguments benefits our quest for truth to develop intellectual humility—admitting errors in our arguments and realizing how much more we have yet to learn—and to build confidence in the truth. “The more you explore the world of mathematics, the more confidence you have in loving the truth” [6]. Virtues that result from seeking beauty and truth in math are invaluable assets to Christians.

In following the command to love God, we should be wise stewards of all that He has created. We need to study and understand how various parts within the whole of creation work and interact with each other. Through our investigations, we inevitably find seemingly countless similarities and patterns. Su states that “mathematics is the science of patterns and the art of engaging the meaning of those patterns” [6]. The ability to do mathematics should be viewed as a gift from God. Through it, we see revelations about God’s nature, such as His subtlety, order, beauty, and variety. We find intimate connections between math and science, among other fields, that highlight the coherence of our Creator. As we enrich our mathematical knowledge of how nature behaves, we can better serve God’s purposes in the world, helping us to fulfill the cultural mandate we have been entrusted with. Math assists our understanding of reality while a belief in creation and a spiritual reality puts the study of the natural world in proper context. When we respond to revelations in

mathematics with awe and joy, revering and thanking God, we are fulfilling a purpose God has for us—to be co-creators with Him as stewards of the world, helping to build cultures and to care for this world [3].

As we work with math, we can even find reassurance for our faith. Math is known to have paradoxes that defy intuition. Additionally, Gödel’s Incompleteness Theorems prove that it is impossible for a system that can express basic arithmetic to establish a set of foundational statements (axioms) that is both complete and consistent. Aspects of what appear to be reasonable, tangible, and incredibly useful axiomatic systems still require faith [2]. If we are unable to fully trust our intuition when it comes to the whole of math, which is generally viewed as incredibly logical and reasonable, how much should we be surprised that to finite human minds we are unable to form a picture of God that is simultaneously complete, consistent, and paradox-free? While this could seem troubling, it can be very reassuring. In no way should an inability to work out for oneself a fully clear, consistent, and complete view of God and His ways detract from faith in Him; faith in God is not irrational. Rationally justified beliefs exist in many areas outside of math and science that cannot be addressed with mathematical proof or the scientific methods [5]. Paradoxes to the human mind are not a roadblock to Christian faith. How could Jesus be fully God and fully man? How does an all-loving God allow evil to exist in the world? How can we resolve Biblical passages that seem to show support for predestination with other passages that seem to support free will in salvation [3]? My lack of clear, consistent, and complete answers here are not indicative of a misplaced faith in God any more than not knowing why electrons can behave like both waves and particles or a lack of intuition about the Banach-Tarski paradox (which states that a single ball can be decomposed into a finite number of disjoint sets which can then be reassembled to form two identical copies of the first ball) indicate misplaced faith in scientific and mathematical results. If mathematics, a carefully defined and logically precise system, contains paradoxical ideas, we should not be filled with angst when we encounter paradoxes in our faith [1].

Math is a creative field. Math is not in need of math memorizers, but instead math explorers. As a math instructor I try to emphasize understanding and asking “what if” over rote memorization of steps that often lack important and flexible insights. Math memorizers cannot react well in unfamiliar situations whereas those who have taken opportunities to explore, ask questions, and play creatively with math can be flexible to adapt to changing conditions. When we, for example, hone our skills of deriving formulas to work in a variety of situations, we can use this learned adaptable thinking to write flexible computer programs to handle many types of inputs or to design products that work for many different people and situations [6]. The freedom we have to explore in math is a vital aspect of our creativity as humans and fulfills a calling toward stewardship.

What can math do for students, even those who supposedly “just aren’t math people”? Letting oneself explore the world of math cultivates important virtues—a vital way of showing our love for God. Exploring math builds confidence in struggle that gets easier to endure over time, thus building perseverance, similar to building up muscles and practicing skills in anticipation of the next sports competition. Even if solutions elude us, we can learn important skills and habits as we struggle and persevere. Exploring math helps strengthen our ability to change perspectives to solve problems as we analyze from many viewpoints. If we share in the struggle, perseverance, and delight of working with others, we also build community with an open spirit in which we see others in a better way [6]. Cultivating these virtues in the world of mathematics allows for easy transfer to our lives at large. We can be better prepared and perhaps find more success as we respond to God’s call in our lives if we have practiced our abilities to struggle, persevere, change perspectives, and build a community with an open spirit.

3 The Great Commission

And Jesus came up and spoke to them, saying, “All authority has been given to Me in heaven and on earth. Go, therefore, and make disciples of all the nations, baptizing them in the name of the Father and the Son and the Holy Spirit, teaching them to follow all that I commanded you: and lo, I am with you always, to the end of the age.” Matthew 28:18-20 (NASB 1995)

We are created with the capacity to do mathematics. How do we use this gift in alignment with God’s will for us?

We as Christians are called to spread the Gospel throughout the nations. That may reasonably conjure images of traveling to foreign countries as a missionary, though it may easily mean seeking out missional or volunteer opportunities without going very far. Whether a part of a mission group, a church, or working more independently, the logistics involved must be managed, often with limited resources. How do we maximize use of resources while minimizing costs? A variety of areas in math are concerned with problem solving and decision making. God appreciates any efforts we put forth to share His love and Gospel with those around us, but how much more valuable are our efforts if we have the capacity to utilize our mathematical resources to extend our reach? And how much will it damage our witness should others see our squandering resources and opportunities that could have been put to good use with just a little effort toward problem solving?

Reaching others for Christ will inevitably involve problem solving. Our training as mathematicians can facilitate a successful spreading of the Gospel. If we have properly prepared ourselves as math explorers, we need not become frustrated when we encounter adverse situations. Rather, as a result of our training, we can demonstrate power over problems by constructing possible strategies for obtaining solutions. If we have practiced an ability to understand an idea or problem multiple ways, we have the freedom and power to choose the view that maximizes our efficiency in solving the problem [6]. While spreading the Gospel, we should aim to be wise stewards of what has been entrusted to us. How tragic is it to miss out on reaching those we could have reached had we not been ignorant or wasteful?

Additionally, we as Christian mathematicians are part of a greater community of mathematicians, many of whom are not Christians. Perhaps we can use our view of rationality and harmony between our faith and our discipline to more effectively reach those within our discipline, including our students, who may not be as open to arguments and reasoning from those outside our discipline. Furthermore, as we work with students, we have the obligation to be good stewards of our time with them, seeking to glorify God by teaching to the best of our abilities. We have valuable opportunities to mentor and build relationships with our students. Through these relationships, we can encourage and strengthen the walks of our Christian students as we also aim to point our non-Christian students toward a relationship with Christ. While there are no doubt many reasons God has led us along the paths we have each taken, who knows whether we have come to our positions for such opportunities as these?

4 The Great Requirement

He has told you, O man, what is good; and what does the Lord require of you but to do justice, to love kindness, and to walk humbly with your God? Micah 6:8 (NASB 1995)

What does it look like to do justice, to love kindness and mercy, and to walk humbly with God in

the realm of mathematics?

I like to show my students examples of misleading graphs and poorly supported conclusions. A popular book on statistics, *How to Lie with Statistics* by Darrell Huff, provides a crash course in analyzing statistical articles, along with many valuable and relevant reminders. While a book like this is not explicitly a Christian math book, it does remind us of the responsibility good citizens have when ingesting and disseminating statistical results (and we as Christians should be good citizens as long as there is no conflict with God's will). Huff includes a paraphrased quote by H. G. Wells: "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write." It's good to remind ourselves that despite numbers or a graph supporting some conclusion, we need to be educated enough to be aware of implicit biases, attempts to deceive, and other ethical concerns regarding statistical results. We especially as Christians have the obligation to diligently try to keep from being easily deceived, to do our best not to spread misinformation, and to be careful to only make conclusions that are properly supported. While we live in a world filled with attention-grabbing headlines that are often meant to get us to react, we need to analyze to determine what's really being supported, which may often be in opposition to what is being said. Avoiding statistical ignorance helps us to ask questions that matter in reaching conclusions. Have we been provided with sample size? How about standard error? *P*-value? What pieces of information are missing or glossed over that are actually important? If we are missing mathematically important information, we should seek out source material to get the whole story [4]. We often define and respond to injustice in society based on numerical data in many forms. How can we as Christians reasonably strive for justice in the world around us if we do not have the complete picture, the whole story the data tells us?

What does kindness look like in math? As with any area, the mathematical field is made up of people—people at various stages of engagement with mathematics. And as with any field, we need to show kindness and hospitality to others, perhaps especially to the newcomers, even if a newcomer is a student who just has to take one math course to get a general education requirement over with. These students may not realize it, but they're at the beginning stages of becoming math explorers, potentially heading in new, unanticipated directions. It is important to be an advocate and encourager for everyone in the field, which can be especially important for those feeling marginalized. In my college math class. Su advises us to "believe that you and every person in your life can flourish in mathematics" [6].

As mathematicians, we must remain humble. While math has proved to be a powerful tool, helping us to explain and make predictions about the world, math also has limitations. Seeing creation through a Christian worldview helps us understand the limits of math. To some, mathematical knowledge can become an idol, giving a feeling of certainty in something that can be found independently of God. In fact, many try to "mathematize" the world, to format all aspects and processes of the world by mathematical insights, techniques, and formulas while rejecting non-mathematical aspects of life as unimportant or nonexistent. Math can be found everywhere in our daily lives, and it can bring an order to our existence and enable us to abstract, model, express, predict, and control behavior in areas that initially may seem quite unrelated to math. Conversely, however, math can be used to obscure necessary aspects of situations and to depersonalize processes and relations, including issues of life and death. We can celebrate the appropriate understanding and control in our lives we can find with math while being careful to avoid the desire to view everything quantitatively. The process of mathematizing the world can naturally lead to developing an arrogant worldview that celebrates independence from God and aims to develop predictive mastery over nature and society through knowledge whose certainty is suggested by mathematics, much like

a modern-day Tower of Babel. As Christians, however, we can affirm math as a good gift from God while simultaneously rejecting the idea that math provides dominion over all facets of creation that exist. We can see a rich variety of aspects within creation that cannot be measured quantitatively, and we should welcome other dimensions of reality, those beyond the limits of mathematical description, as adding to a more complete view of the world. Math has its limits; it does not lay sole claim to truth in the world. God's creation has both mathematical and non-mathematical structure [3].

We were created by God and have been gifted with talents and abilities, which we can use to show love to others. As we strive to fulfill the Great Requirement, the study of mathematics can allow us to be more effective in our efforts to bring about peace, wholeness, and harmony in the world. We, especially as Christians, should be mindful of the mathematics we develop and its possible applications. Specifically, we are to use our analytical and quantitative skills wisely, serving others, showing them we see their worth, and caring for all aspects creation. Viewing the world solely through a mathematized lens reduces life merely down to its quantitative aspects, ignoring important social and cultural aspects of reality—those that may exhibit themselves in art, literature, ethics, religion, politics, economics, etc. This can easily be used to sidestep concerns of personal responsibility, ethics, virtue, and caring. Many vital aspects of life are important because of personal, societal, or cultural values, not mathematical values, and, therefore, are not effectively approached through methods that require impersonality, precision, and abstraction. We must be careful to avoid overvaluing means to an end (like productivity and efficiency) or treating them as ends unto themselves while devaluing or overlooking proper end goals, such as human dignity and worth [3]. Mathematics serves an important purpose in the world, but its place is in service and stewardship, not in impersonally dictating how all aspects of reality should function or be viewed. Loving our neighbors and emphasizing their worth involves using mathematics to best serve others, aiming to fulfill their needs with our gifts and talents.

5 Conclusion

In conclusion, much of what we could consider the technical content of math may seem quite independent from a Christian worldview. However, as Christians who view building the kingdom of God as an ultimate concern, we must ask ourselves many questions when it comes to integrating our subject and our faith. Are we using our full talents, abilities, and opportunities to learn about something God has created and values? Are we attempting to learn more about God and His creation through our studies in all areas? How are we using our gifts to reach others, to serve others, and to be wise stewards of creation? What kind of culture are we building and how are we using math to do that? How is our study of math also building up virtues that are important for daily life? Are we avoiding the temptation to treat math as an idol to solve all of the world's ills? Are there ethical or other value-laden concerns we should be considering as we are developing math or applying it to solve problems? These and other relevant questions may not be easy to answer, but they give us important points to consider in how we use math as Christians.

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The Noodle Problem

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Abstract

This “Classroom-capsule” article describes an assignment given in my Multivariate Calculus class. It is based on the following question: Given n randomly sized noodles (in a bowl of soup and eaten with chopsticks), what is the probability that the first one selected is the longest one?

1 Introduction

Many years ago, our family was eating noodle soup in a Chinese restaurant. We were using chopsticks to reach into the soup in search of (hidden) noodles. My son, who was taking AP Calculus at the time, asked, “If the noodles have random lengths and you only have two of them in your bowl, what is the probability of selecting the longer one?” I had no idea at the time that this problem and related questions would become an important part of my multivariate calculus course from that point forward. In this article I share the assignment I give my students.

2 The Assignment

Suppose you are eating noodle soup, trying somewhat unsuccessfully to find and extract noodles out of the soup using chopsticks. The noodles have random lengths (uniform distribution) between 0 and 1 and there are only two noodles left. Your task is to determine the probability of finding the longer one first. Assume that the probability of obtaining a given noodle equals the length of the noodle divided by the sum of the lengths of all (two, in this case) noodles in the bowl.

1. Create a five-by-five grid (25 cells) that partitions the (x, y) points of a unit square. For each of the 25 cells, determine the probability of finding the longer noodle if the lengths of the two noodles correspond to the values of x and y at the midpoint of the given cell. Average these 25 values to estimate the probability in question.
2. Set up and evaluate a double integral that will find the exact probability of extracting the longer of two noodles.
3. What is the average length of the shorter noodle?
4. What is the average length of the longer noodle?
5. What is the average ratio of the length of the short noodle to the length of the long noodle? (This is different from the ratio of the averages of their lengths!)
6. What is the average of the ratio of the length of the long noodle to the length of the short noodle?

7. For a bonus, write a program to answer Questions 2 through 6, using a 20 by 20 grid, with a random pair of lengths in each cell, as we did in class. This will be much more accurate than just selecting 400 pairs of random noodle lengths. Compare the results from two methods.
8. As an additional bonus, repeat Question 2 with three noodles, using a triple integral, and then generalize your approach to n noodles.

3 Discussion of the Solution

I assigned various versions of this problem as projects several times over the years before it reached its present form. Not surprisingly, most students haven't known where to begin, because they had not yet encountered calculus-based probability problems. To help with this, I eventually scaffolded the problem to include Question 1. In addition, we now discuss the problem at length in class prior to my assigning it. I usually give them a week to work on it individually, but it would work well as a group project, too.

To prepare them for Question 1, I created a five by five grid of 25 congruent cells partitioning the unit square. We went around the classroom with students giving me a "random" (x, y) pair in each of the 25 cells. Letting x and y denote the noodle lengths associated with a given cell, we then calculated the probability that the longer noodle would be selected in each case. After doing this activity, students didn't have much problem with Question 1 of the assignment, though some of them took some prompting to realize they should use a probability of $\frac{1}{2}$ when $x = y = 0.5$. Using the midpoint of each cell for (x, y) rather than random values yields the probability

$$\frac{1}{25} \left(\frac{0.1}{0.1 + 0.1} + \frac{0.3}{0.3 + 0.1} + \frac{0.5}{0.5 + 0.1} + \cdots + \frac{0.9}{0.7 + 0.9} + \frac{0.9}{0.9 + 0.9} \right) \approx 0.678.$$

For Question 2, students could easily express the probability as $p(x, y) = \frac{x}{x+y}$ when $x \geq y$ and $p(x, y) = \frac{y}{x+y}$ when $y \geq x$. We had a lot of experience in Calculus II in turning Riemann sums into integrals in a variety of contexts with density functions. We revisited that here, and I explained that in this problem $p(x, y)$ serves as a density function for the probability calculation. However, they hadn't had much practice in integrating piecewise defined functions over a region, so some of them needed guidance in setting up the integral (see below). Of course, it is also helpful to use symmetry to simplify the calculations, especially since the first integral is considerably easier than the second. Without loss of generality, we may assume that the noodle with length x is the longer noodle. We then integrate over the lower right triangular half of the unit square and double the result. (Note: This approach is especially useful when generalizing the problem, as we will see later. However, in hindsight, it takes a high level of mathematical maturity to truly understand the role of symmetry here. For that reason, I recommend requiring students to set up all probabilities as the sum of two integrals, and let them discover the symmetry for themselves, if they are able.)

$$\int_0^1 \int_0^x \frac{x}{x+y} dy dx + \int_0^1 \int_x^1 \frac{y}{x+y} dy dx = 2 \int_0^1 \left(x \ln(x+y) \Big|_0^x \right) dx = 2 \int_0^1 x \ln(2) dx = \ln 2.$$

Questions 3 and 4 are straight-forward but are used to set up the surprise that appears in Question

6. The average lengths of the shorter and longer noodles are

$$\int_0^1 \int_0^x y \, dy \, dx + \int_0^1 \int_x^1 x \, dy \, dx = \frac{1}{3} \quad \text{and} \quad \int_0^1 \int_0^x x \, dy \, dx + \int_0^1 \int_x^1 y \, dy \, dx = \frac{2}{3},$$

respectively. As before, symmetry can be used to rewrite with one integral in each case.

For Question 5, using symmetry, the average ratio of the short and long noodle lengths is $2 \int_0^1 \int_0^x \frac{y}{x} \, dy \, dx$.

While this is an improper integral, it can be shown to converge to $\frac{1}{2}$.

On the other hand, for Question 6 the average ratio of the long and short noodle lengths is $2 \int_0^1 \int_0^x \frac{x}{y} \, dy \, dx$, which clearly diverges.

These latter two results initially seemed paradoxical to those who assumed the values should be reciprocals (using the false belief that the ratio of averages will equal the average of ratios). Those who pursued Question 7 saw first hand the issues with finding averages when some terms are unbounded. It is worth reminding all students that unbounded integrands may or may not diverge, and, consequently, the average value of an unbounded function over a finite interval may or may not exist.

4 Extension

The problem easily generalizes to $n > 2$ noodles. The probability of selecting the longest noodle is

$$P_n = n! \int_R \cdots \int \frac{x_1}{x_1 + x_2 + \cdots + x_n} \, dx_n \cdots dx_2 \, dx_1,$$

where R is the region in n -space consisting of all n -tuples satisfying $0 \leq x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq 1$.

When $n = 3$, we have

$$P_3 = 6 \int_0^1 \int_0^x \int_0^y \frac{x}{x + y + z} \, dz \, dy \, dx = 3 \ln 3 - 4 \ln 2 \approx 0.523.$$

The first ten values of P_n are approximately 1, 0.691, 0.523, 0.418, 0.347, 0.297, 0.259, 0.229, 0.206, and 0.187.

It appears that $P_n \approx \frac{2}{n+1}$. Indeed, on average, the ordered lengths of n noodles will be in the ratio $1 : 2 : 3 : \cdots : n$, in which case the probability of obtaining the longest noodle is $\frac{n}{1+2+\cdots+n} = \frac{2}{n+1}$. Another way to see this is to note that as n increases, the length of the longest noodle tends to 1 and the average length of the other $n - 1$ noodles tends to $1/2$. The desired probability will therefore approach $\frac{1}{1+(n-1)/2} = \frac{2}{n+1}$.

5 Why I like this Problem

1. It is easily understood, which helps students set up a simulation for an approximation, as in Question 1.
2. Students get a lot of experience with partitioning intervals and using numerical integration with single integrals. Questions 1 and 7 of this problem give them an opportunity to revisit these concepts with a double integral. Most of them do need to be shown that averaging the 25 probabilities in Question 1 is tantamount to calculating a Riemann sum with the 25 $p(x, y)\Delta x\Delta y$ values, using $\Delta x = \Delta y = 0.2$.
3. It is a good example of how integration of a piecewise defined function can be simplified by using symmetry to replace two (or more, if $n > 2$) integrals with one integral.
4. Questions 5 and 6 yield similar-looking improper integrals, but only one of them converges.
5. Though it may be a stretch to call this problem a real-world application, it does provide an interesting example of the need for n -dimensional integration.
6. The heuristic methods used to arrive at the asymptotic approximation $P(n) \approx \frac{2}{n+1}$ are based on the expected value of the n^{th} order statistic of a uniform random variable (see [1]). This estimate matches the first ten values of $P(n)$ from the integrals quite well and implies that

$$\int_R \frac{x_1}{x_1 + x_2 + \cdots + x_n} dx_n \cdots dx_2 dx_1$$

is asymptotically equal to $\frac{2}{(n+1)!}$. For fixed n , the exact value of the integral is an elusive linear combination of natural logarithms, so it seems like an interesting problem to find an independent proof of the asymptotic result for the integral.

6 Acknowledgements

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Intentional Community: Fostering Faith . . . Developing Mathematicians

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Abstract

So often students hinder their mathematical development by distancing themselves from the subject. By simply trying to “meet the requirements,” students not only lose the benefits of exploring and struggling and developing logical explanations as mathematicians, but they also deter their trust in God’s plan and opportunity to serve others. By incorporating rituals and patterns to develop intentional community in the classroom, opportunities to both foster faith and to accept a personal mathematical identity are encouraged. This paper shares the observations made by both students and faculty regarding their development in both faith and mathematics as a result of efforts made to establish intentional community in non-major mathematics classrooms. Our goal is to foster virtues that are both mathematical and scriptural as the community works together.

1 Introduction

Like many Christian institutions, Taylor University emphasizes community. With its catch phrase “intentional community,” Taylor University references the many traditions and routines of student

life with less attention on community in the classroom. After reading Francis Su’s book *Mathematics for Human Flourishing* [10] as a department, we wondered if community enters the mathematics classroom too haphazardly. Four of us tried to experiment with specific practices and rituals to develop community in our classes with greater intentionality.

Su suggests that “mathematics is for human flourishing. *Human flourishing* refers to a wholeness – of being and doing, of realizing one’s potential and helping others do the same, of acting with honor and treating others with dignity, of living with integrity even in challenging circumstances.” Ingrained in his thoughts are aspects of community ... “helping others” ... “treating others with dignity.” There are several factors which may challenge this communal atmosphere in the classroom. Chang [2] claims that institutional culture is one of the many factors that influence students engagement in the sciences. Malcolm [7] posits that the culture of STEM education in colleges and universities separates the personal and academic identities of students thereby impacting both their efforts and achievement. Being purposeful about community in the mathematics classroom is an often unrealized element of the development of effective mathematicians. Some of the essential attributes of mathematicians—individuals who explore possibilities, struggle with challenges, and develop logical and meaningful solutions—are developed best in community.

This communal development of mathematical attributes also fosters aspects of faith—encouraging one another, serving one another, and sharing hope with each other. (James 2:14-26 instructs us that our faith is demonstrated in actions which are often related to community.) The shift from “I” to “we” requires intentionality as emphasized throughout the Pauline epistles. So often students hinder their mathematical development by distancing themselves from the subject and those who study it. By simply trying to “meet the requirements,” students minimize the benefits of human flourishing that develop them both mathematically and as a Christian. Can we help students feel that they belong in the mathematics classroom? Can intentional community lead students to a new understanding of the benefits of mathematics and a connection with their faith?

Winterer [11] recommends the development of rituals in the classroom to develop community. “We carefully conceive and employ structures, routines, and rituals we think will positively influence students’ growth mindset, purpose, and belonging. These structures, routines, and rituals in combination with personal connection and shared responsibility help to create a sense of community in the course.” Seemingly “small” social-psychological interventions—typically brief exercises that do not teach academic content but instead target students’ thoughts, feelings, and beliefs in and about school—have had striking effects on educational achievement [12]. The desire is to foster connection not only between students and the instructor, but between students themselves. The idea is that emotional healthiness allows one to focus more on the class material [6]. Students are often hindered in their learning by being concerned about how other students perceive them. In mathematics especially, students often feel that they do not meet the perceived stereotype (or do not desire to meet the perceived stereotype) of what a “good” mathematician is. Providing an opportunity to dispel the stereotypes by allowing students to get to know each other and the instructor better opens opportunities where students may feel freer to ask question without feeling stupid.

A concept that connects with this emphasis on intentional community is “belonging.” While belonging has potentially different meanings from feeling that one is academically prepared for the course to feeling that one is not “left out” of the culture of the classroom, we took belonging to mean that a student feels seen (recognized for their strengths and contributions [9]) and a part of the community. Belonging, in this sense, is often very difficult, especially in required foundational

or remedial classes. A recent study [5] notes that high-impact teaching practices such as caring and supportive teacher behavior, along with professional competency and communication, improved a student's ability to feel he or she belongs within the classroom. Further studies provide additional details regarding instruction that support belonging. One study concludes that African American students' positive perceptions of belonging are influenced by 1) Instructional styles in which students are more engaged and active in the life of the class, 2) professors who are open and provide a safe and comfortable environment for students, and 3) peers who consider their opinions and viewpoints as important and valuable additions to the class. The results of this study underscore the need for specific instructional techniques and strategies for increased involvement [1]. As students feel that they belong in the class, they develop improved attendance and attention, increased study time, and additional course enrollment in the discipline [8]. The pursuit of intentional community with students developing a sense of belonging in the class could be an important goal for the mathematics classroom.

2 Our Plan

At the 2017 Charleston Southern ACMS conference, Matthew Hawks shared a ritual he had developed in his classes at the US Naval Academy. He called it the "Daily Question" — a time at the beginning of every class where students quickly responded to a simple "ice breaker" type of question. His goal was building student trust and interest in his classes. One of our faculty members incorporated this "Daily Question" into his classes since that time, finding that the response from students was very positive. In seeking to find additional rituals that might encourage intentional community, students were asked to share techniques used in their classes that helped them feel cared for by the professor. One experience that was shared by many of the students was when a professor prayed for them by name. Using this information, a group of our faculty committed to using both rituals in at least one of their classes for the Spring 2022 semester. The classes included Mathematics Fundamentals (a remedial class), Geometry for Elementary Teachers, Calculus I, Introductory Statistics, and Advanced Statistics. The faculty used many of the questions shared by Hawks [4] in the appendix of his article and found other icebreaker questions on line. They also developed a way to randomly select a student to pray for — such as drawing a student name from a deck of name cards.

Each of the faculty members was asked to share his/her hopes and fears in using the rituals. One faculty member had not taught the remedial mathematics in a long time. He wrote, "I am afraid that I will go too fast and not connect with students. [Through using the rituals] I am hoping [the students] feel comfortable to ask questions and receive help. I am hoping that intentionally having a ritual to help students feel they belong, they will be comfortable [to ask questions]." The professor teaching both the introductory and advanced statistics wrote, "[I hope] these rituals will stimulate camaraderie, interpersonal skills, and peer learning." One working with future elementary teachers wrote, "I hope my students will gain a comfort in working with each other and with me. I hope that this comfort will open them up to taking risks, gaining courage, learning through perseverance, and accepting their 'inner mathematician.'" By "inner mathematician," he meant the inner character in each of the students that enjoyed exploring puzzles, finding solutions, and creating patterns and strategies. While there were hopes for positive outcomes from the daily rituals, the professors also expressed concerns. The most repeated concern was the amount of time that would be needed each day to provide such a ritual. There was also a concern that students would see this as childish or a waste of time.

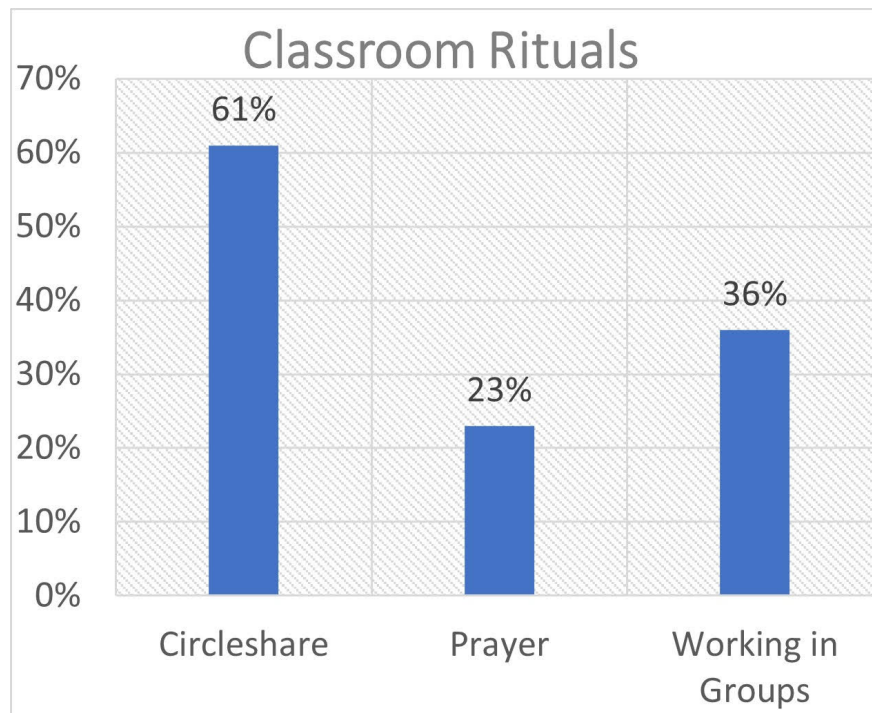


Figure 1: Aspects of class that communicated caring or connection to other students.

3 Results

To get feedback from the students regarding the rituals being used, an anonymous, mid semester survey was promoted within each of the connected classes. There were only two, open-ended items requested in the survey: (1) Please share one or two aspects of class that made you feel cared for or connected to other students. (2) From what you shared in the prior question, in what ways does this help your learning for this class? Of 195 students enrolled in the classes, 64 voluntarily responded (approximately a 33% response rate).

Of those that responded to the midterm survey, 61% mentioned the daily question (which was often called “circle share” since students typically stood in a circle to answer the question), 36% of the students mentioned group work, while 23% mentioned the focused prayer (Figure 1). Here are some responses from the students:

I really enjoyed our circle share time as I was reminded of each student’s name, as well as things they are interested in. Secondly, I really loved that [the professor] took time out of class each morning to pray for us individually. It really made me feel loved and cared for. I was able to get to know my partner for class, because there were many opportunities to work together to problem solve. —*Mathematics Fundamentals Student*

I love doing the circle share every day! I also love how [the professor] prays for a student everyday. I feel very intentionally cared for.—*Geometry for Elementary Teachers Student*

I LOVE the circle shares! They always challenge me to think about my answers, and keep me interested in other people’s responses.—*Introductory Statistics Student*

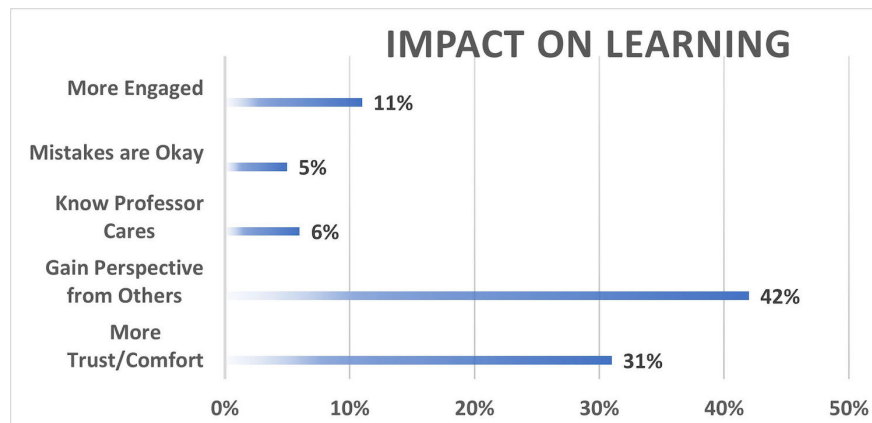


Figure 2: The impact on learning from the intentional community rituals.

Similar to what Hawks [4] found when he started his daily question, 31% of the student responses shared that they developed a greater sense of trust and comfort within the classroom which had a positive impact on their learning. Students also mentioned gaining perspective from others (42%), being more engaged (11%), knowing the professor cared (6%), and that mistakes are okay (5%) (Figure 2). It is interesting to note that many of these responses correspond with ideas shared in the research about belonging and intentional community. Here are a few quotes from the students:

I am more engaged throughout the class period and feel more comfortable sharing with the class or with other individuals. (*Advanced Statistics Student*)

I have never felt like I can do math before but [the professor] made me feel like I can do anything with math if I put my mind to [it]. I am so grateful for my new confidence in mathematics. (*Geometry for Elementary Teachers Student*)

This helps my learning in the class because I am more open to engage with others. Whether it be in group projects, quizzes, or questions about the homework, I feel more confident in approaching my classmates than I normally would. (*Calculus 1 Student*)

In addition to the mid semester survey, we were able to look at some results from common questions asked in the end of course evaluations. One interesting contrast was for a faculty member who tried the intentional community rituals in a remedial math course that could be contrasted with two other courses where the intentional community rituals were not used. Both questions were on a five-point scale from Strongly Disagree (1) to Strongly Agree (5). For the item, “The instructor cultivated a sense of community and belonging through this course (even with pandemic realities),” the instructor received an average of 4.6 for the intentional community ritual classes and an average of 3.8 for the classes not using the rituals. The contrast was not as dramatic for the item, “The instructor communicates his/her care and concern for students,” where the intentional community ritual classes had an average of 4.4 compared to an average of 4.3 for the classes not using the rituals. Some comments related to the rituals from the course evaluations:

The circle shares are the best part of my day on MW!! (*Geometry for Elementary Teachers Student*)

[The professor] is a wonderful professor and man. He shows that he cares for each student in the class by praying for them and assisting them if sees they are struggling.
(*Calculus 1 Student*)

An amazing professor! It is evident you know what you're doing, love what you're doing, and love us. Circle shares every M, W, F were by far the highlight of my semester.
(*Geometry for Elementary Teachers Student*)

While the majority of the responses to the intentional community rituals were positive, we noted some negativity. The professor who used the rituals in both his introductory and advanced classes noted that the students in the advanced classes seemed to lack enthusiasm and would even delay when they would arrive to class in order to miss the sharing. Several noted that students demonstrated little enthusiasm for the routines at the beginning of the semester, but realized the benefits further along. One calculus student shared the following in the course evaluation: "As wholesome as they are, I have to question the value of circle shares, at least for this class. It was a great ice breaker for the first week and maybe it could be reorganized to not be quite as intrusive to valuable class time, but I am not sure that every day was great especially since it's a foundational Calculus course and we need to study as adamantly as possible."

While we started with intentionally promoting community within the classroom, we recognize the need for further explorations and extensions. For example, we could try to determine if differences in student perceptions would be apparent if an instructor taught two sections of the same course, treating one as a control group with no community rituals and another as a treatment group with the community rituals. The responses we collected were completely voluntary so we might incentivize or require participation to obtain more complete data. We wonder if we would see the same type of responses from non-major courses to upper-level majors who potentially take many classes together. We could also collect demographic information to explore whether positive responses to intentional community are associated with gender, major, class time, and other factors. We are also interested in our community as teachers. How much did the collaboration as a community of teachers influence the outcomes within the classroom?

4 Conclusions

Research certainly communicates the value of developing community and belonging within the classroom. While we may hope this community develops on its own, for many of us it requires intentionality, like a classroom ritual. A clear observation from our experience is that students are much more aware of our individual interest in them by investing this small amount of time in each class for something non-content related. Furthermore, a daily ritual provides an opportunity to explain the purpose of the ritual in the context of learning theory. The survey and anecdotal data supply compelling motivation to construct a routine similar to a circle share, to encourage the intentional classroom community and sense of belonging. Of course, our ultimate goal is that each student grows mathematically and in their faith. There is evidence in the survey results that students gained vital attributes of mathematicians. Su paints a picture of a flourishing mathematical community as "People who have joined together in a common mission of exploration and play, bouncing ideas off each other, valuing one another's input, getting excited about the directions their ideas are taking them in, and embodying a wide array of mathematical virtues along the way" [10]. When students come to know one another better, they are able to join together in the

mathematical community that Su describes. It is amazing how students develop confidence as they are able to work with others — especially when barriers that promote superficial community are removed.

One unexpected benefit we observe from intentional community was the reaction by students in the remedial and non-major courses. The development of intentional community in these courses brings about a higher degree of interaction from the students in the course which often results in a greater commitment to higher achievement. Meyers [8] shares, “Supportive relationships between faculty and students are not a potential detriment to instructional rigor, but instead function as a conduit for students to master difficult material.” Our experiences support his conclusions. While there are similar benefits in courses designed for those more invested in mathematics, the additional motivation is not as necessary with those students. Students in mathematics courses related to their majors are often concerned about the time constraints for reviewing and exploring the material. We realize the need to be conscientious of this concern and adapt the rituals to continue to communicate to students the desire for a community within our classroom, while emphasizing efficient use of class time to explore the material. Utilizing small groups is often helpful for these purposes — especially if the small groups require students to meet and work with different members of the class throughout the course. A mini-circleshare within small groups periodically could promote similar aspects of community while maintaining focus on review or exploration of material.

One student wrote, “I look forward to circleshares every day! I feel like I am more excited to come to class each day, and more interested in learning math. I also feel more comfortable asking my peers for help!” Such excitement, interest, and comfort are attitudes that have far-reaching effects for students, mathematically and personally. Marshall Gregory [3] in a book about memorable teachers states, “Education is less about improving the knowledge one owns than about improving the choices one makes.” When students are choosing to be excited to come to class, to be more interested in learning, and to be more comfortable to collaborate with others—they are being educated in the best sense. It is likely that they will not remember a particular lesson or continue to use techniques that are not practiced, but they can remember an experience of mathematical community that contributes to them positively—making them a better mathematician and follower of Christ.

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Faith-Learning Integration in Mathematics: A Beginner's Guide

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Daniel Rodman (Ph.D., University of Iowa—low dimensional topology) is an assistant professor at Taylor University where he received the Joe Burnworth Teaching Award in 2021. Daniel loves getting to know his students, so at Taylor he runs the math club, the math team, and Pi Tea (every Thursday at 3:14 pm). He enjoys running, playing board games, and spending time with friends and family.



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Abstract

Many new faculty at faith-based institutions have no prior experience with faith-learning integration as a professor, or even as a student. Our goal is to provide a practical beginner's guide for faith-learning integration in college mathematics courses.

1 Introduction

As faculty at faith-based institutions, we strive to develop a robust faith-learning integration component to our mathematics program, which starts in each of our individual classes. At our institution, Taylor University, graduating senior mathematics majors consistently say that through reading discussions, special in-class activities, and faculty who display the fruits of the Spirit, the faith connections made during their time in our department were a valuable, important part of their education.

If you are a new faculty member, it can often be difficult to know how to begin when it comes to integrating faith and math in the classroom. Besides the huge job of teaching new classes in a new place, you may also be overwhelmed by your new institution's culture and expectations for teaching, scholarship, and service, not to mention your institution's many acronyms and abbreviations, parking rules, and the many, many names of colleagues you're trying to remember. In the midst of all that, faith-learning integration can easily slip to the bottom of your priority list. We must remember, however, that a Christian perspective in the classroom is one of the primary reasons our students choose our institutions, and it should be permeating our classes from the beginning.

The integration of faith and math can be a nebulous concept to faculty members who did not attend a Christian college or university. Moreover, in our Ph.D. programs, few of us have seen faith-learning integration modeled, let alone received any training in how to do it. Discussions with your colleagues across campus at your current Christian institutions are certainly helpful (for example, through the institution's center for teaching and learning), but such discussions are sometimes too broad or philosophical to be practical, particularly when the discussions involve faculty from different disciplines.

Integrating faith and math well is a process of continuous improvement, and oftentimes the hardest part is simply getting started. Our goal is for this article to serve as a practical beginner's guide for new faculty which presents some faith-learning integration examples you can start doing with your students. This is not a robust, thorough framework for faith-learning integration throughout an entire curriculum. Instead, we will start with some simple pieces of general advice, then describe a few classroom methods, and finally offer some practical examples of faith-learning integration by mathematics topic and a list of resources.

2 Some Advice

2.1 Curate, don't Create

Just like with any other subject, excellent course materials and ideas about faith-learning integration have already been developed by other people. You don't need to reinvent the wheel. Instead of spending vast amounts of time creating new materials and lesson plans from scratch, build on the work others have done. Of course, we hope that some of the ideas in this very article are useful to you, but there are lots of other resources too. In particular, the person who taught your classes before you likely has some faith-learning integration materials or ideas that you can implement. Talk with your colleagues about what they are doing in their classes and what resources they have used for faith-learning integration. Conferences such as the ACMS conference are another great way to gather ideas and learn what others are doing in this area.

2.2 Baby Steps

Be open to starting small and working toward a more systematic approach over time. If you have a few meaningful faith-learning integration discussions or assignments in a course the first time you teach it, that's something you can build on next time. Take the first steps, let the Spirit work in you, and trust God to inspire the rest as you iterate your course.

2.3 Systematic and Simple

Using a book, a list of articles, or a series of videos can be an easy way to kick off faith-learning integration, without you needing to come up with the topics and materials yourself. It also will give you extra motivation to study one of the many books you were probably given in your first week on the job. At the end of this article, we provide suggestions for written and video resources.

2.4 It Will Become Yours Naturally

Faith-learning integration looks a little different for each faculty member, and you will grow in this area as you grow in your teaching. Think about your unique interests and experiences and how that could be a way to get started. For example, if you enjoy history or philosophy or music, etc., then you may be able to use those interests to make faith-learning connections. Mark shares his Bible quizzing experience by giving his students opportunities to think about and memorize Bible verses related to mathematics. Daniel shares about how as a student he had to learn that his identity is not in good grades but in Christ and regularly has his students reflect on their identity in Christ. Derek does an activity with his freshmen about productive failure and how our identity comes from Christ and not our accomplishments. The activity started when he recognized a need in his students and connected that to his own struggles in graduate school.

2.5 Be Real with Your Students

If faith-learning integration is new or uncomfortable territory for you, be honest with your students about that, and communicate that you are excited to grow as a teacher together with them as they grow as students. A little bit of self-deprecating humor goes a long way. Letting students grow comfortable in your classroom through more light-hearted situations will generate social capital with them, paving the way for more serious discussions that come later. Even though some of your initial attempts at integrating faith and math may fall flat in the classroom, students will appreciate your efforts, and you'll improve and grow. Over time, your efforts will have a positive impact.

2.6 Let the Students do the Integration

Instead of having all the answers, our role is typically to ask the right questions and get the students to think about, discuss, or write about their ideas. While you have to be nimble as you guide the discussions and tie different threads together, you don't have to come ready with all the answers. Trust your students and let them explore the connections between faith and what you are learning. As Blaise Pascal said, "We are generally better persuaded by the reasons we discover ourselves than by those given to us by others."

2.7 Fruit of the Spirit Matters

Without downplaying the importance of curriculum that intentionally integrates faith and math, don't forget that living like Jesus and doing your best to display the fruits of the Spirit in your everyday interactions with students in the classroom are at least as important, and students notice when their professors really believe and live out what they're saying! Students are encouraged when they see that their teachers have an active faith. They need to know that you care about them, and they want to see you demonstrating God's love, peace, joy, and grace in the classroom. Some of our students' most memorable instances of faith-learning integration were unplanned moments when their professors were led by the Spirit and were given the right words to say at the right times. This may be the most obvious advice, but it is also the most important: Faithfulness in your own spiritual life will overflow into your classroom, and your students will benefit greatly.

3 Classroom Integration Methods

3.1 Videos and Small Group Discussions

Show a short video that connects the mathematics content you are studying with an important life issue. Write a discussion question on the board, and after watching the video, have the students discuss the question in small groups. Then have various groups share their ideas to begin a larger class discussion. Video and other resources are listed below.

3.2 Articles, Books, and Reflection Papers

Articles and book chapters can stimulate students' thinking about faith-learning integration. A straightforward approach is to have students write a short reflection paper in response to an article and to a few of your own additional discussion questions. For example, Francis Su's book, *Mathematics for Human Flourishing* and his website can be used as a springboard for discussing and writing about ideas such as math and beauty, math and justice, or math and love. For a more comprehensive exploration of faith-learning integration, students could be asked to read a book over the span of all their semesters in the math major. For example, most of the courses in our department require our students to interact with one or two specific chapters of *Math Through the Eyes of Faith*. This of course takes coordination with other department members to ensure students read different portions of the book in each class.

3.3 Papers and Projects

A variety of assignments can easily include faith-learning integration components. For example, students could be asked to research and write about a famous mathematician, and you could instruct them to include the story of this person's religious life (or lack thereof). Students could collect data related to stewardship issues or to living well. Another option would be to ask each student to read a different book or chapter that addresses faith-learning integration topics and then have them present their findings to the whole class.

3.4 Scripture Verses

A simple way to start a discussion is to pick a Bible verse that relates to the class content and then have the students discuss the meaning and application of the verse. This can then be incorporated into group discussions or reflection papers as mentioned above, perhaps weekly. Memorizing relevant verses for extra credit can be a fun motivation to learn verses, and discussing Scripture is a natural way to encourage a sense of belonging for a community of learners.

3.5 Prayer in Class and Community-Building

While our faith-learning integration should go beyond opening class with prayer or spending time on community building activities, both of these still help lay a foundation of safety and belonging that will make faith discussions more natural. Some faculty may find that regular class prayers can be instructive, and praying for individual students can build trust and connections in the classroom. It is also beneficial to pray for guidance before advising meetings and for clarity before homework help sessions.

3.6 Weekly Faith-Learning Integration Time

Setting aside a few minutes each week can be an effective way to systematically incorporate faith-learning integration. For example, you could have “faith-learning integration Fridays” or Monday devotionals to discuss the connections between faith and the mathematical content you are teaching that week. Doing this regularly helps the students begin to see all the connections and life lessons in the course and how faith and math apply to various areas of our lives.

4 Practical Examples of Faith-Learning Integration by Topic

4.1 Pre-Calculus

- A natural life application of exponential growth is compound interest and how being wise stewards of our money should encourage us to try to take advantage of compound interest through saving instead of amassing unnecessary debt. Many biblical Proverbs (such as Proverbs 22:7) encourage wise handling of money. Exponential growth can also be applied to growth in the Kingdom of God (Luke 13:19).
- Students often do not realize the extent to which they have accepted facts without understanding why they are true. Two common examples are the Pythagorean Theorem and the Quadratic Formula. Ask students why they believe these statements are true, and in general, when and why should they trust something without proof (2 Corinthians 5:7). When, why, and in what do we place our faith?
- When students are first introduced to the idea that there are mathematical concepts proven to be unknowable (e.g. the quintic formula), or even the simpler concept of an equation with no solutions, ask them how to connect that to our imprecise understanding of parts of Scripture and God’s lordship over us. Students enjoy discussing various quotations about how much we really know (Ecclesiastes 3:11 and Romans 11:33), and the discussion helps develop epistemic humility.

4.2 Liberal Arts Mathematics Courses

- Beauty in mathematics is a ubiquitous concept that we can highlight in all our courses, but it comes up repeatedly in liberal arts math courses through explorations of Fibonacci numbers and math in nature and through geometric concepts like the golden rectangle, platonic solids, fractals, tessellations, and symmetry. An excellent reason to study mathematics is because it is lovely (Philippians 4:8).
- When studying numbers, have students plot their lives on a number line to put life into perspective. Discuss life expectancy or how much time we spend or should spend on different tasks (Psalm 90:12 and James 4:14). See the “Time you have in Jelly Beans” video below.
- Predetermined chaos and the butterfly effect offer a delightful opportunity to discuss how a small change at the beginning of something can have a major impact at the end, and students can share their experiences of when this has happened in their lives. The life lesson is important in that we should be careful how we live because everything we do may have a significant impact on our lives or on others’ lives (Ephesians 5:15-16).

4.3 Calculus

- When studying higher dimensions, introduce students to Flatland. An anthropomorphized flat square cannot conceptualize a cube or sphere, but that does not mean such three-dimensional objects do not exist and even interact with Flatland. This can be presented as a metaphor for our finitude and God's infinitude. There are many things that are difficult for our limited minds to understand about God (e.g., the Trinity). Considering Flatland helps us see the arrogance of thinking that if we do not understand something, then it is not real or should be rejected. We grow in humility when we realize that while on one hand, we should always be seeking deeper understanding of God, on the other hand we should not expect to completely understand everything.
- Since we often try to maximize or minimize quantities in calculus, a helpful thing for students to think about is what we should try to maximize or minimize in our lives (or not maximize or not minimize). For example, we should not maximize our gain but leave some for others, as is taught in Leviticus 23:22.
- In multivariable calculus, the graphs of three-variable functions are not completely visualizable because they exist in four-dimensional space. However, we can gain understanding and intuition about them by considering the graphs of one- and two-variable functions, which we can visualize. This is somewhat analogous to how we cannot completely comprehend God, but he has provided us with simpler pictures of himself to help us ("lower-dimensional" pictures, if you will). For example, the Bible variously presents the persons of God as a shepherd, a father, a vine, a hen, a lion, a lamb, a king, a vineyard owner, etc., and each of these simpler pictures helps us understand something true about our God.
- In Calculus II or III, it is helpful to point out how often we use the Fundamental Theorem of Calculus without thinking about it or realizing it, certainly without being grateful for it. Point out that this is often analogous to how we treat the gifts of God in our daily lives: his sustaining power, his love, grace, and mercy, his presence, and his provision for our earthly needs on a second-by-second basis, including the very air we breathe. If we're not careful, we can take them for granted like we take the Fundamental Theorem of Calculus for granted.

4.4 Geometry

- Exploration of Euclid's fifth postulate generated new mathematical fields. Whole new geometries could be created by substituting Euclid's postulate with alternative axioms, and this contributed to new discoveries such as the theory of relativity. Even when we think we have understood all results about mathematics (or about God), there are new surprises (Ecclesiastes 3:11).
- In a geometry course, students usually learn how different sets of postulates lead to distinct geometries (Euclidean, spherical, hyperbolic). This can lead to good discussions about how even outside of a mathematical context, distinct conclusions can arise from different premises. For example, two worldviews, one of which contains the premise that God exists and one of which assumes the material world is all that there is will come to different conclusions about any number of philosophical and ethical questions.
- After establishing that infinitely long regions can have finite area (e.g. the Koch snowflake), ask students what it means for God to be infinite and how to reconcile the idea of more than one "type" of infinity (Psalm 147:5).

- As students begin to appreciate the importance of definitions in mathematics, extend the discussion to definitions in the theological context. For example, this is a great jumping-off point for a discussion of Bebbington's Quadrilateral and the meaning of the term "Evangelical."

4.5 Probability and Statistics

- Since many probability concepts come from gambling applications, it can be helpful to discuss how we should respond wisely to gambling both individually and as a community (Proverbs 11:18). It can be interesting to look at the very small probability of winning the lottery and to think about how gambling can affect people's lives.
- Students enjoy exploring how probability can be surprising, and how coincidences or unlikely events have occurred in their lives. Students can consider how God may work through coincidences (Romans 8:28).
- Misleading statistics are often found in advertisements and political statements (Ephesians 4:25). Students can be asked to find examples from the media that demonstrate misleading statistics and think about how relying on our own life experience is a poor sampling technique. You could also discuss the Wesleyan Quadrilateral and why experience is only one piece of the puzzle.

4.6 (Introduction to) Proof Courses

- The sets of rational and irrational numbers are infinitely close to each other in the sense that for any rational r and any distance ε , no matter how small, there are infinitely many irrational numbers within ε units of r (and vice versa). The way that the real line is comprised of these two sets, intermingled and infinitely close, yet also distinct, can be used as an analogy for the relationships between the persons of the Trinity. Each person is *in* the other persons (see John 14:11) in a similar way to how the rationals and irrationals are found *within* each other. You might discuss with your students how analogies for the Trinity are most instructive when one thinks not only about how an analog sheds light on the real Trinity, but also about the ways the analogy breaks down.
- In addition to considering how faith helps us understand math, discuss how mathematical thinking can help us with our faith and understanding of the Bible. When studying propositional logic, point out that there are over 1500 instances of the word "if" in the Bible (NIV). Analyze the meaning of some of these verses (including what they do not say).
- There is a lot of room in proof classes for discussions of how we know what is true, and the nature of truth and reality. A primary purpose of a proof is to convince a reader that something is true. A primary purpose of evangelism is to persuade others of the truth of the gospel (John 20:31). How are proof and evangelism similar and different? Have students read 1 Peter 3:15-16 and discuss the importance of not only what we communicate, but also how to communicate in a godly way.
- When students learn about axiomatic systems, we learn that assumptions that we make immediately have consequences: they imply that various facts (theorems) that follow from them are also true whenever our assumptions are true. The Bible encourages to be avoid hasty judgment (Matthew 7:1-6, Proverbs 25:6-10), and to value understanding over opinions (Proverbs 18:2). Ask students to reflect on what assumptions they may not realize they bring to discussions with peers, and what the implications of those assumptions are.

- When learning about set theory or partial orderings, have students think about how various theological topics could be categorized into sets of “first-level,” “second-level,” and “third-level” issues.

4.7 Outside the Content Areas

- Emphasize that “productive failure” is a Biblical principle, using the example of Peter in the Gospels, Paul’s struggles with sin (Romans 7), and/or the lifelong process of sanctification (Romans 12:2). Model this in your courses by finding ways to incorporate grace without sacrificing standards. For example, this might be achieved through mastery assignments like “mastery proofs” or “gateway exams” for derivatives and integrals, where students are allowed multiple attempts over a long period. It could also be something as simple as the final exam replacing the lowest regular exam score if higher, or giving out practice tests. Emphasize that these are not methods of lowering standards, but ways to foster deeper learning, and that our spiritual lives reflect this kind of progression. Express an expectation of growth (Galatians 6:9) for your students. See Francis Su’s article on grace and videos on a growth mindset below.
- Early in any course it can be valuable to discuss the benefits of working in groups (Ecclesiastes 4:12), especially when studying mathematics. Students can reflect on their own positive and negative experiences of working in groups in the past, what has caused their groups to work or not work well together, and what they are hoping to learn from collaborating with others this semester.
- A fruitful reflective assignment is to have students write about the highs and lows of their own mathematics journeys and the hope they have for a positive semester in this class. Connect this with our pursuit of spiritual growth in Philippians 3:13-14.
- Soon before or after a difficult exam or when the semester is getting stressful, it can be helpful to have students read through Ephesians 1, especially focusing on their identity in Christ. Remind students that their worth and value do not come from their grades or performance, nor from others’ approval of their academic success. Their core identity is a child of God, an ambassador for Christ and his kingdom. They are blessed, chosen, holy and blameless in the Father’s sight, chosen to be for the praise of his glory, included in Christ, sealed with the Holy Spirit. An A in a math class doesn’t increase God’s love for them, nor does a C (or D or F) diminish it.
- In any math course, you can introduce students to philosophical problems of the ontology of mathematical objects (numbers, shapes, functions, etc.). Students can begin to think about whether such things as numbers are part of God’s creation, part of humanity’s creation (as are art or literature), both, or neither: perhaps they are a kind of fundamental truth existing independently from God. Are numbers discovered or invented?

5 Conclusion

For those who are looking for some guidance on getting started with faith-learning integration, we hope that some of these ideas will be helpful launching points for you. As you grow in your teaching skills and spiritual wisdom, we pray that your classrooms become places where students see a vibrant faith uniquely reflected in you and in what you do as a teacher. We also hope

that these ideas can be used by those who are further into their careers yet still looking for some fresh ideas. For all of us who teach mathematics from a Christian perspective, let's work to make faith-learning integration a central and dynamic part of our teaching. Let's prayerfully synthesize ideas and practices from others, make them our own, and take small but fruitful steps to help our students integrate math and faith, to the glory of God.

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Correlation, Causation, Statistics, and Soteriology

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Abstract

Rightly understanding, using, and critiquing statistical inferences that relate to correlation and causation is at the heart of statistical literacy, reasoning, and thinking. The framework of Hartman, Hunt, and Childers [4] for misconceptions of cause and correlation is used in this paper to understand Christian soteriology and its intersection with student's statistical understanding. Bible verses are used to encourage deeper reflection and understanding by students during a statistics course to directly address misconceptions related to cause and effect.

1 Introduction

In 2013, Lindell Ormsbee [7] wrote an insightful piece relating artificial neural networks to the question of free will. Lindell (2013) provided a middle ground between both Arminian and Calvinist theology by defining and mapping out ideas related to God's middle knowledge. Lindell [7] defined this as "God's knowledge of everything that would happen in response to a particular set of circumstances" [7, p. 3] and imposed this knowledge upon the act of salvation of men, soteriology. This paper seeks to continue this discourse as it connects with statistical correlation and causation. The authors hope that readers will be prepared to lead and discuss a faith-based discussion around the topic that encourages students to justify themselves through Biblical interpretation.

2 Correlation or Causation?

Each breath we take, we are plagued with the aspects of cause and effect. How will eating this potato chip result in a waistline increase or will this diet lower my blood pressure? Will walking an extra 1000 steps each day add more days to my life? How will the music I listen to, videos I watch, literature I read, or conversations I have influence my mind? We know these things matter, as the scriptures point us toward this recognition.

Do not be deceived: "Bad company ruins good morals" (1 Corinthians 15:33).

Whoever walks with the wise becomes wise, but the companion of fools will suffer harm (Proverbs 13:20).

The deeper question though we should be concerned with is, do these actions cause their effects?

The heart of man plans his way, but the Lord establishes his steps (Proverbs 16:9).

Is it not from the mouth of the Most High that good and bad come (Lamentations 3:38)?

Our current culture is steeped in statistical misconceptions of correlation and causation. Our newspapers and political leaders press them forward as reasoning for purchasing products, legislative changes, and opportunities to provide a more just and equitable society. The Clinton Foundation, based on research on statistical significance, encouraged the circumcision of men to reduce the expansion of AIDs [1], [6]. Research on the use of masks during the COVID-19 pushed many to mandate the wearing of these for the protection of their citizens. Though research supporting these policies were strong enough for publication, policy makers and others failed to calculate, use, or understand effect size measurements, confounding factors, correlational concerns, and increased risks of type I error rates in large sample sizes. This paper seeks to thoughtfully connect ideas of correlational misconceptions to Biblical passages related to soteriology in order to help statistics teachers purposefully encourage students to integrate their understanding of soteriology with statistical inference.

2.1 Types of Correlational Misconceptions and Ways to Address

Liu et al. [2] identified seven misconceptions of students when learning about and developing understanding of correlation. Of these misconceptions, 20 percent of students within the study had difficulties separating casual and correlated effects. Hartman, Hunt, and Childers [4] identified five different causal logic fallacies: reverse cause fallacy, bidirectional cause fallacy, common cause fallacy, single cause fallacy, and coincidence. The reverse cause fallacy assumed that A caused B without considering the possibility that B is actually the cause of A. The bidirectional cause fallacy assumed that A caused B without considering the possibility that there is a reciprocal relationship between the two where A causes B and B causes A. These misconceptions are easily identified and can easily be addressed with open questions that encourage students to consider the reversal of the independent and dependent variables. Students should be encouraged to describe the controlling factors of the variables to help deduce which variable should be included as a dependent or independent variable.

The common cause fallacy is when a student believes that A caused B without considering the possibility that an external factor C is actually the cause for both A and B [4]. These misconceptions are strongly tied to research methodology, extraneous variables, and confounding factors. Statistics educators seeking to address these issues in class may offer open questions that relate to other factors that relate to the study contextually. They may also use questions that encourage students to consider the research methodology that lack for control.

The single cause fallacy supposed that A caused B without considering the possibility other factors may influence B [4]. These misconceptions may be redirected by providing other variables that may also contribute to the independent variable's variability. Statistics instructors may also encourage more in-depth description of residual plots and residuals to understand the lack of exact modeling and other factors that may contribute to the model. Using multiple variables and understanding the correlation coefficient can also help in understanding the potential of other factors in modeling a situation.

Lastly, the coincidence caused fallacy occurred in students who presumed that A caused B without considering the possibility that the relationship between A and B is merely coincidental [4]. To help students understand these potential issues, teachers may point towards issues in the study that reduce sample size, limit participation in the study, or are related to collection bias. Considering these issues, a change in study parameters may produce different results that are not correlated.

In addition to the recommendations provided, students should have a clear understanding of what is required to show causality in statistical research. According to Hill's [5] criteria for causation, the minimal conditions needed to establish a causal relationship between two items in epidemiological research are as follows:

1. A temporal relationship that identifies and describes how cause always precedes effect.
2. A measure and description of associational strength.
3. A relationship of induced increased or decreased amount of the dependent variable and its change in the independent variable.
4. Replication by other studies.
5. The cause and effect relationship is philosophical or scientifically plausible.
6. Alternate explanations are considered and eliminated.
7. The study is experimental.
8. The variable is specific enough to be seen and shown to relate.
9. The relationship is based on current theory and coherent with current knowledge.

2.2 Correlation and Causation in Soteriology

In philosophy, correlation and causation are described as necessary or sufficient causes. A necessary cause is a condition that must be present for an event to happen. In the health sciences, this may mean that a person must have HIV before they have AIDS. In law, a defendant convicted of a crime must either be proven guilty or plead guilty. A sufficient cause is a condition that, when present, guarantees the occurrence of an event. A sufficient cause in botany may be the death of a plant because of the elimination of light. The elimination of light was sufficient to cause the death of the plant but not necessary. These ideas are important as we move towards understanding the depths of soteriological correlation or causation.

To help statistics educators grapple and lead discussions that tackle common misconceptions in statistical correlation and causation misconceptions, different Bible verses and explanations will be provided that relate to soteriology. The statistical literature provided before are used to situate and include Bible verses and stories for richer discussion in statistics teachers classrooms. To understand these concepts fully, it is important to describe different forms of grace. The Society of Evangelical Arminians describe prevenient grace as that which "prepares and enables sinners to receive the free gift of salvation offered in Christ and his gospel" [9]. Arminians and those parallel to those of this doctrinal thinking believe that prevenient grace is granted to all human beings and allows them to freely accept Christ as their savior. Those opposed to this belief describe grace as with terms common grace and saving grace. Saving grace is an unmerited favor given to the elect

chosen in accord with God's free will. Common grace is the grace of patience and kindness given to all sinners who deserve immediate and justified condemnation. It is important for teachers during this discussion to understand and acknowledge that the very large majority of Christians believe that grace must be given to humans because of the fall of Adam and Even in the garden to be redeemed by the Lord. In the next sections, each statistical fallacy is presented with a balanced lens of soteriological perspective to help teachers address potential misconceptions statistically and in soteriology in their classes.

3 Five Statistical Misconceptions and Verse Associations

Section 3 of this paper provides a connection of statistical and soteriological cause fallacies. Statistics educators focusing on a creating a classroom that fosters statistical reasoning [3] should:

1. Focus on developing central statistical ideas rather than on presenting set of tools and procedures.
2. Promote classroom discourse that includes statistical arguments and sustained exchanges that focus on significant statistical ideas.
3. Use assessment to learn what students know and to monitor the development of their statistical learning as well as to evaluate instructional plans and progress.
4. Integrate the use of appropriate technological tools that allow students to test their conjectures, explore and analyze data, and develop their statistical reasoning.
5. Use real and motivating data sets to engage students in making and testing conjectures.
6. Use classroom activities to support the development of students' reasoning.

Similarly, Christian statistics educators working towards developing a Christian reasoning classroom should intersect these principles with Christian teaching. Teachers should consider using the Bible and history as data sets for understanding the Lord and Christian teaching. In this section, tables are included that connect Bible verses to potential student misconceptions during statistical inference. Teachers can assign these verses for students to engage with at many different times in their course. For instance, if an instructor knows that a particular data set often presents students with bi-directional cause fallacy, they could have these verses written on note cards and complete a sorting activity. Students could be directed to place once set of verses for men working with God in one section, the inability of men to work out their salvation in another, and verses describing the means of grace in another from Table 2. This activity would promote statistical and theological reasoning. After discussing why students placed certain verses in each section and how these ideas work together to create truth, they could then be directed to apply this same concept to a recent problem they investigated. This and similar discussions would help promote discourse that focuses on Biblical truth and statistical arguments that focus on statistical ideas.

In statistics, the reverse cause fallacy can be seen when a student only considers one variable as a cause. For example, a student may think that depression leads to smoking. Thus, they may not consider the idea that smoking may lead to depression. In soteriology, the reverse cause fallacy would be highlighted by someone who focuses only on the responsibility of man or the free will of God. An argument that both work together, does not show a reasonable position of cause based on

Hill's [5] descriptions and relates to the bi-directional fallacy. Students grappling with this fallacy must consider the alternative view of their belief and argue for or against it, making sense of the texts using their own reasoning and sense making. Soteriological proof texts to point students toward with these potential fallacies are represented in Table 1. Teachers may use these examples to bring deeper Biblical insight into who actually causes salvation to occur by offering them up for further discussion.

| Only Considering God's Free Will? | Only Considering Man's Free Will? |
|--|---|
| John 12:32 And I, when I am lifted up from the earth, will draw all people to myself. | John 16:8 You did not choose me, but I chose you and appointed you that you should go and bear fruit and that your fruit should abide, so that whatever you ask the Father in my name, he may give it to you. |
| John 16:8 And when he comes, he will convict the world concerning sin and righteousness and judgment | John 6:65 And he said, "This is why I told you that no one can come to me unless it is granted him by the Father." |
| Titus 2:11 For the grace of God has appeared, bringing salvation for all people, | Ephesians 2:8 For by grace you have been saved through faith. And this is not your own doing; it is the gift of God, |
| Consider assigning Ezekiel 33:1-20 for deeper discussion. | Consider assigning Romans 9 for deeper discussion. |
| Other related verses related to the salvation of the world and all may include John 1:9, 29, 3:16-17, 4:42; 2 Corinthians 5:19; 1 John 2:1-2, 4:14; Romans 5:18; 2 Corinthians 5:14-15; 1 Timothy 2:4-6; Hebrews 2:9; and 2 Peter 3:9. | Other related verses related to God's salvation of a particular people may include Matthew 1:21, 20:28, 26:28; John 10:11, 11:50-53; Acts 20:28; Ephesians 5:25-27, Romans 8:32-34; Hebrews 2:17, 3:1, 9:15, 9:28; and Revelation 5:9 |

Table 1: Reverse Cause

Very similar to the earlier fallacy, is the bi-directional cause fallacy. This fallacy happens when a student does not consider the potential of a reciprocal relationship between the two variables. A good example of two variables working in conjunction are predators and prey in an environment. An increase or decrease in one of the variables will impact the other. This is best represented in soteriology by means of God and man working together in the determination of salvation. Many who have an Arminian or parallel soteriology believe that God's foresight and prevenient grace are the basis of God's decision of who will be saved from all eternity (Ephesians 1:4, Romans 8:29-30, Revelation 13:8). This may be notably connected with Ormsbee's [7] middle knowledge described at the beginning of this article. As students struggle in the soteriological discussion and have bidirectional cause fallacies, consider the verses in Table 2 as ways to encourage deeper prayer and reflection. Encouraging students to consider different perspectives and justifying their beliefs Biblically and philosophically are very important for their faith.

| Men Working with God in Salvation | Inability of Men to Work Salvation |
|---|---|
| Joshua 24:15 And if it is evil in your eyes to serve the Lord, choose this day whom you will serve, whether the gods your fathers served in the region beyond the River, or the gods of the Amorites in whose land you dwell. But as for me and my house, we will serve the Lord. | 1 Corinthians 2:14 The natural person does not accept the things of the Spirit of God, for they are folly to him, and he is not able to understand them because they are spiritually discerned. |
| Deuteronomy 30:19 I call heaven and earth to witness against you today, that I have set before you life and death, blessing and curse. Therefore choose life, that you and your offspring may live, | Ephesians 2:1, 4-5 And you were dead in the trespasses and sins... But God, being rich in mercy, because of the great love with which he loved us, even when we were dead in our trespasses, made us alive together with Christ—by grace you have been saved— |
| Revelation 3:20 Behold, I stand at the door and knock. If anyone hears my voice and opens the door, I will come in to him and eat with him, and he with me. | Romans 8:7 because the mind set on the flesh is hostile toward God; for it does not subject itself to the law of God, for it is not even able to do so, |
| ----- There are very common themes running across this topic because most theologians agree that grace is required to bring humans in relationship with Himself. Some verses to help students understand this fact are: Genesis 6:5, Genesis 8:21, Colossians 2:13, Psalm 51:5, Psalm 58:3, John 3:5-7, Job 14:4, Jeremiah 13:23, Titus 1:15, 1 John 5:19, John 8:34, 2 Chronicles 6:36, Psalm 130:3, Proverbs 20:9, Ecclesiastes 7:20, Isaiah, 53:6, etc. | |

Table 2: The Bi-Directional Cause Fallacy

During most soteriological discussions the relationship of evangelism is brought to the forefront. Arminian theologians and laymen argue that if God has predestined all that should be saved, why should we preach or do missions? Calvinist theologians and missionaries, like their Christian counterparts, see their evangelism work as obedience to commands from the Lord [10], who ultimately provides regeneration and establishes his kingdom (Matthew 28:16-20) and this is a prevalent myth imposed on Calvinist according to Stewart [8]. The discussion appears to be parallel to the common cause fallacy in which there is another factor contributing to salvation, the preaching of the gospel. This fallacy happens when someone believes that one thing caused another, without considering another factor that could have caused both things. For example, someone may have a sore throat and a runny nose. They think their runny nose caused their sore throat. In reality, the doctor discovers that they have strep, which caused both the sore throat and the runny nose.

Arminian and Calvinist theologians strongly agree that the gospel must be preached and much evangelism work has been done by both groups historically. To further extend thinking on these ideas, instructors may encourage students to read and reflect on the verses in Table 3. Proponents of both theologies will note that God is the decisive granter of faith and repentance. Though the bridge that different soteriologies explicate may be seen as short and wide or narrow and long across the chasm that separates humanity from God, unity can be found here. Most theologians agree that we are saved by grace alone through the preaching and teaching of God through humans either orally or by written word.

| Men Preach the Gospel | The Lord Instructs |
|---|--|
| <p>Romans 10:14-17 How then will they call on him in whom they have not believed? And how are they to believe in him of whom they have never heard? And how are they to hear without someone preaching? And how are they to preach unless they are sent? As it is written, “How beautiful are the feet of those who preach the good news!” But they have not all obeyed the gospel. For Isaiah says, “Lord, who has believed what he has heard from us?” So faith comes from hearing, and hearing through the word of Christ.</p> <p>Ezekiel 33:4-6 then if anyone who hears the sound of the trumpet does not take warning, and the sword comes and takes him away, his blood shall be upon his own head. He heard the sound of the trumpet and did not take warning; his blood shall be upon himself. But if he had taken warning, he would have saved his life. But if the watchman sees the sword coming and does not blow the trumpet, so that the people are not warned, and the sword comes and takes any one of them, that person is taken away in his iniquity, but his blood I will require at the watchman’s hand.</p> | <p>Jeremiah 31: 31-34 “The days are coming,” declares the Lord, “when I will make a new covenant with the people of Israel... “I will put my law in their minds and write it on their hearts. I will be their God, and they will be my people. No longer will they teach their neighbor, or say to one another, ‘Know the Lord,...“For I will forgive their wickedness and will remember their sins no more.”</p> <p>1 Thessalonians 4-5 For we know, brothers loved by God, that he has chosen you, because our gospel came to you not only in word, but also in power and in the Holy Spirit and with full conviction. You know what kind of men we proved to be among you for your sake.</p> <p>Philippians 1:29 For it has been granted to you that for the sake of Christ you should not only believe in him but also suffer for his sake</p> |

Table 3: Common Cause Fallacy

Many students of statistics can easily become isolated in believing that a variable can be explained by only one variable. These students do not consider the possibility that other variables may also play a part in the causation. For example, someone may believe that the only factor that could have contributed to a patient having lunch cancer is smoking when in fact it could have been work conditions, genes, or a number of other factors.

As part of the great commission, Christians are charged to share the gospel to every nation. Unfortunately, there are numerous nations in which the majority have generally rejected the gospel or Christianity. Thus, we can easily see that the nationality of a human on earth is strongly associated with their faith. Are there multiple factors that contribute to salvation from scripture as they relate to national acceptance? Having students relate these ideas to other factors in an effort to show causation can be very beneficial. Table 4 is provided as just one example of many potential factors that students may bring out during class related to this fallacy. This is an excellent fallacy to help promote unity in the class because all Christians agree that it is the work of the Triune God of all eternity who brings about redemption and salvation of His people. Looking to other factors that may relate to the spread of the gospel such as nationality, ethnicity, race, sex, apologetics, gospel presentations, etc. are only lens to use to see our own failure and success to fulfil the great commission (Matthew 28:16-20).

| National Softening | National Hardening |
|---|---|
| Ezekiel 36:26 And I will give you a new heart, and a new spirit I will put within you. And I will remove the heart of stone from your flesh and give you a heart of flesh. | Exodus 10:1 Then the Lord said to Moses, “Go to Pharaoh, for I have hardened his heart and the heart of his servants, that I may perform these signs of Mine among them, |
| Ezekiel 11:19 And I will give them one heart, and a new spirit I will put within them. I will remove the heart of stone from their flesh and give them a heart of flesh, | Deuteronomy 2:30 But Sihon king of Heshbon was not willing for us to pass through his land; for the Lord your God hardened his spirit and made his heart obstinate, in order to deliver him into your hand, as he is today. |
| Jonah 3:4-5, 9-10 Jonah began to go into the city, going a day’s journey. And he called out, “Yet forty days, and Nineveh shall be overthrown!” And the people of Nineveh believed God. They called for a fast and put on sackcloth, from the greatest of them to the least of them... Who knows? God may turn and relent and turn from his fierce anger, so that we may not perish.” When God saw what they did, how they turned from their evil way, God relented of the disaster that he had said he would do to them, and he did not do it. | John 12: 39-40 Therefore they were not able to believe. For again Isaiah said, “He has blinded their eyes and hardened their heart, lest they see with their eyes, and understand with their heart, and turn, and I would heal them.” |

Table 4: Other Factors Contributing to Salvation

The coincidence cause fallacy in statistics comes when inference is made that does not consider other parts of the statistical process. In statistics, students will may argue that a sample was not large enough, the sampling process should be broadened, or other factors to from the statistical methodology need to be considered for making causal inference. Though this fallacy commonly appears, it should not be confused with students’ misconceptions of inference that relate to random sampling and assignment.

Unfortunately, many Christians have moved away from a justification of their beliefs through the use of scripture and have moved towards a type of coincidence. These Christians have found difficulty in reconciling seeming differences in Bible verses and have become more deist in nature. The authors of this paper, see differences in Bible verses as not a means towards the eradication of their use but to bring greater clarity to true meaning. The scientific revolution has encouraged many Christians to merge scientific thought and perspectives of randomness or chance into their theological beliefs or doctrine. However, Arminian and Calvinist theologians wholeheartedly agree that our salvation is anything but chance. The following verses in Table 5 may be used to investigate the plan of salvation from eternity as a plan A or plan B work of God. The plan A directive of God implying that the Lord never changes his mind (Hebrews 13:8, Malachi 3:6, Isaiah 40:8, James 1:17, Numbers 23:19) accomplishes all of his purposes (Romans 8:28, Proverbs 19:21, Isaiah 55:11), and directs all things according to the counsel of his own will (Ephesians 1:11) from the beginning of the world to the end ensuring justification. A plan B directive shifting towards God’s purposeful use of his own goodness and freedom of change to interact with the fallen world providing it opportunity for justification.

| Plan A Directive of God | Plan B Directive of God |
|--|--|
| <p>Ephesians 1:4-5 even as he chose us in him before the foundation of the world, that we should be holy and blameless before him. In love he predestined us for adoption to himself as sons through Jesus Christ, according to the purpose of his will,</p> <p>John 17:4-6 I glorified you on earth, having accomplished the work that you gave me to do. And now, Father, glorify me in your own presence with the glory that I had with you before the world existed. "I have manifested your name to the people whom you gave me out of the world. Yours they were, and you gave them to me, and they have kept your word.</p> <p>1 Corinthians 2:7 But we impart a secret and hidden wisdom of God, which God decreed before the ages for our glory.</p> <p>1 Titus 1:3 Paul, a servant of God and an apostle of Jesus Christ, for the sake of the faith of God's elect and their knowledge of the truth, which accords with godliness, in hope of eternal life, which God, who never lies, promised before the ages began and at the proper time manifested in his word through the preaching with which I have been entrusted by the command of God our Savior;</p> | <p>Genesis 1: 27, 31 So God created man in his own image, in the image of God he created him; male and female he created them... And God saw everything that he had made, and behold, it was very good. And there was evening and there was morning, the sixth day.</p> <p>Jeremiah 26:19 Did Hezekiah king of Judah and all Judah put him to death? Did he not fear the Lord and entreat the favor of the Lord, and the Lord changed His mind about the misfortune which He had pronounced against them? But we are committing a great evil against ourselves."</p> <p>Amos 7:3 The Lord changed His mind about this. "It shall not be," said the Lord.</p> |

Table 5: The Coincidence of Salvation

4 Philosophical Arguments

Given the vast scriptures that focus on the sovereignty of God in salvation, our personal belief and faith, and the persistent calls and assurance of faith, repentance, and submission in the scripture we must come to a logical conclusion. Does our faith cause salvation or does God cause our salvation through faith? How much of our rebirth is generated from human willing and how much from the Holy Spirit? Does God provide prevenient grace initially through baptism (Roman Catholicism) or to all men (Wesleyan Theology) in order that they may initiate their belief, faith, repentance, and submission? Does God initiate salvation through regeneration, adoption, rebirth, raising dead men, and opening blind eyes (Calvinism)?

Though Biblical justification and proof texts are important, it is essential to also understand causality philosophically or logically. Students need to walk away with an understanding of the necessary and sufficient soteriological causes. Table 6 provides a few necessary and sufficient statements that can be used to spark deeper conversations in class. These phrases are followed with scripture references that could be used for justification of their statements as necessary or sufficient causes.

Printing these statements on note cards and allowing students to place into their own categories can be useful for encouraging deeper thought and reflection.

| Necessary Cause | Sufficient Cause |
|--|---|
| The Lord must draw each individual to salvation (John 6:44, Romans 8:29). | Believe in the Lord, and you will be saved (Acts 16:31, Romans 10:9). |
| Repentance is required for salvation (1 John 1:9, Acts 3:19). | God grants individuals repentance that leads to life (1 Timothy 2:25, Acts 5:31, Acts 11:18). |
| Grace is necessary for the salvation of all people (Titus 2:11, Romans 2:4). | When the Lord calls, his children answer (John 6:37, John 15:16, John 18:9). |
| All who call on the Lord will be saved (Romans 10:13, Acts 2:21). | Men are unable to come to God by their own free will (Romans 8:7, John 1:13, 1 Corinthians 2:14). |

Table 6: Necessary and Sufficient Causes

5 Conclusion

Though some papers have tried to bring these differences together to make a more united Christian voice [7], the argument in this paper is to use these differences to gain a deeper understanding of our own theology and statistics. Differences in these beliefs will likely not be resolved on this side of heaven, but God intended for us to use scripture to know him more deeply and test truth (Acts 17:11). Moving towards a depth of understanding of differences can actually help shed light on commonalities because there is one truth. That truth is positioned by this organization through the Nicene and Apostles Creeds, that men are sinners and in need of a savior. A savior who descended to men when they were unable to ascend to him.

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Restoring Faith-Based Mathematics

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Abstract

Accrediting visits have led California Baptist University to ask how each of its programs systematically integrates faith into their respective disciplines. In the Department of Mathematical Sciences, we are replacing the word “integration” with “restoration” as we develop a department-wide model for all of our majors. This philosophical replacement specifies our approach to integration. Once our approach is defined, we disclose how our Faith Restoration model accomplishes faith integration via three areas: teaching, research, and culture of the mathematical sciences. In teaching, we incorporate what we call “Faith Themes” into all of our courses. These Faith Themes are meant to restore how we think about our discipline. We then discuss ways to restore research and the culture of the mathematical sciences. The ultimate goal of this model is to produce majors that have restored faith in the thought, practice, and interactions of the mathematical sciences. The philosophy and implementation of Faith Restoration shall be discussed with specific examples.

1 Introduction

Accrediting visits have led California Baptist University to ask how each of its programs systematically integrates faith into their respective disciplines. The Department of Mathematical Sciences is philosophically replacing the word “integration” with “restoration” through a recently proposed model by the author of this paper. This substitution is philosophical and not a call to eliminate the word “integration.” Rather, the use of the word “restoration” emphasizes the faith integration approach of the presently discussed model. This Faith Restoration model aims to renew the worldview in the minds of both students and faculty alike. This paper will discuss both the philosophy and implementation of the Faith Restoration model by building on the foundations of prior work of others for the philosophy, and by providing examples for the implementation.

The present paper will first introduce the concept of Faith Restoration through the analogy of a personal anecdote of the author. To accomplish this, the paper will employ the usage of first and third-person points of view throughout the paper. The narration of the anecdote and accompanying reflective insight is in the first person point of view. The remainder discussion of philosophy and implementation is in the third person. Said discussion will focus on three core categories of the Faith Restoration model: teaching, research, and culture.

2 A Personal Story

I vividly remember my first impression upon viewing the Native-American petroglyphs found in the Barker Dam hike within Joshua Tree National Park. The colors made the petroglyphs stand

out from any other petroglyphs I had ever seen. It was moments later, upon reading a nearby sign, that I discovered that vandalism had added color. The sign reads:

Nearby you will see petroglyphs which have been traced over with paint. This type of vandalism prevents others from seeing the petroglyphs in their original form.

While this was obviously frustrating, I could not help but wonder how these defaced petroglyphs and the accompanying sign provided an analogy of how Christian faculty should integrate faith in the mathematical sciences. Expressly, the strictly secular perspective of our higher education disciplines traces over their original form, truth, and beauty. Just like the color of the vandalized petroglyphs, the secular worldview assumptions of the mathematical sciences induce the learner to assume these are inherent to the mathematical sciences. In the same way that the sign corrected my erroneous impression caused by the perpetrators of this vandalism, Christian faculty should facilitate the same correction to their students regarding the mathematical sciences. That is, Christian faculty should point students to the original faith-based approach of the discipline. This does not mean faith-based math is different from math as we know it. Instead, it implies that how we think about mathematics matters in the context of the renewal of the mind of a follower of Jesus. Indeed, faith should inform how a Christian mathematician thinks about the foundations of mathematics. It should also inform or expand the purpose with which a believer applies or teaches mathematics. Faith should also inform the Christian mathematician that all disciplines can be used as powerful analogies of the Kingdom of God, much like fishers learned to fish people when following Jesus. Thus, faith informs the believer that mathematical sciences can provide deep insight into the principles of the Kingdom of God. As Christian mathematicians are sanctified through the renewal of the mind, they learn to stop compartmentalizing faith from all aspects of life, from the grand ones to the minutiae. Much like the defaced petroglyphs, secularized assumptions and perspectives have tarnished the worldview of mathematical sciences beyond recognition of their original form. Consequently, the role of the Christian professor is to restore carefully the true sacred form of the mathematical sciences. To clarify once more, the word “form” is a limitation of the analogy with the petroglyphs. The “original form” is meant to represent how image-bearers in the garden would have thought about and practiced mathematics.

In this paper, we propose a philosophical foundation for our approach to Faith Integration of mathematics that can possibly be applied to a variety of other higher education disciplines. We summarize our approach and model with the term Faith Restoration. In seeking to restore the *a priori* faith roots of mathematics, this paper proposes three intentional ways to accomplish effective restoration: restoration of the teaching of mathematics, restoration of mathematical research, and restoration of the culture of the mathematical community at large.

3 From Faith Integration to Faith Restoration

Dr. Ted Murcay, director of the Teaching and Learning Center at California Baptist University, voiced his concern regarding a possible misunderstanding that could arise from using the phrase “faith integration” at the 2020 Institute for Faith, Teaching, and Learning at CBU [6]. The Merriam-Webster dictionary defines “integrate” as “to form, coordinate, or blend into a functioning or unified whole” (see [9]). Understandably, this phrase can perpetuate the root of the problem: the secular lens portrays higher education disciplines as naturally disjoint from faith. The purpose of “faith integration” is not to force the union of disjoint areas. Instead, the purpose is to illustrate

faith as an essential, complete, and correct foundation of our discipline. Rightly seen, the secularization of mathematics is the forceful and unnatural separation of faith and mathematics. As David S. Dockery puts it (see [2]), “we seek to sanctify the secular.” Dockery reminds readers through Charles Wesley’s hymn that integrating faith is the act of reuniting “the pair so long disjointed.”

Returning to the analogy of the defaced petroglyphs, Christian faculty are thusly aiming to restore the sacredness of mathematics slowly blemished by secularism. This is similar to our very own progressive sanctification as disciples of Christ. Loosely speaking, sanctification is the process of becoming more like Jesus, the image of the invisible God. Before the fall of humanity, humans were perfect image bearers of God. Ergo, sanctification is the process of restoring secularized or fallen image bearing to its pre-fall form.

In the preservation of art and history, restoration is often necessary to undo the damage of vandalism. The *Oxford English Dictionary* defines the word “restoration” as “the action of returning something to a former . . . condition.” That is the ideal goal of the Christian mathematics professor; teaching and practicing the standard knowledge and skills of mathematics while restoring the defaced sacred elements at the hands of a secular worldview. As Dr. Scott B. Key says (see [5]), we are “to begin the excavation of the deepest assumptions of our disciplines” and to “find the central faith convictions that will illuminate the faith assumptions of our various disciplines.”

4 A Framework for Faith Restoration of Mathematics

The proposed framework will be rooted in the concept of Faith Restoration so far discussed. Ergo, the framework’s outcome should be a faith-rooted understanding of mathematics, mathematical research, and of the culture of mathematical communities. To accomplish this, Christian faculty must clean the blemishes left by secular perspectives of the mathematical sciences. As specified in the second section of the paper, faith should inform how a mathematician perceives the foundations of mathematics. Specifically, mathematicians should reconsider their ontological positions in light of their theological commitments. In fact, the ontological position of a Christian mathematician should become more nuanced in light of certain faith commitments (see the Calculus example below). Furthermore, faith should deepen the reason and the ways mathematicians apply mathematics to the real world. While the enjoyment of applied mathematics is important, the Christian mathematician adopts a good Samaritan mentality by seeking to solve problems that are unnoticed, ignored, or inconvenient. Faith reminds the Christian mathematician that when the beauty of pure mathematics causes awe, it should ultimately lead to the awe of God. Lastly, our assumptions about proper teaching, mentoring, and inclusion should be challenged and illuminated by our faith. It is in these ways that the cleansing should be accomplished.

The cleansing should be done with care to not cause further vandalism in our pursuit of restoration. This balance can be achieved with thoughtful analysis of mathematics from the vantage point of faith as believing mathematicians hold every thought captive to Christ. While this paper is not addressing the practice of discerning between what a blemish is and what a genuine faith mark is within the mathematical sciences, it is worth noting that such discernment deserves its own attention in separate papers or books. At the very least, the distinction between vandalism and original markings should be determined by theological considerations (regarding ontological positions) and Kingdom of God principles (regarding the intention of application and community culture).

Presently, this paper proposes a model that invites Christians to restore the original faith markings

of mathematics. The model invites faculty and students to renew the common assumptions of our discipline through the aforementioned key three areas: mathematics teaching, mathematics research, and mathematics culture.

The approach to restorative teaching via what the author calls “Faith Themes” is partly inspired by the categories of faith integration for Christian higher education proposed by Holmes (see [3]) and expanded by Russell Howell (see [4]). Moreover, at the 23rd Biennial ACMS conference where this paper was presented, the author was made aware of the prior work done by McCoy (see [7] and [8]) via his own presentation at the same conference.

4.1 Restorative Teaching

One way to restore the assumptions students hold is to develop and incorporate a *Faith Theme* into each of the department courses.

Definition 1. A *Faith Theme* is a faith-informed assumption about the mathematical sciences that is periodically discussed throughout the course. The Faith Theme of a course should hold some connection to the course purpose and description.

For the duration of the course, the professor should periodically expound on the Faith Theme through specific faith discussions that argue and underline the overall Faith Theme. Professors should assess a student’s understanding of the Faith Theme and its corresponding faith discussions through a Faith Restoration assignment. It is important to understand that the Faith Theme should be emphasized consistently throughout the semester. The student should have a memorable understanding of the Faith Theme by the end of the semester. The author recommends using 5-10 minutes weekly to reiterate the theme and explore it through a faith discussion.

Faith Themes can be obtained through the exploration of the categories of faith integration for Christian higher education proposed by Holmes (see [3]) and expanded by Russell Howell (see [4]). Specifically, with the 5th category provided by Russell Howell, the following categories can provide inspiration to select appropriate Faith Themes:

1. Foundational
2. Worldview
3. Ethical
4. Attitudinal
5. Pranalogical

Below are two examples of mathematics courses at California Baptist University, their given Faith Themes, and the faith discussions that help to drive the Faith Theme throughout the duration of the course.

MAT 245 Analytical Geometry and Calculus I

Faith Theme: *The ontology and application of mathematics imply God’s existence.*

Explanation: Notice that the Faith Theme is a faith-informed statement about two things mathematicians discuss. These are the ontology and the applicability of mathematics. A secularized perspective of mathematics or even a naturalist worldview would not have a faith-informed conclusion about the applicability and ontology of mathematics. How is this conclusive assertion obtained? The Faith discussions guide the students through various reasons and arguments that build the case of the Faith Theme. First, the professor argues that faith informs the Christian mathematician that Anti-Platonism is a viable alternative for the Christian in light of the tension that exists between Christian doctrine and the independence of mathematical objects under traditional Platonism. Then, the professor argues that Naturalism should not embrace Platonism. Thus, for different reasons, the Christian and the Naturalist may embrace Anti-Platonism. From that point, the professor uses various discussions, listed below only by title, to further argue the main Faith Theme.

Faith Discussions:

- The existence of mathematical objects
 - The abstract nature of mathematics in Calculus.
- Platonism and God
 - The Case for Anti-Platonism
 - The Aseity-Sovereignty doctrine. The problem of God and abstract objects.
 - Naturalism is necessarily Anti-Platonist.
- Unreasonable applications of mathematics
 - Eugene Wigner’s paper “The unreasonable effectiveness of mathematics”
- The applicability of mathematics as an argument for God’s existence by Dr. William Lane Craig (WLC)
 - Alex Rosenberg’s rebuttal to the WLC argument
 - The Naturalist Die
- Defining Inherently Mathematical Physical Phenomena
 - Example of Quantum Mechanics
- Cordero’s Modified Argument for God’s existence based on the applicability and ontology of mathematics.
- Biblical support for the argument
 - Romans 1:20
 - Devotional on Romans 1:20
- Christian versions of Platonism that avoid the Problem of God and Abstract Objects

MAT 145 Mathematical Methods in the Natural Sciences

Faith Theme: *Faith in God is Reasonable within Science and Mathematics.*

Explanation: Notice that the Faith Theme is a challenge to a typical naturalist perspective, namely, that faith and the sciences are incompatible. For the remainder of the semester, the professor continues to develop and defend this statement through various arguments, inspired by the conversion and faith journey of Francis Collins.

Faith Discussions:

- A brief history of faith and the sciences.
- Examples of Christians that are scientists today.
- The conversion of Francis Collins. Use his example to learn various arguments for God's existence.
- Argument for God's existence via the applicability of mathematics to the physical universe.
 - Discuss the ontology of and applicability of mathematics as an argument for God's existence.
 - Romans 1:20
- Kalam-Cosmological argument for God's existence.
 - Big Bang Theory
 - The possibility of a self-causing universe
- Fine-tuning of the cosmological constants-argument for God's existence.
 - One Objection: The Multi-verse
 - Another Objection: Non-Carbon-based life possibilities.
- The historicity of Jesus.
- Christian perspectives on evolution.

Intentionality in Restorative Teaching

Faith Themes developed through Faith discussions keep Christian faculty accountable to restore faith with consistency and excellence. This means that Christian faculty must renew their minds to view the Faith Theme as an essential component of the course; not merely something that is done to meet a requirement. Moreover, faculty cannot merely open the class with an unrelated prayer and/or an unrelated scripture reading. While this is good Christian conduct, it is not thoughtful faith restoration. Rather, a deep understanding of the Faith Theme must permeate through our actions, assumptions, and interactions with others in the mathematical community. Otherwise, the purpose of Christian higher education fades away. Christian higher education cannot simply be higher education taught by Christians who have a great witness and ministry. For that, there is much Christian representation in secular universities. Instead, Christian higher education must be rooted in restoring faith in the disciplines. This requires intentionality, preparation, and commitment.

It is worth noting that there is a challenge to a semester-long commitment to restore faith periodically in our lectures. The challenge is that a faculty member rarely has enough time to cover all the material of a course while providing quality instruction, opportunities for learning, and assessment. Therefore, faculty may be discouraged from periodically restoring faith in the mathematical sciences since the professor's concern is for students to be academically prepared. However, as disciples of Jesus, Christian faculty believe that any academic preparation should be Biblically rooted (see [1]). While most of the time of a faculty member's lecture is devoted to mathematics-specific outcomes, starting a lecture with a very brief but substance-filled faith discussion will remind students that any academic preparation must be rooted in the inspired word of God. This reminder is part of a larger process that moves the minds of Christians beyond the compartmentalization of faith.

4.2 Restorative Research

In keeping with the heart of Faith Restoration, it is important to go beyond the pedagogy of mathematics. Regarding research, mathematicians that have restored faith do not do research in some unknown new "Christian" mathematics. Instead, Restoring research involves restoring the reason one does mathematical research. Often, academics will follow their research interests as an ultimate end. In other words, mathematicians typically become scholars of topics that fascinate them within the mathematical sciences for the pure enjoyment of these. While mathematicians should love their disciplines and passionately pursue the curiosities within, they must cultivate their passion within the cruciform love of God and neighbor. That is, their research curiosities should serve the purpose of loving God and loving their neighbors. This approach should generate research that:

1. Glorifies God through the deep understanding of mathematics that reveals beauty and the intricacies of the created universe (Romans 1:20)
2. Lovingly serves the neighbor via the application of mathematics towards solving real-world problems. (Philippians 2:4)

This means that a Christian scholar should not seek to solve problems that are merely interesting or exciting. Rather, a restored Christian scholar should seek to solve problems because in doing so they consciously exalt God and/or because they compassionately love the sufferers of the problems. It is important to emphasize that one can enjoy research interests. However, the enjoyment should not stop at the object of one's research. The enjoyment of said things should point our affections toward God. Thus, while Christian mathematicians enjoy their research interests, they complete their joy as their research serves others and illustrates God's wondrous glory.

Furthermore, in emphasizing service through research, the community of mathematicians learns to value careers primarily as ministries to individuals in need.

4.3 Restorative Culture

Like other scholars and educators, mathematicians participate in communities and networks of peers. With such communities, one often finds cultural practices that have been passed down from a mentor to a mentee or from a professor to a student. As math majors, graduate students, and faculty share their experiences, they converge on common experiences, idioms, practices, expectations, and approaches. It is likely that most Christian mathematicians have been influenced by these at secular institutions. Much like in the opening analogy of this paper, Christian mathematicians may come

to believe this is the true form of mathematics culture. But Kingdom principles reveal a way to restore the culture of mathematics.

University professors bring a lot of emotional and psychological baggage in the form of discipline culture. For example, one might hear an academic say that graduate school is supposed to be hard and cause severe anxiety. Someone might even witness academic peers tear each other's work apart for the sake of one-upmanship. Whether faculty realize it or not, when they were graduate students trying to survive the academic culture, they assimilated culture and assumptions from their professors. And yet, faculty rarely stop to think about which aspects of the mathematics culture are toxic and likely even detrimental to a student's learning or to fellowship with peers. Even more concerning, some faculty may unknowingly pass these toxic tendencies to their own students. While academia tends to frown on grace, Francis Su (former president of the MAA) implores academics to embrace grace in his highly acclaimed talk on the Lesson of Grace in Teaching (see [11]). Su recalls when his advisor told him that he did not "have what it takes to be a successful mathematician." Fortunately, Su found a different advisor:

Persi Diaconis was an inspiring teacher. More than that, he had shown me a great kindness a couple of years before. The semester I took a class from him, my mother died and I needed an extension on my work. I'll never forget his response: "I'm really sorry about your mother. Let me take you to coffee." I remember thinking: "I'm just some random student and he's taking me to coffee?" But I really needed that talk. We pondered life and its burdens, and he shared some of his own journey. For me, in a challenging academic environment, with enormous family struggles, to connect with my professor on a deeper level was a great comfort. Yes, Persi was an inspiring teacher, but this simple act of kindness—of authentic humanness—gave me a greater capacity and motivation to learn from him, because we had entered into authentic community with each other, as teacher and student, who were real people to each other.

Su went on to say about his new advisor, Persi:

By taking me to coffee, he had shown me he valued me as a human being, independent of my academic record. And having my worthiness separated from my performance gave me great freedom! I could truly enjoy learning again. Whether I succeeded or failed would not affect my worthiness as a human being. Because even if I failed, I knew: I am still worth having coffee with! Knowing my new advisor had grace for me meant that he could give me honest feedback on my dissertation work, even if it was hard to do, without completely destroying my identity. Because, as I was learning, my worthiness does not come from my accomplishments.

The truth is that this realization is not only liberating for students, but also for faculty! Without grace amongst the community of mathematicians, the members of this community will seek validation through achievements. Mathematicians will compete with one another and seek to demolish the achievements of others; for if the value of people is in what they achieve, then those that achieve more will be of more value. But grace frees people to enjoy identity and value in their adoption as sons and daughters of the Father. Consequently, Christians can enjoy the pursuit of the mathematical sciences free from the pressure of having to prove themselves.

Faculty must lead the way to create a safe mathematical community of faculty and students alike where the mind of Christ reigns supreme over their academic endeavors. This common mind would

produce the humility of Christ that frees believers to count others as more significant. What a Kingdom-like community this would be! It would stand in direct contrast to the patterns of worldly academia. When students see that faculty are free from the crushing pressures of academic validation, while also free to pursue the academic endeavor for the sake of the cruciform love of God and others, students will see that their professors know the truth and that indeed the truth has set them free.

Restorative culture should also make mathematicians aware of issues that academia has historically neglected. For example, lack of representation in STEM from various ethnic groups and women is still an ongoing problem (see for example [10]). Moreover, the church provides the most complete case for the need for diverse STEM representation. Christian higher education as such should foster an academic culture that represents every nation, tribe, and language. This is an essential component of the faith since disciples realize that the fullest expression of the union of the body of Christ is found in the diversity of its united members in Christ:

Revelation 7:9-10 ESV After this I looked, and behold, a great multitude that no one could number, from every nation, from all tribes and peoples and languages, standing before the throne and before the Lamb, clothed in white robes, with palm branches in their hands, and crying out with a loud voice, "Salvation belongs to our God who sits on the throne, and to the Lamb!"

Galatians 3:28 ESV There is neither Jew nor Greek, there is neither slave nor free, there is no male and female, for you are all one in Christ Jesus.

5 Closing Remarks

While many Christian mathematicians understand the purpose of Restoring Faith, Christian faculty must make sure that their implementation matches their intended purpose. Without intentionality, some may treat the restoration of faith as inconvenient duty and not as something that is central and vital to Christian Higher Education. Christian professors must understand that they are ministers. Their ministry is the faith restoration of the mathematical sciences. Their passionate liturgy is the methodical and intentional teaching, research, and culture that cleans the vandalism of a secular worldview.

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Discovery, Invention, and the Mathematical Logic Loop

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Abstract

In the present paper we introduce the Logic Loop as a nuanced ontology that weaves perspectives from various individual ontological positions. In Section 2 of the paper, we turn to various case studies as support for the Logic Loop. In Section 3, the Logic Loop is compared with three of the various ontological positions to show none sufficiently illustrates the Logic Loop's makeup. In Section 4 we distinguish three broad categories of ontologies that will help us understand their importance as well as how the Logic Loop can address each of the three. In Section 5 we suggest a list of seven ontological concerns for a Christian mathematician, and through these we formally introduce the Logic Loop. Section 6 argues that a Christian mathematician that subscribes to the Logic Loop will see the practice of mathematics as a habit of grace or a liturgy. Section 7 concludes the paper by highlighting the freedom in the ontology of mathematics that accompanies the Logic Loop.

1 Introduction

The prime occasion from which arose my **discovery** of the method of the Characteristic Triangle ... happened at a time when I had studied geometry for not more than six months ...

Gottfried Leibniz, *letter*, 1680 [8, p. 543]

Then I shall show how this line can be described by a motion that I have **invented**.

Gottfried Leibniz, *Acta Eruditorum*, 1693 [19, p. 282]

In the philosophy of mathematics, a central discussion and concern is the ontological nature and related existence of mathematical objects. Do mathematical objects exist? If so, in what sense? Do we invent or discover mathematics? That is, is mathematics the study of real abstract objects awaiting discovery or is it a useful fiction or construction we have invented to describe the physical world? In this paper, we argue for a simple ontology vis-a-vis what we call The Logic Loop.

The Logic Loop is an ontology that treats mathematics as an ongoing endeavor of invention and discovery within a Christian worldview. We call it the logic loop because we see the rules of logic as the objective canvas on which the loop takes place. That is, if we are to explore and refine mathematics, the only constant is the rules of logic as applied to any selected/invented axioms. We believe that history provides a solid case for these stages of invention and discovery. Sometimes humanity invents axioms. Upon exploring these axioms, we discover that our axioms are insufficient or redundant. When we do so, we refine a new invention. Thus, the two stages of invention and exploration are revisited via processes of exploration and refinement. To use a familiar analogy, mathematical objects become game pieces and axioms are the rules of our game. After we have invented this game we can explore to see whether or not this game is consistent. If we discover flaws in the game, we refine it with a newer and better invention of the game. In this sense, mathematical objects are not abstract objects that exist.

Phrased another way, mathematics is about the *practice of mathematics* and involves stages of invention and discovery that are connected through the processes of refinement and exploration. What Hilbert called a game, we see as a Logic Loop that includes the iterative creation and playing of the game. Whether game or Logic Loop, invention is the stage where new mathematical ideas are formed. Refining is a process of cleaning up and/or distilling existing ideas for clarification. Exploration is a speculative process of asking, “What if?” Finally, the discovery stage is about finding applications for mathematics, discerning patterns, and searching results for consistencies (and inconsistencies). Together these stages and processes form the Logic Loop of mathematics. We contend that this addresses the praxis of mathematics by which we mean the work of the mathematician, the philosophy of mathematical practice, and the ontology of mathematics.

2 Case Studies Introduce the Logic Loop

The very nature of a loop makes identifying its starting point a challenge, but the story of calculus, our first case study¹, traditionally begins with Newton or Leibniz even though mathematicians had been working on extrema, tangent, and area questions for nearly two millennia prior. While Newton revolutionized physics and Leibniz was a philosopher of the first rate, they shared an entirely mathematical context and motivation for their invention of calculus. Their near simultaneous invention and recognition of computational methods connecting the tangent and area questions exemplifies the formation of new mathematical ideas. But what they published was not calculus as we know it today; especially the work of Newton, whose fluxions (derivatives) and reliance on geometric (rather than analytic) methods were difficult to apply. This reminds present day practitioners that raw and unrefined invention is just one step in the Logic Loop of mathematics.

Before proceeding, it is worth asking whether Newton and Leibniz did indeed invent the calculus; perhaps they were its discoverers? This is an especially relevant question given the epigraphs above that show Leibniz referring to both inventive stages and periods of discovery. Are we to accept one and discard the other? Certainly not! Within the context of the Logic Loop both remain true. Newton and Leibniz were also discoverers who found applications for mathematics and discerned its patterns. For that matter, they also refined mathematics and were math explorers. But their role in this case study is as inventors of new mathematical ideas.

What followed Leibniz and Newton’s invention was a rapid era of refinement leading to stages of

¹We have chosen to introduce the Logic Loop via case studies in response to the implicit call for such an analysis in Mancosu. [14, p. 2]

discovery. Noticeably absent were limits and the extremely precise definitions we associate with mathematics. Notations such as the elongated S of Leibniz’s integral being adapted for sums (S being from the Latin *summa*) were refined. Applications were found including countless uses of power series within calculus and differential equations. But what seemed to fuel the explosion was the discovery of numerous applications of calculus in the physical sciences. Newton began this tradition with his work culminating in *Philosophiæ Naturalis Principia Mathematica* (often shortened to *Principia*). This included deeply influential examples such as proving Kepler’s laws for planetary motion from just the calculus and a few basic physical laws.

The introduction of transcendental functions into calculus after their noticeable absence in the first half century of calculus is an example of exploration. While the concepts of logarithms, exponentials, and most notably trigonometric relationships existed, they were generally represented by power series. In particular, the sine was entirely understood geometrically. It wasn’t until 1739 that Euler invented our present-day sine function representation that is continuous everywhere. “Until that time there was no sense of the sine and cosine functions being expressed, like the algebraic functions, as formulas involving letters and numbers, whose relationship to other such formulas could be studied using the techniques of the calculus.” [8, p. 583]² Euler asked, “What if” these seemingly unrelated ideas are actually connected? By bringing trigonometry (and transcendental functions) into mainstream calculus, Euler began a new wave of discovery as cumbersome power series were replaced by their modern representations. More broadly, mathematicians continued to explore their powerful new tools (with only intuitive attention paid to nuance) to discover applications in motion, engineering, hydraulics, and a host of other fields. Anachronisms in our present texts such as the cycloid and hyperbolic functions were discovered to be the source of cutting-edge applications in pendulum clocks and the shapes of suspended cables.

The start of the 19th century saw the discovery of inconsistencies in this laissez-faire logic of calculus such as conflicting views around infinite series, continuity, and convergence. Centering around the soon-to-be foundational concept of the limit, there was a movement to refine the work of the cavalier elders who had simply dismissed and discarded “sufficiently small” terms. Faulty assumptions and sloppy notations were identified, and assumptions challenged, culminating with the invention of Weierstrass’ famous pathological everywhere-continuous, nowhere-differentiable function. This season of fault-finding prompted an era of rigorization and refinement of calculus most recognizable today through Cauchy’s invention of a precise definition of the limit (delta-epsilon).

Thus, calculus illustrates the historical reality of mathematics as an ongoing process. That is, mathematics is a Logic Loop alternating between stages of invention (new ideas formed) and discovery (finding applications and looking for inconsistencies) through processes of refinement (distillation) and exploration (asking, “What if?”). We will now consider additional case studies to clarify what is meant by each of these terms.

2.1 Case Study: Multiple Geometries

Mathematics has long been a refuge for those seeking certainty with no stronger bastion than that of Euclid’s *Geometry*. While Euclid was perhaps more a collector and organizer than originator, his structure of axioms, postulates, theorems, and *proofs* became the gold standard for clear and

²For example, Katz [8, p. 593] relates how Euler solved differential equations using power series methods, but did not recognize a sine or cosine in the resulting answer. It was not until 1739 that Euler realized that the sine function would enable closed-form solutions of such higher order equations to be given. In a paper from March of that year, Euler gives what is perhaps the earliest use of the sine as a function of time.

necessary reasoning. In particular, thinkers were drawn to the irrefutable givens such as, “things which are equal to the same thing are also equal to one another,”³ and “that all right angles are equal to one another.”⁴ These axioms/postulates were irrefutably discovered in nature as we discerned abstract patterns take on physical form and geometry had numerous applications to ordinary living. Geometry was seen as even more secure than the ground we walk upon. These ideas were not just resigned to the ivory tower. Within fiction, famed Russian novelist Fyodor Dostoevsky’s character Ivan Karamazov espoused, “if God exists and He really did create the world then, as we all know, He created it according to the geometry of Euclid . . .” [2, p. 216] This self-evidence and undeniability is an example of what led many to believe in the existence of abstract objects (mathematical Platonism) awaiting discovery by clever math explorers.

However, one of Euclid’s postulates was different. Even to the uninitiated, the fifth postulate stood out being triple the word count of the others. Geometers sought long and hard to reword or replace this black sheep of the postulates. Playfair “simplified” the phrasing to: “Through a given point P not on a line L , there is one and only one line in the plane of P and L which does not meet L .” Still overly complex, mathematicians sought to eliminate the postulate by proving the unique existence of parallel lines from within the other axioms. Both the revised wording from Playfair and the attempts to make it unnecessary are examples of what is meant by refinement. So here, refinement refers to the fine-tuning of wording/technique and repeated attempts to improve geometry and do away with the troublesome fifth postulate.

“What if?” **Geo**-metry: Measuring the earth. Wherever humans looked (whether in nature or in the realm of mathematics), parallel lines behaved exactly as Euclid postulated. To question Euclid was akin to questioning the nature of reality. But about two millennia after Euclid, bold math explorers such as Saccheri, Bolyai, and Gauss began to dare to ask, “What if?” Challenged by the work of Russian mathematician Lobachevsky, Dostoevsky continued the previous passage, “. . . they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity.” What if there are no parallel lines? What if there are multiple parallel lines through a point? Because Euclid’s axiomatic approach had been adopted across the sciences, social science, and theology, the impact of these brave math explorers asking, “What if?” was dramatic. “What if” is the key to exploration.

The necessary absolutes of Euclid could have been otherwise. Specifically, in the 1820’s and 1830’s, new systems were invented; geometrical realms devoid of parallels or, alternately, home to an infinite number of parallels.⁵ Non-Euclidean geometries were a new mathematical idea breaking in and changing the nature of mathematics and the world. In philosophy at large, “one of the basic tenets of [Kant’s] system was the idea that Euclidean geometry is the only possible way of thinking about space.” [18, p. 183] So these mathematical explorers bucked common understanding in mathematics and intellectual culture at large. Needless to say, the invention of non-Euclidean geometries required boldness and was the catalyst of changes across a multitude of disciplines.

A surprising event took place decades into the life of these new non-Euclidean geometries. In the world of physics, Einstein invented the theory of relativity. In fleshing out his ideas, he dared to ask about the geometry of the universe and this explorative “What if?” He connected seemingly unrelated ideas and opened up a new wave of discovering applications in non-Euclidean geometry.

³This is the first of Euclid’s five axioms.

⁴The is the fourth of Euclid’s five postulates. In modern mathematics, we use the term “axiom” and would have said that these five postulates were the basis of Euclid’s axiomatic system and method.

⁵There were earlier hints, but perhaps it is easiest to date non-Euclidean geometries to papers published by Lobachevsky (1829-30) and Bolyai (1832). [9, p. 83]

Einstein also provides a capstone on this case study on the nature of geometry and how the work of geometers manifested each of these four characteristics: discovery, refinement, invention, and exploration.

2.2 Case Study: The Foundational Movements

The “philosophy of mathematics” is sometimes seen as synonymous with the foundational schools formed at the start of the twentieth century. What follows is a sketch of these ideas and events as they pertain to the Logic Loop. It should ring true for those familiar and resources are in the notes for those wanting more detail.⁶

Mathematics is often seen as the most certain of all knowledge providing a perfect one-to-one correspondence to the natural world. As shared in the last section, with its structure of axioms, postulates, and theorems, Euclid’s *Elements* was the North Star guiding the way for over two millennia. But the foundation of geometry was adjacent to, if not resting upon, sand. Questions around the parallel postulate led to the invention of non-Euclidean geometries. The natural world could no longer serve as our test for the truth of axiomatic systems. A new foundation was needed: could logic suffice?

With nature ruled out, Gottlob Frege sought to uncover a new basis for mathematics.⁷ In 1884 and through the concept of “set” adopted from Bolzano, Frege discovered and shared the “foundation for arithmetic” in his book by the same title. These were previously unrecognized patterns and consistencies in the very structure of arithmetic. Through set-theory and logic, Frege believed that the basis of all of mathematics could be secured starting with its most rudimentary topic: logic. This is the origin of the logicist school.

Frege’s ideas were neither widely known nor easily accessible. But Frege only needed one disciple. Bertrand Russell was captivated by the certainty provided by logicism but realized that more refinement was needed. His Barber’s Paradox exposed a fatal flaw in Frege’s program and led to a series of refinements in logicism, each intending to address paradox without sacrificing the self-evidence of the axiomatic system. Only somewhat successful in this, Russell’s labors paralleled the invention of the alternative foundational schools of formalism and intuitionism.

Representative of formalism, David Hilbert outlined 23 unsolved problems in mathematics (for the International Congress of Mathematicians in 1900) which can be seen as a call to explore the limits of the formalist school.⁸ Of all the solutions to Hilbert’s questions, perhaps the most remarkable was that made by Kurt Gödel who, in a pair of papers published in 1931, resolved Hilbert’s second question on the consistency of the axioms of arithmetic. In particular, Gödel proved that the consistency of certain formal systems could not be proved from within the systems themselves. Effectively showing the hubris of Hilbert’s claim that in mathematics there is no, “we do not know and will not know”, Gödel’s discoveries of consistency/inconsistency in the formalist program sparked a new series of refinements as well as the invention of additional ontologies of mathematics beyond the foundational schools.

⁶Two good sources on the foundational schools that are referenced elsewhere in this essay are Kline [9, pp. 216–257] and Hersh [6, pp. 137–164].

⁷In reference to Frege, we have chosen to use the word “discover” to honor Frege’s adherence to mathematical Platonism (about which more will be said later).

⁸Hilbert presented 10 questions in his talk and the remainder were published in the two following years. Solutions to his list of questions were discovered as early as 1900 and as recently as 2002 leaving just three questions completely unresolved.

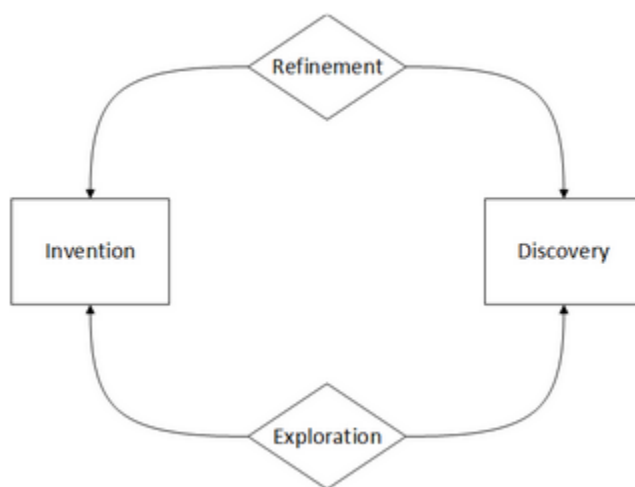
Rewinding momentarily, prior to Gödel’s theorem was a season of exploration. Gödel initially set out to prove Hilbert’s claim just as Russell had wanted to prove and build upon the work of Frege. For both logicians, there came a time when they asked the math explorer’s question: “What if?” Before Gödel’s theorems or Russell’s paradox could be discovered, they made the shift from believing the claim true/valid to looking for a counterexample. Thus, while perhaps quite brief, their discoveries each contain a critical element of exploration.

The final foundational school, intuitionism, is more overt in its ontology of invention. Its oft-quoted forefather, Leopold Kronecker, said, “The integers were created by God; all else is the work of man.” [6, p. 74] This school famously rejected mathematical notions that could not be constructed including a rejection of the axiom of choice as well as the law of excluded middle. The intuitionists are an outlier among mathematical philosophies in their insistence that mathematics is invalid unless we possess a method for positively reaching the result.⁹ They not only hold that math is invented, but dictate the rules by which acceptable new mathematical ideas may be formed. This approach excludes large swathes of mathematical practice, bringing to mind Maddy’s words, “If our philosophical account of mathematics comes into conflict with successful mathematical practice, it is the philosophy that must give.” [14, p. 11]

While this case study focused on the Foundations of Mathematics may span the least time, it again highlights the four phases of mathematics and mathematical practice: The “discovery” of the Foundations of arithmetic followed Russell’s “refinement” of set-theory. Formalism and intuitionism were “invented” and Gödel dared to ask the “What if?” question that marks all math explorers.

2.3 Case Studies and the Logic Loop

Case studies bring life to models and ideas. However historical narratives are rarely as tidy as the neat little diagram we use to model the Logic Loop. Further explanation and guidance is required.



If I may introduce a new metaphor, the rectangles in the diagram indicate that invention and discovery are stages in mathematical practice when a mathematical child is born. Sometimes this mathematical entity is a first-born and we call this invention. Other times it has older siblings in which case we call it discovery.

⁹For example, suppose either A exists or B exists. If we know that B does not exist, then most mathematicians hold that A must exist. However, this requires the use of the Law of the Excluded Middle and so intuitionists would maintain that we do not know whether A exists.

For example, we claim that Newton and Leibniz invented the calculus. Non-Euclidean geometry was invented in the early 1800's. And formalism and intuitionism were entirely new ways of thinking about the nature of mathematics. While a historian of mathematics would likely argue that these ideas rest on the shoulders of their predecessors, we see this as exemplifying the cyclic nature of a loop and press on.

We use three examples to represent the stage of discovery; finding applications for mathematics, discerning patterns, and searching results for consistencies (and inconsistencies). In the 19th century, consistencies and inconsistencies in analysis were discovered which led to the rigorization of the calculus. Patterns were recognized in nature that were critical in developing intuition around geometry (as well as applications of geometry to nature). And Frege focused on discerning patterns in the very building blocks of arithmetic itself.

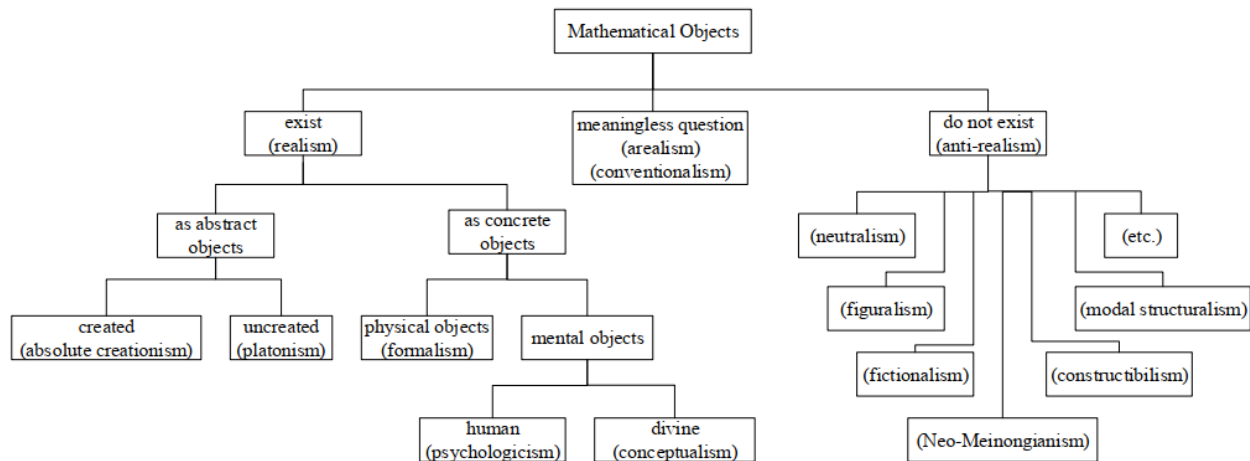
Returning to the diagram, the diamonds of refinement and exploration represent the challenging process of raising mathematical children. Refinement is the day in, day out process of cleaning up and distilling mathematical ideas. We referenced the notation of the calculus, rephrasing of the fifth postulate, and Russell's labor building out Frege's logicism.

The remaining diamond, exploration, comes with a flash of brilliance and the hard labor of a miner. Asking "What if?" is the hallmark of exploration and is the process of making new connections or going in a new direction from predecessors. Euler made a connection between trigonometry and power series and thus was a primary introducer of transcendental functions into calculus. Einstein arrived at a whole new way to geometrize the universe. And Gödel dared to ask if Hilbert might be wrong.

The careful reader might ask if making such a sharp distinction is justified: Who is to say whether a square should be a diamond or this was invented or discovered? While a reasonable concern and one we too ask ourselves, we believe it misses the larger point. Mathematical practice could not proceed without the stages of invention and discovery and the processes of refinement and exploration. They are not wholly distinct and the historical mathematical practice rarely has a clear beginning or end, which is why we call it a loop. The bidirectional arrows remind us that the loop is not traversed in a single direction and, for that matter, we could live in a season of, say, refinement and discovery for quite some time. But while the Logic Loop model may resonate, we have yet to address the nature of mathematics itself: its ontology.

3 The Logic Loop and three Ontologies of Mathematics

Seen from one light, the mathematical Logic Loop is an inclusive ontology with many likely allies. The mathematical Platonist supports its aura of discovery. The formalist approves of the constant refinement. The naturalist nods at the primary role of human ingenuity, and historians of mathematics smile at the creative exploration. At the same time, as a compromise position, the Logic Loop satisfies none of the major ontologies. It disappoints the nominalist by acknowledging the indispensability of mathematics and frustrates the Platonist by denying the existence of abstract objects. We found the following diagram helpful in parsing ontological views in regard to mathematical objects. [5, p. 40] Broadly speaking, the next three sections fall under the categories of Platonism, anti-realism, and the various theistic branches.



3.1 Mathematical Platonism

Mathematical Platonism holds that mathematical truths such as numbers, perfect geometric figures, and the great truths live in Plato’s heaven for abstract objects. This certainly includes the Golden Ratio, Pi, Euler’s number, and the Pythagorean Theorem. It may include every consistent mathematical statement. Thus, each time we “discover” a new mathematical truth, whether awe inspiring like the fundamental theorem of calculus, or seemingly trivial like $1 = 1$, we are simply identifying and isolating one of the members of Plato’s mathematical horde.¹⁰

There are three primary objections to this view. The first is that there is no and can be no physical evidence for a realm of abstract objects. The second is, “it seems that these abstract, eternal, objective things fall outside the range of our human cognitive powers.” [13, p. 354] And, third, even if it exists and is knowable, there is no reason to assume that these abstract objects would have any relevance to the physical world. Mathematical Platonism effectively describes math as 100 % discovery and 0 % invention.¹¹

The mathematical Logic Loop is not mathematical Platonism as it acknowledges that mathematics includes components of both discovery *and invention*. The line between the two lies in murky water, but even the child learning arithmetic has occasion to wonder about the universality of mathematics while also asking, “How did someone come up with this?” In order to walk this tight-rope, within the Logic Loop model, we reject the existence of abstract objects but remain open to concrete realism. Discovery and invention are the reality of the mathematical process. But the axiomatic games of the mathematical mind(s) generate beautiful mathematical realms in need of discovery. Mathematicians of all regions and across all ages are both discoverers and inventors.

3.2 Nominalism

A nominalist paradigm for mathematics denies the existence of abstract mathematical objects (for this reason, it is sometimes referred to as anti-Platonism). While some have argued for physical or psychological explanations for math, the nominalist sees it as a human fiction or useful story humans

¹⁰It should be noted that there are as many versions of Platonism or mathematical realism as there are philosophers. For example, Maddy outlines at least five in her book, *Realism in Mathematics* in an effort to explain her own set-theoretic realism. [12, pp. 20ff]

¹¹This plenitudinous mathematical Platonism wherein every possible coherent mathematical statement and object exists, is called “full blooded Platonism.”

use to explain the world. This can range from the socio-mathematical humanists who practice mathematics within culture to formalists who see it as a game we create and play with meaningless symbols. Nominalists maintain that the patterns we find may be beautiful but certainly are not eternal truths that transcend time, place, and culture. In short, the nominalist holds mathematical work is 100 % human inventive labor and 0 % discovery of an eternal mathematical reality.

There are two primary objections to this brand of mathematics. The first is that it says that any sense of eternal transcendence within mathematics must live in our head: think the Golden Ratio. As if giving the appearance of individual transcendence isn't sufficient, in a discussion of the mathematical beauty, Francis Su writes of the famous formula $e^{i\pi} + 1 = 0$, "there is no obvious reason why five of the most important constants in the universe should appear in the same equation." [20, p. 73] While humans are capable of incredible creativity, in other disciplines, there is a name for duplicate independent artistry: plagiarism. But repeated and independent invention is commonplace in mathematics. Elsewhere Su writes, "When you see the same beautiful idea pop up everywhere, you begin to think that it is pointing to some deeper truth you haven't yet grasped."

The second objection was summarized by Albert Einstein who wrote, "How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason then, without experience, merely by taking thought, able to fathom the properties of real things?" [3, p. 233] Or, as Shapiro writes, "Why are the mathematical games so useful in the sciences? After all, no one even looks for useful applications of chess. Why think that the meaningless game of mathematics should have any application? It clearly does, and we have to explain those applications." [17, p. 146] Returning to Einstein, anti-Platonists like himself (including nominalists) must address this unreasonable effectiveness as explainable, irrelevant, or unsubstantiated.

Our Logic Loop describing mathematical practice is not so dogmatic. It fully embraces the ingenuity of men and women across the ages to invent mathematical ideas and systems. The model allows for the creative and hard work of mathematicians to refine and explore these ideas using their cognitive powers. The loop embraces the explanatory power of language, and it acknowledges the fact that mathematical systems are established inside socio-cultural contexts. At the same time, the loop allows for a sense of discovery and transcendence that lives beyond feelings and perception. Whether discovering something new or simply carving away the excess to invent new mathematics, we are image bearers and co-creators.

3.3 Theism

Writers on the history of the philosophy of mathematics regularly make religious references. Morris Kline drew parallels between the demise of axiomatic confidence in mathematics and the Tower of Babel. [9, p. 71, 95, 207, 226] Mario Livio wrote, *Is God a Mathematician?* and Reuben Hersh penned, "The present trouble with the ontology of mathematics is an after-effect of the spread of atheism." [6, p. 126] While likely not religious themselves, they are tapping into a spiritual theme within mathematics that stretches back hundreds of years. Perhaps the most famous mathematical mystic is Johannes Kepler. His Christian faith was the origin of his belief that the universe is both rational and intelligible. "Those laws [of nature] are within the grasp of the human mind; God wanted us to recognize them by creating us after his own image so that we could share in his own thoughts."¹² Theism isn't a philosophy of mathematics per se; rather it is a metaphysical view

¹²Kepler, Johannes. Letter (9/10 Apr 1599) to the Bavarian chancellor Herwart von Hohenburg. Collected in Carola Baumgardt and Jamie Callan, *Johannes Kepler Life and Letters* (1953), 50. This source is perhaps the origin

of the world that believes that the rationality of our mathematical world is best explained by the existence of a rational entity outside of space and time.

Within mathematics, there seems to be two objections to invoking supernatural elements. The first is the rise of atheism and its dismissal of unscientific explanations as pseudo-answers. Perhaps the most famous example of this comes from a reported interchange involving Napoleon, Laplace, and Lagrange. After reading his treatise *Celestial Mechanics*, Napoleon is said to have shared with Laplace: “You have written this huge book on the system of the world without once mentioning the author of the universe.” To this, Laplace replied, “Sire, I had no need of that hypothesis.” As if this gem isn’t sufficient, the story continues that when later told by Napoleon about the incident, Lagrange commented: “Ah, but that is a fine hypothesis. It explains so many things.” (MacTutor) This anecdote is the inspiration for numerous books and articles (both in support of and discounting) on the God hypothesis.

The second concern is that divine explanations for mathematics are unclear and ill-defined in an area where we are seeking absolute certainty and precision. A novelization on Bertrand Russell captures this nicely. “Nothing in my life was quite the same after that first meeting with Euclid. In his work, I found what I had vainly sought for in Grandma’s faith! . . . Reason [and] knowing something with total certainty.” [15, p. 57] We struggle even in providing a coherent and consistent definition of faith, but within mathematics we have “proof.”

The Logic Loop is more flexible, acknowledging that the ontology of mathematics extends well beyond mathematics into philosophy and theology. Lagrange and Laplace were overly dogmatic. At the same time, it may not be as simple as claiming *all* of mathematics is ideas in the mind of God.¹³ The Logic Loop may not avow abstract objects but allows space for other forms of realism.¹⁴ If you will, the Logic Loop may be incompatible with “either-or” theism but should be acceptable to “both-and” belief systems.

However, mathematical Platonism, nominalism, and theism are not the only presuppositional positions one might take toward mathematics. As we explore the question “What is mathematics, really?” (Hersh), we quickly learn the dead ends and questions along each path.

4 The Threefold Problem of the Ontology of Mathematics

Ontology is the branch of philosophy studying the nature of being or existence at a first principles level. In terms of mathematics, the ontologist asks: what is mathematics about? What are numbers, geometric figures, or theorems and proofs? As noted historian of mathematics Judith Grabiner shared, “Is the square root of two a noun or adjective?”¹⁵ Are the natural numbers objects, properties, or something else entirely?

Histories of mathematics and readings on the philosophy of mathematics (within both mathematics and logic) tend to describe a wide range of views. While there is agreement around the foundational schools of logicism, formalism, and intuitionism, each had predecessors and has descendants. (This

of the unattributed Kepler phrase, “Thinking God’s thoughts after Him.”

¹³This phrase has roots in Augustine. See <http://readingthesumma.blogspot.com/2010/05/question-15-ideas-in-mind-of-god.html> for more information.

¹⁴13. See for example [5, pp. 1ff].

¹⁵Personal conversation between Dusty Wilson and Judith Grabiner (October 22, 2021) wherein Dr. Grabiner quoted her husband Dr. Sandy Grabiner.

is further complicated by the propensity of authors to coin their own nomenclature. For example, regarding her truth theory of realism in mathematics, Penelope Maddy writes, “This sort of theory has various names - redundancy theory, disappearance theory, deflationary theory - but I’ll call it a disquotational theory.” [12, p. 17] All this poses a challenge for locating the Logic Loop faithfully within the greater historical *and* philosophical narrative. But both are important. In the words of Imre Lakatos, “The history of mathematics, lacking the guidance of philosophy, has become *blind*, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics has become *empty*.” [10, p. 2]. With case studies grounding our studies and the guidance of past and present philosophers of mathematics, the Logic Loop is neither empty nor blind.

Some may believe ontological concerns are irrelevant for the daily working of mathematicians. But we take to heart the words of Irish mathematical physicist John Synge who wrote, “[E]ach young mathematician who formulates his own philosophy - and all do....” (Kline) As is often quoted, ideas have consequences and the Logic Loop is formed, at least in part, to push back against absolutism within mathematical paradigms. But what are the ontologies that the loop is meant to address? We frame the Logic Loop in the context of three broad ontologies of mathematics. They are ontologies that root existence within math/logic itself (intrinsic explanations), explanations from nature, culture, or the mind (extrinsic), and those that go beyond the mind and empiricism (transcendent explanations).

4.1 Intrinsic Explanations

An intrinsic explanation for mathematics is one that comes from within mathematics (or logic). Classic examples include Euclid’s geometry and the foundational schools of logicism and formalism. What these have in common is that they are predicated on the idea that there is a collection of (mathematical) objects, rules, or a structure (such as the axioms of Zermelo-Frankel) from which the rest of mathematics may be derived.

Looking inside mathematics for its own ontology runs afoul of two primary objections. The first are the logical paradoxes of Russell and incompleteness theorems of Gödel. While we must be careful not to overstate their impact, broadly speaking both seem to say that the more carefully mathematical systems are constructed, the more clearly the self-contradictory nature of intrinsic explanations becomes. This Tower of Babel culminated in Gödel’s work which formally proved that mathematics was not internally complete.¹⁶

The second objection stems from the applicability of mathematics to the natural world. Even were intrinsic explanations logically consistent, there is no obvious reason that it should reflect the patterns of nature. “[H]ow can the fact that one mathematical statement follows from another be correctly used in our investigation of the physical world?” [12, p. 25] Or phrased another way, there is an “unreasonable effectiveness of mathematics in the natural sciences” for “a game played according to certain simple rules with meaningless marks on paper.” [7, 21]

4.2 Extrinsic Explanations

For outsiders to mathematics, it may seem strange that mathematicians would have looked to internal narratives whereby math was explained from within mathematics. The observer can scarcely

¹⁶There exist theorems which are *true* within a finite set of axioms, but *not provable* within those same axioms.

avoid visions of the Hindu mythology whereby the Earth rests on the back of four elephants who are themselves supported on a giant turtle. What supports the turtle and how can math explain math? These (as well as many philosophers and practitioners of mathematics) see things otherwise and believe mathematics is both inspired by and generated based on the natural world. As Galileo famously wrote, “The universe cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.” (Galilei) Math is simply the language we create to describe the natural world. It is neurons firing a certain way in our heads. Math is a cultural agreement to see things a certain way.

Here again there are three points of contention. The first is that an extrinsic ontology requires that mathematics be a miracle drug wherein the same concept (say the derivative or integral) applies across numerous branches of science (and social science). “What accounts for the connectivity of mathematics? How is it that concepts developed in one part of mathematics suddenly turn out to be connected to apparently unrelated concepts in other areas?” (Corfield in Mancosu) It may be that if we have only a hammer, every problem appears as a nail. But if that is the case, then there are a lot of nails (applications) and multiple simultaneous inventors of the hammer (mathematical techniques). But if the hammer and nail have such unreasonably effective applicability, then this naturally prompts us to explore a related issue: universality.

Universality has two forms. The first is that the same mathematical concepts have been “discovered” in multiple times/places; Pascal’s triangle, the Pythagorean theorem, calculus, non-Euclidean geometries, . . . the list goes on. The second part of universality is that there are concepts that keep appearing; pi, the golden ratio, Euler’s number. It’s hard to escape the feeling that mathematical questions are too broad to explain with such a small set of tools.

The final concern with an ontologically extrinsic explanation for mathematics is perhaps less philosophical and more experiential. Many practitioners of mathematics feel a sense of discovery. They feel that they have stumbled upon or unlocked a pattern that goes beyond purely humanistic explanations. Putnam is quoted as writing, “Quine . . . for years stressed both the indispensability of [talk about] mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes.” [12, p. 30] Why double-speak in how we refer to our own work? If mathematics is just a structure, language, or fiction that we create to describe the world, why is it seemingly impossible to escape the sense of wonder and awe in mathematics?

4.3 Transcendent Explanations

Transcendent explanations are of (at least) two forms. The ontology of mathematics may simply “transcend” intrinsic and extrinsic explanations including self-existence *and* connection to the spatiotemporal world. Alternatively, mathematics may owe its being to a transcendent Being who brings mathematics into existence and/or provides humans the capacity to understand/interact with mathematical objects.

This latter explanation may include but doesn’t require the adoption of mathematical Platonism. That is, there are theists who are Platonists and theists who are not Platonists. There are naturalists who are Platonists and there are naturalists who are not Platonists. Furthermore, theists inclined toward realism in abstract objects should be cautious, knowing that there are se-

rious theological concerns with the idea of mathematical Platonism.¹⁷ So, while we use the term “transcendent” broadly, please understand that it includes more breadth than simply mathematical Platonism.

Case in point: Penelope Maddy, one of the few female philosophers of mathematics is herself a set-theoretic realist. About her motivation, she writes, “my motivation ... is naturalism; set-theoretic realism¹⁸ seems to me the most promising approach for bringing mathematical ontology and epistemology into line with our overall scientific world-view.” [12, p. 78] While perhaps more explicit in her motivations than most, Maddy’s desire to rule out non-material explanations for the ontology of mathematics is throughout the literature.

However, all forms of transcendence are prone to the labels of “unscientific” or “pseudo-explanations.” Simply put, every mathematician to have ever lived has been human. Books like Carl Sagan’s *Contact* aside,¹⁹ there isn’t a shred of physical evidence that mathematicians exist who aren’t human. So, to provide a transcendent explanation for the efficacy of labor performed exclusively through the work and resolve of men and women is to devalue the work of mathematicians. If you will, there are earthly explanations for all of mathematics thus making ethereal ontologies unnecessary.

So, if the known ontologies of mathematical objects and philosophies of mathematics are not satisfactory, perhaps the synthesis of ideas that allows for discovery and invention betwixt periods of refinement and exploration will allow us to move forward.

5 The Ontology of the Logic Loop

Thus far, we have been introduced to the Logic Loop model through historical case studies and in the context of broad philosophical paradigms. It is time we see the philosophical underpinnings of the Logic Loop. To develop this philosophy, let us unravel it through a brief mental simulation of how we are typically brought to ask ourselves whether math is invented or discovered. The moment we properly define mathematical objects and equip said objects with certain axioms, the magic of mathematics unfolds. Results, properties, and in a sense new objects, seem to be immediately generated. A rich mathematical land suddenly lays in front of us, delighting our mathematical curiosities. As we further investigate, we find that different mathematical lands are intricately connected (topology, algebra, etc.). Coupled with certain unreasonable successes of these objects with regards to natural world applications, we are faced with an ontological dilemma; wondering whether we are mathematical inventors or simply discovered these mathematical objects.

Faced with this ontological dilemma of mathematics, we seek to provide yet another ontological perspective. We understand that more than enough has been written about the ontology of mathematics. As such, we are not innovating so much as compiling a narrative and synthesis of various perspectives, and we have a profound sense of humility and gratitude towards the rich history of the ontological debate. Without this history, the Logic Loop would not exist.

¹⁷This will be explained further in Section 5.7 where we discuss theological priority, but in brief, realism about abstract objects establishes an uncreated, necessarily existing, eternal world apart from any concept of the divine.

¹⁸Mathematical Platonists are often called “realists.” However, this is not perfectly accurate. Some realists hold that mathematical objects are concrete rather than abstract. So while all mathematical Platonists are mathematical realists, there are some realists who aren’t Platonists.

¹⁹I was introduced to Sagan’s story through its 1997 film adaptation. In “Contact”, aliens reach out to Earth by making their presence known through a series of prime numbers. Thus, in Sagan’s meta narrative, there definitely are non-human mathematicians.

Furthermore, why should we bring another perspective? Is there a need? Our motivation for this narrative is to address various necessities of the Christian mathematician. We want an ontological perspective or narrative that speaks to various concerns and necessities of a Christian mathematician: Ontological simplicity, intrinsic formality, applicational transcendence, extrinsic universality, ongoing refinement, cultural awareness, and theological priority. Let us examine each of these in detail and see how the Logic Loop addresses them.

5.1 Ontological Simplicity

Our ontology of mathematics should be simple. It should not require the invocation of a platonic realm, or some sort of vague description of the terms “existence” and “truth” as applied to mathematical propositions. The game description provided in mathematical formalism²⁰ as well as through Maddy’s arealism²¹ helps us. That is, in the Logic Loop we need not assume realism and the independent existence of abstract mathematical objects.

5.2 Intrinsic Formality

Instead, mathematical objects and axioms are like game pieces and rules in the game of mathematics. The view is that mathematical characters have no more meaning than the pieces on a chessboard. [17, p. 144] That is, they are concepts. As concepts, they do not exist in a metaphysically heavy sense as independent and eternal abstract objects. This means that we **invent** mathematics much like we invent a game. After inventing the game, we play it and explore its consistency. Is the game too trivial? Is the game full of contradictions? As we **discover** flaws, undesirables, and untapped potential, we **refine** it towards a v2.0 invention, where our definitions and axioms have been tweaked. Ergo, the loop continues. We provide more detail on refinement, below.

5.3 Ongoing Refinement

Our conceptual framework allows for a philosophy of mathematical practice that is reflective of the historical reality of mathematical endeavors. Math, as we see it today in texts and articles, is the result of centuries of refinement. It is a process. We develop mathematics, and after exploration of our creations, we discover mathematical consistencies, untapped potentials, and maybe some undesirables. This leads to a refinement process that allows us to distill and clarify axioms, definitions, theorems, and methods. As we have argued, historically the mathematical process is much more organic than just writing definitions and then proving results. Nonetheless, this is the impression our students get in the classroom. The rigorous mathematics we know today is the result of a refining process. Take for example the publication of real analysis texts. Even as new Real Analysis books are published, we find new approaches or ways of setting up or defining objects and assuming axioms that enhance learning. Moreover, refinement is true to how students learn new concepts. Often, students must first explore axioms and definitions in order to make sense of

²⁰There is a lengthy explanation of mathematical formalism in [17, pp. 144–148].

²¹We keep with Maddy in choosing “arealism” over “the more usual term ‘anti-realism’ because it is most often used for positions that rest on principled, often a priori, objections to realism. [Maddy’s] arealist is not against mathematical entities any more than she is against unicorns; she just has no evidence for the existence of either.” [12, p. 364]

what is built upon them. Otherwise, they may end up learning proofs to misunderstood propositions. As students explore and discover, they may find a need to refine their own assumptions and understanding.

Here is one example of refinement as a process from discovery to invention: Recently, one of the authors was teaching an introduction to Real Analysis course to students. In the first weeks, the course builds the real line by starting with a field and building further assumptions on said field. After ordering the real line through Positivity Axioms, we then proceed to explore and discover what our field looks like. We discover that our field has sets such as the natural numbers and that this set through the very field axioms generates other sets such as integers and rational numbers. But our exploration comes to a shocking conclusion as we discover that there is no real number (as our ordered field currently stands) whose square is two. This leads to a necessary refinement: the Completeness Axiom. By assuming the Completeness Axiom, we refine our previous set of real numbers into a new invention: the completed real line. This process of exploration is a common technique in introductory real analysis or advanced calculus.

Recently, a further example of the exploration and refinement processes was observed in the same series of lectures. We had finished discussing the field axioms with the non-triviality assumption. After briefly proving that the product of any number with the additive identity is zero, a student proposed that the non-triviality assumption could be thrown away and obtained through the remaining axioms and even wrote a mathematical proof to support his claim. The student was exploring our field invention and decided to explore whether or not we could obtain non-triviality from the other axioms. We proceeded as a group to explore the student's proposed proof and to carefully consider its assumptions. After some thought, we *discovered* that the student had simply created a *trivial* field, made only of the additive identity (zero). Like past math explorers who asked "What if?" about Euclid's fifth postulate, we realized that the student's field needed refinement through the appropriately named *non-triviality axiom*. It was a simple though powerful learning experience that revealed how mathematical learning and innovation occur.

5.4 Application Transcendence

Yet the ontological simplicity, intrinsic formality, and the refinement of our axioms and narrative insufficiently account for the applicability of mathematics to the physical universe. Case in point are the mental gymnastics nominalist naturalists undergo to explain the indispensability of fictional mathematical objects in applications to the natural world. That is, if abstract mathematical fictions are necessary in explaining the dynamics of physical phenomena, naturalism cannot provide an adequate answer to Wigner's unreasonable effectiveness of mathematics to the natural sciences.

The nominal naturalist position is overly simplistic, seemingly unable to address indispensability arguments. As Einstein is attributed to have said, "Everything should be made as simple as possible, but not simpler." Incorporeal mathematical objects perfectly suited to a corporeal universe require at least a deist position to explain this applicational transcendence. God designed the laws of the universe mathematically. In this sense, while we retain an anti-Platonist position, we do not subscribe to naturalism. Wigner's dilemma is not a problem if we assume the existence of God. Taking this a step farther, our ability to seemingly unlock the mathematical secrets of the universe is rooted in our image-bearing qualities as the special creation of a personal God.

5.5 Extrinsic Universality

Our Logic Loop should account for the extrinsic universality of mathematics. Previously, we stated that universality has two related forms. The first is the discovery of properties or theorems in different times and places. The second is the appearance of special numbers across various disciplines.

Within the game analogy of our Logic Loop, the first form is addressed as follows: If we explore the same or virtually identical games, we will find identical or similar strategies. Examples of this in the world of playing cards occur frequently in games such as Rummy and Phase 10 that focus on runs and sets. Another example from the world of playing cards is that similar strategies succeed in the trick-trump card games Spades and Bridge. Similarly, we see what we call “Pascal’s Triangle” used in ancient India (2nd century BC) with a binary system, Persia (10th century AD), and China (11th century). Even in Europe, numerous mathematicians used Pascal’s triangle to calculate the binomial coefficients prior to Pascal’s publication (17th century). Although in different eras, regions, and even with different numbering systems, mathematicians have found similar strategies for solving combinatorial games. ²²

The second form of universality is addressed through God’s divine influence in our minds as image-bearers. We practice mathematics in the Logic Loop as influenced by God’s perfect mathematical mind. God builds mathematics in interconnected ways with axioms that show connections that are unexpected to us. Since we are his image-bearers, we choose to build fields, groups, geometries, etc. with axioms that mirror God’s mathematical practice. We then find repeated and unexpected appearances of famous special numbers. While these appearances are surprising to us, they are not surprising to God who knows the appropriate games and rules to build in order to reveal the connections.

5.6 Cultural Awareness

Many mathematical practitioners work under the assumption that there is (and has been) only one mathematics. While peripherally aware of its history, mathematics is thought of as monolithic and the product of steady progress independent of context. But the Logic Loop includes an awareness of the way mathematics has been influenced by its historical and cultural contexts.

While examples abound, here are two brief examples stemming from our first case study: calculus. Contemporary introductions to calculus begin with limits (including its precise delta-epsilon definition) and only much later touch on power series. However, this masks that power series were used heavily by Newton/Leibniz in the invention of the calculus while the careful use of limits was the result of refinement and exploration over a century later. Power series can be traced back to India. Its partner binomial theorem with which Newton manipulated series has roots in Persia and elsewhere. This demonstrates how the math of invention/exploration can evolve substantially through the refinement and discovery components of the loop.

Culture can have a positive or negative impact within the Logic Loop. A sad example comes from the development of calculus in Europe in particular. As is well known, after just two letters, Newton cut off all correspondence with Leibniz believing that his ideas had been stolen. Mathematicians and scientists in England followed Newton’s lead in what has been called The Great Snub. As

²²Another example: It does not surprise us that the Mayans had a number system that has a (mostly) consistent base even though they used base 20, dots, lines, and shells where we use base 10 and the Hindu-Arabic characters. We are both representing numbers and as such find similar strategies effective.

a result, mathematics in England stagnated for a century whilst thriving on the continent. This awareness of the role of context and culture in the Logic Loop of mathematics aligns with an ontology of mathematicians simultaneously *imago dei* across time and place as well as broken, prideful, and short-sighted.

5.7 Theological Priority

We give the Logic Loop theological priority by presupposing that our ontology must align with orthodox theologies. In the book, *Beyond the Control of God?: Six Views on The Problem of God and Abstract Objects*, Paul Gould explains the so-called problem of God and abstract objects. In essence, Gould states that should God exist, and abstract objects exist (in our case, mathematical objects), there is an inconsistent triad of premises. In order to avoid contradiction, one of the premises in the following triad must be removed:

1. Abstract objects exist.
2. If abstract objects exist, they exist independently of God.
3. If abstract objects exist, they depend on God for their existence.

The first two premises are Platonist assumptions. The third is an assumption via the Aseity-Sovereignty doctrine of God which holds that all things depend upon God, but God depends on nothing. Scripture has various passages to support the Aseity-Sovereignty doctrine of God. For example, consider Colossians 1, verse 16: “For by him all things were created, in heaven and on earth, visible and invisible, whether thrones or dominions or rulers or authorities—all things were created through him and for him.” (ESV)

Thus, if mathematical objects do exist and fall under the described “things” of this verse, they would have been created by God. Of course, some may interpret the scope of “things” in this verse to exclude abstract objects. For this reason, some may select to remove premise 3. Others may choose to remove premise 2 thereby modifying Platonism. Given the lack of evidence of a platonic realm, and given the metaphysical speculation required to sustain the co-existence of abstract objects and God, we submit the simplest solution is to remove premise 1 by denying the existence of abstract objects. This gives theological priority to God since we have no mathematical reason to suspect the existence of abstract objects would necessarily depend on God’s existence.

Nevertheless, numerous mathematicians, including many Christian mathematicians, gravitate towards Platonism. We should examine this trend. More generally, what in Platonism appeals to mathematicians? We argue that two of the primary attractions of Platonism are:

1. The universality of mathematical content.
2. The objectivity of mathematical statements.

We have already addressed the first attraction and argued that the Logic Loop provides an explanation for both forms of the universality of content. Regarding the desire for objectivity, the Logic Loop reveals that objectivity is possible, but only within the parameters of selected axioms. As Kline has argued, the objectivity of mathematics is not unshakeable. [9] Within the walls of selected axioms, mathematical statements are objective due to the rules of logic applied to the selected axioms. Returning to the game analogy of the logic loop, the moves, strategies, and results of a game

are objectively true, as permitted by the rules or axioms of the mathematical game. Moreover, the objectivity of mathematical statements does not necessitate an ontological commitment. When a chess position is equivalent to a checkmate, the other player has objectively lost. This does not mean that the rook, pawn, bishops, etc. are real game pieces nor is the fate of the defeated king metaphysically real in some platonic realm. But the statement is objectively true.

It seems that Christians have historically been drawn to Platonism via a modified version known as Christian Platonism.²³ This modified version views mathematical objects as ideas in the mind of God. However, this quasi-Platonism is not problematic with the Aseity-Sovereignty doctrine of God since it seems to remove the independence of mathematical objects by categorizing these as thoughts in the mind of God. However, said position does leave some concerning loose ends that deserve attention. For one, if mathematical objects are ideas in the mind of God, are we placing the objectivity of mathematical statements on God's mind? Is $2+2=4$ true because God thought so? Furthermore, are divine ideas metaphysically existing objects? If so, are they abstract or are they concrete? If the latter, what is the nature of concrete divine ideas? While these questions are not deal-breakers²⁴, they do reveal that the Christian mathematician must think deeply about what is meant when they assert the existence of mathematical objects in relation to God. In the present paper, while both authors differ on some of these questions, both authors agree that an easy solution to the problem of God and abstract objects is to assume mathematical objects do not exist as abstract objects. Thus we agree on the Logic Loop which seems to allow for either a Fictionalist position or for a concrete realism position.²⁵

While it may seem that the present ontological discussion favors formalism over concrete realism, a final note on the ontology working behind the scene of the Logic Loop may help balance this impression. While the authors have used the so-called Logic Loop as a blanket term for an ontology and for the history of invention and discovery, the reality is that the Logic Loop refers primarily to the way humans have practiced mathematics. But does God experience this so-called Loop? Theologically, one should assume God does not discover flawed game dynamics in need of refinement. Moreover, God does not discover any game dynamics. His first mathematical invention is perfect and known to God to the fullest extent. God invents perfect games and is not surprised by the mathematical discoveries within these games. Furthermore, when did God invent these games? Did he not know he would eventually invent the games and what they would look like? Thus, these perfect games have always been in God's mind. It is here where concrete realism can help us.

As this concrete realism may seem to be a form of Platonism, the reader should be reminded that traditional Platonism assumes the existence of abstract objects instead of concrete objects. In this sense, the authors invite the reader to consider a divine quasi-formalist ontology in which God's mathematical ideas are either abstract concepts in the mind of God that do not metaphysically exist or concrete concepts that exist in the mind of God.

6 Mathematics as a Habit of Grace

Even if the existence of abstract mathematical objects is sacrificed, transcendence remains via the unreasonable effectiveness of mathematics in the natural sciences. Indeed, the naturalist finds it unreasonable, but the theist embraces the fascination of a Divine designer that fashions the fabric

²³For example, [1, pp. 204ff]

²⁴These issues are discussed at length in Gould.

²⁵See Gould on concrete realism.

and behavior of the universe mathematically. As such, applicational transcendence is a catalyst of awe and the worship of God. While presumptuous to claim that the Apostle Paul had mathematical objects in mind, one could argue that mathematical objects are included in Romans 1:20:

For his invisible attributes, namely, his eternal power and divine nature, have been clearly perceived, ever since the creation of the world, in the things that have been made. So they are without excuse.

Paul seems to suggest that the magnificence of the created universe reveals God's eternal power and divine nature. While Paul wasn't thinking of the mathematical structure of the universe, this passage reveals that the more we study and understand the created order, the more we are able to perceive God's aforementioned invisible attributes. Mathematics allows us to perceive his invisible attributes as we continue to unravel the mathematical nature of our universe. Therefore, if glorifying God is the chief aim of humanity as image-bearers, the practice of mathematics becomes a habit of devotion or discipline in the form of intentional mathematics. The pursuit of pure mathematics invites us to mimic the Divine creativity that forges a created order. We are to think up mathematical possibilities as a manifestation of our co-creator status as image bearers. We explore these inventions to see whether what we have fashioned is deemed good, much like God sees that what He has made is good in Genesis 1.

If the practice of pure mathematics can be a habit of worship of the Godhead, the application of mathematics can be a form of neighbor love. The practice of applied mathematics can become a service to a rapidly changing world in need of new solutions. As God redeems the world from the power of sin and death, we act as His hands and feet as we compassionately apply mathematics in fresh ways to understand and heal a broken world. This is especially true in a rapidly changing world that consistently generates new problems. While one could indeed argue that all problem-solving is worthy of being labeled "service," we take said service a step further by looking to apply mathematics to problems that have been neglected or ignored. That is, we ask "what world problems lack a mathematical perspective and solution?" The Christian applied mathematician thus emulates the good Samaritan as she steps out of the comfort zone for the sake of serving the destitute. In this way, our application of mathematics emulates Jesus.

But the practice of mathematics is not inherently a habit of grace. Indeed, some may apply mathematics out of self-service and not out of compassion to others. Furthermore, idolatry and hubris can creep into the pure mathematician's heart. Blaise Pascal gave us such a warning when he stated that science can satisfy, "a greed or lust for learning, a profligate appetite for knowledge." To Pascal, "such a study of science springs from a priori concern for self as the center of things rather than a concern for seeking out, amid all surrounding natural phenomena, the presence of God and His glory." [9, p. 46]

It is worth clarifying that one can and should enjoy mathematics for itself. Moreover, to complete this joy and avoid idolatry of self and mathematics, one should see the enjoyment of mathematics as a means to an ultimate end that is not mathematics itself. The ultimate end is the love of God and neighbor. While it can be challenging to remain aware of God within the practice of mathematics, we should seek to cultivate this awareness throughout mathematical endeavors. The payout of this effort is a renewed mind that does not compartmentalize faith from our disciplines but instead can see cruciform beauty within our mathematical liturgy.

7 Choose your Ontology

Our proposed Logic Loop was constructed from the seven priorities of the Christian mathematician that we previously addressed. While the Logic Loop does step away from Platonism, it allows room for two distinct ontological commitments. On one hand, a nominalist position fits well within the Logic Loop since we argue mathematics can still be indispensable without attributing existence to mathematical objects, vis a vis a creator. On the other hand, one can subscribe to a divine conceptualist framework where mathematical objects exist as ideas in the mind of God. Per William Lane Craig, this would constitute a different ontology altogether than mathematicians are used to. Specifically, he would say that these conceptual objects are concrete (as opposed to abstract). At present the authors are still seeking to understand this form of concrete realism. But as this demonstrates, there are multiple ontologies (outside of Platonism) of mathematics that also seem to satisfy the Aseity-Sovereignty doctrine which was one of our concerns. Thus, one author takes an anti-realist position that still sees mathematics as a game but knows the limitations of formalism, while another author is more akin to a conceptualist position that sees mathematical objects as concrete divine ideas. At the present time, we are not tasked with distinguishing between concrete and abstract objects and leave that for a subsequent paper.

The authors of the present paper have two different positions as described above. This was both necessary and intentional as this joint effort came to a conclusion. For one, the authors understand that we are adding just a small grain of sand to the ongoing conversation of the ontology of mathematics. Humbly, we recognize that we can't propose an all-satisfying answer, especially within the grand and rich history of ontological discussions. Moreover, we have simply assembled an ontology that honors the legacy of the philosophy of mathematics. Rather than solve the ontology of mathematics, we hope our proposed Logic Loop would stimulate thought and dialogue the way the Church represents a diversity and unity of members under the banner of faith in Christ. Just as we can have liberty and charity in the plenitude of faith beyond the creeds, there must be freedom and grace in the philosophy of mathematics.

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The Intersection of Mathematics and Creative Habits

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Abstract

Since employers value creativity in the rapidly changing twenty-first century, creativity has become an intentionally incorporated and named part of the higher education curriculum through courses, programs, and faculty titles. Further, as Christians, creativity connects to our identity as image bearers of God the Creator and our mission as his children. Although as mathematicians we may view ourselves as highly creative, others fail to understand the creative aspects of our field and the role mathematics can play in developing students' creative skills. Due to the current importance of creativity, we suggest mathematics educators learn the terminology of creativity experts so that we can prepare our students to transfer their creative practices to other non-mathematical contexts and to effectively communicate their creative mathematical experiences to prospective employers. We provide an example mapping between creativity as described in Don Perini's text *Emerge* and math instructional practices.

1 Introduction

Creativity is among the soft skills currently in demand by the employers of college graduates. An October 2020 survey of 496 employers (executives and hiring managers) by the Association of American Colleges and Universities (AACU) found that for creative thinking, and each of thirteen other essential learning outcomes of a liberal education, at least 90% of employers considered each to be somewhat important, including more than 50% who considered each very important for success in the workforce [3, p. 6]. Further, employers under age forty were significantly more likely than employers fifty and above to consider creative thinking as very important (60% to 43%) [3, p. 26]. In a chapter aptly titled “An Education that Lasts: Thinking Creatively and Globally,” Ostrander explains that in the twenty-first century we have shifted from the Information Age, in which white-collar workers produced and processed information, to the Conceptual Age; due to increased automation and outsourcing:

... creators, designers, and collaborators command the highest value. The right brain—the part that synthesizes information, sees the big picture, envisions new scenarios, and empathizes with others—is as essential to the modern economy as the left brain. Writing code for an iPhone is one thing; but it takes a different sort of mind to envision the need for and value of an iPhone in the first place, and to design one that has the mysterious quality of ‘feeling right’ to the user. Creativity, not just computation, is the important trait of the future [8, pp. 144-146].

Technologists, scholars, practitioners, strategic thinkers, and education leaders canvassed by the Pew Research Center in 2016 suggested workers nurture creativity as one of the “unique human skills that artificial intelligence (AI) and machines seem unable to replicate” [10, p. 4].

Higher education has picked up the creativity mantra in curriculum and faculty titles. For example, the panelists at a 2022 *The Chronicle of Higher Education* forum on “Lessons on Creativity and Communication” included a professor of creativity and innovation, a professor in education innovations, and a vice provost for creative inquiry [11]. Our own institution requires a sophomore seminar titled “Courage, Creativity, and Calling” and includes creativity in two of the thirteen learning outcomes of our Foundations Program: students will “integrate, synthesize, and apply knowledge and information creatively” and students will “engage the created world and human culture with curiosity and creativity” [12, p. 62]. The AACU emphasizes the importance of vocabulary in discussing workforce preparation: “employers and educators are largely in agreement when it comes to the value of a contemporary liberal education—provided it is described using language that is common and accessible to both stakeholder groups” [3, p. iii]. Suggested practical action steps from the AACU’s employment research include that educators should “equip students to name and reflect upon the skills that matter” and should “make mindsets and aptitudes [such as curiosity and persistence] an explicit part of learning to help students better understand what they can contribute as professionals” [3, p. 32]. As educators whose mission includes preparing students for professional life after college, the current cultural significance of creativity should inspire us to increase our familiarity with the discussions and research on creativity so that we may be intentional in communicating with our students and in incorporating creativity into our classrooms.

In his text on creativity *Emerge*, Don Perini, Professor of Creativity and Innovation at Cornerstone University, defines creativity as “works that are both new and useful” [9, p. 15]. This definition seems to include that action has been taken so that a finished product now exists interacting with the world, that this product must be novel or original, and that this product must be of use to effectively serve a purpose. “Creativity isn’t a genie in a bottle. It takes determination, process, and good habits to come up with great results” [9, p. 11]. The good news, according to Perini and other teachers of creativity, is that we each can learn and develop our practice of the habits and processes that lead to the completion of creative works.

2 The Value of Creativity in Christianity

From a Christian perspective, the importance of creativity goes far beyond essential job skills, connecting to our identity and purpose as human beings as described in Genesis:

So God created man in his own image, in the image of God he created him; male and female he created them. And God blessed them. And God said to them, ‘Be fruitful and multiply and fill the earth and subdue it, and have dominion over the fish of the sea and over the birds of the heavens and over every living thing that moves on the earth’ (Gen. 1:27-28, ESV).

The cultural mandate of verse twenty-eight serves as the “instructional manual” for the “playground” of Earth that God has given humankind for our enjoyment and flourishing [9, p. 24]. The concept of *imago Dei* and the cultural mandate in these two verses establish the inherent worth and significance of each human being, how we are different from the rest of creation, and our unique mission; Perini describes who each one of us really is:

I am much more than a bunch of cells. I am God’s image bearer, sharing in God’s rule and administration over the earth, and I have been given a mandate to harness the natural world and develop the social world. God has given me a unique set of talents to assist my efforts in ruling the world, fulfilling the mandate, and making culture [9, p. 37].

The focus of God during the full arc of the Biblical narrative, according to Fujimura, is on creating: “the consummation of God’s plan as it unfolds in the Bible is not a utilitarian restoration but an imaginative New Creation,” for the imagery of Revelation is a city, new Jerusalem, and not a return to the garden of Eden [4, pp. 29-30]. “Restoring a broken world is a noble goal, and yet biblical promises go further than even that ideal” to a “generative, expansive promise of the New” [4, pp. 22-23].

The cultural mandate calls us to participate in God’s plan for new creation. Our activities of justice, beauty, and evangelism, according to Wright, “are not simply ways of making the present life a little less beastly, a little more bearable, until the day when we leave it behind altogether... They are part of what we may call building for God’s kingdom” [13, p. 193]. The promise of 1 Corinthians 15:58 is that our labor is not in vain:

God’s recreation of his wonderful world, which began with the resurrection of Jesus and continues mysteriously as God’s people live in the risen Christ and in the power of his Spirit means that what we do in Christ and by the Spirit in the present is not wasted. It will last all the way into God’s new world. In fact, it will be enhanced there [13, p. 208].

We can only imagine how this will work. Fujimura tells a parable of a child building a sandcastle washed away by high tide. The child’s father, an architect, chooses to build a real castle based on the child’s work, possibly even transforming the sand into the new reality of a permanent building. One lesson from this parable: “God takes far more seriously than we do what we make, even in ‘inconsequential play,’ and everyday realities can be enduring materials through which the New Creation is to be made” [4, p. 36].

When our creative endeavors glorify God, we are bringing his kingdom and reflecting glimpses of new creation, but Perini warns that due to sin our behavior tends toward the immoral:

We believe the best way to achieve personal worth is by gaining significance and obtaining security... Your own efforts to find personal worth are always temporary. They give the illusion that significance and security can be found through fame and fortune [9, p. 26].

We are tempted to be like the builders at Babel: “Then they said, ‘Come, let us build ourselves a city and a tower with its top in the heavens, and let us make a name for ourselves, lest we be dispersed over the face of the whole earth’” [Gen. 11:4]. These attempts to be God, through self-aggrandizement and trying to establish our own identities, lead to “inevitable despair at our constant failure” [13, p. 114]. Instead we should embrace ourselves as creatures remembering that we “are significant, as image bearers, and secure, because of [God’s] unconditional love” [9, p. 26].

The “creatives” who imagine innovative solutions do important work and are highly prized, but the risk is that this comes at the cost of demeaning others. The work of creatives alone is not sufficient: for example, project managers shepherd ideas to implementation, and there would be no workplace without the service staff completing tasks viewed as menial. As we experienced during the 2020 pandemic, the formerly invisible grocery store clerks, truck drivers, and sanitation personnel became essential. Christians are called to value the contributions of each: “But God has so composed the body, giving greater honor to the part that lacked it, that there may be no division in the body, but that the members may have the same care for one another” [1 Cor. 12:24b-25]. While we

celebrate unique talents, our focus should be on glorifying God through the success of the whole and the inherent worth of each member. Brent Waters, in discussing his book *Common Callings and Ordinary Virtues*, said that we should not expect on judgement day to be held accountable for being interesting but instead for being faithful stewards as required in 1 Corinthians 4:2 [6].

As always, we should be cautious of the ingrained cultural perspectives we have inherited. New is not always better. Wright argues that Western culture's modernist myth of progress, that human improvement is unlimited and humankind is inevitably marching toward utopia, fails to deal with evil because it did not stop the great evils of the twentieth century, it cannot address the moral issue of making right the evil of the past, and it underestimates the nature and power of evil [13, pp. 84-87]. A presumption of natural progress forgets the nature of humankind and confuses human power for the fruit of transformation by the Spirit: "For I know that nothing good dwells in me, that is, in my flesh. For I have the desire to do what is right, but not the ability to carry it out" [Rom. 7:18]. Warning against the overemphasis of usefulness, Fujimura argues that the "bottom-line utilitarian pragmatism" passed down since the Industrial Revolution leads us to value, even as we view ourselves and humanity, "only what is most efficient, what is practical and industrial," whereas "the essence of humanity under God is not just utility and practical applications; the essence of humanity may be in what we deem to be 'use-less' ... but essential," including the works of mercy and beauty he believes contribute to New Creation [4, pp. 18-19]. The Christian understanding of new and useful in creative work should be shaped by the characteristics named in Philippians 4:8: true, honorable, just, pure, lovely, commendable, excellent, and worthy of praise.

As Christians, these warnings and corrections should lead us to acknowledge that either our definition of creativity is somewhat different than employers and academics or that our understanding of the cultural mandate is more expansive than creativity as defined, or both. Those glorifying God through faithful service in the ordinary, often anonymously, are building for New Creation, too. And yet, these steadfast servants also benefit from an imagination to look ahead to New Creation and should steward their God-given talents through development. Further, the students called to our college classrooms are likely to pursue twenty-first century careers that require innovation. Thus, using the cultural definition of creativity as works that are both new and useful is helpful, as long as it is accompanied by discernment to ensure the glorification of God in our endeavors and by care as we invoke the cultural mandate to motivate creativity curriculum.

3 Math's Public Relations Problem

Creativity experts suggest studying the habits of the successful to discern processes that lead to accomplishment [9, pp. 141]. Based on a survey of top modern mathematicians in 2002, combined with historical quotes, the ingredients for mathematical discovery include hard work, experimentation, insight, tenacity, technical skill, mistakes, intuition, and good problems: "Creativity and mathematics may be an oxymoron to the non-mathematical but it certainly isn't to productive mathematicians" [2, p. xi]. But then, discussions of creativity at this highest level of math are likely quite esoteric for the average non-mathematician who has never been asked and cannot imagine being able to uncover a new mathematical discovery. Even at the undergraduate level, many of the problems students encounter have a solution already known to the professor so that the students' work seems to not satisfy the requirement of being new in the definition creativity. Pólya, from his 1945 text *How to Solve It*, speaks to how even these problems connect to mathematical creativity:

A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and

brings into play your inventive faculties and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery [2, p. 108].

Since this paper discusses the preparation for undergraduate students for professions after college, we will focus on this type of creativity found in the everyday work of college math classes.

Unfortunately, the mathematical experience of too many students in the United States has been memorization rather than conceptual understanding and thus, beyond college math departments, colleagues and prospective employers for our students may not have experienced the triumph of discovery Pólya describes. Based on their own negative experiences, they may confuse a greater memory for students' actual inventiveness. On the 2012 Programme for International Student Assessment, a problem-solving test for fifteen-year-olds, American students came in last or close to last; students whose approach to learning focused on memorization, and countries including the United States with high numbers of memorizers, were lower achieving than students who used a relational approach (connecting new ideas to what they already knew) or a self-management approach (identifying what they knew and what they needed to learn), and the countries with high numbers of these students [1, pp. 159-161]. In the hopes of improving math education and fostering a broader appreciation of the creativity in mathematics within society, it is encouraging that the National Council of Teachers of Mathematics includes among the required teaching practices many items that challenge students to go beyond memorization to deeper and more flexible understanding of material that allows creativity: promote reasoning and problem solving, connect mathematical representations, build procedural fluency from conceptual understanding, support productive struggle, and elicit evidence of student thinking [7, p. 3].

Additionally, the grain of discovery in solutions is often hidden because the final draft of written mathematics is the communication of an argument, not the record of the solution process:

Mathematical publications obscure mathematical creativity. Journal articles, books, and monographs do not present math in action, but show the work in finished form, with the footsteps of discovery (invention) carefully erased. But mathematics rarely, if ever, emerges in finished form. The logicity of polished mathematics differs enormously from the more mysterious process of mathematical discovery. The more artful aspects of mathematics are elusive.

Although this may be describing professional mathematics, we teach our students to do the same. Homework submissions often require students to submit rewritten work, including only the necessary information. An example may be given to demonstrate a pattern but the longer list of examples used to search for a pattern do not make the final draft. For proofs of limits or series convergence, the constant multiple of epsilon or of the comparison series seems to appear at random since the side work to find the correct constant satisfying the required relationships has been scrapped from the final proof. Pólya described mathematics as having two faces: in finished form math is purely demonstrative, but the creative work of mathematics is like that of experimental sciences involving observation, analogy, conjectural generalizations, and mere guesses [2, p. 109]. While learning formal communication of mathematics is important, we should also require our students to practice explaining their reasoning process for they may be called to guide others to understanding in future professional situations and be required to testify to their own problem-solving inventiveness.

Society recognizes the importance of applied mathematics, but abstract mathematics often seems irrelevant to non-mathematicians so that they may see it as failing to meet the usefulness criterion

of the definition of creativity. For part of the twentieth century the mathematics community encouraged this view: through the emphasis on pure mathematics, mathematicians “tended to isolate themselves from practical affairs” and claimed irrelevance to the broader culture to ignore responsibility for the negative consequences of the mathematization of culture [5, p. 219]. To change this perception, as a community we should continue to share with our students and more broadly the reasons we value mathematics, whether that is guiding others to see the beauty of this part of God’s good creation, celebrating the surprising power of mathematics to help us understand our world, or the ways in which mathematical exploration can develop the skills, mindsets, and aptitudes that are in demand.

4 Creative Habits in the Classroom

Perini’s philosophy is that everyone can be creative: since highly successful people are not simply born that way, we can learn and follow their habits to creative achievement [9, p. 12-13].

God has given humans a mission, a playground, and talents. Our talents need catalysts that enable a new entity to emerge in order for culture-making (Creative Emergence) to occur. These catalysts are our creative habits [9, p. 39-40].

Practicing creative habits develops our God-given talents, increasing the effectiveness of our talents and, hence, the likelihood of the emergence of new and useful work [9, p. 32]. Not meant to be an exhaustive list, Perini identifies nine habits he discusses in detail and several others he mentions in passing. We may find some more consistent with our classrooms than others. Our hope is that by mapping our activities to Perini’s creative habits, our students are better able to share the creative habits they could implement in other contexts because they have regularly practiced them to build solutions to problems, in particular solutions to the novel-to-the-student problems where the complete path to the answer is not immediately clear.

One habit is making habits: “Enthusiasm, dreams, and novelty will only motivate us in the short run. What we need are the habits that keep up moving forward when we lose that feeling of grandeur. What we need is something to activate us when we want to quit ” [9, p. 109]. Perini includes several steps in habit formation: creating triggers to start, rewards, accountability, and repetition. Mathematics would be easier to teach if it merely required fact transfer, process memorization, and easy-to-find solutions. Instead, math instructors often create strong daily and weekly rhythms with problem-solving an integral part of each class and assignment, requiring students to engage even on the days they do not feel particularly inspired. The external structure compels students to participate in the habit formation cycle, establishing group work patterns and triggers to start homework. Hopefully, students experience the satisfaction of finishing difficult problems. Of course, just as habit formation is an act of will founded on the belief that the new habit is good, right, and in our best interest [9, p. 114], persistence develops when students buy-in to the benefits of regular productive struggle.

Possibly more than the other habits and compared to other fields, mathematical training strengthens the habit of ideation, the generation of ideas. Perini’s suggestions to generate ideas include getting started, ensuring we capture moments of inspiration, and allowing the work to marinate. A math instructor’s response to still students, closed mouths, and blank papers is often to tell students to simply start, reminding them that, as with the idea for any new and useful work, solutions to complicated problems “rarely come to us whole and complete” [9, p. 58]. We may teach students to identify the knowns and the question, to review other problems looking for an idea to imitate,

and to take a first deductive step by connecting the problem to previous knowledge. We encourage them to continue down a path until solution or dead end, for even an unsuccessful argument may provide pieces that when rearranged, as with a puzzle, reveal the answer. We suggest students write down all their ideas until they figure out which are relevant, since seeing two ideas listed together may spark progress or a fact may be waiting ready for insertion later in the problem solution. On homework, we send students away, after a nudge in the right direction, to incubate and hopefully to return to the problem with a new perspective. The most successful problem-solvers are disciples of Thomas Edison's perspective: "I have not failed, I've just found 10,000 ways that won't work" and "The most certain way to succeed is always to try just one more time" [9, p. 59].

Practicing the habit of mastery through 10,000 hours of practice, practicing with a purpose, and integrating other knowledge leads to talent development through a progression of developing an interest, honing technical skills, and finally mastery and artistry [9, pp. 84-85]. Artistry may be beyond the scope of a course, but well-designed curriculum does include a purposeful progression that if made explicit for students can give them a transferable model for learning. For example, work for a section may start with basic skill exercises, but then problems progress from typical questions and interpretations to novel issues requiring a synthesis of earlier work. Next, we require students to complete problems with multi-parts, that benefit from the greater problem-solving abilities of a group over any individual, and that require a discussion of process or an expression of conceptual understanding. Eventually we may challenge students with open-ended problems. Not only the tasks but the ordering of the tasks lead to a more complete understanding as learning spirals, revisiting earlier topics to dive deeper and connecting them to new material. Further, as students progress we challenge them to generalize from specific examples and contexts to the characteristics and properties of underlying abstract concepts.

Perini's emphasis with the habit of simplicity is reducing physical and behavioral clutter, including removing unimportant tasks so that there is space for creativity. Although the space and time affected is narrower, we teach our students techniques that reduce distractions and make space for the essential ideas leading to a solution. Some of these techniques focus on the physical page, such as copying ideas onto a single sheet of paper instead of continuing to reference the book or refocusing on the central path by finding a clean board or a fresh sheet, as testified by the mathematician's ever-present notebook and stack of scrap paper. But simplicity also inspires translating from verbal descriptions to symbols or diagrams, so that we can hold in view more pieces of the problem and see the connections between them, and inspires starting with small or simpler examples to build up to the given problem, allowing us to identify the key properties and to test hypotheses.

The creative habit of truth replaces lies with healthy scripts. The emphasis within the mathematical community on growth mindset demonstrates a commitment to shift student-thinking, replacing false scripts on potential both through direct discussion and by incorporating policies like mastery-based grading that allow for differently-paced growth. Sharing our own experiences of mathematical struggle normalizes development: in our department we tell our students a common story that our undergraduate course which introduced proofs seemed harder than more advanced courses, as learning to write proofs and to evaluate the validity of an argument—particularly within our own proofs—took practice. Similarly, the habit of messiness focuses on learning from failure. "The fear of failure paralyzes us. It keeps us from taking those first few steps, from experimenting with ideas, or from finishing our creative endeavors" [9, p. 99]. Instead, we hope students learn to see errors as a step in the process to a solution and thus to take risks. Further, as required solutions progress from answers to arguments, students increasingly benefit from messy scrap work that sketches key points or from rewriting messy first drafts.

Perini's vision of a village includes encouragers, mentors, consultants, and collaborators who can provide both praise and criticism: "If we only hear praise, we will never receive what we need to do our careers and endeavors well. If we only hear criticism, we are likely to give up and quit" [9, p. 87]. Mathematicians are quick to name the benefits of collaborating with others, our own histories—even from elementary school—filled with fond memories of assignments with friends, comparing answers and helping each other when we got stuck. Individuals within the group notice different points, ask different questions, and remember different facts; the synthesis of these varying perspectives often leads the group to accomplish more than any one individual member could complete alone. Talking through an idea with others helps to clarify an explanation or reveal a logical gap. In math, collaboration is usually encouraged, often rewarded, and sometimes required. Group work during class makes space for the instructor to engage regularly in small group and even individual mentoring. The instructor can catch and correct errors in understanding early so that students can make additional progress outside of class, but it also provides an opportunity to coach students through the problem-solving process. As students' skills develop, assistance can often progress from problem-specific hints to more generic questions such as "What are we trying to find?" and "What are we given?" and then pointing out that they can be their own coaches since these questions do not require knowledge of the solution. One quality of great math teachers is the ability to verbalize their thought process and to help students to describe their own. Math can also be a good subject for learning to accept criticism since it may feel less personal: when the emphasis is placed on providing a valid proof, a good-faith critique is not of your opinion, interpretation, or viewpoint. It is instead a question of the sufficiency of your argument and early feedback gives you an opportunity to fill the gaps.

Our habitat, including colors and sounds, influence our thinking. In math, we often discuss space as a way to increase collaboration during group work. Whether it is sitting students around a table instead of in rows or having students group around a whiteboard or giant notepad, the bringing together of multiple perspectives can lead a team to piece together a solution that no one individual could solve alone. Background music ensures that no one must speak into silence. Occasionally, disrupting the routine by first disrupting the space re-energizes collaboration. By forcing students to write on a large sheet of paper on the table in front of them instead of their own notebooks, the process of discovering a solution may become less linear with the writing working out from the center; their teammates' work appearing sideways or upside down may compel them to move themselves or the paper changing their engagement with the work and each other. Even changing the sizes of the teams for a day can shift the conversation structures. It is also natural to inquire about students' habitat and village habits outside the classroom to see what choices they make on their own.

Perini's final habit is usefulness. He contrasts what we should be, good stewards of our God-given talents to help humanity move closer to God, to the warning to the Laodiceans of spiritual uselessness in the book of Revelation. He warns us of three current cultural influences (consumerism, entertainment, and technology) that often distract us from creative endeavors [9, p. 116]. Consumerism encourages us to satisfy our desires through the shortcut of buying happiness. The centrality of entertainment allows us to escape the hard things; placing our appetite before purpose, we forgo the sense of meaning we get from work and the persistence we develop from pressing on amidst difficulty. Technology is a powerful tool, but when its use is out of balance we can trade reality for living in the virtual world and satisfy our desire for new things through virtual engagement. Technology is useful and powerful in math as well, allowing us to check our work or to calculate answers to questions that are difficult or impossible to do by hand, such as finding zeros of functions or estimating definite integrals for functions without an elementary antiderivative. But

to ensure that students develop fluency in the mathematical concepts, we explicitly establish acceptable uses of technology. For example, in calculus students may be allowed to use their graphing calculators to find the zeros of function and the sign of the output, but not to determine the intervals where the function is increasing or the concavity of the function. Instead, they must demonstrate their understanding of the relationship of first and second derivatives to these characteristics of the function. Because the goal of the work is not merely to find the correct answer, we need to discuss the proper use of technology to support our primary goal of learning. Recognizing the need for and then implementing boundaries in line with our priorities is important for stewarding any of our talents.

Good mathematics instruction involves explicit sharing of the thought process for finding solutions to difficult problems and requires intentionality in designing activities and an environment where students can develop problem-solving skills. Practicing creative habits in mathematics classrooms not only facilitates students' discovery of problem solutions, but can also develop habits and skills transferable to other contexts. In the twenty-first century, creativity is an important skill required for the Conceptual Age careers of college graduates. Further, Christians are called to participate in building for New Creation in all our vocational settings. We hope that studying the research of creativity experts like Perini will give educators and students additional language for discussing creativity within mathematics courses and then communicating these experiences, skills, and habits in a manner that is more accessible to non-mathematicians.

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Surviving a Pandemic Using Alternate Assessments

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Abstract

The COVID-19 pandemic is one of the most unexpected global challenges of the early 21st century. It has been devastating to healthcare systems and provided unprecedented difficulties for educators worldwide as on-ground classes were suddenly transitioned online. In addition to navigating an unknown virus, educators also faced the challenge of determining how to continue to teach and assess students during this new normal. In this paper, we will share several assessment practices we used in online and hybrid courses. We will include lessons learned from transitioning "traditional" assessments, group projects, alternative/mastery exams, and oral exams to online and hybrid classes. We will also provide a comparison of student surveys from pre- and post-March 2020, which include questions about test anxiety, growth mindset, and student perceptions of learning mathematics.

1 Introduction and Literature Review

The COVID-19 pandemic has been one of the most difficult global challenges of the twenty-first century. It has been devastating to healthcare systems and provided unprecedented challenges to education systems worldwide as classes were suddenly and unexpectedly shifted online. In addition to needing to adapt learning and testing strategies, this abrupt transition to online learning and quarantining caused increased anxiety for students, faculty, and staff [1, 4, 30, 37, 43, 47]. To mitigate exposure to the virus while maintaining quality educational support for students, many educators adopted new teaching strategies. However, not all initial implementations went as anticipated, requiring updates and adaptations in the subsequent years. While many higher education institutions are returning to pre-pandemic, generally in-person learning and instruction methods, the educational strategies and lessons learned during the height of the pandemic should not be forgotten nor their practices ended. In this paper, we will highlight some of the lessons learned as well as intentional practices that could continue to be incorporated moving forward.

With the unexpected transition of courses to online learning, students and teachers faced numerous additional stressors [37]. Fear of the unknown, concern for not only their own health and well-being, but also that of their family and students, and adjustment to new classroom expectations while online were only a few of the worries arising from the pandemic. New research [13, 28, 29] reports that levels of depression and anxiety in students increased as the pandemic continued. Mendoza et al. [37] found that university mathematics students reported high levels of anxiety from March through July 2020 using the Inventory of Anxiety Situations and Responses-ISRA-B. Debowska et al. [13] found that female students reported higher levels of depression, anxiety, and stress, but noted that previous research [21, 36] has shown that women generally report higher levels of depression and anxiety.

Even before the onset of the COVID-19 pandemic, many instructors attempted to counteract mathematics anxiety by encouraging students to adopt a growth mindset. Moreover, instructors have created means to allow students to learn new ideas more deeply using a family of assessment models commonly referred to as *mastery grading*, *tiered grading*, *proficiency-based grading*, or *competency-based grading* [3, 10, 18, 23]. In the past, *mastery grading* has been used as an umbrella term for multiple grading for growth techniques, but there has been movement away from this term [46]. Since *mastery grading* was the term used during the time of this study, the authors have decided to use this phrase, but recognize its limitations as discussed in [46].

Campbell et al. [8] define a mastery grading scheme as any grading system that does the following:

1. Provides students with a clear list of learning objectives and bases course grades on how many of these objectives are met;
2. Grades student work based on mastery rather than according to points and provides feedback by learning objective instead of by assignment; and
3. Determines grades by whether students demonstrate learning *eventually*, rather than at any particular moment in time, usually by providing students multiple opportunities to demonstrate their learning.

Although there are many ways to implement mastery grading schemes, the grading method discussed in Section 2 divides course learning objectives into a list of discretely assessable concepts,

and student work is graded according to objective. Generally, course grades are assigned based on the number of course objectives students have demonstrated as being learned by the conclusion of the course. Either the number of mastered concepts is converted to a percentage and used as a portion of the final course grade computed via a weighted average or each grade level may require a certain number of mastered concepts [3, 23]. Similar grading schemes are also being adopted in other areas of academia including chemistry, engineering, nursing, and physics [5, 9, 16, 22, 35].

Many mastery grading practitioners in mathematics education have reported that mastery grading results in lower stress and anxiety for students [19, 24, 25, 31, 32, 34, 45]. Because students have multiple opportunities to demonstrate their understanding of concepts in mastery grading, these assessments have lower stakes in general [2, 24, 33]. Lower stakes assessments, in turn, have been shown to help alleviate test anxiety [50]. Thus, these alternative assessment techniques can benefit students in a myriad of ways.

The unexpected and rapid move to remote learning provided an opportunity to analyze a cross section of mathematics courses on the impact of mastery grading approaches, especially as related to student anxiety during the first few years of the COVID-19 pandemic. Since we have already seen some evidence in support of these alternate grading methods in reducing test anxiety, this study was intended to investigate how alternate grading methods would fare with the additional anxiety added by an unexpected global pandemic.

In Section 2, we discuss the changes we made to our pedagogy after we shifted online and how these changes could continue to be implemented as we return to in-person classes. Specifically, we mention technological changes, assessment changes, alternative assessment techniques, and resulting challenges. Then, in Section 3, we describe the design of our surveys used to compare mastery graded and traditional courses as well as the specific courses we examined while the results are broken down in Section 4. Finally, the implications of our results are discussed in Section 5, with concluding remarks in Section 6. Sections 7, 8, and 9 contain rubrics and surveys as additional materials.

2 Pedagogical Changes

In response to an abrupt modality shift, many adaptations were implemented in both the daily class experience as well as general course structure. Here we will highlight some of these transitions with an emphasis on some of the lessons learned in the process. In particular, we feature adjustments made concerning technology, group work, presentations, and assessments.

2.1 Technological Changes

The need to quarantine necessitated adjusting the everyday learning environment to remote learning, which further required alterations to course structure and policies. Due to the online modality, teaching via a tablet or document camera became common practice and highlighted the versatility of this teaching technique for hybrid modalities. This adaptation facilitated the process of posting notes to the course website or learning management system (Blackboard, Canvas, D2L, Moodle, etc.) for students who needed to miss class or those with poor internet connection. Moreover, the use of a tablet aided in seamless grading where an instructor could continue to leave feedback on assignments comparable to an in-person counterpart.

Another intentional choice which began pre-pandemic, but became notably beneficial once learning was moved online, was to incorporate Open Educational Resources (OERs). Using these texts and other free resources not only diminishes the financial barrier of textbook costs to our students, but it enhances accessibility for students since many such resources are available in PDF, HTML, and print formats. OERs have been shown to contribute to academic success by improving course grades and lowering drop-fail-withdraw rates. Further, it has been shown that students enrolled in OER courses choose to take a higher number of credits in subsequent semesters [11, 20, 27]. Research completed by Delgado, Delgado, and Hilton [14] showed in the calculus context that OERs have a positive impact on Pell Grant-eligible students as well as international students while simultaneously resulting in similar final exam performance when compared with traditional textbooks.

Transferring active learning and group work to the online setting proved more challenging, but the intention was to prioritize consistency in students' learning experiences. Breakout rooms allowed students to work together while online, and tools such as Google Classroom allowed the instructor to easily create individual assignments for each student. Students could then share their screen to work collaboratively, with each student still remaining responsible to complete their individual work for a grade. Instructors could then grade the activity using rubrics through Google Classroom. The BigBlueButton accessed through Canvas proved especially useful for group work as the instructor could join and listen in on all of the breakout rooms simultaneously, most closely mimicking the in-person classroom.

Students shared their preference in having consistent groups, which can be expedited by assigning groups in advance or by having users self-select to join their group. Although online groups seem to take longer to complete assignments in comparison to that of in-person group work since students can be especially hesitant to contribute online, designating roles such as the group facilitator, secretary, or task manager can help ease these interactions.

Other tools used include shared Google Docs for groups to record their discussions and Google Forms to have students submit Exit Tickets. Alternative options such as Jamboards or Padlets have also proved useful for organizing online discussion, both in the classroom as well as in workshops and conferences. Specifically, Jamboards worked well for having student groups share their work so that the instructor could review it quickly without going in and out of Zoom rooms, while Padlets were useful for general comments and collaborations, especially across universities.

While the need to use breakout rooms is removed during in-person instruction, many of the aforementioned resources remain applicable. Students continue to request that course notes are posted online, which has remained feasible with the continued use of tablets to take notes while projected in the classroom. Additionally, students may continue to share Google Docs, Jamboards, and Padlets to combine course notes, ideas, and solutions, which have become increasingly helpful as students prepare for tests or exams. Student comfort levels in using video conferencing software have increased, making software such as Zoom or Microsoft Teams useful in hosting online office hours or quick student check-ins. As a whole, an increased use of technology has made instructors more readily available to their students.

2.2 Traditional Assessment Changes

Regardless of the modality of a course, it is imperative that an instructor assess students, both formally and informally, for their understanding of course objectives. When learning was moved online, certain assessment strategies transitioned more easily than others.

For classes in which presentations were a part of the typical semester work, we continued these presentations online using a video conferencing platform such as Zoom. In several of our courses, including an interdisciplinary freshman seminar and a group work-oriented general education mathematics course, student presentations are expected to include an interactive element, encouraging engagement with the class. To facilitate online engagement, students often used an online, game-based learning platform called *Kahoot!* [12, 15, 38, 41] to help increase class participation. Feedback from student presenters revealed that the experience of hosting virtual presentations was eye-opening and more challenging than they had realized. Overall, this presentation technique was viewed as a positive learning experience among student presenters. In addition to presenting, students were expected to provide feedback on their peers' presentations. To implement the peer review, Google Forms were created and shared for students to provide feedback, and they were later used by the instructor to inform the overall grade for the presentation. One unforeseen benefit to using Google Forms for peer-reviews is that typed comments from classmates were easily shared with the presenting group. In the in-person counterpart, either the instructor needed to type all of the feedback or unfortunately, the feedback would not be shared. Questions from our peer review forms are included in Section 7 along with sample rubrics used for presentations. These online forms remain useful for in-person classes to quickly gather and share feedback with student presenters.

Changes were also necessary to update exams, group work, and other assessments. In general, submitting and grading online was made easier with a tablet and stylus, and with the ability to type extended feedback instead of trying to scribble in a margin. Requiring assignments to be submitted as a single PDF in the correct orientation simplified the grading process, and students appreciated flexible deadlines (e.g., at 11:59 pm) in lieu of submitting homework at the beginning of class.

For traditional exams, the online setting created both academic integrity and student privacy concerns. As seen in Figure 1, one solution is to use a two-camera proctoring system together with a lockdown browser to ensure that students can only access approved materials while taking their test or quiz. This setup can be applied for both synchronous (live Zoom proctoring) or asynchronous (recorded Zoom proctoring) classes. However, traditional assignments can also be given through the learning management system or Google Classroom.

2.3 Alternative Assessment Techniques

One option to combat the difficulties of transitioning a course to a variety of modalities is to use *mastery grading* or *grading for growth*, as described in Section 1. In hybrid courses, the in-person meeting(s) can be used for assessments and reassessments, while completely online sessions can find convenient times for reassessments using the two-camera proctoring system mentioned earlier. Other options for courses that are entirely online include open-note but restricted time quizzes or “Convince Me” meetings during which students can answer questions in office hours that align to given objectives. Some of these courses used a three-tiered grading scale in which students are only awarded credit when they demonstrate a complete understanding of the objective being assessed in the question. The student earns the top score of “mastery” once they demonstrate understanding of the objective, and then they do not need to be assessed again on this objective. The middle tier is used to signal to students that they showed partial understanding, but were missing a key part of the objective. The lowest tier is given when a student only demonstrates minimal understanding. Course grades are based on how many of those objectives students have demonstrated they have learned by the end of the course usually through exams, quizzes, or retesting opportunities outside of class [3, 23].

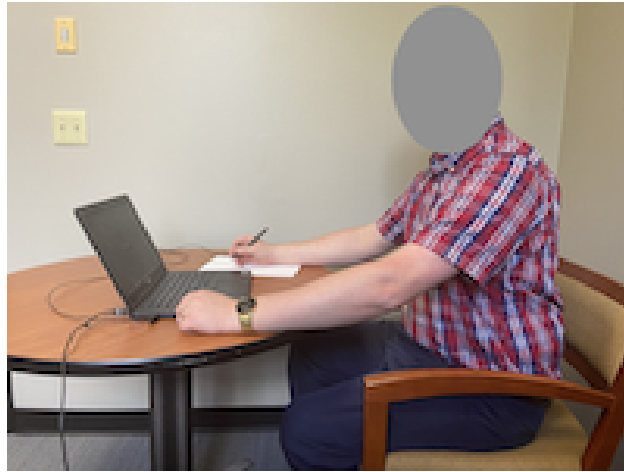


Figure 1: An example of a view from the secondary device of a two-camera proctoring system for use in online exams. Students are given setup instructions before the beginning of the exam that specify permissible resources and proper camera orientation. Students show their entire workspace to ensure that they are not using unauthorized materials.

For alternative assessment techniques such as oral exams, finding an online solution required more creativity. Oral exams have been found to have pedagogical advantages and have been used in a variety of courses including business, computer science, geology, mathematics, and statistics [6, 7, 26, 39, 40, 42, 48, 49]. Instructors have reported that these oral exams allow them to better determine student thinking [6, 42, 48, 49]. Oral exams were used in Real Analysis to assess more challenging proofs than what could be asked in an in-class mastery exam. Before the pandemic, this class assigned students three to four proofs to work on each week. The students and instructor then scheduled a time outside of class to present one of the proofs, chosen by the instructor. This assessment strategy also works well in other upper-division proof-based courses. Sometimes when students work on proofs, they make incorrect assumptions which derail the logical argument, making it challenging for the instructor to assess student understanding, particularly when the assumption trivializes the proof. If the instructor were to grade these proofs as take-home exams, they would need to determine how many points to take off without being sure what the student really understands. During an oral presentation, however, student-teacher interactions become a dialogue. It is easy for the instructor to ask “How do you know?” or “Why?” to have the student justify their reasoning. If they make a mistake or wrong assumption, the instructor can assess how they respond to questions. Additionally, if the student gets stuck on the proof, they can ask for a hint. This allows the instructor to take off a percentage for the hint but then assess their understanding of the course concepts by seeing how they use the hint. The student feedback and exit interviews from Real Analysis have been favorable, and students say that they learn more from this oral method of assessment than if the instructor were to write comments on their take-home exams, which supports Sabin et al. [42]. Oral exams also allow students to take the time to work on more challenging proofs. For example, the instructor can ask questions related to having a Darboux integrable function bounded below and ask the student to prove the reciprocal function is also Darboux integrable. Such a question may take too much time for an in-class exam. Another example could be to introduce the definition of *pseudo-Cauchy* and have students explain the difference between sequences with this property and Cauchy sequences. The instructor could then ask them to give an example of a divergent *pseudo-Cauchy* sequence. One suggestion is to perhaps rename properties so that it is harder for students to find the answer online.

During the in-person oral exams, students would work and write up their proofs at the board in the professor's office. When moving these assessments online, students shared their screen to present their proof. Usually these proofs were written beforehand in L^AT_EX. This worked, but there were challenges because students would read their argument instead of presenting it when they were online. It was also challenging for students to rework their proof. They could add text and annotate the shared screen or share a L^AT_EX document, but this took much more time than in-person oral exams. Moreover, it was difficult for students who did not have a stylus to write on the virtual white board. Even though there were some small inconveniences for doing online oral exams, the pedagogical benefits that come from using oral exams, such as creating a dialogue between instructor and student, allowing students to practice communicating logical reasoning, and reframing mistakes as learning opportunities, were still worthwhile. Two sample rubrics which can be used to assess oral exams are included in Section 8.

3 Methodology

Given all of the pedagogical changes that were necessitated by the pandemic, questions about students' mathematical anxiety, mindset, and reception of alternative assessment techniques became even more relevant. We had already begun studying the impact of mastery grading and expanding upon the initial studies in [24] and [25] when the unexpected move to remote learning in 2020 occurred. The COVID-19 pandemic provided an opportunity to analyze a subset of mathematics courses that used mastery grading by surveying the impact of mastery grading for remote or online classes. As mentioned in Section 1, there has been some evidence that alternate grading methods can help with reducing test anxiety, and we wanted to study the impact of using these grading schemes from the Spring 2020 through the Spring 2021 semesters. This paper reports on the following research questions which naturally occurred during the pandemic:

1. Was there a difference between students' anxiety in mastery graded courses throughout the course during the pandemic?
2. Did mastery graded courses help students develop a growth mindset in comparison with traditionally graded courses during the pandemic?
3. How much control over their grade/situation did students feel they had in mastery graded courses during the pandemic?

For this study, data were collected in eight different undergraduate mathematics courses at Lewis University and Marian University ranging from Calculus 1 through Real Analysis. In order to analyze the effectiveness of alternative assessment as a result of the pandemic, these data represent 146 students who responded to our survey from Spring 2020 through Spring 2021 and were enrolled in courses that implemented some type of mastery-based testing, standards-based grading, etc.

Twice during the semester, students completed an anonymous survey about their thoughts on assessment and the testing process. To maintain anonymity while still allowing for data pairing, each student was asked to use a six-digit identification number that consisted of the two-digit month and two-digit day of their birthday followed by the first letter of their mother's first name and the first letter of their father's first name. The survey asked to what extent the students felt levels of test anxiety throughout the semester, whether they went back and studied past concepts, and whether or not they felt that they better understood the material after studying the topics multiple

times. The survey also included questions to help gauge the student’s mindset about learning taken from Dweck’s Mindset Survey [17]. Survey questions are provided in Section 9.

Since we had pre- and post-data from mastery courses and post-data from traditional courses, we decided to use two main types of hypothesis tests for our analysis. When comparing the pre- and post-data, we were able to use the identification numbers to create paired data for two-sample Student’s t -tests for paired data. When comparing the post-data between mastery and traditional courses, we utilized two-sample Student’s t -tests for independent samples.

4 Results and Interpretation

In this section, we present our findings from the analysis organized into three categories: mathematics anxiety level, growth mindset, and perceptions of learning in the classroom.

4.1 Mathematics Anxiety

When analyzing the statement “I felt anxious before taking exams in this course,” a Student’s t -test for independent samples was unable to find a significant difference between students in traditional versus mastery courses (with a p -value of approximately 0.479). When surveying students in mastery courses, 64.5% of students felt less anxious about mastery exams than they did about traditional exams. Figure 2 provides the percentages of students in mastery graded courses who agreed or disagreed with the statement, “I felt anxious before taking exams in this course,” at the beginning of the semester.

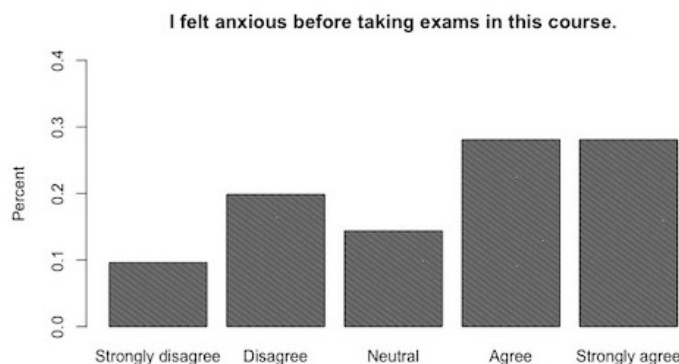


Figure 2: Percentages of students in a mastery graded course and their beliefs on the statement, “I felt anxious before taking exams in this course.”

4.2 Growth Mindset

In order to determine significant differences, two separate hypothesis tests were run for each question. A Student’s t -test with independent samples comparing the post-surveys from the traditional versus mastery courses was used to determine if there were significant differences between the groups after completing their courses. Additionally, a Student’s t -test with paired samples using the identification system described in Section 3 was used to determine if the students’ mindsets in mastery courses had changed between their pre- and post-surveys. Table 1 provides the p -values from these hypothesis tests with the p -values deemed significant in bold font.

| Statements | Paired Mastery Pre- vs. Post-Survey | Post-Traditional vs. Post-Mastery |
|--|---|--------------------------------------|
| “You have a certain amount of math intelligence, and you can’t really do much to change it.” | 0.911 | 0.724 |
| “I tend to give up when I make mistakes or fail a task.” | 0.029 | 0.484 |
| “When something is hard, it just makes me want to work more on it, not less.” | 0.003 | 0.545 |
| “To be honest, you can’t really change how intelligent you are in math.” | 0.496 | 0.090 |
| “You can learn new things, but you can’t really change your basic math intelligence.” | 0.150 | 0.663 |
| “I feel confident when doing mathematics.” | 0.005 | 0.029 |
| “To tell the truth, when I work hard, it makes me feel as though I’m not very smart.” | 0.717 | 0.526 |

Table 1: Results from the two-sample hypothesis test on growth mindset. Results that are in bold font and highlighted in blue represent significant p -values with a threshold of 0.05.

Only four of the hypothesis tests led to significant p -values, allowing us to support the following conclusions:

- Students in mastery courses agreed more with the phrase, “I tend to give up when I make mistakes or fail a task,” near the end of their class when compared to the beginning.
- Students in mastery courses agreed more with the phrase, “When something is hard, it just makes me want to work more on it, not less,” near the beginning of their class when compared to the end.
- Students in mastery courses agreed more with the phrase, “I feel confident when doing mathematics,” near the beginning of their class when compared to the end.
- Students at the end of mastery courses agreed more with the phrase, “I feel confident when doing mathematics,” when compared to students at the end of traditional courses.

The first three results above seem to suggest that the growth mindset of students was hindered in mastery courses since students would come in with more confidence and a harder work ethic. However, the fourth result suggests that student confidence at the end of mastery courses was greater than student confidence at the end of traditional courses. Combining these results with what teachers noticed in the classroom, it seemed as though students were burnt out by the end of the semester, which may be the true culprit that led to a lessened growth mindset.

4.3 Perceptions of Learning

There were four statements whose post-data from traditional and mastery courses were compared using Student’s t -tests for independent samples. The three statements, “I felt that I had control over the grade I earned in this course,” “The in-class assessments deepened my understanding of the ideas in this course,” and “The results of my in-class assessments accurately reflect my knowledge,” all did not have significant results, with p -values of approximately 0.337, 0.132, and 0.711, respectively. However, the hypothesis test on the statement, “I felt the grade that I earned reflected my understanding of the course content,” had a p -value of approximately 0.026. Since this

was less than 0.05, we investigated it further and found that students in traditional courses agreed with that statement more than students in mastery courses, indicating that students in traditional courses felt that their grade was a better reflection of their understanding than students in mastery courses did. This does not, however, mean that mastery students felt as though they had no control over their course grade.

As seen in Figure 3, 82.2% of respondents in mastery graded courses agreed or strongly agreed that they felt they had control over their grade. Much of this belief is likely strengthened by students' ability to retest concepts to show understanding of the material. Additionally, 58.6% of students in mastery-graded courses agreed or strongly agreed that their course grade reflected their understanding of course content.

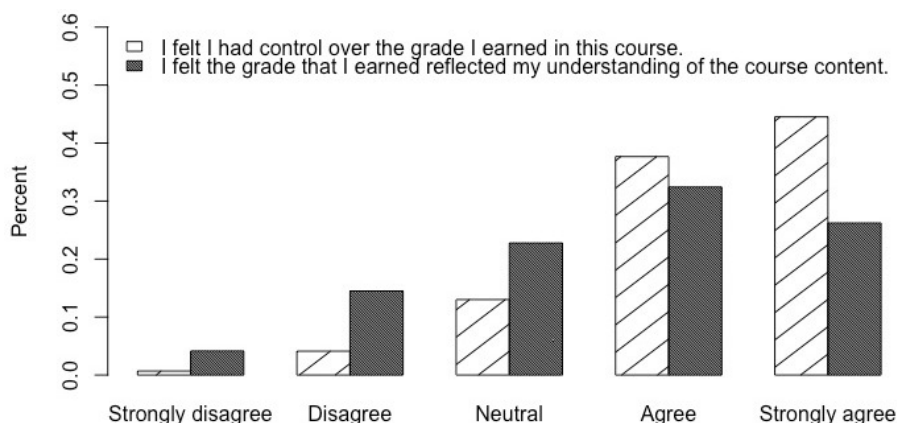


Figure 3: Percentages of students in a mastery-graded course and their beliefs on the statements, “I felt that I had control over the grade I earned in this course,” and “I felt the grade that I earned reflected my understanding of the course content.”

This was further supported by 73.3% of students responding that the in-class assessments deepened their understanding of the ideas in the course while 63.4% agreed or strongly agreed that their in-class assessment results reflected their knowledge of the course content, as shown in Figure 4.

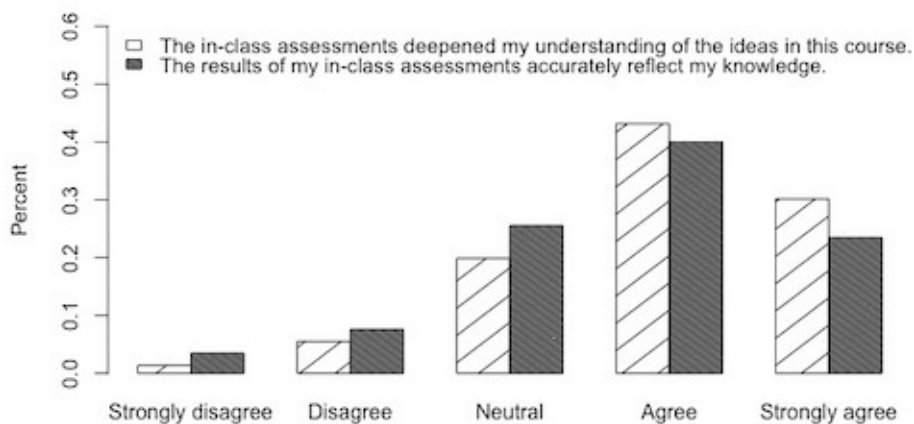


Figure 4: Percentages of students in a mastery graded course and their beliefs on the statements, “The in-class assessments deepened my understanding of the ideas in this course,” and “The results of my in-class assessments accurately reflect my knowledge.”

Much of the qualitative student feedback that we received at the end of the semester aligned with the data above. One student mentioned their role in a mastery graded course, “I love mastery based grading and I feel like it gives me a **real choice in my grade.**” Another student recognized how mastery grading forced them to learn the material: “I really feel mastery based testing is the best way to learn math. It forces one to **actually understand the process** of problem solving and **not just memorize** a small bit of information to regurgitate onto a test.” Finally, we must recognize that not all feedback was positive. Students have become accustomed to certain types of assessment in their secondary education. One student mentioned, “Being new to a mastery based class might be the reason why I struggled a bit because I’m so **used to traditional grading**, but at least I tried.” In a mastery-graded course, it is important to emphasize the reason and benefits of using this method of assessment. Multiple articles [3, 8, 10, 23, 24, 44] report student satisfaction in this assessment method and improvements on test anxiety and confidence. Students report appreciation for having extra time to practice objectives and that this assessment method allows them to learn the material more deeply [3, 24, 25]. Generally, student buy-in is easy to obtain, but sometimes, they miss their old routine of testing. We point the readers to Kelly’s article [31] which specifically outlines suggestions for how to build buy-in for this type of alternate assessment.

5 Discussion

In the abrupt transition to online learning, we found it imperative to keep courses as similar to their in-person counterparts as possible. Students were dealing with stress and other difficulties outside the classroom. If we were able to maintain “normal” in the mathematics classroom, we hoped this would aid in minimizing their stress levels. Students mentioned that breakout rooms helped to maintain the in-person interactions (“She made the transition seamless. She is still interactive with us and the class **still feels normal.**”) and that the new course set-up was comparable to in-person meetings (“This **transition went well.** Nothing was lost in the move to zoom meetings instead of in-person class.”). They recognized the intention behind keeping the class the same (“It has been the same because she is trying to teach the same. The class was already easy to begin with, but she has done a great job of **keeping the class consistent.**”), and some even found online to be more beneficial for their learning (“I think I may have been the only one that found this course **better online.** I think this was the best class of mine to have online this semester. I learned a lot and loved how the course was laid out. The mastery learning grading was very different and a little hard to get used to but in the end I loved it.”). As a whole, students were appreciative of the flexibility behind the transition to online learning.

There were two major items to note in our data. The first was having a very small pool of traditional students (only seven respondents for the post-survey and none in pre-survey) made data analysis difficult. Many of our hypothesis tests comparing traditional and mastery classes did not have significant results, and we were unable to run a pre- versus post-analysis on traditional classes without having any pre-surveys. The second was that there was a trend in student growth mindset being lessened by the end of the semester. We can see from our mastery pre- versus post-results that students were less confident and wanted to put in less work by the end of those courses. Anecdotally, we noticed that students in general seemed to be burned out by the end of the semester, especially given the context of living in a world at the height of a pandemic.

6 Conclusion

As instructors, the measures we took in order to merely survive teaching during a pandemic have benefits that extend beyond these tumultuous times. Many of the adaptations outlined in Section 2 are practices that continue to be useful moving forward, both in-person and online. Such pedagogical changes include having “blended” classes in which we maintain an online environment that is similar to the in-person experience and offering online tutoring or office hours to engage students who are unable to attend in-person. Traditional in-person courses can also benefit from online instruction when either the student or instructor is unable to attend class. Another pedagogical change is the uploading, collecting, and grading assignments online. This allows instructors to provide more immediate and helpful feedback as well as the possibility of end of the day assignment deadlines. Students indicated turning in online assignments provided more flexibility. Finally, providing students with consistent groups during class time allowed them to develop better teamwork skills working with the same peers throughout the semester. These new changes, however, bring to light new challenges that must be considered. From the student perspective, there were difficulties accessing stable internet and a quiet workspace. There were privacy issues that resulted from technological changes such as the lockdown browser Respondus, and the two-camera proctoring system for tests. From the faculty perspective, there were concerns about increased workload, student cheating, and providing equitable and fair opportunities for reassessment. The student feedback in Sections 4 and 5 suggests the pedagogical changes were overall helpful in student learning.

From students, the adaptations seemed to be appreciated. While they seemed to have slightly lessened test anxiety with mastery grading, it was harder to judge the impact that mastery had on their growth mindset and their perceptions of their mathematical learning. On one hand, a small sample size of traditional courses made analysis difficult. On the other, students seemed very burned out from their multitude of shifting responsibilities throughout the pandemic. These combined made it difficult to fully understand the direct impacts different teaching and testing styles had on students.

As a whole, the pandemic taught us that as instructors, we are always learning as much as our students. While students focus on the content, our focus is in how to *best* present material to them. There are teaching techniques that we try and find do not work for our audience. Other times, changes are necessitated by worldwide shutdowns. Regardless, it is our responsibility to reflect upon these strategies and lessons to determine how they can help us moving forward.

7 Sample Group Project Rubrics

Below is a sample rubric for group projects that include two components: a formal presentation and a class discussion or activity. This rubric has been used in both major and non-major courses. We frequently use it in general education mathematics courses where a final group project is more appropriate than a final exam.

- **Preparation for presentation (5 pts)**

1. Attended the last two class sessions to work on the presentation.

- **Presentation (30 pts)**

1. Introduction about the story (3 pts)

2. Introduction about the author (3 pts)
 3. Fun Facts or interesting tidbits about story/time period (2 pts)
 4. Includes analysis of literary concepts discussed in class (armchair detective, film noir, etc.) (5 pts)
 5. Includes analysis of deductive reasoning concepts discussed in class (logic, deductive reasoning, inductive reasoning, Holmesian deduction) (5 pts)
 6. Use of technology in presentation, visual is interesting, not just text (3 pts)
 7. Organization and flow of presentation/don't just read slides (3 pts)
 8. Verbal explanations (3 pts)
 9. Each member of the group is involved in the presentation or activity (3 pts)
- **Class Discussion or Activity (10 pts)**
 1. Well-planned and organized (2 pts)
 2. Engaging and interesting activities or discussion questions (4 pts)
 3. The group members did a good job of facilitating the activity (4 pts)
 - **Peer Evaluations (5 pts)** (To be completed by students in class, see Rubric [7.1](#))
 - **Own Group Evaluations (10 pts)** (To be completed by students in class, see Rubric [7.2](#))
 - **Attendance during presentations (5 pts)**
1 point for each presentation you are not presenting.
 - **Final Grade for Presentation (65 pts)**

7.1 Group Project: Peer Evaluation Form/Rubric

Immediately following a group presentation, we ask the students in the class to provide feedback on the presentation. Selected comments are shared with the group to provide peer feedback. Below we provide the evaluation questions.

1. The presentation was interesting, engaging, and held my attention.
strongly disagree 1 2 3 4 5 6 7 8 9 10 strongly agree
2. The presentation was understandable and easy to follow.
strongly disagree 1 2 3 4 5 6 7 8 9 10 strongly agree
3. The presentation was organized and flowed well.
strongly disagree 1 2 3 4 5 6 7 8 9 10 strongly agree
4. The group seemed to have put a lot of thought and effort into this presentation.
strongly disagree 1 2 3 4 5 6 7 8 9 10 strongly agree
5. Overall how would you rate this presentation?
poor fair good very good excellent
6. Additional Comments:

7.2 Group Project: Own Group Evaluation Form/Rubric

After their group projects, we ask students to provide feedback on each of their group members and **their own** participation as a group member. Below we provide the evaluation questions.

| | |
|---|--|
| Participation | |
| Group member did not participate, wasted time, or worked on unrelated tasks. | <div>1 2 3 4</div> <div>Group member participated fully and was always on task.</div> |
| Listening | |
| Group member did not listen to others and often interrupted them. | <div>1 2 3 4</div> <div>Group member listened carefully to others' ideas.</div> |
| Feedback | |
| Group member did not offer constructive or useful feedback. | <div>1 2 3 4</div> <div>Group member offered detailed, constructive feedback when appropriate.</div> |
| Cooperation | |
| Group member often treated others disrespectfully and/or did not share the work load. | <div>1 2 3 4</div> <div>Group member treated others respectfully and shared the work load fairly.</div> |
| Time Management | |
| Group member wasted time, was distracting, and did not complete tasks in a timely manner. | <div>1 2 3 4</div> <div>Group member stayed on task and completed tasks in a timely manner.</div> |

8 Sample Oral Exam Rubrics

Both sample rubrics in Tables 2 and 3 were modified from several resources from the Mathematical Association of America’s Project NExT and Inquiry-based Learning Networks. We would also like to thank Dr. Jessie Hamm, Associate Professor of Mathematics at Winthrop University, and Dr. Cory Johnson, Associate Professor of Mathematics at California State University San Bernardino, for their support and feedback on the logistics and rubrics used for oral exams. Below are two sample rubrics which have been used for oral exams.

| | Excellent | Good | Fair | Poor |
|---|---|--|---|--|
| Correct Mathematics: 30 points | Correct computations, demonstrates clear understanding of problem and concepts, no errors, no hints needed (29-30 pts). | Correct computations, good understanding of problem and concepts, needs one hint, but was able to continue from there (25-28 pts). | Few errors in computations, shows slight misunderstandings of problem and concepts, needs more than one hint, but was able to continue after hints (21-24 pts). | Many hints and prompts needed and was not able to complete proof without prompting, shows misunderstanding about problems and concepts (0-20 pts). |
| Correct Communication of Mathematics: 30 points | Explains every step without prompting, shows clear understanding (29-30 pts). | Explains most steps, but after prompting, gives clear explanations which demonstrate understanding (25-28 pts). | Explains most steps, but after prompting, gives explanations which demonstrate slight misunderstanding of problem and concepts (21-24 pts). | Does not explain most steps, cannot sufficiently explain reasoning, even after prompting (0-20 pts). |
| Correct Notation of Mathematics: 20 points | Uses correct notation and correct quantifiers (20 pts). | Uses correct notation and quantifiers except for 1-2 minor errors (17-19 pts). | Makes multiple minor or 1-2 major errors in use of notation/quantifiers (14-16 pts). | Does not use correct notation, misuses or does not use quantifiers (0-13 pts). |
| Correct Structure of Proof: 20 points | Uses good structure of proof, clearly defines all terms, includes connection statements such as “thus, so then, therefore, ...” (20 pts). | Organized proof, but may not define all terms or miss a connecting statement (17-19 pts). | Proof is disorganized, but reasoning and steps can be shown to follow from previous statements; variables are not all defined (14-16 pts). | Proof is very disorganized and hard to follow, next step does not clearly follow from previous step (0-13 pts). |

Table 2: Sample of oral exam rubric.

| Reasoning | Excellent (6 pts) | Acceptable (4 pts) | Repairable (2 pts) | Unacceptable (0 pts) |
|--|---|--|---|---|
| Hypothesis and Structure (direct, induction, contradiction, etc.) | Structure and hypothesis are appropriate and correctly applied. | Correct hypothesis applied, but with needless complication or minor flaw(s). | Inappropriate or seriously flawed structure (requires many corrections), hypothesis incorrectly used. | No identifiable or incomplete proof structure, proves different result, hypothesis ignored. |
| Definitions, Theorems, Terminology | Each used or referenced correctly. The proof uses accurate and appropriate mathematical notation and terminology. Symbolic notation is used where it clearly simplifies the discourse, and avoided when English will better serve the reader. | Minor terminology or definition error. Notation and terminology are correctly used, but there may be instances where the discourse would benefit from either more or less use of symbols versus English. The proof accurately invokes all needed definitions, through they may appear other than precisely where needed. | Terminology confused, distracting notation. Some relevant definitions or theorems are missing or misstated, but proof is otherwise understandable. Most of the notation and terminology used accurately. Errors are identifiable and correctable by a reader with experience similar to author. | Terminology incorrect or notation indecipherable, complete misuse of definition or theorem. Notation and/or terminology is frequently misused. The writer may use personal rather than standard notation. |
| Logic and Reasoning | A clear, complete, concise, cohesive, and properly ordered chain of deductive steps leads from the hypothesis to the conclusion. | Correct and reasonably well-organized, all variables defined. The chain of deductive steps is complete and correctly ordered. | Contains flaws or omissions, at least one undefined variable. One or more intermediate deductive steps are missing or unclear, but the correctness of the proof is not compromised. | At least one fatal flaw (e.g., assumed what is being proved or omits key step) or the stated chain of deduction does not lead to the stated conclusion. |
| Presentation | Excellent (3 pts) | Acceptable (2 pts) | Repairable (1 pt) | Unacceptable (0 pts) |
| Style and Clarity | The proof is well-organized and succinct, easy-to-read, without inclusion of irrelevant definitions or theorems. The proof moves seamlessly between symbolic notation and standard English. | Unnecessary detail(s) or awkwardness, but start and conclusion are clearly marked. The author generally avoids digressions, but may repeat some ideas unnecessarily. | The proof is well-organized, but includes extraneous steps, definitions, theorems, or unnecessary repetition. Disjointed, arrows to additions, and scratch outs are included. | Totally confused (e.g., scattered about the page or unnecessary statements). |
| Grammar | Grammar and spelling correct; equations are part of the sentence structure. | Structure apparent, but with sentence fragments. | Spelling and grammar (or lack of) are distracting. | Difficult to decipher. |
| Hints (oral exam only) | No hints needed. | Needs one hint and is able to continue from there. | Needs more than one hint or a huge hint; able to continue. | Many hints and prompts needed; not able to continue. |

Table 3: Another sample of oral exam rubric.

9 Surveys

9.1 Pre-Course Questionnaire

The results of this questionnaire will be anonymous. We appreciate your honest feedback. Please show how much you agree or disagree with each statement by marking the box that corresponds to your opinion. There are no right or wrong answers. We are interested in your ideas.

To ensure that the surveys remain anonymous, participants will use a 5-digit identifying number, given by the month and day of their birthday followed by the last digit of their birth year. For example if you were born on May 19, 1990, your number would be 05190.

Identification Number: _____

Instructions: Mark the response that most accurately reflects your opinion

1. **You have a certain amount of math intelligence, and you can't really do much to change it.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
2. **I like my work best when I can do it really well without too much trouble.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
3. **It is important to me that I learn the main ideas of this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
4. **I expect that I will have control over the grade I earn in this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
5. **I tend to give up when I make mistakes or fail a task.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
6. **I expect that the grade that I earn will reflect my understanding of the course content.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
7. **When something is hard, it just makes me want to work more on it, not less.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
8. **To be honest, you can't really change how intelligent you are in math.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
9. **I like work that I'll learn from even if I make a lot of mistakes.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
10. **I am excited to take this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
11. **You can learn new things, but you can't really change your basic math intelligence.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
12. **I like my work best when I can do it perfectly without any mistakes.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
13. **I feel confident when doing mathematics.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
14. **To tell the truth, when I work hard, it makes me feel as though I'm not very smart.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
15. **How many hours per week are you spending on this course outside of class time?**
☐ 0-2 hours ☐ 3-5 hours ☐ 6-8 hours ☐ 9-11 hours ☐ 12-14 hours ☐ more than 14 hours
16. **I am usually anxious before taking exams.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
17. **I feel anxious before exams in a majority of my other courses.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response

18. **Which study methods did you use? Mark all that apply.**
- ☐ Practicing additional problems from the book
 - ☐ Completing problems on review materials
 - ☐ Reading the textbook
 - ☐ Reading my notes
 - ☐ Redoing homework problems
 - ☐ Trying examples from the textbook or notes on my own
 - ☐ Group studying
 - ☐ Discussions with my instructor
 - ☐ Recopying notes from class
 - ☐ Watching videos online
 - ☐ Math Study Tables
 - ☐ Private Tutor
 - ☐ LARC or CASE Tutoring
 - ☐ Other: _____
19. **Demographic information:**
- ☐ Male ☐ Female ☐ Other ☐ Prefer not to answer
20. **Are you White, Black or African-American, American Indian or Alaskan Native, Asian, Native Hawaiian or other Pacific islander, or some other race (check all that apply):**
- ☐ American Indian or Alaskan Native
 - ☐ Asian
 - ☐ Black or African-American
 - ☐ White
 - ☐ Hispanic
 - ☐ Latinx
 - ☐ I prefer not to answer
 - ☐ Other
21. **Additional comments regarding assessment in this course:**

9.2 Post-Course Questionnaire

The results of this questionnaire will be anonymous. We appreciate your honest feedback. Please show how much you agree or disagree with each statement by marking the box that corresponds to your opinion. There are no right or wrong answers. We are interested in your ideas.

To ensure that the surveys remain anonymous, participants will use a 5-digit identifying number, given by the month and day of their birthday followed by the last digit of their birth year. For example if you were born on May 19, 1990, your number would be 05190.

Identification Number: _____

Instructions: Mark the response that most accurately reflects your opinion

1. **You have a certain amount of math intelligence, and you can't really do much to change it.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
2. **I like my work best when I can do it really well without too much trouble.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
3. **It is important to me that I learn the main ideas of this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
4. **I felt that I had control over the grade I earned in this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
5. **I tend to give up when I make mistakes or fail a task.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response

6. **I felt that the grade that I earned in this course reflected my understanding of the course content.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
7. **When something is hard, it just makes me want to work more on it, not less.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
8. **To be honest, you can't really change how intelligent you are in math.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
9. **I like work that I'll learn from even if I make a lot of mistakes.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
10. **I enjoyed taking this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
11. **You can learn new things, but you can't really change your basic math intelligence.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
12. **I like my work best when I can do it perfectly without any mistakes.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
13. **I feel confident when doing mathematics.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
14. **To tell the truth, when I work hard, it makes me feel as though I'm not very smart.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
15. **How many hours per week did you spend on this course outside of class time?**
☐ 0-2 hours ☐ 3-5 hours ☐ 6-8 hours ☐ 9-11 hours ☐ 12-14 hours ☐ more than 14 hours
16. **I felt anxious before taking exams in this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
17. **I feel anxious before exams in a majority of my other courses.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
18. **I feel less anxious before mastery-based exams than traditional exams. (Skip if you have not used mastery-based exams.)**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
19. **During the course of the semester, my anxiety level...**
☐ I had no anxiety ☐ decreased ☐ stayed the same ☐ increased ☐ no response
20. **The assessments in this course test our understanding of key concepts.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
21. **Studying for the exams in this course helped me to learn the material.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
22. **The results of my in-class assessments accurately reflect my knowledge.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
23. **The in-class assessments deepened my understanding of the ideas in this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
24. **I relied mostly on memorizing solutions to earlier problems to prepare for in-class assessments.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
25. **Throughout the semester, I often revisited old ideas that I hadn't fully understood.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
26. **I have retained past concepts and ideas learned earlier in the course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
27. **I have a good understanding on how material learned earlier in the semester relates to material learned later in the semester.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response
28. **I feel prepared to approach a wide range of problems from this course.**
☐ strongly agree ☐ agree ☐ disagree ☐ strongly disagree ☐ no response

29. **Which study methods did you use? Mark all that apply.**

- ☐ Practicing additional problems from the book
- ☐ Completing problems on review materials
- ☐ Reading the textbook
- ☐ Reading my notes
- ☐ Redoing homework problems
- ☐ Trying examples from the textbook or notes on my own
- ☐ Group studying
- ☐ Discussions with my instructor
- ☐ Recopying notes from class
- ☐ Watching videos online
- ☐ Math Study Tables
- ☐ Private Tutor
- ☐ LARC or CASE Tutoring
- ☐ Other: _____

30. **Additional comments regarding assessment in this course:**

31. **Demographic information:**

- ☐ Male ☐ Female ☐ Other ☐ Prefer not to answer

32. **Are you White, Black or African-American, American Indian or Alaskan Native, Asian, Native Hawaiian or other Pacific islander, or some other race (check all that apply):**

- ☐ American Indian or Alaskan Native
- ☐ Asian
- ☐ Black or African-American
- ☐ White
- ☐ Hispanic
- ☐ Latinx
- ☐ I prefer not to answer
- ☐ Other

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Using Graphic Novels in the Teaching and Learning of Mathematics and Physics

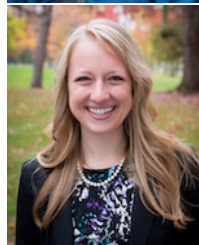
Jason Ho (Dordt University), Dave Klanderma (Calvin University)
Sarah Klanderma (Marian University), James Turner (Calvin University)



Jason Ho is an Assistant Professor of Physics and Engineering at Dordt University. His research focuses on properties of hadrons within the framework of quantum chromodynamics, including investigating exotic states of matter, and analytic techniques such as QCD sum-rules. His interests also include physics education, outreach, and facilitating research experiences at the undergraduate level.



Dave Klanderma is a Professor of Mathematics Education at Calvin University. Prior to that he spent many years as a faculty member of Trinity Christian College. His interests include mathematics education, the scholarship of teaching and learning, and the integration of Christian faith in the teaching and learning of mathematics. He recently completed his term as president of the ACMS.



Sarah Klanderma is an Assistant Professor of Mathematics at Marian University. Her research interests include computations related to topological Hochschild homology, studying students' transition to proof-writing courses, connections between mathematics and other disciplines, and her work with undergraduate research students at the intersection of number sequences and graph theory.



Jim Turner received a B.A. in mathematics from Boston University in 1988 and a Ph.D. from MIT in 1994. He was an NSF-NATO fellow at the IHES in France in 1996. Jim has taught at the University of Virginia, Holy Cross College (Worcester, MA), and Purdue University. He has served as a professor of mathematics at Calvin University since 1999. His research area is in homotopy theory.

Abstract

Are you looking for innovative teaching strategies for geometry or other mathematics and physics courses? In this article, we offer a discussion of several graphic novels and their potential for successful teaching and learning at the high school and university levels. We describe how engaging stories, combined with mathematical and scientific meaning found in both text and image, can help to excite students, enrich learning, and explain mathematical concepts. We report on recent data collected from multiple mathematics and physics classes that extend prior research on the use of graphic novels to teach English Language Arts (Boerman-Cornell and Kim, 2020) and will inform a book focused on the STEM disciplines that is currently in development (Boerman-Cornell, Ho, Klanderma, and Klanderma, *in press*).

1 Introduction

When you reflect on your experiences learning mathematics or physics in high school and college, what non-textbook readings impacted your understanding of and love of the discipline? Are there similar resources that you have implemented to teach mathematics or physics at these levels? Common responses to these questions include Edwin A. Abbott's *Flatland*, George Polya's *How to Solve It*, a biography of a famous physicist such as Albert Einstein, William Dunham's *Journey Through Genius*, or most recently Francis Su's *Mathematics for Human Flourishing*. We argue that these non-textbook readings can help to explain concepts, enrich understanding, and excite students to pursue careers in mathematics and physics as well as other disciplines within Science, Technology, Engineering, and Mathematics (STEM).

In this article, we expand the scope of non-textbook readings to include a genre known as *graphic novels*. Graphic novels leverage text, images, and the interplay between these two media in longer works than traditional comic books (see Figure 1 for a sample page). In particular, we argue that graphic novels can be a useful supplement to traditional textbooks for learning in both the mathematics and physics classrooms. Further, we offer exemplars of graphic novels that you might find useful in the teaching and learning at the university level.

2 Literature Review & Methodology

Although research into the use of graphic novels in teaching and learning is still emerging, there are several recent studies that document their potential for use in the classroom. Though not exclusively novels, graphic novels are generally book length and employ text, images, and the interplay of the two to convey meaning. Cook (2014) did an experimental study with high school students in which he compared students reading graphic novels with conventional texts containing the same information. Cook found that both grade levels and gender have a significant effect on how well students learn from graphic novels.

In particular, while most students had improved reading comprehension through the use of graphic novels, females and seniors benefited even more than their male and younger counterparts. Meier (2012) identified specific affordances that graphic novels can provide science teachers and their students, including embedding pictures in text to encourage students to engage more deeply and using anthropomorphizing objects of study to help students connect with the material. While research in the use of graphic novels in the teaching and learning of mathematics is not yet available, multiple studies have documented the positive impact of integrating literature into mathematics classrooms (e.g., Siebert and Draper, 2012; Koellner, Wallace, and Swackhamer, 2009).

Because vocabulary can be a high barrier to entry in STEM disciplines, graphic novels provide a helpful context for new ideas within the interplay of text and image. For instance, Boerman-Cornell, Kim, and Manderino (2017) discuss questioning, hypothesizing, model building, and many of the

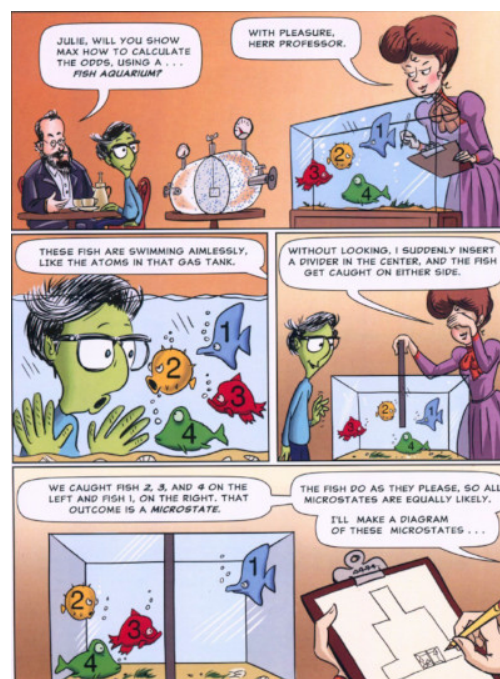


Figure 1: Page 55 from the graphic novel *Max the Demon versus Entropy of Doom* by Assa Auerbach and Richard Codor, used with permission.

higher-order skills necessary for work in these fields, as well as their use in graphic novels. They also identify three habits of practice in science, including constructing models of scientific processes, writing scientific explanations, and reading nonlinguistic scientific representations. They provide examples of graphic novels from both science and mathematics that demonstrate each of these habits. More recently, Boerman-Cornell and Kim (2020) have analyzed the use of graphic novels in the teaching and learning of English Language Arts, and Boerman-Cornell, Ho, Klanderma, and Klanderma (in press) are currently working on a similar analysis of the use of graphic novels in the teaching and learning of science, technology, engineering, and mathematics (STEM) at the middle school, high school, and university levels.

In this article, we limit our focus to two specific STEM disciplines: physics and mathematics. We offer examples of specific graphic novels that could be used in college and university classrooms while noting that some of these same graphic novels could be used at the middle school or high school level. Further, many graphic novels are versatile enough to be able to be used in the context of a variety of courses and highlight the interdisciplinary nature of many real world applications. Therefore, especially for readers who may not be familiar with graphic novels and their use in the classroom, this paper is designed as a jumping-off point for creative use of graphic novels in education more broadly.

This research team designed a study to assess the potential impact of specific graphic novels in the teaching and learning of physics and mathematics at the undergraduate level. Jason Ho selected two different graphic novels, one for use in his first-year physics course and one for use in an advanced physics course, each taught at Dordt University during the 2021-2022 academic year. Sarah Klanderma selected the graphic novel *Who Killed Professor X?* for use with her pre-service secondary mathematics teachers enrolled in her advanced geometry course at Marian University during the fall 2021 semester. Jim Turner agreed to use this same graphic novel in his advanced geometry course at Calvin University during the spring 2022 semester. Finally, Dave Klanderma selected the same graphic novel for a writing prompt on the take-home final of his middle school mathematics methods course taught at Calvin University during the spring 2022 semester. Approvals from the appropriate Institution Review Board from each of these three universities were obtained that allowed students enrolled in these courses to voluntarily provide their feedback on each of the graphic novels. For the methods course, students also gave permission for their writing prompts to be included as data for the study. In the following sections, we provide an overview of our experiences with the use of these graphic novels in the teaching and learning of mathematics and physics at our universities, and we include representative student comments to provide one qualitative measure of the effectiveness of this pedagogical strategy.

3 Graphic Novels for Physics

In his first-year physics course at Dordt University, Jason Ho created a laboratory activity based upon the graphic novel *Max the Demon versus Entropy of Doom* written by Assa Auerbach and illustrated by Richard Codor. After noticing that students in a first-year physics course typically found the concept of entropy to be challenging, he developed a set of activities designed to be completed during a three-hour laboratory period. The activities explored multiple definitions of entropy, including as an equation, a statistical phenomenon, unavailable energy, and the “arrow of time.” Students read and discussed the chapters of the graphic novel as they examined these paradigms of entropy used throughout history. He secured permission to reproduce portions of the graphic novel for use in his classroom. He notes that the recent release of *Tenet*, a film by

Christopher Nolan, may have increased student engagement with the “arrow of time” paradigm for entropy. The learning activity concluded with a hands-on exploration of entropy as a statistical phenomenon using 6-, 8-, and 12-sided dice, an activity inspired by LoPresto (2010). This activity has been included in full in Appendix 1 for those who would like to see more.

After implementing these activities, Ho found that students were highly engaged in this laboratory session based upon the graphic novel. Several students even stayed late after class to read the entire graphic novel with one remarking “Why can’t all textbooks be comic books?” He also asked students to complete a short survey based upon this laboratory session. Out of 22 students, 19 either agreed or strongly agreed with the statement “I would recommend an appropriate graphic novel in the teaching and learning of other STEM concepts,” and 20 either agreed or strongly agreed with the statement “I found the graphic novel a helpful approach to learning the STEM concepts more deeply.” While noting that this is a convenience sample, we nonetheless argue that there is strong potential for the use of graphic novels in the teaching and learning of physics as well as other STEM disciplines. Comparing a graphic novel to other textbook and non-textbook readings, 18 either agreed or strongly agreed with the statement “The graphic novel format makes the story more meaningful for me.” This response is not surprising and perhaps underscores the disciplinary reading skills that are necessary to understand difficult concepts in more traditional physics textbooks.

During a subsequent semester in his advanced quantum physics course, Ho used the graphic novel *Suspended in Language: Niels Bohr’s Life, Discoveries, and the Century He Shaped* by Jim Ottaviani and a team of five illustrators. This was a small class of five students, and he decided to place one copy of the graphic novel on reserve in the library and purchased another copy to leave in the physics lab for students to borrow as needed. This graphic novel links to a related work by Allison Fleshman, *Engaging Students in Quantum Theory Using a Graphic Novel About Niels Bohr*. In this physics course, Ho assigned weekly readings from the graphic novel and hosted discussions during class that aligned with specific concepts covered in the course (see Appendix 1 for sample questions that were integrated in these activities). Incorporating this graphic novel expanded the course content in an unanticipated way: the students were very attuned to the justice and ethical aspects of the novel, and they unanimously commented on how humanizing it made the content. Students generally noted how much it added to the course, without being an overly burdensome activity. He also notes that the novel provided an anchor during the lecture to draw on topics that shaped the development of quantum theory, such as the fleeing of Jewish physicists due to the rise in antisemitism and policies of genocide under the Nazi regime, the reception of refugee scientists by other countries, the nuclear arms race, and the resulting push for nuclear disarmament led by nuclear physicists.

Among the comments offered by his students are the following:

- “This comic-novel book gave me good insights of the life of Niels Bohr, and his discoveries play an important role in recognition of the physical world. This book provided me a steady foundation, a step stone in my understanding of physics history along with the magnificent impact from early physicists to our daily life.”
- “...even though [Bohr] seemed to get an audience, in the end his pleas for peace and communication across allied lines were to little avail.”
- “Niels didn’t have the words to accurately communicate the ideas that he wanted to express in lectures. No one had the words to describe quantum mechanics in a way that was widely accepted.”

Through this work, Ho's students were highly engaged in the reading and discussion of this graphic novel, and it resulted in a deeper more contextualized understanding of the related physics concepts.

4 Graphic Novels for Mathematics

In her advanced geometry course at Marian University, Sarah Klanderman used the graphic novel *Who Killed Professor X?* written by Thodoris Andriopoulos and illustrated by Thanasis Gkiokas. This book details the investigation into the mysterious death of the fictitious Professor X. Each of the principal characters is a famous mathematician, and thus the alibis detailing their innocence are communicated using mathematics, including skills from algebra, pre-calculus, and geometry. Klanderman's class included a total of five secondary mathematics majors, and she assessed that these pre-service teachers would benefit from a deeper discussion of geometric proof, including topics typically taught at the high school level. To that end, each student was given a copy of the graphic novel to read at their own pace and then selected characters to present on, both as real historical figures, detailing their biography as well as mathematical contributions, and to justify their innocence in the story by explaining the geometric proof for their associated witness statements. A description of this project is included in Appendix 2.

After their presentations, students suggested ways in which they could incorporate the graphic novel in their own teaching in the future. These ideas included weekly reading assignments from the graphic novel, discussion of a specific mathematician's life, small group work to solve a witness statement linked to a geometric proof, and weekly quizzes in which their students would be required to prove the "claim of the week." The advanced geometry students also identified a critical shortcoming of the graphic novel related to the sample of mathematicians from history that appear in the story as potential suspects: all are Caucasian and all except Sophie Germain are male. Overall, each of the five pre-service high school mathematics teachers enjoyed using the graphic novel and expressed an interest in using this and similar graphic novels in their future teaching. They also appreciated the opportunity to learn more about famous mathematicians and their key mathematical contributions and voiced curiosity about contemporary mathematicians and their research.

At Calvin University, Jim Turner used the same graphic novel in his advanced geometry course. A total of fifteen students were enrolled in the course, including eight secondary mathematics education majors. The course incorporated weekly lab sessions, most of which focused on Geogebra-based learning activities. However, in place of Geogebra during two of these lab sessions, he used the graphic novel and had students working in groups of three or four to analyze witness statements linked to geometric concepts. In addition to the stated Euclidean geometry context, students were asked to view the proof of the witness statement from a neutral geometry perspective in which hyperbolic geometry as well as Euclidean geometry would be valid options. Recall that neutral geometry assumes the existence of parallel lines (though not the specific number) passing through a point not on a given line; it also assumes the validity of the SAS theorem for congruent triangles. In particular, either hyperbolic or Euclidean (but not elliptical or spherical) geometries meet these conditions. Turner notes that the results of these lab sessions were mixed. While students enjoyed using the graphic novel and analyzing the witness statements, most were quick to believe that certain strictly Euclidean propositions (e.g., that similar triangles have proportional side lengths) were also neutral propositions, but were unable to provide plausible arguments to that effect. On their course evaluations, multiple students mentioned their enjoyment of the graphic novel and its connections to the course material.

In Dave Klanderman's methods of teaching middle school mathematics course at Calvin University,

students were provided with a copy of the same graphic novel as part of the take-home final exam. Like the pre-service teachers at Marian University, several students identified issues related to the lack of ethnic and gender diversity. One student noted the inclusion of Muhammad ibn Musa al-Khwarizmi in the graphic novel, but he was not included as one of the witnesses and therefore did not have a mathematical alibi statement. Klanderman earlier had provided students access to two posters with a total of over 20 women and non-white mathematicians that had been compiled by another Calvin student as part of a summer research project funded by the National Science Foundation. Some of the students noted that incorporating a more diverse set of mathematicians as part of a biography assignment would offer one way to address this concern. Among the written responses from these students are the following excerpts:

- “I would love to be able to show students these problems, possibly as a once a week activity where I put the book up on a document camera. If I wanted to keep the book hidden from my students, I could use the problems as a challenge problem, maybe at the end of tests, then students have a fun problem to work on while finished with the test.”
- “I would want to incorporate this in my math classroom because of how creative and different it would be for a math classroom. I never experienced anything like this. I think I would rather use it in geometry than algebra, but both would be great. The book acts as an extra resource for students to understand the content in a different, creative way.”
- “This is a great way to bring in review from algebra and see how it intertwines with geometry.”
- “For geometry it would be very easy to go through reading the book, but then stop at the specific geometry examples that would be applicable to one’s own class and curriculum.”
- “Formative assessments would come in the form of the teacher observing students as they work independently and with others. The discussion would also be a form of a formative assessment. At the end of class students could turn in an exit ticket explaining why the mathematician was innocent or guilty along with supporting work as a summative assessment.”
- “The teacher could have the students explain how they solved the problem, but it is still possible that they just read and understood the back of the book rather than actually figuring it out on their own. One way this could be helped is if the author published both a teacher’s edition of the book, which had the answers, and a student’s edition of the book, which did not have the answers.”
- “I also want to connect the ideas that this is a graphic novel about and using mathematics – a direct connection to my own background as an English major. I want students, regardless of how many books we have, to see the value of combining disciplines. Completing the math in this book was fun because it had story connected to it that was more than just surface level. I could have copied and pasted the problems without the story, and the problems would have been normal, albeit a little more advanced, than we normally complete. But when we combine the elements of English literature and Mathematics, we can create something new and exciting that opens a whole new realm of possibilities.”

It is interesting that the students identified a variety of ways to use the graphic novel, including both formative and summative assessments, and at least one highlighted the potential interdisciplinary nature of the graphic novel that blurred the boundary between literature and mathematics. Overall, all three classes utilized *Who Killed Professor X?* in university level mathematics and mathematics education courses. Students were engaged with the graphic novel, and the future high school

teachers saw potential for its use in the teaching and learning of both algebra and geometry. It is also worth noting that a high school mathematics teacher in the Chicago Public Schools (CPS) system recently used this graphic novel in his geometry class. He reported a high level of engagement with the graphic novel and the mathematical witness statements.

Other graphic novels may also play an important role in the teaching and learning of mathematics at the undergraduate level as well as other grade levels. One category of such graphic novels are Manga books that focus on specific topics in mathematics or statistics, including calculus, linear algebra, and introductory statistics. As an example, Shin Takahashi's *The Manga Guide to Statistics* offers readers an engaging discussion of many concepts that are typically covered in an introductory statistics course. Although we do not have data from the use of this graphic novel at the college or university level, we have feedback from a former student who used the book with both middle school and high school students in Chicago Public Schools during his student teaching internship. He used *The Manga Guide to Statistics* during lessons in which measures of central tendency and different data types were introduced. He reported that several students shared his own interest in Manga books, and these students were especially engaged in the use of this graphic novel. Interestingly, he also discovered that middle school students were generally more willing to spend the time analyzing the images and the supporting text than the high school students.

5 Conclusion

This article has identified multiple graphic novels that could enhance the teaching and learning of both physics and mathematics at the university level. Some of these same graphic novels might prove useful to middle school and high school teachers of the same disciplines. These graphic novels can enhance the understanding of specific concepts, explain some concepts in greater detail and in a way that links text and images, and can excite readers to pursue careers in these same disciplines. As mathematics and physics teachers, we should focus on the big ideas, key concepts, and necessary skills of our respective disciplines in our classroom teaching. Although graphic novels provide helpful modalities for engaging the readers and promoting deeper understanding, we should be cautious when selecting these materials for classroom use to ensure that larger pedagogical goals and objectives are met. Nonetheless, these graphic novels offer the potential to positively impact the learning of our students. This classroom-based research will inform a larger writing project that documents the potential role of graphic novels in the teaching and learning of science, technology, engineering, and mathematics (STEM) at the middle school, high school, and university levels (Boerman-Cornell, Ho, Klanderman, and Klanderman, *in press*). Interested readers are encouraged to visit a related website gnclassroom.com for more resources.

Appendix 1: *PHYS232L: Exploring Entropy* Activity (Jason Ho)

Discovering Entropy

Objectives

- To explore different definitions of entropy.
- To simulate entropy using dice sets and analyze the outcomes using a histogram.
- To understand the history of entropy through the lens of historical physicists and chemists.

1 Introduction

There has long been discussion around what entropy is conceptually, not just mathematically. Here, we'll spend some time thinking about some different definitions you may encounter. This lab will be more of a problem-solving session than a traditional lab. Please write your reflections, observations, and data down in your lab book or on a separate sheet of paper—these will be handed in and counted towards your grade for this lab.

2 Materials

- Excerpts from *Max the Demon vs. Entropy of Doom*
- Microsoft Excel
- A set of dice

3 Reading & Reflection

1. Read the following thought experiment proposed by James Clerk Maxwell in 1867:

... if we conceive of a being whose faculties are so sharpened that he can follow every molecule in its course, such a being, whose attributes are as essentially finite as our own, would be able to do what is impossible to us. For we have seen that molecules in a vessel full of air at uniform temperature are moving with velocities by no means uniform, though the mean velocity of any great number of them, arbitrarily selected, is almost exactly uniform. Now let us suppose that such a vessel is divided into two portions, A and B, by a division in which there is a small hole, and that a being, who can see the individual molecules, opens and closes this hole, so as to allow only the swifter molecules to pass from A to B, and only the slower molecules to pass from B to A. He will thus, without expenditure of work, raise the temperature of B and lower that of A, in contradiction to the second law of thermodynamics

With your group, answer the following questions:

- Maxwell describes that individual molecules travel at different velocities, but their average is always uniform. What measurable quantity is he describing here?
- Draw a set of diagrams that depict the experiment that Maxwell describes.

- How does this thought experiment violate the second law of thermodynamics? What would you observe if the experiment could be carried out? (if you're having trouble, think in terms of temperature.)
2. On Canvas, read the excerpts from the graphic novel *Max the Demon vs. Entropy of Doom* by physicist Assa Auerbach and illustrator Richard Codor. We're jumping into the middle of the story, but you can grab the full copy from the library (it's on reserve for this course) or borrow my copy in the lab if you'd like to read the full work. Here's the synopsis:

Max, a character based on the mythical Maxwell's Demon, is sent on a mission to help save earth from environmental disaster. First, Max must learn about energy, heat, and entropy from the historical giants of science. Along the way there is villainy and near catastrophe.

4 Entropy as a Mathematical Relationship

3. Entropy S is fundamentally defined by the equation

$$dS = \frac{dQ}{T}$$

where dQ is a small amount of heat in joules, and T is a temperature in Kelvin. Using this differential definition of entropy, consider the special cases of the first law of thermodynamics we considered in class. In each case, what would you expect the entropy to be? Assume an ideal gas, and *fully simplify* in each case (i.e., do the integrals when possible).

- (a) Adiabatic processes: $dQ = 0$, $dE_{\text{int}} = dW$
- (b) Isochoric (constant volume) processes: $dW = 0$, $dE_{\text{int}} = dQ$
- (c) Cyclical processes: $dQ = 0$, $dQ = dW$
- (d) Free expansions: $dQ = dE_{\text{int}} = dW = 0$

5 Entropy as Unavailable Energy

4. One of the earliest definitions of entropy came from Lazare Carnot and his son, Sadi Carnot (who is referenced in *Max the Demon*). In thinking about mechanical systems (for example, pulleys and inclined planes), Lazare Carnot observed that energy is always lost through friction or other dissipative forces. Sadi Carnot later extended this idea in his development of engines, and hypothesized that even in an idealized engine, some heat would always be lost

One way we can think about entropy then is as unavailable energy. Imagine a piston containing a gas at a certain temperature T_0 . If thermal energy is added to the gas, raising its internal temperature to T , the gas will do work on the piston and start pushing it upwards. If you had the power to arrange the direction and velocity of the particles in the gas in order to maximize the efficiency of the gas pushing on the piston, how would you arrange them? How do you think they are arranged in reality? Sketch each situation as you discuss it with your group.

6 Entropy as the Arrow of Time

5. Entropy is also sometimes looked at as a way to anchor the direction that *time* flows. One of the yet unanswered questions in physics is, why does time have a direction? At the microscopic level, if time were to reverse, it is thought that most of the laws of physics would remain the same—the mathematics that describe the microscopic level would not change. However, at a macroscopic scale, we would immediately know if time were reversed.

Many forces have a symmetry in time; for example, gravity is a time-reversible force. If you took a video of a ball being thrown up in the air and being caught again, the physics of that experience would not look very strange if you watched it in reverse. But, if we were to drop a ball on the ground and record a video of it bouncing, the physics of that experience would look odd if we were to watch it in reverse. What other examples of time-symmetric forces can you come up with? What examples of time-asymmetric forces (or phenomenon) can you come up with?

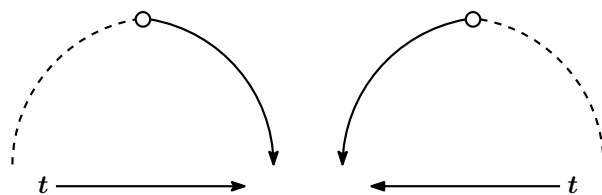


Figure 2: Based on a figure from a [MinutePhysics video by Sean Carroll](#)

6. Given the two snapshots of a system of a gas in a closed and insulated box below, which way is time flowing? Justify your decision with your group. How does this idea connect with the definition of entropy as chaos or as unavailable energy?

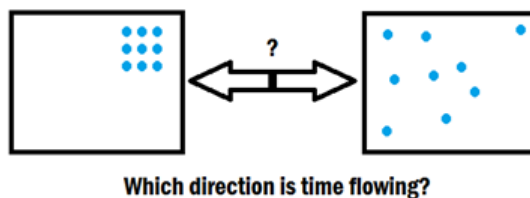


Figure 3: Taken from *Energy Education* (<https://energyeducation.ca/encyclopedia/Entropy>)

7 Entropy from a Statistical Perspective

7. Entropy can be described as the amount of disorder or chaos in a system, but this doesn't quite get at some of the nuances of what entropy represents. In this activity, we'll explore the statistical side of entropy. In **statistical mechanics**, the thermodynamics we are investigating in this course are explained through the motion of atoms and molecules (when thermodynamics was first being developed, the concept of an atom had yet to be developed) Consider the coin toss described in the excerpt from *Max the Demon*. A single coin toss carries a random probability, but if we flip the same coin many times, we reach a predictable result—half the time the coin will land on one side, and half the time it will land on the other. What other examples can you come up with where an individual outcome is unpredictable, but a large number of outcomes is predictable?

8. In statistical mechanics, entropy S is defined in terms of the number of **microstates** W of a system,

$$S = k_B \ln W.$$

How is a microstate defined in *Max the Demon*?

9. Consider a single dice (1d4, 1d6, or 1d8). What is the probability of any one number being rolled?
10. Take a pair of four-sided, six-sided, or eight-sided dice (2d4, 2d6, or 2d8). Using your method of choice (whiteboard, paper, Excel, etc.) tabulate all the possible values you can get by adding the digits of the two dice together. Determine the *multiplicity* (or *degeneracy*) of each value: how many different ways can you roll each possible value? What do you think the most likely values are? Which values are the least likely?
11. Using the multiplicities, calculate the probability for each outcome you determined in the previous step. If the dice behave exactly as you expect them to, what would a histogram of 64 trials look like? 128 trials? By hand or in Excel, create a histogram showing the theoretical prediction of the dice behavior.
12. Using the two dice you chose, roll the pair of dice 64 times. Record the sum of each roll.
13. Plot a histogram of your results (either by hand, or in Excel). What is your most common value? Is it what you expected? Check for any unexpected behavior in your histogram. Would you expect the results to change if you took more data?
14. Roll your pair of dice another 64 times (perhaps let your partner do it this time). Record the sum of each roll.
15. Plot your combined results from steps 7 and 9 in one histogram. What changed?
16. Ludwig Boltzmann was the first to define entropy in terms of microstates. As shown in Figure 4, his equation is engraved on his tombstone in Vienna. Boltzmann's construct determined that a highly ordered state (for example, one with all molecules traveling in the same direction) was the most improbable configuration of a system. In other words, the more "disordered" or chaotic a system was, the more likely it was to occur. How does this connect with the other definitions we've discussed? How much energy would be unavailable to be used as work in a highly ordered system?



8 Conclusion

To finish the exercise, fill out this short survey on how you reacted to the incorporation of a graphic novel into this lab activity: <https://forms.gle/QoBgjZwFGtEr99Y8>

Figure 4: Ludwig Boltzmann's tombstone. (Source: [Daderot](#) at [English Wikipedia](#))

Appendix 2: *MAT-330: Graphic Novel Project* (Sarah Klanderman)

Purpose: In this class we will be using the graphic novel *Who Killed Professor X?* to see how geometry can be used to solve the mystery of the book, learn about the history of various mathematicians, and explore ways in which alternative media such as graphic novels can be incorporated in your future teaching as well.

Process:

1. Each student will choose two mathematicians from the lists below. You are welcome to read through the graphic novel prior to choosing if you would like to make an informed decision (although there are no bad choices).
2. Read the graphic novel, spending time to determine how to do each of the problems given for every defense.
3. Create a presentation of the solution of your first mathematician's defense using geometry. Make sure that you are able to answer any questions you may receive about the solution.
4. Research the life and history of both of your mathematicians to provide historical vignettes about each mathematician in your presentation.
5. Create a lesson plan or proposal of how you could utilize this graphic novel (or other graphic novels) in your own teaching in the future. Focus on how you can use it for enrichment and motivation for your own students to study math.

Lists of Mathematicians

Select one of the following mathematicians. Then present the geometry involved in that mathematician's defense in the graphic novel, and also provide an historical vignette of that mathematician's life.

- | | | |
|---------------------------|--------------------|------------------|
| • Constantin Carathéodory | • Pierre de Fermat | • Pheidias |
| • Blaise Pascal | • Isaac Newton | • René Descartes |

In addition, choose one of the following mathematicians and outline that person's biography (but not necessarily provide the solution to their *Who Killed Professor X?* defense).

- | | | |
|--------------------------------|------------------------|-----------------------------|
| • Leonhard Euler | • Carl Friedrich Gauss | • Bernhard Riemann |
| • Marie-Sophie Germain | • Évariste Galois | • Gottfried Wilhelm Leibniz |
| • David Hilbert | • Kurt Gödel | • Augustin-Louis Cauchy |
| • Jean Baptiste Joseph Fourier | • Siméon Denis Poisson | |

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A Mathematical Analysis of Large Numbers in the Book of Numbers

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Abstract

For more than a century, both Christian and non-Christian scholars have questioned large numbers mentioned in the Bible, especially those in the book of Numbers about the population of Israel in the Exodus. Many of them view those numbers as fictitious because of the seemingly unreasonably large size of them. There are also Christian scientists, such as Sir Colin Humphreys [6], believing that the scriptures were misinterpreted and trying to use mathematics to “fix the problem” about these large numbers in the Bible. This article provides a different angle viewing the numbers of population to justify the likelihood of the seemingly unlikely large numbers using the exponential growth model of population. This attempt, on one hand, may open more space for the teaching and learning of intro-level mathematics courses in college. On the other hand, it shows that mathematical knowledge and skills can be powerful for us to debunk the “this-is-not-possible” challenges, rather than doubting, about what is written in the Bible.

1 Introduction

So all those listed of the people of Israel, by their fathers’ houses, from twenty years old and upwards, every man able to go to war in Israel—all those listed were 603,550. (Numbers 1: 45-46, ESV)

It seems much easier to claim the errors of the Bible than really believe it. One of the questions bothering many people is about the large numbers in the book of Numbers. As shown in verses 45 and 46 in Numbers 1, the number of war men (twenty years old and upwards, who are able to go to war) in the first census of the Exodus is 603,550. Later on, after forty years of wandering in the wilderness, the second census recorded in Numbers 26 shows a similar number (601,730) of the population of Israelite war men. It is commonly accepted that, the number of about 600,000 war men implies a total population in the Exodus (including men, women, and children) of at least two million [6]. However, as Bishop J. W. Colenso argued in his influential book, *The Pentateuch and Book of Joshua Critically Examined* (published in 1862), the number of two million seems “well beyond the bounds of what may be regarded as historically probable” for many readers of the Bible [2, p. 450]. Following Bishop Colenso, the conservative biblical scholar J. W. Wenham once addressed more bluntly,

It is notorious that the Old Testament in many places records numbers which seem impossibly large [10, p. 19].

Too large to be true—this indeed has troubled many people including Old Testament scholars and Christian scientists for a long time. J. W. Wenham and many other Christians have specifically examined this issue to defend their faith about God, His Word (the Bible), and the Christianity. Besides the commonly accepted claim of possible “textual corruptions” of the Holy Scriptures copied from generation to generation in thousands of year [10, p. 21], another persuasive argument emerging is that the extremely large numbers probably came from the misinterpretation of a Hebrew word root, “*a-le-f*”¹ in the two census listed in Numbers 1 and 26. As argued by Wenham, this Hebrew word has at least two different meanings, one of which is the exact numerical value of “thousand”, and the other is used in Old Testament as a social unit, such as “family”, “clan”, “troop”, etc. Proposing that in the two censuses “*a-le-f*” should mean a social unit with a much smaller size than one thousand ², Wenham concluded that the reasonable number of the total population of the Exodus should be about 70,000.

Assuming the same possible misinterpretation of this Hebrew word, the Christian British physicist, Sir Colin J. Humphreys has conducted a mathematical “decoding” on the “apparent error” of large numbers shown in Numbers 1 and 26 [6, p. 196]. His analysis drew on several more deduced assumptions, such as the average number of males in a regular Israelite family at the time of the Exodus, the ratio of Israelite male population to Levite male, and the ratio of Levite male to first born Israelite male at that time. After a seemingly rigorous mathematical analysis, he concluded that “as a round number, the total of men, women and children at the Exodus was about 20,000” rather than the figure of more than 2 million suggested by the census mentioned in the book of Numbers [6, p. 211].

Besides Sir Humphreys’ work on this so-called “mathematical conundrum” [2], a group of biblical scholars have inherited the way of interpreting the Hebrew word “*a-le-f*” not as the exact number of 1000 but a smaller social unit and claimed that we should do a “hyperbolic interpretation” [4] of those large numbers, considering them as a rhetorical way of speaking commonly found in ancient Near Eastern. For instance, David M. Fouts, a Bible professor, claimed that we should interpret the large numbers in this way for the sake of removing the “stumbling block for accepting the Biblical accounts as legitimate records of history” and stopping people “questioning the integrity of the record” [4, p. 387].

Including aforementioned, there are more than several ways of interpreting the seemingly unreasonably large numbers in the book of Numbers through the years after Bishop Colenso’s work in the 1860s (e.g., [2], [7]). No matter what ways they were taking, the arguments aforementioned were built on the same assumption: the large numbers can not and should not be true because they are so unlikely to be true and so unreasonable that they have to do some work to fix this problem. However, I would argue that the work they have done were not just “fixing” the problem; but, more importantly, they were affirming the existence of the problem by trying hard to solve it. In the sections below, I try to challenge this assumption by asking: *Are these numbers too large to be true? What if we assume that they are possibly historical records? Are they really unreasonable?*

Guided by these questions, I designed a problem of exponential modeling (In Section 2) using the numbers in the Bible for the intro-level college algebra course I taught in Fall 2021. Mathematically, the exponential model itself is surely not the best model for the estimation of the population in

¹I translate the Hebrew word to English letters following the way that the Jewish mathematician Haim Shore does in his book, *Coincidences in the Bible and in Biblical Hebrew* [9].

²In fact, according to a personal conversation with a Hebrew professor, Dr. Charles Lu, when the “*a-le-f*” shows after exact numbers in Hebrew (as it is in Chapter 1 of the book of Numbers), it does mean “(one) thousand”.

400 years. Pedagogically, however, it would be a good start to develop students' critical thinking on mathematics modeling and the *reasonableness* of a mathematical solution when taking into account more information of real life. Following that, I argue that the analysis on reasonableness not only provides more opportunities for students to authentically experience exponential models; it also helps setting up the necessity of a more complicated mathematics model—a logistic model (in Section 3). Furthermore, the possibility of analyzing the same story in different models provides an opportunity for us to reflect on and then challenge the reasoning people have made based on *reasonableness (or not)* heavily relied on personal experiences. Is it really possible for us to justify either the reasonableness or unreasonableness of the numbers in the Bible? The last section of this paper I try to go back to the author of the Bible, the LORD, and ponder on the purpose of the Holy Scripture. After that, I conclude this paper with the power of mathematics as a tool to stretch our personal feelings and draw us closer to the Almighty God.

2 Connect the Big Numbers with Exponential Model

It should not be a surprise that the big numbers are usually put on the spot when people are questioning the inerrancy of biblical record of the historical event of the Exodus [2], although this was not my personal experience with the Bible in the last six years as a new re-born Christian. It is probably because I was too amazed by the stunningly rich contents of the Bible to really scrutinize the numbers in details in the first five years of being a Christian. Fortunately, as a mathematics educator, I got asked and forced to think about those large numbers. It was in Fall 2021, while I was in the midst of teaching college algebra at a public university, which is the lowest level of credit-bearing college mathematics course for non-STEM (Science, Technology, Engineering, and Mathematics) majors. Someone in a Chinese bible study group brought forth the following question: *Is it possible for the population of Jacob's family growing from 70 people³ to more than 2 million after 430 years⁴?*

This reminded me of the exponential model of population growth in my college algebra class. It is not hard to make up a typical practice question of this model based on the information about Jacob's family given in the Bible, as shown below:

Assume that the growth of the population of Jacob's family after t years of the start of its record can be modeled by $P(t) = P_0 e^{rt}$, where P_0 is the initial population of the family.

- 1) If the initial population of this family was 5, and after 53 years, the family had grew up as a family of 70⁵, what was the annual rate of growth of its population in these 53 years?
- 2) The family went down to the land of Egypt after these 53 years with a population of 70. If hereafter the population of this family kept growing at the same annual rate of growth, after how many years would the family reach the population of 2.5 million?
- 3) If hereafter the population of this family kept growing at the same annual rate of growth, what would be the population 430 years after they wend down to Egypt?
- 4) If after 430 years, the family population has grown from 70 to a population of 2.5 million, what should their annual rate of growth be?

³Exodus 1:5

⁴Exodus 12:40

⁵Genesis 47:9

In Part 1) above, Jacob's family is assumed as originally having five people: Jacob, his two wives, and two concubines, which gives the initial population $P_0 = 5$. In Genesis 47:9, we are told that Jacob's family had 70 people down into Egypt, which gives $P(t) = 70$. The years for the family developed to 70 people is assumed as $t = 53$, the difference between the age of 130 when Jacob went down to Egypt with the family (Genesis 47:9) and the age he fled from home, which is commonly accepted as 77⁶. Substitute these information into the model, soon we can solve the average annual rate of growth is $r = 0.0497$ ⁷ = 4.97%. Deduced from a commonly accepted mathematical model and two fairly small numbers of population (5 and 70), I would argue that this result should be considered as a reasonable one, although there may be some arguable points about the potential over-simplification of an exponential model.

Once the reasonableness of the annual growth rate of the family, 4.97%, is settled, I may use it to estimate the population of the family after 430 years based on the exponential model. But before doing so in Part 2) and Part 3) , I should justify if the population growth of this family would be able to keep following the same model as described in Part 1), even after they went down to the land of Egypt. It is worth noting that, before they went down to Egypt, the life of Jacob's family in Canaan was far from "prosperity" if we consider the scandalous events that happened in the family one after another, such as the slaughter at Shechem (Genesis 34), the deaths of Judah's sons (Genesis 38), Joseph sold by his own brothers (Genesis 37) and so on. As summarized by Jacob when he talked to the Pharaoh, "few and evil have been the days of the years of [his] life" (Genesis 47:22) before they went to Egypt during the severe famine in all the land. Now if we take into account the verses in the Bible saying that the Israel "multiplied greatly" (e.g., Genesis 47:27, Exodus 1:7, 20) after they settled in Egypt, where they possessed "the best of the land, in the land of Rameses, as Pharaoh had commanded" (Genesis 47:11), it should not be unreasonable to assume that the growth of this family's population would keep the same, if not faster, in Egypt as before. That is, using $r = 4.97\%$ would not overestimate the population of the family based on the exponential model.

According to the exponential growth model with $r = 4.97\%$, the result of Part 2) is about 211 years. That is, if we agree that in Egypt Jacob's family would be growing larger and larger in a speed at least not slower than before, it would only take the family about 211 years (less than half of the 430 years) to increase to a population of 2.5 million, which I purposefully set as a number "over 2 million". Now evidenced by the analysis according to this model, it is very likely that the population of a family could increase to a "unreasonably large" number of over 2 million after hundreds years. Following this, the Part 3) is more about my own curiosity and amusement with a mathematics model: What would the model tell us about the population if the prosperity of Jacob's descendants really lasted for 430 years? The answer of it is 1.33×10^{11} , or 133 billion! Comparing this humongous number produced by the mathematical analysis done above, it may be more natural to question, "*Why the population in the Bible is NOT this large?*", rather than simply assuming the numbers recorded in the Numbers as too large to be reasonable.

One response to the question of "*Why the population in the Bible is NOT this large?*" may be on

⁶Here is one of many resources about how to get this number using the information given in Bible: <http://hiswordsarepure.com/TheLifeOfJacob.htm>. There are some other views on the age of Jacob when he fled home. However, even if we take the age of 40 (Genesis 26:34) as the time Jacob fled home, and then assume the time for the family growing to 70 people as $130 - 40 = 90$ years, according to the exponential model, the annual growth rate would be $0.0293 = 2.93\%$. If we compare this to the result of Part 4), it is still large enough to justify the likely-hood of the over 2 million population of Exodus.

⁷To not overestimate the population, the decimal is rounded down to the ten-thousandth place. For the decimals hereafter are the same.

the growth rate⁸. For example, *Is it reasonable to keep the population growth rate as 4.97% for such a long time? If not, what is the average annual growth rate that could make the 2.5 million population in 430 years?* Part 4) was designed with the question on the growth rate of 4.97% in mind. If we accept the population of 2.5 million at the time of the Exodus⁹, which is after 430 years of growth, and we may ask: What should the annual rate of growth be to make the population reach 2.5 million? Mathematically, this is exactly same type of question as part 1). The information of $P_0 = 70$ and $P(430) = 2,500,000$, gives us $r = 0.0190 = 1.9\%$. This means that, to increase to a population of 2.5 million in 430 years, the growth rate only needs to be 1.9%. Compared to the growth rate of population in the world today, which is 1% [8], the rate of 1.9% seems very large. However, if we look back a little further at the last five decades, according to the data from Dr. Max Roser and his colleagues [8], we may easily find more than several countries in the world having reached a growth rate over 2.0% for a period at least about 10 years. For example, China from 1964 to 1974, India from 1958 to 1995, Bangladesh from 1975 to 2000, just name a few. Even the United States peaked the growth rate in the 1950s as about 1.6%.

With regard to the reasonableness of a growth rate of 1.9%; I cannot say much about other countries; but I do know that the period of 1964-1974 in China definitely cannot be viewed as a good time for Chinese people in a country taken over by the Communist Party organizing political movements one after another all over the country from 1957, the start of Anti-Rightist Campaign, to 1976, the end of the Great Cultural Revolution. If the growth rate of the population in China was able to maintain 2% for about 10 years in that tough societal environment, why this could not happen for the Israel people before the Exodus? To this point, the mathematical analysis using exponential model shows very well that, the “unreasonably large” numbers recorded in the book of Numbers might not be as large as it should be if the population really follows the exponential model. In other words, according to the exponential growth model, the claim of the “unreasonableness” of these numbers may be supported by the fact that they are not large enough, rather than too large.

3 Further Exploration and the Pedagogical Meaning

The topic of exponential growth has been documented as an important but difficult one in the literature of mathematics teaching and learning [3]. Researchers have noticed that the co-variation of the parameter (annual rate of growth in this case) and the population increasing exponentially is not as obvious as that of the slope and the change of the height of a line. Even worse, when the dependent variable like population increases exponentially, it is very common that an instructor (or author of a textbook) adjusts the units on the two axes to make its graph more presentable. That is, even the graphic representation of an exponential growth is not so accurate as a line representing a linear growth. The learning of exponential and logarithmic functions became so hard when even the graphic expression cannot offer much clue for students to experience these functions, especially for those in the lowest level of credit-bearing mathematics courses (such as College Algebra).

For the students who just made it through the developmental mathematics contents and got into college algebra class, it is not easy to understand how an exponential growth would be significantly different from a linear growth. It is even harder to realize how the little change of the parameter can make a huge difference in the population in a longer term. The problem in Section 2, especially the comparison of Part 3) and Part 4) may naturally draw students' attention on the feature of

⁸Surely there are a lot more to discuss, such as the capacity of the land and the wilderness of Sinai. But this is beyond the scope of exponential growth model and I will discuss this briefly in Section 4 of this paper.

⁹Hopefully this may not be considered so unreasonably large compared to the 133 billion.

exponential model. We may remind students by asking them, *Is it okay to use the same growth rate of the first 53 years to model the next 430 years? Why or why not? How much would the error of the population be if we choose different growth rates?* These questions may open up more space for a multi-angle discussion on the exponential model.

More than just focusing on the content of exponential model, at the time when I made up the problem in Section 2 as a practice problem, I intended to use it to show my students the limitation of a simple mathematics model¹⁰. There is a large body of literature in mathematics education critiquing how far the final, perfect product of mathematics knowledge is from the real life of the real world. With this in mind, the design of the problem was not for getting some correct historical numbers at all.¹¹ On the contrary, encouraging students playing with those numbers using the exponential model, I attempt to focus on how the problem may bring students more opportunities to gain a deeper understanding on the exponential growth model of population in particular and mathematics modeling in general.

When talking about mathematics modeling, the examples provided in textbooks are usually very well designed for a purpose to show how well the mathematics model may perform in a certain situation. This is especially true in lower level mathematics courses such as college algebra. Differently, the problem presented in the last section ends up with some unpredictable numbers for students and may well challenge their previous understandings of the model and the reasonableness of numbers. That is, the practice of using the exponential model in that problem is more about helping students realize that we should be very cautious with the accuracy of a seemingly pretty mathematical model, especially when we attempt to describe the change of a variable in a longer term. Furthermore, it also may draw students attention on the point of “reasonableness”: What do we usually mean by “reasonable”?

In Part 1) of the problem, the growth rate of population, 4.97% may seem very reasonable because it is deduced from a well-developed mathematics model and two fairly small and then reasonable numbers. However, does the same growth rate still seem reasonable after we solved Part 3) with a number of 133 billion? Probably not any more. Similarly, does the growth rate of 1.9% seem reasonable? The answer may be YES, if it is only compared to the 4.97%. The answer may be changed if students are reminded that today in the United States the growth rate of the total population is only 0.4%! However, they may change their idea again to claim that 1.9% is very reasonable if they consider the argument I made in the last section about the 2% growth rate of Chinese population during 1964 to 1974. The back and forth here is aimed to show how much the judgement of “reasonableness” is relied on our personal experience with given information. And therefore, we want to be very careful when we claim some result is “unreasonable” based on our feeling or personal experience.

On the other hand, would the result deduced from mathematics reasoning always be more reliable than personal experience? Not really. For example, the number of 133 billion we got from Part 3) of the problem sounds obviously unreasonable if we compare it to the world population, which is close to 8 billion today. But, why this is “obviously unreasonable”? Although it is beyond the scope of teaching and learning exponential model, it is worth noting that the feeling of “obviously

¹⁰I only use this problem as an extra out-of-class practice because the university that I currently teach is a public state university.

¹¹I had never thought the numbers in the Bible as not real historical numbers before I read Sir Humphreys’ article arguing that the population of the Exodus should be 20,000 [6]. So, I truly felt and still feel that this would be a very good example to show how real life usually does not happen exactly as a mathematics model would predict due to many other complicated factors surrounding.

unreasonable” comes from the feeling (rather than a solid understanding) of an important concept: Carrying Capacity K , which is the maximum population size that a land may sustain. So, this may be a great moment for an instructor to bring forth the logistic model of population growth to take into account the parameter of carrying capacity of the land and make a better model. Pedagogically, the capture of the feeling of either reasonableness or unreasonableness may always be a great catalyst of new knowledge.

Interestingly, many scholars aforementioned in the Introduction claimed the “unreasonableness” of the large numbers of the Exodus population based on the exact same feeling—the doubt of the capacity of the land for the Israel family settled in (e.g., [4]). However, this does not justify the “unreasonableness” they claimed. The carrying capacity K of a logistic model is usually assumed based on one’s observation, precious experience, or scientific experiments. On the carrying capacity of a certain land, someone from New York City may have different idea based on his/her life style from those in a different living environment. By saying this, I by no means want to undermine the strength of the logistic model or any mathematics model for prediction or description of certain social and/or scientific events. On the contrary, my point is that it could be more powerful if we are more careful with the choice of the parameters.

Nevertheless the point here is that, rather than claiming that the large numbers are unreasonable and never using them as trustworthy information, the acceptance of them in fact opens much more space to sustain beneficial discussions on the co-variation of the exponential model in particular and then the fundamental understanding on mathematics modeling in general. That is, the purpose of discussing the population of the Exodus is not to prove that the numbers are the real historical fact about how many Israel people really were in the event (which probably no one in the world can really prove), but to open up the possibility of the existence of a seemingly unreasonable event.

4 Significance of Mathematics for Faith in God

Here in this paper discussing the exponential model and logistic model of population growth is by no means an attempt to “prove” the accuracy of the numbers recorded in the book of Numbers. After all, the Bible is a book about Him and His miraculous work in the history of Israel and in the world. As pointed out by the Dutch theologian Herman Bavinck¹²:

Holy Scripture has a purpose that is religious-ethical through and through. It is not designed to be a manual for the various sciences. [...] In all the disciplines that are grouped around Scripture, our aim must be the saving knowledge of God. For that purpose Scripture offers us all the data needed. [1, p. 444]

That is, the large numbers in the book of Numbers are there to serve the purpose of revealing necessary and sufficient knowledge of the Almighty God, rather than part of an encyclopedia. So, Bavinck reminds us that we should not use the criteria that we have used in scientific disciplines (such as mathematics, astronomy, chemistry and so on) to examine the Holy Scripture, where the LORD reveals what we should know and conceals what we need not to know. As it was said, “The secret things belong to the LORD our God, but the things that are revealed belong to us and to our children forever” (Deuteronomy 29:29).

Surely there is no way for us as human beings to “prove” the accuracy of the numbers mentioned there using mathematical skills or knowledge. Does this mean that we should never make connec-

¹²I would like to express special thanks to the reviewer who brought the work of this theologian to me.

tions between mathematics knowledge and the contents in the Bible? I would argue that, although “the Scripture does not satisfy the demand of exact knowledge in the way we demand it in mathematics” [1, . 444] , the incapability of human knowledge is not a reason for us to separate His Holy Scripture from disciplines such as mathematics¹³. Even though I don’t agree with the result of the mathematical work done by Sir Humphreys [6], he does remind me of what I can do as an educator using mathematics as a powerful tool.

Nowadays, after God has been canceled out from public education for about a half century, it is more important to open our next generations’ mind toward the possibility of the existence of an Almighty God rather than to “prove” it to them, which in fact is an impossible mission for us as human beings if we truly believe in Him as the Almighty God that is beyond our comprehension. The next generation must be willing to explore the possibility and know Him by themselves to make the faith *their* faith, instead of their *fathers’* (or *teachers’*) faith. Nowadays, not only those aforementioned scholars trying to “fix problems in the Bible for God”, many scholars doing so-called biblical studies assume that these “biblical narratives” were just man-made narratives impacted by traditional activities and religious ideology (Ronald Hendel’s work is one of the examples [5]). This happens especially in public institutions, where a large number of the next generations from Christian families drift away from their parents’ faith.

This ideology of *God as a man-made religious figure* has been adopted widely in the world including China through public education. I still vividly remember how astonished I was when I found Jesus in a history textbook that my son brought home from his middle school¹⁴. It was the Winter of 2014, and I was not a Christian yet at that time. I said to myself, “*Wow, I thought Jesus was someone like Monkey King in the famous traditional Chinese fiction novel!*” Thanks be to God, for He found me and changed my heart indeed. Anyway, fortunately, the ideological portraits of the Bible stories do not touch the numbers there. Maybe for them it is very obvious that the numbers cannot be true and then there is no need to even argue about it as what the aforementioned Christian scholars did. If we can point to our next generation that even the numbers there may very likely be true, then why is it not possible that the Almighty God does exist?

More importantly, the exploration along the way described in the last two sections really humbles me: as human being my understanding is always limited so much by my own previous experience and it is almost impossible to describe accurately what we see, what we hear, and what we experience; let alone to describe the Almighty God who is well beyond the scope of what we can see, what we can hear, and what we can experience. With this in mind, it is very important to recognize that the arguments about the “unreasonably large” numbers of the population of Exodus are mainly made based on someone’s personal feeling of “hard to imagine” or “inherently improbable”. Furthermore, when we let these feelings be the standard of reasonableness, more and more “unreasonableness” in the Bible would follow up. For example, if we feel 20,000 more reasonable as the population of the Israel people in Exodus, why would the Pharaoh be concerned so much by them “joining our enemies and fight against us and escape from the land” (Exodus 1:10) at a time that Egypt had a population of several millions? [11]

Once we try to use our own feelings to judge the narratives in the Bible as reasonable or not, it is never a hard task to find plenty of unreasonable and questionable narratives in the Bible. They are

¹³Especially, think about what is said in Colossians 1:16 (*For by Him all things were created, in heaven and on earth, visible and invisible, [...] all things were created through Him and for Him.*); God is not someone who cannot satisfy the mathematical commands but the creator of them. We are usually the ones who have failed to comprehend his work using mathematics or science.

¹⁴He was in a public school and he told me that his teacher did not teach that part at all.

called “*miracles*”. When people, whether Christians or non-Christians, try to “fix” the problem of unreasonableness, what they really convey to their audience is, “*No. There is no way God can do this.*” Differently, I believe that “*For nothing will be impossible with God*”(Luke 1:37). At the same time, as a mathematics educator, I believe that mathematics has a kind of power helping people break the boundary of our personal, bodily experience, and then draw us closer to the Almighty One using the ability of reasoning, which is also given to us by Him. Among all the subjects in school nowadays, mathematics may be the most important subject in terms of its transcendental power to break through our flesh-bonded human feelings and experiences.

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Innovative Assessments via Creative Projects

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Abstract

After taking more traditional (standardized) tests as students, we tend to rely on such forms of assessment as teachers. Over the years, I have been trying to find alternate forms of assessment that afford my students a richer “exam” experience that is also more enjoyable. Ultimately, I want my students to form deeper connections between something they know really well and/or enjoy very much and the material we are covering in class together. So I, as a mathematics professor, started transitioning my traditional final exam to a final project instead. In this paper, we will go through my journey creating and adapting assessments that promote and encourage student creativity. I will share rubrics and samples of student work as well as suggestions for creating effective rubrics.

1 Introduction

Traditionally, one of the most common forms of assessment in a mathematics classroom is a comprehensive final exam. These finals are timed high-stakes assignments that do not support all of our students equitably. Throughout my teaching career, I have further observed a majority of my students cram too much studying into too small a window and, as a result, reach unnecessary levels of stress. The National Council of Teachers of Mathematics (NCTM) encouraged K-12 teachers to supplement exams with alternate assessment techniques such as portfolios and open-ended questions [1]. While attending conferences, I also started hearing from more and more colleagues about using real world projects as an alternate form of assessment. I appreciated that projects would alleviate the stress coming from the timed aspect of the traditional cumulative final. However, I did not feel well equipped to find real world problems that would matter to my students. So I decided to implement a creative comprehensive final project instead. I welcomed creativity in the form of original problems and/or mediums that suited the individual’s interests. By giving students choice in format, creative projects provide more entry points and make room for varied solution strategies. So this form of assessment aligns with the NCTM Teaching Practice to *implement tasks that promote reasoning and problem solving* [2]. Furthermore, students fit what they know into a different context while creating their projects. So this type of assessment aligns with the NCTM Teaching Practice to *pose purposeful questions*. Instead of spending time studying for a traditional final, students spent time assembling what they had learned throughout the semester into creating products that exemplified their knowledge. As a result, collecting and assessing creative projects also aligns with the NCTM Teaching Practice to *elicit and use evidence of student thinking*. Like those common cumulative final exams, each student created, and submitted, their own project for all the ones discussed here.

All of my work implementing creative projects as alternate forms of assessment took place during my six years of service at Lenoir-Rhyne University (LR). LR is a small private liberal arts college.

Classes tend to be smaller to encourage deeper connections among faculty and students and promote a more personalized learning experience. The classes discussed here ranged in size from 5 students to 23 students.

2 Initial Approach: Holistic Final

In my first round of implementing creative projects in place of the more traditional comprehensive final exam, students completed comprehensive final projects. This means that students were expected to highlight everything they had learned throughout the semester.

To discourage last minute cramming and panic attacks, students wrote a proposal worth 100 points towards the Project category of their final grades. As part of the proposal, students stated what type of format they would like to use and why they selected it. Several medium suggestions were explicitly provided in the assignment details. Students were also asked to *articulate how you envision including the information required for the project as detailed in the rubric*. Students were expected to write 1-2 pages in Fall 2018. To encourage students to address more pieces of the project rubric and, *in particular, specify what connections exist between the material and how you plan to leverage these connections to make the project more concise*, students were then expected to write at least two pages in Spring 2019. During a project proposal meeting, which contributed an additional 100 points towards the project category of their final grades, we discussed proposal submissions one-on-one to brainstorm ways for each student to earn full credit on her/his final project.

The remaining 800 points in the project category for their course grade came from the project itself. Four categories worth a total of 200 points are common to these rubrics. They are included in Table 1. The remaining 600 points make up 75% of each rubric and are content specific. These categories will be specified below. Combining understanding of content and the content specific categories accounts for more than 80% of the project score. So the individual creative aspects contribute less than 20% of the final project.

2.1 Fall 2018: Excursions in Mathematics

Throughout the Fall 2018 semester, we covered twelve different problem solving strategies. These strategies were make a systematic list, eliminate possibilities, draw Venn diagrams, analyze the units, visualize spatial relationships, use matrix logic, create a physical representation, look for a pattern, guess and check, work backwards, and evaluate finite differences. Students were able to earn up to 50 points for each of these strategies. Students earned a score of 20 when they included a strategy with little to no description, a score of 30 when they included a strategy with a description that was mostly accurate, a score of 40 when they included the strategy with a full description of what makes that particular strategy unique, and a score of 50 when they went above and beyond the required elements by including examples. In the rare case where a student left out a strategy, they earned a score of 0.

When asked to create a final product which also showed off particular strengths and interests, the 23 students came up with a wide variety of mediums. The final projects included an array of different scrapbooks, displayed drawings, a collection of paintings, a story, a film script, virtual set designs, a booklet, and children's books.

2.2 Spring 2019

The comprehensive final project worked so well and received so much positive response from students in Fall 2018 that a comprehensive final project was incorporated into two more classes in Spring 2019. While both of these classes were divided into 6 units, the material was vastly different. Discrete Mathematics I consisted of mathematical logic (axioms, models, inference rules, propositional logic, operations, truth tables), more mathematical logic (quantifiers, proofs), set theory (set notation, definitions, set proofs – element chasing), induction and recursion (sequences, recursive definitions, proof by induction), relations (representing relations, operations on relations, equivalence relations), and graphs and counting. Meanwhile, antiderivatives (constructing accurate graphs of antiderivatives; the Second Fundamental Theorem of Calculus; integration by substitution), integration techniques (integration by parts; other options for finding algebraic antiderivatives; numerical integration), some applications (using definite integrals to find area and length; using definite integrals to find volume); more applications (density, mass, and center of mass; physics applications: work, force, and pressure; improper integrals), sequences and series (sequences; geometric series; series of real numbers), and more series (alternating series; Taylor polynomials and Taylor series; power series) were the topic groupings for Calculus II.

In both classes, students were able to earn up to 100 points for each subdivision of the respective material. Whenever students accurately incorporated all of the key ideas for a particular topic grouping, they earned full marks. Whenever students incorporated all of the key ideas with little error or incorporated most of the key ideas with no error, they earned 80 points out of the 100 possible. Whenever students incorporated many key ideas with a few errors, they earned 60 points. Whenever students incorporated some key ideas accurately or had many errors, they earned 40 points. Students earned no points if they failed to mention anything about a particular unit.

Due to the sheer volume of content in both classes, students were more likely to score low on a couple of specific content categories than in the previous semester. In these cases, topics were barely incorporated or never even mentioned. Yet the projects students created were still incredibly impressive. Furthermore, students who excelled on their final projects in every content category spent far more time creating their projects than I had intended them to take.

Discrete Mathematics I

There were a wide variety of project mediums utilized in Discrete Mathematics I during Spring 2019. One student wrote a sample final with a solution key. This student used a central theme of tennis to connect the problems to personal interests. Showing off a love of exercise, another student designed an exercise schedule for one week of training. A number of students wrote for their projects. These written projects included song lyrics, a song parody, a play, a comedic story book, and a short book on how a college student in a Discrete Math course goes about solving different discrete problems. Another student created a comprehensive Discrete Math scrapbook. One page is shown in Figure 1.

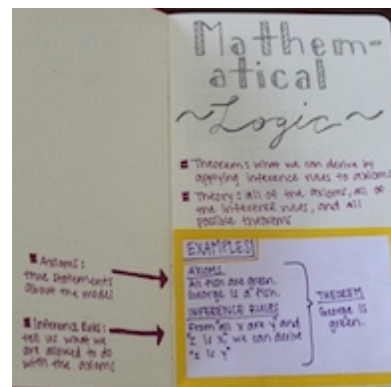


Figure 1: This page gives a glimpse of the comprehensive Discrete Math scrapbook that was created in Spring 2019.

Students also created a wide variety of games ranging from computer programs to tangible board games. A couple of students created different jeopardy review games using their programming expertise. Other computer science majors created their own *Who Wants to be a Millionaire: Discrete Math Edition* games. One student added their own questions to the game of Life. Someone else created their own version of Chutes and Ladders from scratch. While climbing and sliding movements match the rules in the traditional version, players must answer questions correctly before moving. The answers to most of the questions appear on the back of the respective card whenever possible.

Some projects were designed with a more intentional view of how topics were related across the different units. One such project displayed a series of Venn Diagrams across a thick poster board. Inside the pocket found in each region is a note card with detailed information on the respective category. Another student wrote a newspaper article *Breaking it Down: Alex Honnold Climbs El Capitan Without a Rope!*. In this article, the student illustrates how much was learned and how the topics are related through the lens of a passion for climbing.

Calculus II

Like the 17 Discrete Mathematics I students, the 20 Calculus II students created quite diverse projects. Handmade games, 3D printing, detailed and decorated notebooks, and posters are only a few examples of these projects. Four particular projects will be highlighted here.

One of the homemade games was Math-opoly. The game booklet was completely made by hand. The rules and special places reflected different aspects of our classroom culture. Competitors had to correctly answer the question that corresponds to the spot on which their piece landed to earn the right to buy property for that particular space. Following the schema of Monopoly, the student associated later properties with problems that were more challenging in her eyes. The student even sought the help of a math tutor to 3D print the pieces she had designed for her riff on Monopoly.

A superhero fan carefully crafted a trifold poster board. From finding the area between the curves that make up the bat signal to the work needed for the crane to move building pieces to outwitting the Joker with power series manipulation, this student stayed true to a superhero theme.

Another student illustrated a series of four children's books with multiple stories. Readers interacted with these activity books by coloring, answering questions, and more.

Someone else sketched the color by number cartoon fish shown in Figure 2. As expected, the problems and solutions let people know what color to use to fill in a particular piece of the fish. Notice that the problems intentionally span many topics from Calculus II.

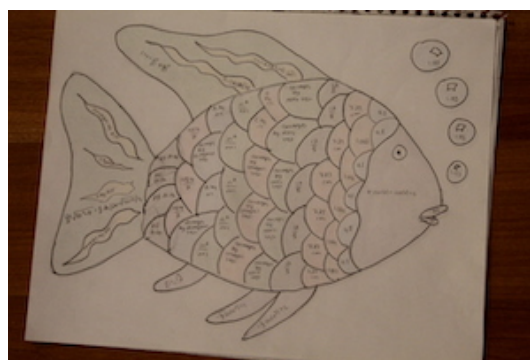


Figure 2: Here is a color by number cartoon fish from the Calculus II Spring 2019 comprehensive final project.

3 Take Two: Depth over Breadth

Redesigning Excursions in Mathematics over the Summer of 2019 allowed me time and space to reflect on this first year of comprehensive final project implementation. As mentioned previously, students struggled to create something special while incorporating every topic covered throughout the semester. To be more cognizant about student time and other resources, I intentionally focused the expectation of the creative project(s). Instead of addressing everything covered in class, students were responsible for diving more deeply into less material.

Along with this shift to a more focused project, the 200 points for writing a proposal and meeting to discuss that proposal slimmed down to 20 points for writing a proposal that is at most two pages and no points for brainstorming proposal ideas during class time. In addition, the massive 800 point cumulative project rubric evolved into rubrics dependent on free form comments (instead of specific rating categories) as evidenced in Tables 2, 3, and 4. There was an intentional shift in the weight of content versus the weight of the more creative aspects. The original rubrics had close to an 80% content, 20% creative split. To encourage more originality, these new rubrics have a 60% content, 40% creative split.

The rubrics from Tables 2 and 3 show that the categories and descriptions for Excursions in Mathematics and Calculus II are almost identical. However, there are a couple of distinctions. The audience for projects created by Excursions in Mathematics students included younger students as well as classmates. Differences in the understanding of content category reflect the differing nature of these two courses. While Table 4's rubric appears wildly different from the other two, the categories are simply rephrased to match the assigned medium. Furthermore, the comparative weights of each category remain constant. In actuality, the biggest difference is that Discrete Mathematics I students were tasked with incorporating 3 to 4 problems with solutions instead of the 2 to 4 problems required for the other classes.

3.1 Excursions in Mathematics

Several aspects of the Excursions in Mathematics course changed as part of the course redesign process. After getting brief introductions to the 17 strategies in our text, students voted on which ones appealed to them the most. The 5 problem solving strategies we covered were based on a majority voting system. For their creative project, students focused on 1 of these 5 strategies. Every student ranked the 5 strategies. Then the strategies were assigned based on these rankings in such a way that each one was covered by someone.

Fall 2019

In Fall of 2019, students elected to solve problems using the following strategies: eliminate possibilities, look for a pattern, guess and check, solve an easier related problem, and create a physical representation. Several students wrote lesson plans around eliminating possibilities. These lesson plans were accompanied by original problems aimed for younger students, a *Guess Who* game featuring the faces of our class, and worksheets. There was also a PowerPoint with two cookie counting problems for which actual homemade cookies were used to model eliminating possibilities. Other students found patterns via writing original poetry, sketching Pacman and Tetris artwork examples, several examples from written music and playing the saxophone, and a fourth grade lesson plan PowerPoint with worksheet handout. Other students illustrated the guess and check

problem solving strategy by catching shiny Pokemon, writing a short play, designing an interactive photo album, constructing an original board game, and creating a couple of lesson plans (one featuring geometric problems and the other featuring a ladybug game). The only student who chose to solve easier related problems for the creative project designed a lesson plan PowerPoint with three examples from her life. The remaining students designed a miniature scrapbook, a lesson plan for elementary school students, and four miniature artwork paintings to highlight the strategy of creating a physical representation.

Fall 2021

Draw a diagram, eliminate possibilities, look for a pattern, guess and check, and create a physical representation were the five chosen problem solving strategies in Fall 2021. Transitioning from 19 students the previous year to 8 students this year meant fewer projects. Only 6 students of the 8 enrolled consistently engaged in the course and completed assignments. One student created a slide show with a Christmas themed TikTok video to illustrate drawing a diagram. Another student designed an eliminating possibilities lesson plan with examples for elementary-aged students. Two students created projects around looking for patterns. One student found patterns in the writing, composing, and reading of music; meanwhile, the other student painted the Christmas pattern shown in Figure 3 as a visual aid for elementary school aged children. To demonstrate guess and check, one student created problems based on her *My Little Ponies* figurine's spending habits and another student designed a PowerPoint with karaoke examples. The last student created manipulatives as a physical representation for peers to use as they solved original problems.



Figure 3: There is a clear pattern in these Christmas ornaments.

3.2 Calculus II

Unlike Excursions in Mathematics, there were enough topics and few enough students that everyone in Calculus II was responsible for completing 2 distinct creative projects. The content for this course was divided into the three units; sequences and series, applications of integrals, and evaluating integrals. Every student selected three topics, one from each unit. And each topic was selected by one person or no one. Since students were required to create two projects, they also completed two proposals. Both proposals were due near the end of class in Fall 2019 and Fall 2020. When reflecting, the latter students shared that completing one of the proposals earlier in the semester would encourage them to start thinking and planning for the corresponding project sooner and, subsequently, alleviate some pressure. Therefore, the first proposal was due shortly after midterm in Spring 2021. Any student who wanted to earn extra credit had to complete the third project. They could elect whether to discuss their ideas for this extra project with me or not, but there was no formal proposal required.

Fall 2019

Students shared their sequences and series projects at our first exam meeting. There were home-made scrapbooks on sequences and the limit comparison test. A third scrapbook illustrated the

alternating series test with an analogy to the real life behavior of ocean waves. Geometric series were illustrated via a hands on banana cutting experiment. A student featured power series in a slide show. Another student wrote a poem on the ratio test. As a final project from this unit, a sequence of growing 3D printed squares provided a memorable example of the divergence test.

We kicked off our second exam meeting with several projects featuring applications of integrals. One poster displayed information on the area between curves while three smaller posters highlighted improper integrals. Washer method examples were carefully explained on personal white boards. Arc length and mass and density were brought to life via a 3D printed disc and a science experiment, respectively. Homemade magnetic grid and rods brought center of mass problems to life. Work problems were illustrated via CAD images included in one PowerPoint, and another PowerPoint illustrated the disk method via physics examples. A couple of additional PowerPoints for topics from evaluating integrals wrapped up this second exam, and final, meeting.

Fall 2020

During the 2020 Thanksgiving break, the administration informed everyone that the last week of classes was canceled and all final exam meetings would be conducted remotely. Luckily, Zoom provided a convenient way for students to share projects. Given this new context, a few topics went from a more creative proposed project to original work in slide shows. There were still several more creative ideas incorporated into these slide shows and other types of projects shared.

One person created sequences based on different chords in music. Another student shared a hand made booklet containing a plethora of information on power series and interval of convergence. In addition to a history of the integral test and improper integrals in an original newspaper titled *The Paper of Calculus*, there were problems in the form of trivia questions. Peers were asked to role play characters 0, 1, and r from the story book *Math is Just a Walk in the Park* which highlighted the divergence test, limit comparison test, and ratio test with original artwork.

Based on photographs featuring curves taken and found by one student, we found the area between those curves by modeling them with functions. One such image is shown in Figure 4 on the following page. We also measured the lengths of arcs based on recordings of track wins. We played an original game with colored spaces associated with density and mass questions. We then explored physics applications through participation in a *Desmos* activity using various examples based on kitchen containers. A baking video provided a new way to analyze substitution. We even studied numerical integration techniques by running examples through an interactive, student-generated integral calculator and analyzing its related code.

Spring 2021

When we returned to in person finals in Spring 2021, the individuality of all 9 Calculus II students shone forth amongst the various projects created. A golfer challenged everyone to solve Taylor polynomials and Taylor series problems at each hole on a fabricated golf course. Someone else was excited to share how much time was spent recording three stop animation sequences. A future mathematics teacher created a sequencing of problems in such a way that the letters corresponding to entries on an integral table answered a clever integration riddle. A member of the LR rocket team demonstrated force ideas by launching a rocket for everyone. A music lover demonstrated the alternating series test by playing composed pieces on an online keyboard. Center of mass

problems even featured images of class teammates. A meme enthusiast had everyone crying from a deliberate use of memes to review the limit comparison test. One of these memes is shown in Figure 5. Approximations of the speed of a family hedgehog when walking versus running towards food were used to construct relevant work problems. Another student designed posters featuring objects signifying her favorite summer time hobbies to review finding volume using the washer method.

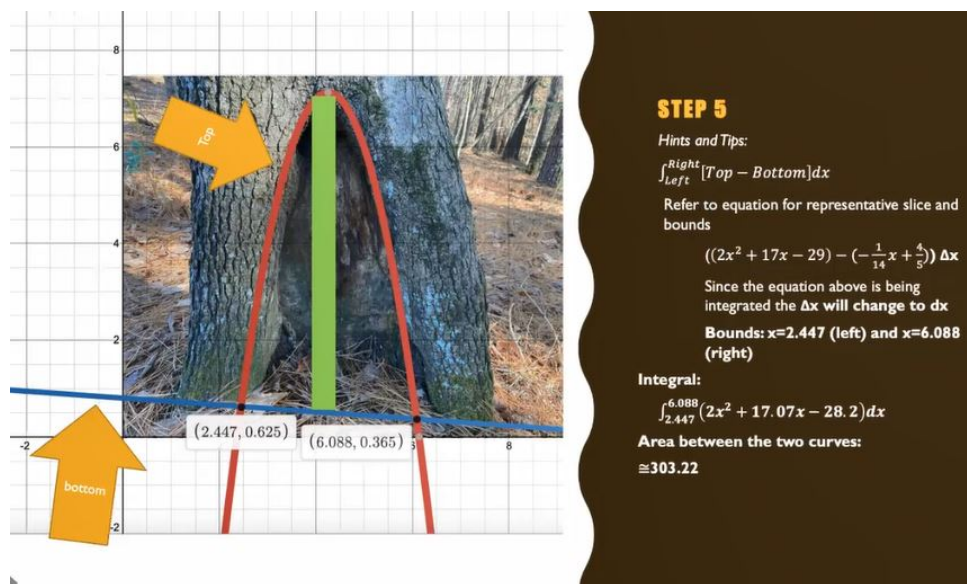


Figure 4: This figure features several steps from the process for finding the area between curves.



Figure 5: This figure shows two series who are essentially the same as twins.

3.3 Discrete Mathematics I: Spring 2020

Course material was divided into mathematical logic, relations, set theory, proof, induction and recursion, and graphs and counting units in Discrete Mathematics I in Spring 2020. When COVID led to remote asynchronous instruction following Spring Break, instructors were tasked with paring down the remaining material by only keeping essential assignments. While students were still expected to produce a creative project on 1 topic, they all had to creatively produce a section of our class graphic novel on an assigned topic. Topics were assigned based on student responses to the following prompts:

- Rank the six units in order of personal preference.
- Explain your ranking in three or more sentences.
- Identify any topics that you particularly enjoyed studying. Briefly explain.
- Identify any topics that you have not particularly enjoyed studying. Briefly explain.

Students had been reading Eric Goessett's *Discrete Math: The Graphic Novel, (Second Edition)* throughout the entire semester, so this medium was a natural choice. Snippets from a handful of finished products included in Figure 6 showcase a variety of the styles used, from hand drawn sketches to different online templates.



Figure 6: There is one panel each from 5 different comic strips.

4 Concluding Remarks

Work submitted by students for projects showcased understanding of content, creativity, individuality, and so much more. The samples from projects described and pictured throughout this article highlight the strengths of projects over more traditional cumulative final exam tests. Implementing, and adapting, final creative projects allowed me to gain confidence in coaching students to successfully create products. Increased confidence in implementation as well as great joy over student ownership of their work together encouraged me to implement project assessments more holistically. In fact, Discrete Mathematics II students completed three projects throughout the Spring of 2020 as their only forms of assessment. The 5 students in this upper level course produced such impressive projects and responded so positively that I once again replaced all forms of assessment in the lower level Discrete Mathematics I course in Spring 2022. Since there are 6 units, each of these 22 students created a total of 6 projects.

When first implementing projects for comprehensive creative projects, it was grueling to grade. Looking for breadth and depth in projects took longer than preferred. Furthermore, recognizing that students who were not able to include all the topics were scoring lower than I would have hoped was disappointing. When grading more traditional exams, it is easier to blindly grade using a solution key. With final projects, I am more emotionally invested throughout the entire grading process. Adapting the assignment details and rubric for creative projects that focused on depth not only provided space for students to be more successful in creating masterful projects but also provided me more space for grading projects in a healthier way. Students were expected to dive even deeper into specific topics, and they delivered quality products. How well a student understands a particular topic was even more clearly evident in creative projects than in those more traditional assessments. Projects did not allow students to simply memorize and regurgitate steps to a problem type. They had to move past remembering and understanding related material

to examine and interpret material from a different perspective to produce an original product. So grading more focused projects with the current rubrics is a more enjoyable and speedy process. Familiarity with my current project rubric allowed me to grade projects as quickly as I had graded more traditional tests in previous semesters. The progression of rubrics emphasizes the importance of including enough detail to help students understand your expectations while leaving enough room for students to shine in ways that are already meaningful to them. This balance allows instructors the freedom and constraint to grade in a faster and fairer way.

In the future, I plan to share more details about the well over 200 projects created. In particular, I would like to analyze the two classes in which all forms of assessment were replaced with creative projects. I would like to share how I incorporated creative projects with mastery grading as well. Having seen the benefits of creative projects as an alternate form of assessment firsthand, I would like to initiate a longitudinal study of the benefits of such implementation. Personal experience shows me that there are many positive results. Collecting more qualitative and quantitative data will better allow us to determine the effectiveness of alternate forms of assessment.

Since Fall 2018, I have been blown away by the products produced by over 172 students. I am beyond grateful for how much I have grown with, and for, my students throughout this journey around creative projects. I am particularly thankful to every student who gave me permission to share their work, including all of the anonymous students as well as those listed in Section 9.

5 Common Rubric Categories for Comprehensive Final Projects

| Criteria | Ratings | | | |
|--------------------------------------|--|---|---|--|
| | 50 points Full Marks | 40 points Very Good | 30 points Competent | 20 points Needs Work |
| Creativity | Exceptionally clever and unique in showing deep understanding | Thoughtfully and uniquely presented; clever at times in showing understanding of the material | A few original touches enhance the project to show some understanding of the material | Shows little creativity, originality and/or effort in understanding the material |
| Neatness and attractiveness | Exceptionally attractive and particularly neat in design and layout. No grammatical or mechanical mistakes in the project. | Attractive and neat in design and layout | Acceptably attractive but may be messy at times and/or show lack of organization | Distractingly messy or very poorly designed. Does not show pride in work. |
| Understanding of Content | Shows a sophisticated understanding of all the different problem solving strategies | Shows an understanding of several of the problem solving strategies. | Displays a somewhat limited understanding of the problem solving strategies. May have a few misinterpretations. | Does not show an understanding of the problem solving strategies. Misses several of them and has quite a few misinterpretations. |
| Overall Effectiveness and Completion | Project is engagingly organized and presents material that is captivating for the viewer. | Project is somewhat organized, complete, and holds the attention of the viewer. | Project is disorganized and incomplete at times and is somewhat able to hold the attention of the viewer. | Project is incomplete and not easy to follow. |

Table 1: Here are all the criteria and ratings that were shared among courses in which a comprehensive final was assigned.

6 Excursions in Mathematics (Individual) Strategy Project Rubric

| Criteria | Description | Points Possible |
|--|---|-----------------|
| Appropriate Audience | Presentation of material is easily understood by non-experts (peers in our class and/or younger students). Furthermore, the material is captivating for the intended audience. | 20 |
| Understanding of Content | Shows a sophisticated understanding of all the key ideas associated with the particular problem solving strategy. Highlights key phrases found in problems which may be solved using the given strategy. Incorporates the distinguishing characteristics of that particular strategy. Explains how to apply the strategy effectively when solving problems. | 60 |
| Accuracy | There are 2-4 problems with solutions clearly articulated without errors. | 60 |
| Creativity | Presentation of material is well thought out and unique to the strengths and interests of the student. | 20 |
| Aesthetically Pleasing | Exceptionally attractive and particularly neat in design and layout. No grammatical or mechanical mistakes in the project. | 20 |
| Overall Effectiveness and Organization | Project is engagingly organized and demonstrates clarity of ideas. | 20 |

Table 2: This rubric was used to grade every strategy project submitted in Excursions in Mathematics for Fall 2019, Fall 2021, and Spring 2022.

7 Calculus Rubric for Final Projects

| Criteria | Description | Points Possible |
|--|---|-----------------|
| Appropriate Audience | Presentation of material is easily understood by peers in this class. Furthermore, the material is captivating for the intended audience. | 20 |
| Understanding of Content | Shows a sophisticated understanding of all the main ideas associated with the particular topic. Highlights key steps used when solving related problems. Incorporates tips for studying this particular topic. Explains how this topic fits into the bigger Calculus II story line. | 60 |
| Accuracy | There are 2-4 problems with solutions clearly articulated without errors. | 60 |
| Creativity | Presentation of material is well thought out and unique to the strengths and interests of the student. | 20 |
| Aesthetically Pleasing | Exceptionally attractive and particularly neat in design and layout. No grammatical or mechanical mistakes in the project. | 20 |
| Overall Effectiveness and Organization | Project is engagingly organized and demonstrates clarity of ideas. | 20 |

Table 3: Here is the complete rubric used to grade each final project submitted in Calculus II for Fall 2019, Fall 2020, and Spring 2021.

8 Discrete Mathematics I Rubric for Final Graphic Project

| Criteria | Description | Points Possible |
|---|---|-----------------|
| Appropriate Audience | Presentation of material is easily understood by non-experts (peers in our class and/or younger students). Furthermore, the material is captivating for the intended audience. | 10 |
| Understanding of Content for Assigned Topic | Shows a sophisticated understanding of all the main ideas associated with your assigned topic. Provides the background information necessary to study the topic. Clearly articulates all related definitions. Explains how the associated objects are related to one another. | 30 |
| Problem Solving for Assigned Topic | Clearly state 3 to 4 problems which provide insight into the breadth of your topic. Provide accurate solutions to each problem. Highlights key steps used when solving these problems. Incorporates tips for studying this particular topic. | 30 |
| Story Creativity | Presentation of material is well thought out and unique to the strengths and interests of the student. | 10 |
| Effective Visual Aids Appropriate for Graphic Novel | Uses drawings (by hand, computer-generated, stickers, etc.) and dialogue 'bubbles' to convey your information. Effectively incorporate these visual aid(s). *Check out our class text Discrete Math: The Graphic Novel, (Second Edition), by Eric Gossett. | 10 |
| Story Effectiveness and Organization | Presentation of material is engagingly organized and demonstrates clarity of ideas. | 10 |

Table 4: MAT 150 Spring 2020

9 Acknowledgements

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Mathematical Minds: A Senior Seminar for the Mathematically-Minded

Kim Regnier Jongerius (Northwestern College)



Kim Jongerius, Professor of Mathematics at Northwestern College in Orange City, IA, enjoys teaching, reading, writing, and sarcasm, not necessarily in that order. Co-authoring three chapters in *Mathematics Through the Eyes of Faith* remains one of her favorite professional experiences, as does her work on connecting the linguistic and mathematical abilities of C.S. Lewis. She attempts to entertain her students by singing Broadway show tunes with mathematical lyrics.

Abstract

This paper describes a successful senior seminar course that leads to interesting and fruitful conversations, allows and encourages enrollment by non-majors, provides opportunities for meaningful program assessment, and could easily be adapted to between one and four credits.

1 Introduction

Capstone courses or projects have long been recognized as, if not essential, certainly beneficial aspects of undergraduates' collegiate experiences. Rash and Weld, in "The Capstone Course: Origins, Goals, Methods, and Issues," share 12 papers arising from a conference session (sponsored by *PRIMUS* at the Joint Mathematics Meetings in 2012) on capstone courses. They note that the courses described often connect to institutional mission and share goals of promoting life-long learning and "enhanc[ing] mathematical thinking and communication skills among students" [8]. In *Designing and Teaching Undergraduate Capstone Courses*, Hauhart and Grahe make substantial use of numerous studies to provide a summary of the history and practice of, and research into, capstone experiences. They find that capstones are usually disciplinary courses with a major project for each student that leads to a peer-reviewed paper and its oral presentation, though capstones can take a wide variety of forms. A common goal, however, is the integration and application of material [7].

One somewhat tenuous conclusion in *Designing and Teaching Undergraduate Capstone Courses* is that private, undergraduate colleges are more likely to offer capstone experiences than research institutions. Though noting that the claim is preliminary due to the need for more comparative studies, Hauhart and Grahe proceed with their assumption about the schools most likely to offer capstones and conjecture that faculty at private, undergraduate colleges have more time and resources to apply to capstone courses since they are not required to focus on graduate education [7]. Hauhart and Grahe do not appear to factor in the heavier teaching loads typical at many private institutions, however, and in recent years the risk of elimination of traditional programs seems to be increasing as small colleges in particular face enrollment struggles [4] and may seek to prioritize programs seen as more financially viable [5]. Faculty teaching in small programs can find it hard to convince themselves of the value in assigning to a capstone course personnel and credits that might benefit the program in other ways. In particular, project-based capstone courses are unlikely to attract non-majors and thus seem likely to negatively impact departmental statistics used in administrative decisions about program futures. This paper describes a successful senior

seminar course that leads to interesting and fruitful conversations, allows and encourages enrollment by non-majors, provides opportunities for meaningful program assessment, and could easily be adapted to between one and four credits.

2 Requirements and Goals

About five years ago, a new rendition of the general education program (NWCore) at Northwestern College added senior seminars as a required element for the first time in several decades. To facilitate faculty buy-in during the approval phase, this aspect of the program was designed, after significant study of the wide variety of senior seminars available nationwide, with a lot of flexibility. Senior seminars can be re-tooled discipline-specific courses, interdisciplinary (though still somewhat narrowly-focused) courses, or multidisciplinary courses intended for students from virtually any major. The courses can be offered for two, three, or four credits, and students can take them any time in their last three semesters of college with an option for the start of their junior year if that is the only time the course fits the student's schedule. While generally a Northwestern major cannot require students to take more than 36 credits within a single department, majors can require a specific senior seminar, even if it is essentially a disciplinary course, without counting the course's credits against that limit.

There are components, however, that every such course is required to feature. Each senior seminar must:

- Serve as a capstone to each student's entire collegiate experience.
- Be writing- and speech-intensive.
- Require students to reflect on their experiences during their college years, focusing on the extent to which they have met the college's NWCore goals.
- Require students to reflect on how they might implement what they have learned.
- Assist in institutional assessment of the NWCore.

Northwestern's Core goals are not significantly different from those at other small, regional, Christian colleges whose faculties emphasize the liberal arts. Our intent is that students will:

- Integrate Learning (integrate faith and learning and connect knowledge across disciplines and to life outside the academy)
- Seek Knowledge (develop intellectual curiosity and explore major themes of the Bible and theology, of human cultures and human nature, and of the natural world)
- Demonstrate Skill (practice multidisciplinary thinking and integrative habits of mind)
- Live Responsibly (participate in God's redeeming work, developing a comprehensive view of ethical responsibility that encompasses both individual behavior and responsible action in community)

Though not already offering a required senior seminar that could be revamped, the math department did have the convenient pairing of a professor who was eager to develop such a course and an easily-adaptable course ("History of Mathematics") with class discussions that already had the feel and

content of a senior seminar. Aside from the title (now “Mathematical Minds”), the most significant change was to the prerequisite: calculus II replaced linear algebra, opening the course to more than just departmental majors and minors and thus enabling an annual offering of the course with an enrollment of 12 to 15.

3 Framework and Texts

It’s important to note that, while faculty and staff at Northwestern are required to profess and demonstrate an active Christian faith, students are not. Faculty approach this in a variety of ways when it comes to the decidedly Christian goals of the NWCore. In this course, students know from the start that while the professor’s hope is that each of them has developed or enhanced a vibrant Christian faith by the time they graduate, she also realizes that they’re all in different places on the journey and that some may be on a different path entirely. They are not asked to proclaim a faith they do not have—and they are not graded differently if they make such a profession in a paper or speech—but they are expected to be able to discuss issues from a Christian perspective, which they may introduce using phrases like “Some Christians believe” or “Reformed Christianity asserts.”

The NWCore focus of the course and its associated emphasis on the expression of Christian perspectives together call for texts with the potential to lead to deep thinking and meaningful discussion that encourage students to draw on a wide variety of their experiences in and out of the classroom both before and during college. Francis Su’s book *Mathematics for Human Flourishing* seems tailor-made for setting the framework for the course, and William Dunham’s *Journey through Genius* provides excellent explication of important mathematical ideas and their historical contexts. The ACMS book *Mathematics through the Eyes of Faith* is also a key text, with literally every chapter introducing students to significant questions about the relationship of mathematics to Christian faith. Rounding out the current list of texts is *The Curious Incident of the Dog in the Night-Time* by Mark Haddon. This short book tells a compelling story and makes a particularly nice Thanksgiving-week assignment. Students appreciate the point of view of the main character and narrator, Christopher, a young man on the autism spectrum whose take on life and logic leads to fruitful classroom conversation starting with the professor’s and students’ similarities to Christopher and continuing through discussions about meaningful interactions with and the integration of people with exceptionalities. The final content of the course is “The Man Who Knew Infinity,” the 2015 movie about Ramanujan. Watching the movie during class instead of reading texts outside of class eases students’ workload in the last weeks of the semester when they are working on their final paper and its presentation. This particular movie leads to more conversation about “otherness,” about being seen (and our tendency to not see people who are different from ourselves), and about the importance of rigorous proof.

4 Course Details

The current rendition of the course earns students three credits and meets twice a week in 75-minute sessions. Since *Mathematics for Human Flourishing* and *Mathematics through the Eyes of Faith* are for the most part not too mathematically-intense and often connect well, chapters from those books are usually paired for one day’s discussion. Each chapter of *Journey through Genius* gets an entire class period, however, primarily to make sure that there is time to clear up any questions students have about the proofs.

Another aspect of the course that helps break up the read-discuss-journal pace of the bulk of the semester are “framing discussions.” These involve pre-class journaling and in-class discussion about why mathematics is important in general and to each student in particular, why the college requires all those general education courses, and whether or not the students feel a sense of calling, with special emphasis on a call to use their analytical gifts. The framing discussions and their prep work also form the basis for much of the final paper and presentation, which spreads some of the end-of-course workload throughout the semester.

Project work also helps keep the class moving, along with occasional just-for-fun time spent on challenges from the ends of the chapters in *Mathematics for Human Flourishing* or on problems from *Mathematics Through the Eyes of Faith*, and students get some class time—on multiple occasions—to work in groups on a larger project that counts for 10% of the course grade.

The remaining 90% of the grade is evenly divided among class participation, journal entries reflecting on the key ideas from the week’s discussions, and the goals paper and presentation. The class participation portion is broken further into prepping and leading class discussion on assigned days, submitting one question and one insight from each chapter to assist colleagues in leading discussion on their days, responding to occasional journal prompts that address program goals for the math department’s various majors, contributing periodically to the classroom conversation, and presenting a proof to the class.

5 Writing Prompts

While the production of meaningful student writing often requires detailed writing prompts, past students in this senior seminar have expressed desire for and appreciation of prompts that are simplified and somewhat open-ended. The weekly journal prompt below has served well to balance both student and instructor desires:

- Discuss two or three insights that you found particularly helpful or interesting.

On the other hand, the framing discussion prompts are intended to give students a head start on their final essay and are necessarily more detailed:

- Discuss one or two moments or events in your life that helped you see mathematics as a discipline worth studying. You can contribute to this discussion even if you are not a math major—why did you take as much math as you did? What about math led you to take this course?
- Discuss one or two things you learned about the value and importance of taking classes outside your major discipline. How has the NWCore contributed to your overall education? Examine the NWCore goals one by one. Has the program met those goals? Also, get copies of your curricular and co-curricular transcripts, and reflect on the content and sequencing of your college experiences—how have these things contributed to who you are and how you will live in the world?
- Reflect on how you have changed since your early days of college, and take note of experiences, ideas, or assignments that contributed significantly to your development into the person—and, in particular, the thinker—that you are today. Have they led to a specific calling or perhaps several callings? Do you expect to feel the same way in 5 years? In 10? At the age of 50?

Assessment prompts connect to some of the department's specific goals for student learning and provide quick data to inform assessment efforts:

- Discuss two or three real-world applications of mathematical concepts that you have learned about during your college career.
- How would you articulate a Christian perspective on the claim that mathematical structure reflects God's character present in Creation? Give reasons for your response.
- To what extent, if any, would you agree with the claim that mathematics is a creative discipline? Give reasons for your response.

Finally, the goals paper prompt asks students to reflect on their entire collegiate experience.

- Your paper should ...
 - Note each NWCore goal and examine your progress in reaching it, referring to at least 3 assignments, ideas and/or or experiences per goal in tracing your progress over time.
 - Explore significant turning points in your development.
 - Reflect on your experiences outside the classroom and how they contributed to your progress toward meeting the goals of the Northwestern Core.
 - Name and explain your personal commitments as a prospective contributor to your chosen field. This is a good place to explore your development in seeing a relationship between your faith commitments and your future work.

These prompts have usually been sufficient for generating thoughtful student responses, with detailed grading rubrics and careful and speedy feedback on the first of each type of writing (including the first draft of the goals paper) helping to clarify expectations.

6 Feedback

Student feedback—usually offered in course evaluations but also in journal entries, class discussion prep, and the goals paper—has been the catalyst for many modifications that have led to a more successful course. These suggestions have resulted in more class time to work on projects, including time to collaborate with other groups; replacing a required text with a movie to ease the workload at the end of the semester; streamlining expectations for journal entry length and content; and spending more in-class time on the more difficult mathematical topics in the course texts.

While the NWCore goal of demonstrating skill is assessed in multiple ways throughout the curriculum, the following data (from Campus Labs' IDEA course evaluations over the past 4 offerings of "Mathematical Minds") suggest that the course helps students make progress on the goals of integrating learning, seeking knowledge, and living responsibly:

| | Frequently | Always | Total |
|---|------------|--------|-------|
| Found ways to help students answer their own questions | 39% | 58% | 97% |
| Helped students to interpret subject matter from diverse perspectives | 30% | 61% | 91% |
| Asked students to share ideas and experiences with others whose backgrounds and view-points differ from their own | 33% | 67% | 100% |
| Asked students to help each other understand ideas or concepts | 24% | 64% | 88% |
| Related course material to real life situations | 30% | 67% | 97% |
| Introduced stimulating ideas about the subject | 33% | 64% | 97% |
| Encouraged students to reflect on and evaluate what they have learned | 33% | 67% | 100% |

Written comments in course evaluations, in journals, in student-submitted pre-discussion questions and insights over the readings, and in the NWCore goals paper also suggest that the course is encouraging students to reflect on what they have learned and to recognize the boundaries of their learning, helping them connect faith and math, and leading them to ask and wrestle with big questions.

Course evaluation comments, in particular, illustrate the integrative nature of the class:

- I was able to really understand how I can use my gift of math to benefit and serve the kingdom of Christ. This course was probably one of the best courses I have taken when it comes to faith integration.
- [This course] has made me think more about God's complexity because of the books that we read about mathematics and how [it relates] to God.
- I really appreciated the meaningful discussions we had during class. These discussions helped me gain a broader perspective of different topics that helped me strengthen my faith.
- This class has caused me to ask questions of how mathematics, God, and my faith relate, and it has been a great growing experience. This semester I've asked myself questions that I would have never thought of if not for this class.

Journal entries point further to students' recognition of their own progress in understanding mathematics, understanding faith, and seeing new connections between the two:

- One form of permanence Su did not necessarily mention is the permanence of mathematical processes within our memories . . . I may forget the exact terms and definitions for things, but after simply reviewing a definition or two when going back to a concept, I have found that the mental pathway is usually still there. With this, math is not about developing a growing number of skills. We are simply always building on one skill—analytical thinking—which is a skill that stays with us permanently.
- Before, I was thinking of mathematics as simply the properties that exist in the world themselves, which would exist whether or not someone proves them. Now, I see the label of mathematics as including the work done to explain these ideas, meaning that there are indeed parts of math that are discovered and parts that are invented.

- Our class discussion made me recognize that much of what I learn in math classes is learned by initially accepting the procedure and then formulating meaning months or even years down the road. My education professors have drilled into me that every lesson has to involve meaning-making, but I have to remember that sometimes it is good enough that students just have access to meaning, rather than a full understanding of meaning.
- In *Mathematics Through the Eyes of Faith* I thought that it was interesting to think about trust and mathematics being related. It's so easy to take for granted everything that came before us that we didn't have to work toward. I've certainly never worked through the proofs of many things that I have used in math, however, I still trust that they are accurate and use them with faith. I really appreciated that connection back to faith.

Questions submitted for classroom discussion, like course evaluation comments, demonstrated students' increasing ability to ask and begin to wrestle with meaningful questions:

- What is a practical way of dignifying humans (to promote flourishing) with Mathematics, when convention would say Mathematics dehumanizes people by often reducing them to a number or statistic?
- Is it possible that God helps us in ways that we never even consider, such as getting us out of the loop of moving to the halfway point ... infinitely many times and thus not going anywhere at all?
- In what ways can we use infinity to better explain the omnipotence and omniscience of God to those who are not of Faith?
- Are there ways that we think that we are being logical, like Christopher [the narrator in *The Curious Incident of the Dog in the Night-Time*], that actually make no sense whatsoever? Should we be more prepared to step back and examine our understanding of things?

Student-submitted reading insights (used in class discussion) often illustrated deep connections to the ideas in the texts and a desire to explore some of those ideas in more depth:

- The idea that mathematical ideas can be conveyed via story is new to me and something I have personally never experienced, although it does make sense in theory. Patterns are best conveyed through stories, where patterns are obvious, so if mathematics is the science of finding and evaluating meaning in patterns, then stories would be the best way of instructing students about new ideas. I can definitely confirm the claim that the chapter makes about student learning, however—the times I have purposefully ignored the fine details of what I'm reading and instead looked at the big picture and compared my reading to previous topics have been much more fruitful than those where I simply memorized for the sake of finishing homework.
- The concept of dimensions fascinates me ... The idea that God is a higher dimensional being that can interact with our world in ways that do not make sense to us because they are truly out of the realm of what is comprehensible to us. But Him interacting in our 3 dimensional world, we would only see what is possible to be seen in our world.
- I think sometimes when we have a success in an area we forget that we are building on prior knowledge that we have taken in and internalized due to the breakthroughs of others. It is easy to look and say yes that was all me and I figured it out on my own when in fact anything

that we figure out is not on our own but we are building on previous knowledge and ideas we can take for fact because of what everyone else has done before us. Same goes in our faith lives and journeys.

- I liked the way this chapter explained the “freedom” of mathematics. From my experience, math was always seen as stiff & restricted, and my classmates and I always felt math was just “do it this way to solve this problem.” Expanding further into mathematics helps me realize how everything is connected, and any problem can be solved a number of ways. There’s freedom to explore and understand math, two of the points Francis Su touched on in this chapter ... Freedom of understanding instills so much more hope and curiosity into the mind of a mathematical thinker.

Finally, student goals papers suggest further ways in which the students are incorporating NWCORE goals into their future plans:

- In [one] class, [I] gave a final presentation about ... Rwanda ... and a practical application towards how to help. I ... came up with a business idea that helps recycle plastics that can be used in roads and other construction projects. [This helped] me to develop a heart to live out my faith in whatever job I go into.
- [Contemporary Moral Issues] taught me a lot about how to talk to others with different beliefs than me, especially about topics that are controversial ... To be able to pass the class, I had to learn how to hear and understand opinions and worldviews that are different from mine. Through the various readings that approached a topic from all different sides and by watching the way my professor interacted with my class, I learned how to have a conversation about controversial topics instead of just arguing over them.
- Being a knowledge seeker means not shying away from [difficult] questions but instead embracing them and the conflicts they raise in our minds, relishing the tensions that eventually give way to cognitive closure, and those who investigate these questions are bettered through it.
- During [the statistics instructor’s] lecture, he would often be reminded of something that happened outside of the classroom where he used his experience ... These were parts of class that I may remember more than much of the mathematics that I learned in class because it was a helpful example of how we as mathematicians could help others in our work as an actuary or as a member of a church.

As this collection of student thoughts illustrates, the course is indeed meeting the college’s goals for the senior seminar, in particular by requiring students to reflect on their experiences during their college years and on how they might implement what they have learned.

7 Potential Modification

While the course works well in the three-credit format, it could easily be modified to support a load of four credits. One approach might be to add a substantial research project. Another might be to include an additional text and related assignments. Seife’s *Zero: The Biography of a Dangerous Idea*, for example, one of the course texts prior to the publication of Su’s book, led to fascinating discussions about faith aspects of mathematical thinking throughout history.

Removing a text or two, on the other hand, would allow a very similar course to be taught in a two- or even one-credit format. Programs with a strong faith integration component would be served well by pairing *Mathematics for Human Flourishing* with *Mathematics through the Eyes of Faith*, while those with a strong research component might supplement *Journey through Genius* with individual and group research projects.

8 Conclusion

As demonstrated above in the table presenting key results from the quantitative portion of student course evaluations, “Mathematical Minds” helps students make progress on three of Northwestern’s Core goals (integrating learning, seeking knowledge, and living responsibly). An impressive 88 to 100 percent of students reported that the course “frequently” or “always” helped them answer their own questions, seek and respect diverse perspectives, connect learning to real life, and reflect on and evaluate their learning. Student comments and statements in a variety of course contexts reinforce the conclusions suggested by the numbers in the table, and taken as a whole this feedback provides evidence that the course meets the college’s objectives for a senior seminar.

As anticipated, a more modest course prerequisite of calculus II has led to more diversity among the students, with each roster to date including not only actuarial science, mathematics, and statistics majors but also two to four computer science majors as well as the occasional student majoring in biology/health professions, chemistry, economics, or history (not counting those double-majoring in one of these disciplines and a math department major). These students have made significant contributions to the class both during discussions and in their various written submissions, and, while their responses are separated out when it comes to formal departmental assessment efforts, they nevertheless have provided interesting feedback—some of it represented in the comments above—that has made its way to department meetings and influenced the thinking of department faculty about what and how we teach. It is also worth noting that these additional students, in this small-college context, help ensure the survival of the course.

Each semester, student comments (some noted above) suggest that the classroom conversations in “Mathematical Minds” are deep and varied. Such conversations—according to anecdotal student reports—often continued in the cafeteria and in campus housing settings, drawing in students who were not participants in the class. One semester, two students in the course were roommates, and they enjoyed posing interesting math problems to friends who dropped by. Given a response of, “Well, I’m not very good at math,” one of the math majors exclaimed, “No! No! It’s not about being good at math! It’s about the journey! That’s the point of our whole class!” When he shared this exchange in class the next day, his colleagues agreed wholeheartedly: mathematics is about the journey, and this course helps students understand how to navigate their paths.

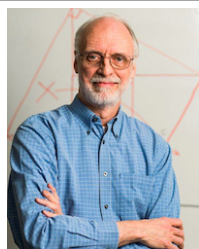
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Incorporating Perspectival Elements in a Discrete Mathematics Course

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Calvin Jongsma is Professor Emeritus of Mathematics at Dordt University. His interest in history and philosophy of mathematics and logic is evident in the many book reviews he has written for *MAA Reviews* and *Perspectives on Science and Christian Faith*. An essay recapitulating his dissertation on Whately's revival of logic in 19th-century Britain is the opening chapter in the forthcoming book *Aristotle's Syllogism and the Creation of Modern Logic*.

Abstract

Discrete mathematics is a vast field that can be explored along many different paths. Opening with a unit on logic and proof and then taking up some additional core topics (induction, set theory, combinatorics, relations, Boolean algebra, graph theory) allows one to bring in a wealth of relevant material on history, philosophy, axiomatics, and abstraction in very natural ways. This talk looks at how my 2019 textbook on discrete mathematics, focused in this way, came to be, and it highlights the various perspectival elements the book includes.

1 Introduction

A concern of the Association of Christians in the Mathematical Sciences from the very beginning has been how to connect our Christian faith with our professional interests and work in the mathematical sciences. As one of this year's ACMS pre-conference workshops and a number of speakers demonstrated, an important way to implement this is by attending to historical and philosophical perspectives. What I'd like to do in this essay is show how I incorporated perspectival elements in the discrete mathematics text I wrote for an intermediate-level course that I taught for years at Dordt. I'll share some of my mathography, explain how my text came to be, and briefly outline what the text covers and how it touches on perspectival matters.

2 Interest in Mathematics and Perspectival Issues

My love of mathematics goes as far back as I can remember, but in college I discovered that I also enjoyed history of philosophy. While a junior, a philosophy professor helped me deconflict what my major should be: he told me to stick with mathematics! That's not quite the advice you might think it was. He thought there were plenty of Christian academics seeking to work out their faith in philosophy, but fewer of them in mathematics with a historical and philosophical bent. Also, a mathematics professor familiar with my multiple interests counseled me not to philosophize without a solid grounding in mathematics proper; else, he said, no one would listen to me.

Armed with these two sage pieces of advice, I went off to graduate school to study mathematics, alert for ways to enrich my program with history and philosophy of mathematics. After satisfying comprehensive doctoral exams at the State University of New York at Buffalo in algebra and analysis, I was hoping to do a historically weighted dissertation on some topic in algebra or foundations. But a potential supervisor told me that if I did that I would never find a job, and he had no interest in overseeing such an undertaking. This time I ignored my professor's advice. After attending a

talk on Gauss by the well-known historian of mathematics Ken May, I decided to transfer to the University of Toronto, where I could do a joint Ph.D. in mathematics and history of mathematics under him, specializing in the history of logic. And, as it turned out, it *was possible* to find a job teaching college mathematics with that kind of professional training.

My association with what came to be called ACMS after 1985 goes back to its inception in the early '70s. Like Bob Brabenec and some others, I thought Christian mathematicians should be more intentional about connecting their faith to their academic work and that this might be helped by engaging with the history, philosophy, and foundations of mathematics, broadly conceived. In addition to attending and speaking at the early conferences, Gene Chase and I collaborated to compile and annotate a *Bibliography of Christianity and Mathematics* (1983).¹ Later, I contributed two historical chapters on mathematization trends in the pre-modern and early modern eras in the book authored by a number of ACMS members, *Mathematics in a Postmodern Age* (2001). And in 2006, I gave the keynote address “Mathematics: Always Important, Never Enough” at a Dordt conference for Christian mathematics educators on the topic of a Christian approach to mathematics and mathematics education. In these ways I was able to draw upon my background interests and professional training in history and philosophy of mathematics to share my ideas with a broader academic audience.

3 Teaching, Textbook Writing, and Perspectival Issues

My teaching career at Dordt College beginning in 1982 encompassed a full range of standard undergraduate mathematics courses. One course I inherited at the outset, though, was a bit of an odd duck. Since Dordt’s calculus sequence served engineering and computer science students as well as mathematics majors, our department decided to focus on the theory of calculus in a transition course for our students. To prepare them for navigating rigorous ϵ – δ limit arguments and proving the basic theorems of differential and integral calculus, we first studied how to construct mathematical proofs. This gave me an opportunity to tap into my knowledge of developments in logic, though I didn’t initially see how to inject very much perspectival material into such a technical course. I did, of course, concentrate on those sorts of issues in my alternate-year capstone course on the history of mathematics, but beyond having students read Judy Grabiner’s *Who Gave You the Epsilon: Cauchy and the Origins of Rigorous Calculus*, I didn’t pay much attention to perspectival issues in our sophomore-level transition course. Except, a significant undercurrent soon pushed me to do more.

What I had learned about the history of logic and the foundations of mathematics in my doctoral program made it difficult for me to accept some of what was being asserted about logic and mathematical proof in the pamphlets I initially chose to use as a text (first Bittinger’s *Logic, Proof, and Sets*; and then Solow’s *How to Read and Do Proofs*). I found myself pushing back on some of their key viewpoints, but I discovered that students didn’t much appreciate being told that what they were reading was problematic—it’s the textbook, after all!

At the same time, I was teaching the introductory logic course for Dordt’s philosophy department, using a text (Bergmann, Moor, and Nelson’s *The Logic Book*) that presented logic from a twentieth-century natural deduction perspective—the very thing, I soon realized, that my mathematics students needed for learning how to do proofs. The alternative Frege-Hilbert-style approach, better known to mathematicians, focused on tautologies (logical axioms) more than rules of inference

¹For more information about this project, see my tribute to Gene on page 324 of this *Journal and Proceedings*.

because that had been the basis for implementing the logicians' program of reducing mathematics to logic. Recognizing the divergent goals of these two approaches to logic caused me to revamp the first part of my transition-math course and think about writing my own booklet on mathematical proof. It also made me realize that I could tailor my research interests to enrich my mathematics teaching.

At the time (1990-1991) no publishers among the 25 I contacted were willing to risk adopting my approach to mathematical proof construction. But I was seeing its value in my students' upper-level coursework, so I continued to invest appreciable time and energy in developing a classroom text for the course. This gradually morphed from bare-bones outlines to paragraphs of mathematical prose, using Knuth's recently available typographical system \TeX (plain \TeX at the time) to produce a publication-quality manuscript.

I also used some of this material on logic in a discrete mathematics course that I taught to computer science majors and engineering students. Thus, when enrollments in our mathematics transition course fell below an acceptable level in the early 2000s, we decided to combine these two courses. I dropped out the baby real analysis from the transition mathematics course and the finite state machines material from the other, redesigning the course to suit our particular clientele. This also gave me an opportunity to try to integrate perspectival elements where there was a natural fit.

Over the years, then, I gradually developed a text in *Discrete Mathematics* for the hybrid course I was teaching on an annual basis. Upon retiring years later in the mid-2010s, I posted chapters of my text to Dordt's Digital Collections to continue to make them available to Dordt students—and, as it happened, to whoever else might be interested in them. To my great surprise, they began to be downloaded all over the world. Within a year, over 9000 chapter-downloads were made by people in some 600 institutions and 90 countries, the most popular chapter being, amazingly enough, the rather technical seventh chapter, *Posets, Lattices, and Boolean Algebra*, geared to students learning about Boolean functions and logic gate circuits—downloads of this chapter proceeded at the rate of around 18 per day for months.

Pleasantly encouraged by this unexpected response, I decided to see whether a publisher might now be interested in my text. Two of the five publishers I sent a new prospectus to (Springer and Wiley) now expressed interest; a third (MAA) declined consideration because they already had two discrete mathematics texts in the pipeline. By the time *Introduction to Discrete Mathematics via Logic and Proof* was published by Springer in late 2019 as part of their *Undergraduate Texts in Mathematics* series, downloads of early chapter-drafts reached almost 80,000 by people connected to 2500 institutions in over 150 countries. Since then, another 20,000 digital chapter-downloads of the published textbook have been tallied by Springer through their licensing program with libraries.

4 Content and Perspectival Elements

I'll now briefly describe the material contained in *Introduction to Discrete Mathematics via Logic and Proof* (see the [Appendix](#) for the Table of Contents), and explain how perspectival elements are embedded within it.

The book opens with a somewhat leisurely study of logic, which (as the book's title indicates) is a coherent integrative thread running through the text. *Logic* is a field with its own content and concerns, but it clearly also has important links to mathematics and computer science. Logic forms the proper basis, I believe, for a genuine understanding of how proofs work—under-the-hood, as it

were—in mathematics and elsewhere. I know this is disregarded by some mathematics practitioners who believe that guided practice at constructing proofs is sufficient and who think that logic introduces extraneous and distracting content. Having suffered under this benign-neglect approach as a student, however, I beg to differ, even though I agree that practice is absolutely necessary. I do find this negative attitude understandable, however, given mathematicians’ experience with logic as a specialized subfield of mathematics dealing with truth tables and tautologies as well as foundational matters such as consistency and decidability. My different purpose for including logic is methodological, which requires an alternative approach to deduction. I employ the natural deduction approach due to Jaśkowski and Fitch, an approach that emphasizes suppositional inference rules such as *Conditional Proof*, *Proof by Contradiction*, and *Cases*. These are the life-blood of mathematical proof. When haven’t you seen a mathematical proof declare, “suppose such and such”?

In addition to supplying a good basis for learning how to construct proofs, the replacement rules of *Propositional Logic* have close connections to results in elementary *Set Theory*, the topic of the fourth chapter. And this system of logic is, of course, also the theoretical basis for discussing Boolean functions and logic circuits, a main focus of the seventh chapter.

The interconnected roles played by logic are pointed out as the text proceeds, but logic is also the focus of a few historical and philosophical discussions scattered throughout. I survey the historically evolving role of logic at the outset, noting that logic’s initial link to mathematics was as the deductive instrument (Aristotle’s *organon*) for organizing and developing Euclidean geometry and Pythagorean number theory. Logic was completely unconnected to computation and algebra until the nineteenth and twentieth centuries when the work of Boole and Shannon became foundational for logic and computer science. Around the same time, the relation between logic and mathematics was tipped upside down by Frege, Dedekind, and Russell, who reshaped logic to be a content foundation for (parts of) mathematics. These logicist developments are elaborated in remarks integral to the text where germane.

I also make explicit connections to the other main foundational and philosophical developments in the early twentieth century. Brouwer’s intuitionistic position on mathematical argumentation along with some push-back by Hilbert and others is mentioned, and differences between intuitionist logic and classical logic on issues involving negation are pointed out, particularly with respect to *Proof by Contradiction*.

With regard to Hilbert’s formalist philosophy of mathematics, students are pressed to think through the meaning and value of formalizing mathematics. A formal approach is important for allowing multiple interpretations or models of mathematical theories, something germane to fields such as *Abstract Algebra*, but this is also important for parsing out the logical properties of and connections between a theory’s axioms, as was seen in the rise of non-Euclidean geometry. Knowing these things encourages students to distinguish between taking a formal approach for technical mathematical and metalogical considerations and adopting a formalistic viewpoint on mathematics as a whole. Formalization is also relevant for computer-aided or computer-generated proofs, such as those developed for the *Four-Color Theorem*, a topic in Chapter 8’s *Topics in Graph Theory*.

Closely connected to these philosophical schools of thought and various nineteenth- and twentieth-century developments is the praxis of axiomatization with its attendant methodologies of formalization and abstraction.

Axiomatization first comes up in the text for a theory having a familiar intended interpretation:

Peano Arithmetic. Computational axioms were first isolated and stressed in mid-nineteenth century British algebra (De Morgan, Hamilton), being later systematized by Dedekind and Peano in the late 1880s. This brought ordinary arithmetic into the axiomatic fold. Students are often surprised that properties of the computational apparatus they've used all their lives can be deduced (with some work) from a few definitions and five axioms characterizing the counting process, including the seminal *Axiom of Induction*. As a student once observed to me about this whole process, "calculus is easy; it's arithmetic that's hard!"

Axiomatization next appears in the context of studying *Set Theory*. After learning about Cantor's and Dedekind's work in elucidating the nature of infinite collections, students are informally introduced to axiomatic *Set Theory*. This was first proposed by Zermelo primarily as a way to organize and deductively justify some key set-theoretic results (such as the *Well-Ordering Theorem*) but also in order to avoid *Set Theory*'s paradoxes (such as *Russell's Paradox*). Its use as a foundation for mathematics is illustrated by noting how *Peano Arithmetic* and other mathematical ideas and results can be modeled (or coded) inside *Set Theory* (e.g., by von Neumann in 1923), and its impact on later developments in mathematics education such as the *New Math Revolution* of the '60s is pointed out. Metalogical concerns such as Gödel's and Cohen's independence results for the *Continuum Hypothesis* are also cited, though obviously not argued.

Finally, axiomatization is introduced with respect to exploring Boolean lattices, generalizing and systematizing its results as *Boolean Algebra*. This leads students deeper into contemporary mathematics, stretching them to think more abstractly about mathematical axioms and models, something they will experience in spades in certain upper-level mathematics courses.

As I'm more of a historian than a philosopher, my default is to include historical material wherever relevant. Thus, the text highlights the significance of scores of people—for *Logic* (Aristotle, Boole, Leibniz, De Morgan, Frege, Russell, Tarski, Shannon, Jaśkowski); for *Set Theory* (Cantor, Dedekind, Venn, Zermelo, Gödel); and for *Graph Theory* (Euler, Hamilton, Kuratowski, Kempe, Heawood)—about 70 mathematicians in all. Naturally, I can't do much besides mention most of these people, but doing this in context is more than name-dropping: it gives students and professors a hook for further historical exploration.

So, the text raises issues both of philosophical and historical interest. What about connections to religious outlooks? Are there even any, and if so, can such things be broached in a commercial textbook? This question can be answered along several different lines. Obviously, Christian doxological reflections on God as the Source and Sustainer of mathematical and logical realities would be deemed out of bounds by a secular publisher like Springer. Such comments can be integrated into an instructor's use of my textbook, however: certain parts of the text provide a stage for adding such sentiments in a rather natural way. And raising various foundational issues provides an information platform for evaluating prominent philosophical perspectives from a Christian viewpoint. I think aspects of all three or four major early twentieth-century foundational philosophies can be challenged from the vantage point of a Christian worldview, even while recognizing legitimate advances in logic and mathematics due to their outlook. Rejecting the reductionist tendency of the logicist program (or of the related set-theoretic foundation), for instance, comports well with a Christian philosophy that emphasizes the rich multidimensional character of creation. This non-reductionist outlook is partly what makes me choose a natural deduction approach to logic over that of Frege and Russell—another reason being that a stress on inference rules as governing valid argumentation fits better with an emphasis on God's sovereignty over truth and validity. I believe that similar sorts of reflections can be made about intuitionist and formalist philosophies

of mathematics.

There is one point in the book where a connection to Christian theology is explicitly made: Cantor's view of infinity was formed through both mathematical and theological reflection on this notion. I cite this fact because of its historical influence, not because I believe Christian mathematicians should anoint Cantor as God's prophet of the infinite. Yet noting this connection provides an opportunity to ponder mathematics' appropriation of infinity from theology and philosophy, and it also explains why some mathematicians at the time thought Cantor had lost his way and wandered off into metaphysics.

5 Concluding Remarks

I hope it's clear how I've used perspectival elements to integrate, contextualize, and enrich some fairly standard discrete mathematics material. Still, I know that my particular emphasis and choice of topics isn't for everyone: *Discrete Mathematics* is a bit of an amorphous grab-bag that can be taken in many different directions. I've chosen topics that connect with one another in natural ways. There is some flexibility in the book to tailor topics to fit one's interests or needs, though, and instructors can always proceed at their own pace. Some sections can be combined by omitting certain subsections, and extensive exercise sets encourage exploring topics in more depth if that is wanted; an instructor's *Solution Manual* provides complete solutions to half of the book's 1500 problems.

In conclusion, while I have emphasized the historical and philosophical features of my book, treating it as a case study in how one can incorporate perspectival elements in a technical mathematics text, I believe that anyone who examines the text will recognize that it adheres to the advice I was given as an undergraduate: do your philosophizing and historicizing in the context of substantial mathematical considerations.

6 Appendix

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The Nature of Reasoning in Theology, Philosophy, and Mathematics

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Abstract

This article supports the epistemological claim that sound human reasoning about ultimate knowledge is either foundational or circularly justified. In particular, questions which naturally arise in theology, philosophy, and related disciplines, to the extent that they rationally treat ultimate knowledge, are necessarily supported in one of these ways. Comparisons with, contrasts to, and analogies from mathematics are given to illustrate and enhance this central claim.

1 Introduction

One of the most basic goals of any rational human enterprise is to intellectually justify one's truth claims. This article will investigate the nature of that justification process. We have in mind what we will call "propositional justification," that is, justification *of* propositions *by* propositions (truth-valued statements). Words like "support" or "justification" will refer to this notion of propositional justification unless specified otherwise.

Our aim is to examine the *structure* of rational justification, not to arbitrate rationality itself. The question we seek to answer is not "is proposition P rationally held?" but rather, "*given* a theory of warrant,¹ what is the nature of the structure of the rational support of proposition P ?"

It is natural to relate these epistemological issues to mathematics as the latter is commonly regarded as occupying rarefied air, perhaps even uniquely, in its theoretical completeness. Famously adored as the "queen of the sciences"² with the "most uncontroversial examples of proof," [11, p. 56] if something is observed in mathematical thinking, it is often expected to apply to other (maybe even all) rational endeavors. Examining the interplay between mathematics and ultimate justification for truth claims is therefore worthwhile and will be taken up in this article.

This article is split into three sections. Section 2 states and argues for the main thesis. Section 3 uses mathematics as a case study to make connections between the notions of foundational knowledge and circular reasoning, particularly noting how circularity is avoided by the major schools of thought in the philosophy of math. Section 4 briefly addresses some questions which naturally follow.

¹A set of (and relations between) standards or criteria by which rationality is measured; it goes by other names as well.

²While applying this term to mathematics is attributed to Carl Friedrich Gauss, math was not its original object. It referred to theology (of all disciplines!) during the Middle Ages. See [4, p. 213].

2 The Argument

2.1 Assumptions and Definitions

It would be a Herculean (or impossible) task to fairly and incontrovertibly list *every* assumption; complete, untainted epistemological self-awareness might be only a theoretical ideal. We will nonetheless attempt to identify the ones most fundamental and most relevant to the present topic.

Our primary object in this section will be the “hypothetical belief set” (*HBS*) of a fixed human agent *H* at a fixed time *t*. It is “hypothetical” because we allow the set to include beliefs that *H* *would* hold if prompted appropriately, but which *H* might not otherwise believe or even conceive of. The set *HBS*, sometimes written *HBS_t* to emphasize the dependence on *t*, along with the relations among its elements, is the *noetic structure* of *H*. [20, p. 48] We will assume *H*’s thoughts are not past eternal and that it takes some positive amount of time for *H* to form a thought. Coupled with the dependence on *t*, we may thus assume that the cardinality of *HBS_t* is finite.

We will assume the laws of logic and general principles of rationality which are too many (and, likely, too familiar) to enumerate. But as they will be essential terms, let us define them a bit more thoroughly.³ Here, we will consider logic to be formal, classical logic.⁴ Definitions include

- “a formal, scientific method of examining or thinking about ideas;”⁵
- “a science that deals with the principles and criteria of validity of inference and demonstration: the science of the formal principles of reasoning;”⁶
- “the science of thinking about or explaining the reason for something using formal methods.”⁷

We consider deduction to be the process of drawing conclusions by applying the rules of logic to one or more statements (elements of *HBS*). Rationality is

- “the quality of being based on clear thought and reason, or of making decisions based on clear thought and reason;”⁸
- “the quality or state of being agreeable to reason;”⁹
- “the fact of being based on reason rather than emotions.”¹⁰

³We take the notions of rationality and logic to be *rationality* foundational (as seems to be required of rational human endeavors), but not “semantically foundational.” As will be discussed throughout the article, there are two sets in any rational human endeavor to which our thesis applies: rational support and definitions. An element which appears in both sets need not be ultimate (in the sense defined later in the section) in both sets, modulo the caveat discussed below that definitions can be converted to truth claims. In this article, “rationality” is considered ultimate in the former set, but not the latter. Nonetheless, the definitions we give should be taken to be informal.

⁴As opposed to the informal use of the word “logic,” which can refer to any kind of reasoning, and as opposed to nonstandard formal logics. If one insists on a nonstandard logic, it is likely to be unproblematic for this article.

⁵<https://dictionary.cambridge.org/us/dictionary/english/logic>, accessed June 20, 2022.

⁶<https://www.merriam-webster.com/dictionary/logic>, accessed June 20, 2022.

⁷https://www.oxfordlearnersdictionaries.com/us/definition/english/logic_1?q=logic, accessed June 20, 2022.

⁸<https://dictionary.cambridge.org/us/dictionary/english/rationality>, accessed June 20, 2022.

⁹<https://www.merriam-webster.com/dictionary/rationality>, accessed June 20, 2022.

¹⁰<https://www.oxfordlearnersdictionaries.com/us/definition/english/rationality>, accessed June 20, 2022.

Accordingly, we will call an intellectual endeavor or claim *rational* if it is ultimately supported by reasons. What constitutes a “reason” is a criterion which we do not attempt to define in this article (beyond noting that a reason must be a truth-valued statement). That is, we will assume that some theory of warrant has already been accepted; there is much epistemological debate on such topics on which we (attempt to) remain mostly silent. We simply consider whether a claim is supported by reasons or not, without regard to whether they are “good reasons.” Note also that there is a contrast with *rationalism*, the philosophical view that “regards reason as the chief source and test of knowledge.”¹¹

We consider, then, *irrationality* to be the negation of rationality and *irrational* the negation of rational; in particular, for brevity, the notion of “arationality” (being outside the arena of reason, even if not in conflict with it) is subsumed under an umbrella term of irrationality in this article. Academic disciplines such as theology (the study of religious belief), religion (the pursuit of the supernatural or supreme existence), and philosophy (the study of the foundational nature of reality) will be referred to in broad strokes.

By *circular reasoning*, therefore, we will mean the support of a proposition in HBS_t for which the conclusion is among the premises. We will use the term “circularity” for brevity to refer to circular human reasoning. It is sometimes referred to in the literature as “begging the question” or “petitio principii,” though some authors make a distinction among these terms. A proposition will be considered *foundational* if it has no propositional support in HBS_t .

We will assume an agreed-upon meaning¹² of all terms used, with only a few key terms being explicitly defined. Of course, this is a nearly universal assumption, and any meaning-carrying dialogue depends on it, so it is not overly ambitious. Nonetheless, given the epistemologically fundamental nature of the content that follows, adequate epistemological self-awareness dictates that this assumption be stated, along with the acknowledgement that semantic misunderstandings are still possible regardless. It is the author’s hope in defining a few key terms that any such persisting misunderstandings are insignificant to the central thesis.

We fix a claim, $P \in HBS_t$, of genuine knowledge about reality, i.e. P is (agreed upon to be) true. Likewise, we assume a goal of rational investigation of objective truth about reality, as opposed to statements grounded ultimately in opinion, preference, feeling, or other subjective standards. In particular, issues raised by the philosophical positions of skepticism, which doubt whether truth can be known or if there is any objective reality at all, will not be considered herein. Thus, we will assume that H can justifiably claim to know *something*. Note also that definitions could be rephrased as truth claims in the form “term X has meaning Y ”;¹³ P might be as such.

It is useful to refine the epistemic *level* of P : either P is ultimate knowledge or it is not. In this article, *ultimate knowledge* is the most basic knowledge that H possesses. We will often refer to such a proposition as *an* ultimate, or ultimates in plural (using the word “ultimate” as a noun). An ultimate is something for which H does not have a deeper knowledge *level* to undergird it. Likewise, we will call questions about ultimate knowledge “ultimate questions.” What we call “ultimates” in this article are given many other names depending on the context and author: presupposition, basic commitment, [11, p. xvii] (properly) basic belief, [20], [11, p. 59], starting point, innate belief, final authority, foundation, foundational belief, assumption, and so on (and, unfortunately, sometimes

¹¹See <https://www.britannica.com/topic/rationalism>, accessed June 20, 2022.

¹²Or “approximate meaning.”

¹³A similar phenomenon can be seen in mathematics, when sometimes definitions are referred to as axioms; indeed, “there is little distinction between a mathematical axiom and a definition.” [4, p. 216]

these terms are used with different meanings).¹⁴ It is tempting to equate this notion with the mathematical notion of axiom (or “hypothesis” to some), but as Christian theologian John Frame discusses, [11, pp. xxxi-xxxii] and as we shall see shortly, there is a distinction.

The aforementioned assumptions are not necessarily themselves ultimates, but are a practical necessity to focus the discussion and keep it to a manageable length.

2.2 Primary Thesis: Ultimate Justification for a Truth Claim

What we discuss below is a version of Agrippa’s trilemma, credited to the philosophical skeptic Agrippa and the Pyrrhonists.¹⁵ It is one manifestation of the problem in epistemology regarding ultimate justification for a truth claim. The nuanced differences of its disparate treatments will not be essential here.¹⁶

Our primary thesis is the claim that

(FC) HBS_t contains elements which are foundational or are circularly supported.

We will begin with P to conclude (FC). One can ask for proximate, rational justification of P : why, or on what rational authority (empirical evidence, logic, the authority of the source from which P was discovered, or whatever reason might be legitimate according to H ’s theory of warrant), can we say that P is the case?¹⁷ Suppose a distinct supporting reason, Q , is given. One could equally ask for the distinct, proximate, rational justification of Q , call it R . The pattern continues, descending into a sequence of “why” questions and answers: P, Q, R, \dots We will call this the *support sequence*.¹⁸ Under our assumptions, there are only two options¹⁹ with regard to the termination of this sequence (and hence, for the foundations of H ’s noetic structure) which we will explore in turn:

1. the sequence never terminates, continuing to give distinct supporting reasons; or
2. the sequence terminates in finitely many steps, ceasing to give distinct supporting reasons.

¹⁴At t , we consider the set of foundational propositions to be a subset of the set of ultimates.

¹⁵The essentials of the problem go by other names as well, such as Münchhausen’s trilemma, or Fries’ trilemma. The term “Münchhausen’s trilemma” is credited to [1]. The present article uses the framework of this trilemma, while coming to different conclusions than Agrippa and the Pyrrhonists.

¹⁶For example, in a work by Greek philosopher Sextus Empiricus, five possible outcomes to the question of ultimate justification are given. Three of those five are contained in Agrippa’s trilemma, with two additional ones which are disallowed by our current assumptions: dissent (in our context, this would be to assume that the proposition P is not agreed to be true) and relation (P appears to be true from the point of view of a certain individual, but that does not necessarily constitute a statement of the inherent truth of P ; in other words, it is the conclusion that truth does not refer to an objective reality). See [8]. For a readable overview of the issues in Agrippa’s trilemma, refer to the Stanford Encyclopedia of Philosophy’s entry on Skepticism; see <https://plato.stanford.edu/entries/skepticism/> (accessed July 31, 2022).

¹⁷We can consider such a question as a “Chisholm-like question” without the misunderstanding of how such a question should be taken as in [20, p. 51].

¹⁸Could P have a distinct support sequence, say P, E, F, G, \dots ? We allow for that possibility. The analysis which follows will fix a particular sequence, though the conclusions apply to any support sequence which one desires to choose. The salient characteristic is that P has only finitely many distinct support sequences, which follows since HBS_t is assumed to be finite in size. Thus, fixing just one support sequence suffices.

¹⁹The reader may wonder where the third horn of Agrippa’s trilemma is. Two of them are together contained in a single option: the finite termination of the sequence.

2.3 Option 1: The “Endless Why”

This option implies there is an infinite regress of supporting reasons for claim P , a truly endless sequence of “why” questions and answers. In Agrippa’s trilemma, this is the “infinetism” horn. This contradicts our assumption that HBS is finite at t , and therefore can be discarded.

Additionally, perhaps enlightening is the observation that, even if the finiteness assumption were dropped, it is questionable if P could be considered to be genuinely rationally supported. If the argument supporting P is never complete, is P really supported after all? Is an in-progress proof really a proof (in any theory of warrant)? One particularly troublesome potentiality: the support sequence could eventually contain something that breaks our assumptions, and, even if that hasn’t occurred yet, we may not have sufficient confidence that it won’t occur at one of the future steps. Such observations demonstrate that, even under slightly more general assumptions, an infinite support sequence for P is dubious at best, and possibly irrationality “in disguise.”

2.4 Option 2: Foundational or Circular Support

Therefore, the support sequence cannot be truly endless in rational support of H ’s genuine knowledge; it must terminate in finitely many steps. That is, some supporting reason Z ²⁰ must be the last distinct supporting reason at t (where we allow for the sequence to terminate in 0 steps, *i.e.*, $P = Z$). That Z , by definition, has no deeper or more fundamental level with distinct reason(s) supporting it at t , so it is at the ultimate justificatory level. How does Z relate to other elements of HBS_t ?

That question is answered in two different ways in Agrippa’s trilemma, the first being *foundationalism*.²¹ Here, Z is accepted as foundational without further *propositional* rational defense (though it may be defended some other way). The support sequence merely stops at Z . Depending on the nature of Z and the theory of warrant, Z could be considered knowledge without being inferred from *any* proposition, including from itself. [20]

The other horn of the trilemma is referred to as *coherentism* in which Z is part of a mutually supporting, self-contained theory. This requires a “cycle” of reasons;²² this is circularity.

Therefore, there are only two non-rejected options for the ultimate justification of P :²³ foundational support and circular support.²⁴ Thus we conclude (FC).

Most connections to mathematics will be saved for Section 3, but one is appropriate here. We may impose a finite directed graph structure on HBS_t since we view it as a finite, discrete set with nonsymmetric relations among (some) pairs of elements. Call this the “knowledge graph” of H and label it G .²⁵ Nodes of G are propositions (elements of HBS), and arrows point from one node to

²⁰The termination of the support sequence of P is likely to be a set of reasons rather than one single statement. For ease of notation, and since the individual elements of such a set need not be separated for our purposes, we will simply notate the termination Z and treat it as a single reason.

²¹The related stance of positivism, [10] in which Z is simply unjustified (presumably relative to the chosen theory of warrant), is discarded as it contradicts our assumptions. In being unjustified by its own admission, it represents a choice that is not rational truth-seeking.

²²Note that Z is the final *distinct* reason. It could be the case in HBS_t that the reason given for Z is some other proposition R already listed in the support sequence, thereby initiating cyclical traversal through the sequence.

²³That is, only two options which do not contradict our assumptions.

²⁴The interplay between them is examined deeply in [14].

²⁵This may be a metaphor to add to the list of “raft” and “pyramid.” See [23].

the others which support it. Every node in a connected²⁶ component of G has at least one arrow pointing to it or emanating from it. A theorem²⁷ in graph theory tells us that every such graph contains a cycle (which would indicate circular reasoning in our analogy) or a sink (a node with no arrows emanating from it; in our analogy, this is a foundational proposition).²⁸ If one accepts the knowledge graph analogy, then the proof of this theorem offers an alternative proof of (FC). Perhaps this mathematical analogy can be mined to provide further insight into epistemology.

2.5 Why entertain the plausibility of circular reasoning at all?

Nearly all rational thinkers eschew any form of circular reasoning as logically fallacious. Philosopher Douglas N. Walton, with an entire book devoted to circular reasoning, [34] explains succinctly why this is the case: “Arguing in a circle becomes a fallacy by basing it on prior acceptance of the conclusion to be proved. So the fallacy of begging the question is a systematic tactic to evade fulfillment of a legitimate burden of proof.” [35, p. 66], [11, pp. 254-5] As philosopher S. Morris Engel puts it, “if the supporting premises merely repeat or rephrase what is stated in the conclusion, as in all cases of begging the question, the argument contains no premises and is therefore fallacious.” [9, p. 147], [11, p. 255] According to theologian Joseph E. Torres, “[i]f question-begging is embraced, fideism is implied. Fideism is the rejection of a rational [argument].” [31], [11, p. 255]²⁹ Even in everyday discourse, labeling a putatively intellectual argument to be circular is a death blow.

Why, then, allow for circular reasoning as a plausible option at all? The short answer is that the rationality of circular (or any other kind of) reasoning is left up to the theory of warrant, and in this article, we attempt not to evaluate such theories. But there is another layer worth examining.

A variety of circularity is discussed, and evidently promoted, by Christian presuppositional apologists, a school of thought expounding on the works of theologian Cornelius Van Til. For example, John Frame defines “circularity” as “an argument in which the conclusion justifies³⁰ itself. All arguments seeking to prove the existence of an ultimate or final authority are circular in this sense.” [11, p. 291] Philosopher and apologist Greg Bahnsen, often considered the primary popularizer of Van Til’s work,³¹ says that this kind of circularity is “involved in a coherent theory (where all the parts are consistent with or assume each other) and which is required when one reasons about a precondition for reasoning.” [3, p. 518n122] Theologian R. C. Sproul concurs: “That all reasoning is ultimately circular in the sense that conclusions are inseparably related to presuppositions is not in dispute.” [24, p. 70], [11, p. 258] Seeing circularity as a consequence of the finiteness of humanity is expressed succinctly by Van Til himself, who writes, “[w]e hold it to be true that circular reasoning is the only reasoning that is possible to *finite* man” (emphasis added). [33, p. 12], [11, p. 254]

How can serious thinkers seem to accept an argumentation tactic that is clearly fallacious? It seems that they refer to different types of rational support. When seeking *propositional support*, as in this article, there is broad agreement, even among presuppositionalists, that circular reasoning is

²⁶Or, more commonly, “weakly connected.” See [5, p. 163].

²⁷[5, p. 89], Theorem 4.3 proves this result for an undirected graph. The directed version can be found in Lemma 1 at <https://www.math.cmu.edu/~af1p/Teaching/GT/CH10.pdf>, accessed June 15, 2022.

²⁸Note, then, that a foundational proposition cannot be part of a cycle, but an ultimate might be as per (FC).

²⁹Fideism, beliefs taken on faith alone without any rational support, is widely seen as anti-intellectual.

³⁰It is not entirely clear if Frame’s notion of “justification” is the *propositional* variety. Similarly, his notion of “ultimate” seems to refer to a standard or criterion in the theory of warrant rather than to a proposition.

³¹Rev. Dr. K. Scott Oliphint, expresses this in his comment of [2]. Oliphint says, “For those who want to understand Van Til, whether to agree or disagree, at least two things are both essential and too often neglected. The first is to read Van Til, the second is to read Greg Bahnsen.” (Back cover).

fallacious. The variety of circularity endorsed by presuppositionalists is of a different type, a type we will call *authoritative support*,³² referring to the *criteria* which one uses to judge veracity or rationality. Frame recognizes the necessity: “[W]hen one is arguing for an ultimate criterion, . . . one must use criteria compatible with that conclusion. If that is circularity, then everybody is guilty of circularity.” [11, p. 11]³³ In other words, rhetorically, by what standard can an *ultimate criterion* of knowledge be aptly declared suitable, if not by itself (for, if not by itself, then the criterion is not ultimate)?³⁴ By what criteria of knowledge shall we judge our criteria of knowledge but themselves? To emphasize the distinction between fallacious circular reasoning of propositional support and purported circularity of authoritative support, Torres terms the latter “circular coherence.” [31], [11, p. 256]³⁵ This is the version of circularity endorsed by the presuppositionalist school.

It is worth emphasizing that this type of circularity and the question begging fallacy (*petitio principii*) are distinct notions. According to Torres, a presuppositionalist, “Van Tillians, at least implicitly, distinguish between circular coherence and begging the question, embracing the former and rejecting the latter.” [11, p. 258]³⁶ The circularity of presuppositionalists is a requirement of the theory of warrant to use only reasons within argumentation (arrows in G) that comport with the ultimates (each would appear as a node in G , either a sink or as part of a cycle); thus it is some relation between arrows and ultimates (specific nodes) which does not appear explicitly in G (in particular, presuppositional circularity does not require a cycle in G). A type of “circle” appears in that ultimates and the theory of warrant are self-consistent,³⁷ but this is *not* a cycle in G . Question begging is a relation between various nodes of G and *is* evident by the presence of a cycle in G . As the presuppositional notion of circularity is distinct from the familiar concept of circular reasoning, criticism of presuppositionalists for ambiguity in language may well be warranted.³⁸

Thus the contrast of support type (propositional versus authoritative) demonstrates how the presuppositional notion of circularity is distinct from and not contradictory to the common understanding which rightly disparages circular reasoning. As authoritative support reveals itself in evaluating rationality and not the structure among propositions, questions of authoritative support reside in the theory of warrant. Hence, we do not judge such positions here, but underscore the point that permissibility of circularity in *authoritative* support is not inconsistent with fallaciousness of circularity in *propositional* support.

³²Perhaps the term “foundational support” would be equally descriptive.

³³Indeed, “reasoning in a vicious circle is the only alternative to reasoning in a circle.” [32, p. 24], [11, p. 257] In the words of philosopher Richard Pratt, “in argumentation, reasoning should be linear,” but circularity of the “*process* by which finite minds attain knowledge to be used in arguments” is “inescapable.” [22], [11, p. 257] (emphasis added).

³⁴It is in this sense that ultimates occupy a “privileged position” in rational argumentation in that they exhibit special epistemological properties not seen in non-ultimates. See Bahnsen’s lecture “Reasoning With Unbelievers,” beginning at 25:20: <https://www.youtube.com/watch?v=6RBz-zAWoKk>, accessed December 8, 2021. See also the discussion of the “meta” relationship that ultimates have to the support sequence in Sections 3.7, 4.1.

³⁵More from Torres on this (and related topics) can be found at “Presuppositionalism and Circularity... Again?” at <https://apolojet.wordpress.com/2012/03/15/presuppositionalism-and-circularity-again/>, accessed June 20, 2022. A similar bifurcation of types of ultimates, propositional versus authoritative, might be in order as well.

³⁶Other terms are used for the distinction between circular coherence and question begging: Frame calls the former circularity “broad” and the latter “narrow.” [11, p. 14] Torres labels them “virtuous” and “vicious,” respectively. [31], [11, p. 255] As Torres explains, “the main distinguishing mark between these two forms of circularity is how one handles the burden of proof, or evidential priority.” [31], [11, p. 264] Thus, the question-begging fallacy is “vicious,” whereas circular coherence is seen as “virtuous.” (cf. Russell’s “vicious circle principle” discussed in Section 3.)

³⁷That is, the ultimate criteria underlie the reasoning used, even in (linear) argumentation for their own legitimacy, in a way that non-ultimates do not. This “circle,” however, may be non-propositional in nature, and hence not be circular reasoning as traditionally understood.

³⁸Frame recognizes the risk: “I don’t care very much whether the Christian apologist accepts or rejects the term *circular* to describe his argument. There are obvious dangers of misunderstanding in using it...I am more inclined now to say to my critics, ‘Granted your definition of circularity, I don’t believe in it.’ ” [11, p. 11n19]

3 Connections to Mathematics

3.1 The Epistemic Characteristics

It is justifiable to bring mathematics into this discussion so long as we assume that at least one statement of mathematics is genuine knowledge (to play the role of P from Section 2). We add that assumption for this section.

Mainstream mathematics today follows the axiomatic method.³⁹ It begins⁴⁰ with a set of (formally undefined) terms⁴¹ and axioms (asserted-but-unquestioned assumptions, taken to be true without proof),⁴² and uses logical deduction, enhanced by intuition,⁴³ to justify new statements. For convenience, we will give names to two epistemic characteristics exhibited by this process of mathematics. Mathematics

1. (EC1: starts somewhere) begins with foundational principles, recognizing that mathematical knowledge must start with something;⁴⁴ and
2. (EC2: restricts the scope) restricts what is properly a “mathematical question” and what is not. Some questions, even some which refer to indisputably mathematical objects, are not mathematical questions. In particular, there are logical arguments which are not part of mathematics (and are usually placed within the realm of theology and/or philosophy) because those arguments refer to nonmathematical questions or objects.

The content of mathematics is whatever can be concluded (according to the methods allowable in the given philosophical school, as discussed below) from the starting point(s) of EC1 and within the bounds of EC2. The process of mathematics can be seen through the lens of propositional support as discussed in Section 2. On the basis of the elements of EC1 (principally axioms and definitions), mathematics is frequently seen as being foundationalist, [4, p. 215], [16, pp. 51-2] even if artificially so vis-à-vis EC2.

EC1 implies that the “why” support sequence for a statement of mathematical knowledge terminates in finitely many steps at some (set of) reason(s), X , which is (are) the most fundamental *when viewed inside the theory at hand*. X is a set of one or more axioms. The restriction accomplished by EC2 is made manifest in practice by choosing X *not* to be an ultimate and only asking questions of the objects involved which do not require the intake of ultimates. Thus, for the philosophical schools considered below, mathematical inquiry halts its inspection before reaching the epistemic level of ultimates. As ultimates are not part of mathematics, mathematics remains mum on any issues regarding circularity in the authoritative support (cf. Section 2.5). Thus, any circular reasoning arising in mathematics is indeed rightly seen as problematic.

³⁹The discipline settled on this more structured approach “since the free use of our conceptions... led to disaster,” specifically the paradoxes that arose from Cantorian set theory, discussed below. [17, p. 40]

⁴⁰That is, the theoretical construct begins in the way described. The process in which humans engage – the process of doing mathematics – rarely proceeds in this order chronologically.

⁴¹This idea has a long heritage in mathematics. As far back as Aristotle it was acknowledged that some terms “must be undefined or else there would be no starting point.” [18, p. 20] (Note the lurking foundationalism!) Interestingly, Euclid’s *Elements* infamously attempted to define all concepts used, despite there being historical indications that he was aware of Aristotle’s works. Subsequently, the work of Euclid had an impact on “practically all the mathematicians who followed him for two thousand years.” [18, pp. 101-2] We will comment further on this in the subsection on formalism.

⁴²Formally, one could view an axiom as proving itself.

⁴³One might prefer the word “creativity” instead, or any of several related ideas.

⁴⁴This is expressed nicely by 18th century Scottish philosopher Thomas Reid in [7], esp. p. 148.

We will demonstrate for various philosophical schools how EC1 and EC2 are exhibited.⁴⁵ We will give special attention to ZFC set theory and the “Big Three.”⁴⁶ Our goal is not an exhaustive study of the many philosophies of mathematics, but a brief introduction to show how each one considered expresses the ECs. In particular, every philosophy discussed below shares the same question for which the examination would violate EC2, namely, “once the foundational elements of EC1 are chosen, *why* are they the correct ones?” One could defend the choice of “these axioms instead of those,” but the point is that such a defense is a philosophical argument rather than a mathematical one. Additional comments on EC2 will be mentioned below as appropriate. Perhaps our discussion will also give the reader a sense for humanity’s intellectual limitations.⁴⁷

3.2 ZFC Set Theory: Object Language, Metalanguage, and Russell’s Paradox

The default approach to mainstream mathematics today is termed ZFC set theory, the acronym referring to a list of 8-9 (depending on the specific formulation) Zermelo-Fraenkel axioms, along with the axiom of choice.⁴⁸ The genesis of set theory as its own subfield is often credited to Georg Cantor and his pioneering work on infinite series, work in which complications were quickly discovered. [28, pp. 1-2] Cantor’s work made implicit use of the axiom of abstraction which asserted the existence of a set whose members are determined by sharing any given property. A 1901 letter from Bertrand Russell to Gottlob Frege brought to light a fatal flaw with the use of this axiom. To see the flaw, form the set S whose members share the property that the members are not members of themselves. Is S a member of itself? If so, then by its own defining property, S is *not* a member of itself. And if not, then, again by the defining property, S is indeed a member of itself. In both cases, S is simultaneously a member of itself and not a member of itself. This contradiction was termed “Russell’s Paradox” and forced mathematicians to concede the logical flaw in the seemingly innocuous, intuitive realization of set theory afforded by the axiom of abstraction. [28, pp. 5-8]

Russell himself gave an explanation for this state of affairs in 1905. The problem lies with the use of *impredicative* definitions, a term first used by Poincaré in 1906, “wherein an object is defined (or described) in terms of a class of objects which contains the object being defined. Such definitions are illegitimate” since they are circular; hence the related name, Russell’s “vicious circle principle.” [18, pp. 204-7], [25, p. 116], [17, p. 42]⁴⁹ Notions such as a “set of all sets” are impredicative and hence barred.⁵⁰ A subtle but powerful modification, introduced by Ernst Zermelo in 1908, led to a resolution of Russell’s Paradox and was included in ZFC as the “axiom schema of separation.” It required that a given set first be known to exist; then, members of that set with a given property can be collected in a subset. Other paradoxes (or contradictions) were found in early set theory as well, the details of which we leave to other works. [28, pp. 5-12], [18, pp. 204-7], ch. IX.

What are ZFC’s intellectual foundations (EC1)? The axioms and formally undefined terms such

⁴⁵We give only very brief introductions to the schools of thought. For more, see [18]; [29], especially chapters 17–20; [25]; [17], especially chapter III; [12], Part 5.

⁴⁶Term courtesy of [25], Part III.

⁴⁷Compare the discussion in [21, pp. 103-1].

⁴⁸We leave an enumeration of the axioms to the references as it is technical and inessential to the current article. For details, see [28]; [26]; [18], especially chapters VIII–XII; [12], especially Entry 5.3; [17], especially chapter III; [30]. For a less technical introduction, see [21], App. E.

⁴⁹Circularity in such definitions is indeed problematic, even in view of Section 2.5, since propositional support is our goal.

⁵⁰Some mathematicians and philosophers have defended the use of impredicative definitions. This debate is beyond the scope of this article. See [25, pp. 128, 180], [17, pp. 42-5]. The Burali-Forti paradox regarding the set of all ordinals is another closely related concept too technical for our coverage. See [28, pp. 8, 133].

as “set” are taken as given and unquestioned.⁵¹

How does ZFC exhibit EC2? Through an *object language/metalanguage distinction*.⁵² Such a distinction is what allowed the paradoxes to be resolved. The *object language* is the language in which we discuss the mathematical entities themselves, whether they be sets or numbers or anything else, and the *metalanguage* is the “language in which we talk about the object language.” [28, p. 11] The metalanguage includes the object language, but not vice versa. Indeed, this relationship is what allows us to leave the notion of set formally undefined in the object language and yet still talk about and understand it in the metalanguage. Whereas axioms or definitions are *treated* as ultimates (here, synonymous with “foundations”) when doing mathematics, they are only ultimate *with respect to the object language*, not the metalanguage.⁵³ This distinction makes it possible to “get outside” of set theory (and thus mathematics) to discuss mathematical objects/notions without *doing* mathematics (and, hence, without being unwittingly restricted by its methods or limitations). Indeed, much of the history of mathematics since the late 19th century demonstrates the utility of making this distinction. As Suppes summarizes,

Every one of these paradoxes arises from having available in the language expressions for referring to other expressions in the language. Any language with such unlimited means of expression is perforce inconsistent. Consequently it is important to distinguish between the object language – here the language in which we talk about sets – and the metalanguage, that is, the language in which we talk about the object language... In other words, we avoid these paradoxes by severely restricting the richness of our language. [28, p. 11]

Let us illustrate this with our P , Z , and X notation from above. If P is a statement of knowledge in mathematics, a complete support sequence of P (a proof of P) could be written like “ P because Q because R because ... because X because Y because Z .” The supporting reasons up to X would be considered part of mathematics, and at that point, the mathematical proof (the use of the object language) would stop.⁵⁴ Y and Z would be statements in rational human endeavors which fall outside of mathematics proper, and thus would be expressions in the metalanguage which are outside of the object language.

3.3 Logicism

Once the set theory paradoxes were discovered, the question of the consistency (freedom from contradiction) of mathematics was brought front and center, with the logistic school being the first serious attempt to address it. [18, p. 216] The school credits its founding to Frege in the late 1800s (though its main idea “can be traced back to Leibniz” [18, p. 217]), and it saw much development in the early 1900s from mathematicians of renown like Peano, Russell, and Whitehead. The magnum opus of the school was Russell and Whitehead’s *Principia Mathematica*, or *Principia* for short (which, in particular, included the famous theory of types by which Russell and Whitehead avoided falling victim to Russell’s Paradox). [26, p. 237] “The purpose of logicism was to show that classical

⁵¹Similar comments apply equally well to Russellian type theory, von Neumann-Bernays-Gödel set theory, and other, alternative set-theoretic foundations.

⁵²Some authors use the term “syntax language” instead of metalanguage. [17, p. 63]

⁵³Kline offers an analogy at [18, p. 250]. “If one wished to study the effectiveness or comprehensiveness of the Japanese language, to do so in Japanese would handicap the analysis because it might be subject to the limitations of Japanese. However, if English is an effective language, one might use English to study Japanese.”

⁵⁴Formally, X would be one or more axioms, though in practice most mathematical proofs stop short of this.

mathematics is part of logic.” [29, p. 122] This would transfer any foundational questions about the nature of mathematics into the realm of logic, the consistency of which seemed firmly established.

Principia “may be considered as a formal set theory” analogous to ZFC. [29, p. 122] Thus the primary goal became to show that each of the axioms of that set theory, from which flow all of classical mathematics, are themselves part of (first order) logic. This venture failed since “at least two of these axioms, namely, the axiom of infinity and the axiom of choice, cannot possibly be considered as logical propositions.” [29, p. 123]⁵⁵ Mathematicians accept the axiom of infinity, the statement that infinite sets exist, “in virtue of its content [i.e. from presupposed familiarity with infinite sets] and not in virtue of its syntactical form,” the latter of which would be necessary to consider it as a logical proposition. [29, p. 124]

Logicism exhibits EC1 by accepting first-order logic as mathematics’ starting point. Moreover, “Frege *proved* that each natural number exists, but his proof is impredicative, violating the type restrictions [of Russell’s type theory]. Russell had to *assume* the existence of enough individuals [type 0 objects which are neither classes nor sets] for each natural number to exist.” [25, p. 119]

Though logicism is considered to have failed in its primary goal, it remains highly influential in modern mathematics. “In fact, it was logicism which started mathematical logic in a serious way.” [29, p. 124]⁵⁶

3.4 Intuitionism

Lying in stark contrast to logicism is intuitionism, which sees mathematics, even classical mathematics, very differently. Here, “mathematics should be defined as a mental activity and not as a set of theorems (as was done... [in] logicism).” [29, p. 126] “Intuition determines the soundness and acceptability of ideas, not experience or logic.” [18, p. 235] In fact, intuitionism even questions whether the principles of logic have absolute validity. [17, p. 46] Moreover, to be considered legitimate to intuitionists, the mental activity must be *constructive*. Objects can only be considered to exist if one has engaged in every mental step to formulate *completely* that object, beginning with, for example, the “primordial intuition” for the number 1. [29, p. 125], [17, p. 49] A precise definition of “constructive” is debatable, but, for example, “Kleene saw that he could translate it as ‘compute,’ which he *had* precisely defined.” [17, p. iv]

Intuitionism traces many of its ideas back to Kantian philosophy. [25, Ch. 7, Sec. 2] Additionally, “just as logicism is related to [philosophical] realism, intuitionism is related to the philosophy called ‘conceptualism.’ ” [29, p. 128] Mathematicians such as Descartes and Pascal had advocated for a leading role of human intuition, and the mantle was taken up by such luminaries as Kronecker, Borel, Lebesgue, Poincaré, Baire, and, especially, L. E. J. Brouwer. [18, pp. 230-1, 234]

Many mathematicians consider intuitionism unnecessarily restrictive on the tools available to them. For example, the law of the excluded middle (the principle that either a statement or its negation, but not both, is true) is considered invalid mathematics by many intuitionists. [29, p. 126], [18, p. 237], [25, pp. 173-4], [17, pp. 47-8] The constructive restriction likewise eliminates the validity

⁵⁵In Russell’s formulation, it was the axiom of reducibility which was contentious. “The axiom of reducibility for propositions say [sic] that any proposition of higher type is equivalent to one of first order.” [18, p. 222] Russell and Whitehead admitted that this “axiom has a purely pragmatic justification: it leads to the desired results, and to no others.” [17, p. 45]

⁵⁶For a treatment of mathematical logic, see [30]. More on the logistic school can be found in [18], chapter X; [17], chapter III; [25], chapter 5.

of many existence proofs in mathematics. [18, pp. 238-9] Further, constructive proofs are often so much lengthier and more complex that opponents see the burden as too great. Even more troubling, there are theorems which are true in intuitionistic mathematics but false in classical mathematics.⁵⁷ This does not bother intuitionists: while the logicians sought to ground all of classical mathematics, intuitionists do not consider all of classical mathematics legitimate, and the illegitimate parts need no grounding. Instead, intuitionists have the goal of giving “a valid definition of mathematics and then ‘wait and see’ what mathematics comes out of it,” fully expecting that some of classical mathematics will never be realized this way. [29, p. 126] “Brouwer recognized that intuitionistic mathematics is not a mere restriction of classical mathematics, but is incompatible with it.” [25, p. 184] More generally, “as a substitute for classical mathematics [intuitionism] has turned out to be less powerful, and in many ways more complicated to develop.” [17, pp. 52-3] For reasons such as these, this is a philosophical school that the majority of mathematicians have repudiated. And, as with virtually all philosophical schools, disagreements within intuitionism exist as well. [18, p. 244]

“According to intuitionistic philosophy, all human beings have a primordial intuition for the natural numbers within them... we have an immediate certainty as to what is meant by the number 1...” [29, p. 125] and thus EC1 is manifest in intuitionism. As Leopold Kronecker famously quipped, “God made the integers; all the rest is the work of man” (quoted in [18, p. 232]). Brouwer took mathematical notions like addition and mathematical induction to be intuitively clear, hence foundational, also. [18, p. 235] Even so, intuitionism places mathematics at a higher epistemic level (i.e. closer to ultimate) than do most other philosophies of mathematics. Since mathematics is “identical with the exact part of our thinking [since it is a mental activity]... no science, in particular philosophy or logic, can be a presupposition for mathematics. It would be [fallaciously] circular to apply any philosophical or logical principles as means of proof, since mathematical conceptions are already presupposed in the formation of such principles.”⁵⁸ [Mathematics has] no other source than an intuition, which... [is] immediately clear” (see [17, p. 51], quoting a work from Heyting).

Intuitionism’s influence upon the understanding of mathematical existence, as well as on the related school of constructivism, remain significant today. [18, p. 244], [25, p. 184] Also, “the possibility of an intuitionistic reconstruction of classical mathematics... is not to be ruled out.” [17, p. 53]

3.5 Formalism

As a critic of both the logistic and intuitionist schools, with special vitriol for intuitionism, [18, p. 246] David Hilbert is the primary figure in the formalist school. (That said, many of the basic notions existed before his “Hilbert program” was set in motion around 1910.) The objective of formalism is, as the name suggests, to formalize a given axiomatization using methods which Hilbert called “finitary,” a somewhat subtle notion which we need not attempt to define here. [29, pp. 128, 131], [18, p. 250] The formalization process involves creating a first order language for the axiomatized theory by enumerating a vocabulary: symbols for items like quantifiers, the equality relation, connectives (“not,” “and,” “or,” etc.), variables, and the undefined terms particular to the theory. While this process envelops the theory with rigidity, perhaps constraining free-flowing creativity, it has the advantage of allowing one to study the theory itself via the tools of mathematical inquiry (an undertaking appropriately termed “metamathematics”).⁵⁹ Hilbert’s purpose was to prove *mathematically* that mathematics itself was consistent, [29, pp. 128-30] and even further, to

⁵⁷For example, in intuitionistic mathematics, the statement that “every real-valued function which is defined for all real numbers is continuous” is true. [29, p. 127]

⁵⁸Intuitionists might therefore revise EC2, though not dispose of it.

⁵⁹For a thorough discussion of metamathematics, see [17].

prove the completeness of the mathematical enterprise; that is, that “the axioms... are adequate to establish the correctness or falsity of any meaningful assertion that involves the concepts” of mathematics. [18, p. 158] For Hilbert, these twin ventures would establish “the certitude of mathematical methods” (quoted in [25, p. 158]).

Philosophically, logicism relates to realism, and intuitionism relates to conceptualism, with the philosophical leaning of formalism being nominalism. [29, p. 131] Nominalism’s focus is on the “names” of the objects it treats, as its Latin root word suggests. For a formalist, mathematical objects need not have any existence beyond the name (term, symbol) given to them within the theory. One consequence is that “*anything at all* can play the role of the undefined primitives... so long as the axioms are satisfied.” [25, p. 151]⁶⁰ No intuitive meaning is needed. Hilbert, alluding to Euclid’s inadequate definitions,⁶¹ noted that the “meaning” of the undefined terms is given only in “the *relations* of points, lines, and planes to each other – via the axiomatization. All we can provide is an implicit definition of the terminology” by specifying their properties through the stated axioms. [25, p. 156]⁶² The formal names can then be manipulated according to the rules set forth in the first order language, not unlike the way one plays the game of chess by moving the different pieces according to the game’s rules. [29, p. 131] This link is strong enough to have birthed a subschool referred to as “game formalism.” [25, p. 144] It is interesting to note that Hilbert developed these notions specifically to avoid circularity in mathematical reasoning, and for that he required that foundational axioms and definitions be asserted (in the axiomatization). This suggests an implicit agreement with the current article’s thesis that the rational alternative to foundationalism is circularity.⁶³

The failure of formalism to prove the consistency and completeness of mathematics is now famous. Kurt Gödel’s Incompleteness Theorems of the 1930s comprised the fatal blow. [29, p. 130] By 1930, some corners of mathematics had been proven to be consistent and complete (some were even proved by Gödel himself),⁶⁴ but these results did not extend to the whole of mathematics. His incompleteness theorems state that, “if any formal theory *T* adequate to embrace the theory of whole numbers is consistent, then *T* is incomplete. This means that there is a meaningful... [and] true statement of number theory which is not provable and so is undecidable.” [18, p. 261] Putting the Big Three philosophical schools in its crosshairs, “the consistency of any mathematical system that is extensive enough to embrace even the arithmetic of whole numbers cannot be established by the logical principles adopted by the several foundational schools.” [18, p. 261] In other words, “no axiom system for mathematics as we know it is powerful enough to lead to a proof of its own consistency.” [29, p. 134], [25, p. 167]

⁶⁰In [18], Kline includes a similar discussion of undefined terms in mathematics in the 19th century. The axioms in a deductive system give an “implicit definition” of the terms by telling us how they can be used and what one knows about them. Thus, the undefined terms in a deductive system “can be interpreted to be anything that satisfies the axioms,” introducing “a new level of abstraction” in mathematics; see [18, pp. 191ff.].

⁶¹For example, Euclid defined a “point” as “that which has no parts.” [25, p. 156] For Hilbert, such vagueness in definition does more harm than good. In such a situation, “everything gets lost and becomes vague and tangled and degenerates into a game of hide and seek.” Quoted in [25, p. 156]. Regarding Euclid, it was also observed (before Hilbert) that in his *Elements*, “flaws had arisen because Euclid has been misled by his visual imagery... [H]e had inadvertently assumed certain properties of [points and lines], without stating them as axioms.” [26, p. 243]

⁶²As Hilbert put it: “I have become convinced that the more subtle parts of mathematics... can be treated with certainty only in this way; otherwise one is only going around in a circle.” Quoted in [25, p. 157]. Regarding the development of logic and arithmetic, he advocated for a “partly simultaneous development,” essentially a set of co-foundational principles, in order to avoid “a vicious circle and... paradoxes.” Quoted in [29, p. 111].

⁶³It is even referred to as “the circular definition problem.” [4, p. 122]

⁶⁴Notably, first order predicate calculus. See [18, p. 260].

The implications to the mathematical enterprise were far-reaching and led to further unsettling results: a mathematical proof might have pristine use of logic internally, but there was now no guarantee that mathematics more broadly was logically meaningful at all. [26, p. 244] It is reported that Hilbert became furious when he learned of Gödel’s work, but that fury has given way in the mathematical community either to unease or, more commonly, indifference toward the philosophy of the foundations of mathematics. [26, p. 245], [29, p. 132] A fuller account of these issues (crises!) will be left to the references.⁶⁵

What about the epistemic characteristics’ relationship to formalism? On EC1, formalism begins with an axiomatized theory (a set of axioms and undefined terms), so the items in such a theory are deemed to be supplied beforehand.

Regarding EC2, “one has to talk about the [first order] language L as one object, and while doing this, one is not talking within that safe language L itself, ... [but in] ordinary, everyday language... [In so doing,] there is of course every danger that contradictions, in fact, any kind of error, may slip in.” [29, pp. 130-1]⁶⁶ Hilbert’s solution was the aforementioned insistence on finitary reasoning, a restriction which demonstrates EC2. EC2 is displayed perhaps even more clearly by Gödel’s rebuke of the formalist school. Whereas Gödel proved that a mathematical system cannot show *its own* consistency, at least some of the undecidable statements in a formal system can be shown to be true by *informal* arguments. [18, p. 263] As Gödel put it, “it is necessary to go beyond the framework of what is, in Hilbert’s sense, finitary mathematics if one wants to prove consistency of classical mathematics.”⁶⁷ And again, “the methods which we must trust in the proof [of consistency] must include some which lie outside the collection of the methods formalized in the system.” [17, p. 211] Natural, interesting, and even crucial questions about mathematics must use tools which transcend the discipline. Since such tools evidently exist, Gödel’s theorems substantiate the claim that mathematical knowledge is not ultimate knowledge.

Despite the formalist’s failure in their initial goal, the school has had immense, ongoing impact on the way mathematics is done today. Fields such as metamathematics or proof theory,⁶⁸ as well as “modern mathematical logic and its various offshoots, such as model theory, recursive function theory, etc.” [29, p. 130] owe an immeasurable debt to formalism. The emphasis on stripping objects of their “unnecessary” features (whatever is not pertinent to the logical structure), and the recognition that physical intuition can be misleading, help explain why “so much of the abstraction of early 20th century mathematics stemmed from Hilbert’s viewpoint.” [26, p. 243], [17, pp. 54-5]

3.6 Other Philosophies of Mathematics

For each of the philosophical schools given above, there remain adherents of (modified versions of) them,⁶⁹ but they don’t exhaust mathematicians’ and philosophers’ present-day views. We will make passing mention of some other selected philosophies of mathematics, some of which overlap the above (and each other). As described by theologian and mathematician Vern Poythress, “Platonism says that numbers and mathematics belong to a realm of abstract ideas, a realm that exists before mathematicians begin to study it.” [21, p. 153] An alternative view, empiricism, seeks to found

⁶⁵See [18], chapter XII; [25], sections 6.4, 8.1; [17], section 42. The notion of (un)certainly in mathematics is a broader theme of many of these works; cf. [4, pp. 212-6].

⁶⁶Notice the object language/metalanguage distinction.

⁶⁷Quoted in [25, p. 167].

⁶⁸These terms are introduced in [17, p. 55].

⁶⁹Shapiro provides examples throughout [25].

mathematics in the experiences of the senses. [21, pp. 154-5] Predicativism “accepts the natural numbers as given. . . either by our intuition or by a Platonic realm or by both,” and then seeks to rebuild as much of mathematics as possible while avoiding impredicative definitions. [21, p. 159]⁷⁰ These three views each begin mathematics proper with (at least) the natural numbers, and thus all exhibit EC1.⁷¹ William van Orman Quine’s philosophical naturalism advocates for science as the guide for what constitutes acceptable mathematics, displaying EC1 by assuming results of scientific inquiry before mathematics can commence. [21, p. 160]

Shapiro’s treatment of some contemporary philosophies of mathematics tells a similar story. [25, Part IV] The core difference between the philosophers and their associated schools (e.g. realism vs. anti-realism) is the ontology of mathematical objects. But in each case, *something* is provided to mathematics (frequently, the natural numbers) before it can begin. EC1 thus appears in each. As two examples: in structuralism, the structures themselves which mathematicians study are the givens. [25, pp. 258ff.] In Hartry Field’s fictionalism, specifically his realization of Newtonian gravitational theory, “points” and “regions” are (among) the givens. [25, pp. 229ff.]

It is worth noting that Poythress’s *Redeeming Mathematics* proposes (with remarkable succinctness and in lay terminology) an alternative philosophy of mathematics which could, in some sense, be considered an exception among the philosophies of mathematics. He grounds the tools of mathematics in the Christian God, who is ultimate in the Christian worldview. Poythress criticizes the “reductionisms” of the aforementioned philosophies of mathematics⁷² and lays out a more holistic approach which might require refinement of the ECs.⁷³

3.7 (The Lack of) The Epistemic Characteristics in Theology and Philosophy

If mathematics has been able to employ EC1 and EC2 with great success (as have other academic disciplines), can fields such as theology, philosophy, religion, or worldview studies emulate it? These latter fields are of a fundamentally different character to the extent that duplicating the pattern is not possible. Many, though not all, of their questions are inherently about ultimates (recall we refer to these as “ultimate questions”). For the ultimate questions in these fields (or in whatever field one places ultimate questions), there is no way to exhibit both EC1 and EC2.

Most significant for the present discussion is in regards to EC2.⁷⁴ Limiting the scope of ultimate questions is not possible, for to do so would be to render them non-ultimate. In this case, an object language/metalinguage distinction is impossible; that is, the object language also includes the metalinguage, and thus the two are equal (recall that the reverse inclusion holds for mathematics and any other discipline in which the distinction is utilized).⁷⁵ There is not, and indeed cannot

⁷⁰For a thorough, favorable treatment of predicativism, see [27]. Information available at <https://philpapers.org/rec/STOADO-2>. Accessed July 12, 2022.

⁷¹Their ontological claims about the nature of mathematical objects differ, but that is immaterial to our discussion.

⁷²“Each of these [philosophies] has a preferred starting point. . . [which] becomes the preferred platform for explaining everything else in mathematics.” [21, p. 151] Note the close connection of this observation with EC1.

⁷³The comments in [21, pp. 104-5] are prime examples. In Poythress’ view, God, and hence ultimate knowledge, is inextricably linked to mathematics. The pivotal feature is then the internal coherence of the system, not which “starting point” is chosen.

⁷⁴To briefly address EC1: while different philosophical or theological schools of thought disagree on the starting points of EC1, each school presumably has some such foundational set. For example, see [13] for a comparison of starting points for generalists and particularists, and [6] for a thorough treatment of the deep epistemological issues faced in determining the elements which comprise EC1 (note the assumption that such elements exist).

⁷⁵This is illustrated in simple terms in [29]: “when the logicians tell us what they mean by a logical proposition, . . . they use philosophical and not mathematical language. They have to use philosophical language for that purpose

be (for our human agent H), a greater context in which to discuss them. Since these questions arise naturally in theology and philosophy, there is no way to “get outside” of those disciplines when investigating such questions; one cannot attempt to answer ultimate questions in theology without theologizing. There is nothing in the line of reasoning that falls outside of the umbrella of, or transcends, the discipline. As Van Til put it, “[w]e must go round and round a thing to see of its dimensions and to know more about it, in general, unless we are larger than that which we are investigating,” [32, p. 24], [11, p. 257] and in the case of ultimate questions, we are by definition not “larger” than they are. In this sense, theology and philosophy are intellectually all-encompassing. Evidently it is this observation, in part, which motivates worldview scholar Nancy Pearcey to refer to her Christian worldview as “Total Truth.” [19] EC2 cannot be exhibited for ultimate questions.

Ultimates thus have a “meta” relationship to other statements of knowledge. They don’t simply lead to or propositionally support other statements; rather, they form the framework in which an intellectual discipline is done. Intelligibility itself, a precondition of any intellectual pursuit, is determined by the ultimates.⁷⁶ Since it is fields such as philosophy and theology which handle ultimate questions, the ECs cannot both be always exhibited therein.⁷⁷ One particular conclusion, in view of (FC), is that if circular support is acceptable anywhere, it would be so in fields such as theology and philosophy by virtue of their inclusion of ultimate questions.⁷⁸

3.8 Analogies

As an epilogue to the primary thrust of the article, we will make brief mention of a potpourri of analogies, both within and without mathematics.

First, the epistemic distinction between ultimates and non-ultimates bears a resemblance to the distinction in properties between the boundary and interior of certain mathematical objects, such as a manifold. A planar disk (a circle with all of the interior points) is an example. The interior has different properties (most notably, it is 2-dimensional) than the boundary (the 1-dimensional edge of the disk). As ultimates are at the “boundary” of human knowledge, it is not surprising that their epistemological properties differ qualitatively from the “interior,” non-ultimate knowledge.

Consider another mathematical analogy: a local/global distinction. A local property is satisfied only at or within a neighborhood of a particular point/object (for example, any open neighborhood of a point on a 2-sphere smaller than the 2-sphere itself, is isomorphic to the real plane), whereas a global property in mathematics refers to the set of all points (objects) under consideration (the 2-sphere, taken as an entire object, is *not* isomorphic to the real plane). The realm of epistemological influence of a non-ultimate proposition can be thought of as local, whereas an ultimate has global influence.

Many of the above concepts are not unique to mathematics, so it can be illuminating to see an “everyday” example. One appears in dictionaries: a word cannot be defined without the use of

since mathematics simply cannot handle definitions of so wide a scope.” [29, p. 124] Similar comments for intuitionism can be found in [29, pp. 127-8].

⁷⁶In theology in particular, ultimates are often referred to as presuppositions. On this topic, Frame says, “The ‘pre’ [in presupposition] should be understood mainly as an indicator of eminence. . . not temporal priority.” [11, p. xxxii]

⁷⁷Note that some of the questions which are asked in theology, philosophy, and related disciplines *can* be handled in such a way that both EC1 and EC2 can be exhibited. The main point is that, unlike mathematics, *not all* questions in theology or philosophy can exhibit *both* epistemic characteristics (cf. the discussion of EC1 in subsection 4.3).

⁷⁸Recall that it is precisely the exhibition of the ECs which allows mathematics to categorically reject circular reasoning; see Section 3.1, and compare the nuance in the meaning of “circularity” in Section 2.5.

other words. Since there are only finitely many words, this means that, eventually, some word has to be left either undefined (deemed so foundational as to be left out of the dictionary), or there is a sequence of words which are used in each other's definition (which is circularity; examples can indeed be found in modern dictionaries). (FC) applies to definitions as well as propositions.

The object language/metalinguage distinction can also be seen outside of mathematics. Consider a theatrical script: it employs stage directions (metalinguage) in addition to the lines of the show (object language). Similarly, some government or legal documentation says, "This page intentionally left blank." Those words themselves cause the page not to be blank, but they are considered part of the document's metalinguage rather than the document's content (object language).

4 Follow-up Questions

4.1 Does this article itself use foundational or circular reasoning?

This article uses foundationalism in its propositional progression. We assumed several notions at the outset, such as the use of logic and rationality in our truth-seeking. These were taken as ultimate (without further defense) *for this article*. This particular assumption illustrates the "meta" relationship that ultimates have with respect to other statements: had we not assumed anything regarding logic and rationality, by what standards would we have proceeded to answer any questions or make any claims? We could not reason about reason without using reason! The rationality assumption established a precondition for the remainder of the article.

4.2 Could different individuals choose different ultimates, resulting in diverse but self-consistent worldviews that are mutually exclusive?

In mathematics, this circumstance (a panoply of axiomatic theories) was essentially endorsed by David Hilbert. "Literally, Hilbert claimed that if a collection of axioms is consistent, then they are true and the things the axioms speak of exist." [25, p. 156] Axioms in mathematics can be arbitrarily chosen so long as they are consistent - a version of coherence epistemology in mathematics. In fact, it is this line of reasoning which allows for the parallel⁷⁹ existence of multiple geometries (Euclidean and non-Euclidean) by asserting different axioms. Mathematicians are comfortable with this state of affairs, so can we apply the same reasoning to worldviews, inclusive of ultimates?

A priori an affirmative answer to this question seems possible, and by our assumptions it seems to be so *in this article*. However, let us momentarily suspend our moratorium on evaluating theories of warrant. It is the author's belief that, in practice, all truth-conscious individuals operate, implicitly or explicitly, under some shared presuppositions⁸⁰ which eliminate some worldviews from rational contention; in particular, the possibility of global mutual exclusivity is removed. These presuppositions include shared ultimate standards (e.g. for rationality and morality) by which everything else is evaluated. The commonality between persons' presuppositions assuages fears that two individuals could have perfectly consistent and veracious worldview theories and yet have nothing on which they agree; in reality, the two don't actually inhabit disjoint systems.

For example, any theological or philosophical system that includes the principle of noncontradiction,

⁷⁹No pun intended.

⁸⁰Presuppositions in this context are ultimates, but as the former term is used more commonly when discussing worldviews, we will adopt it for this follow-up question.

as is assumed here, requires internal consistency. A necessary condition for all other ultimate criteria would be that they not refute anything already in the system of ultimates. This condition alone can indeed rule out many worldviews which might otherwise seem to be viable a priori. Details and a more thorough treatment of how to choose the “right” presuppositions, or what it might mean to be the right presuppositions, would be the topic of another work.

4.3 Does God need foundationalism, circularity, both, or neither?

One would first have to define what is meant by “God.” Specifically in Christianity and the other Abrahamic faiths, the conception of God is traditionally as an *ultimate* God (in particular, He is or determines the ultimate standard for rationality and morality). In other words, by definition, there is no higher standard by which God can be judged. His word can be taken as foundational, and His authority (and anything in His nature) as self-verifying; the notion of circularity in *authoritative* support seems to be consistent with God’s nature in these faiths. This concept is part of what theologians refer to as divine aseity,⁸¹ a concept hinted at in Hebrews 6:13;⁸² God has no standard to appeal to above Himself! Thus, anything that is known to be part of God’s nature need not (indeed cannot) be further justified.⁸³ We must be careful, though, because it is quite possible that the assumptions of the current article do not all necessarily apply to God’s “divine knowledge.” In particular, the finiteness and time-bound nature of humanity are evidently not limitations for God. Thus, we do not claim that the argumentation of the current article applies to divine knowledge, nor is a claim made excluding the possibility of unimagined spiritual or supernatural realities about which humans have never received revelation nor have other means of gaining knowledge.

One final comment on the relationship between Christian faith and human knowledge is in order. The Bible, especially the wisdom literature (such as the book of Proverbs), affirms that human knowledge, mathematics included, finds its final foundation in (reverence for) God. Job 28:28; Psalm 111:10; Proverbs 1:7, 2:6, 3:5-6, 8:22, 9:10; and Colossians 2:2-3 are examples.⁸⁴ At risk of over-enthusiastic hermeneutics, verses like these seem to suggest that, in a Christian worldview, EC1 will eventually be exhibited by all human knowledge, with the starting point being God’s own

⁸¹“Something ‘exists necessarily’ if it cannot fail to exist. Something ‘exists contingently’ if it can fail to exist. Whether or not something contingent exists depends on factors outside itself. Therefore, ‘contingent existence’ (such as ours) is ‘dependent existence.’ Necessary existence is aseity or self-existence.” [11, p. 116n36] For more on divine aseity, see [11, pp. 265-78]. For more on the relationship between divine aseity and mathematics, see [21, pp. 63ff.]

⁸²“For when God made a promise to Abraham, since he had no one greater by whom to swear, he swore by himself. . .” [15]

⁸³The sticking point is knowing which characteristics are part of His nature. There is not agreement there, even among devoted practitioners of a particular faith tradition. But the point is that if the qualities of His nature are assumed to be known, then those qualities ground completely the authoritative justification. A common objection is that grounding a standard like rationality or morality in God’s nature is “kicking the can down the road,” suggesting that there is another step of reasoning beyond God’s nature which requires explanation. Such an objection violates the conception of God as being Himself ultimate (by definition), or it seems to make the implicit assumption that finite termination of the support sequence is impossible. Either is problematic.

⁸⁴They read as follows. “And he said to man, ‘Behold, the fear of the Lord, that is wisdom, and to turn away from evil is understanding’ ” (Job 28:28). “The fear of the Lord is the beginning of wisdom; all those who practice it have a good understanding. His praise endures forever!” (Psalm 111:10). “The fear of the Lord is the beginning of knowledge; fools despise wisdom and instruction” (Proverbs 1:7). “For the Lord gives wisdom; from his mouth come knowledge and understanding” (Proverbs 2:6). “Trust in the Lord with all your heart, and do not lean on your own understanding. In all your ways acknowledge him, and he will make straight your paths” (Proverbs 3:5-6). “[Wisdom says] ‘The Lord possessed me at the beginning of his work, the first of his acts of old’ ” (Proverbs 8:22). “The fear of the Lord is the beginning of wisdom, and the knowledge of the Holy One is insight” (Proverbs 9:10). “. . . that their hearts may be encouraged, being knit together in love, to reach all the riches of full assurance of understanding and the knowledge of God’s mystery, which is Christ, in whom are hidden all the treasures of wisdom and knowledge” (Colossians 2:2-3). Scripture quotations are from [15].

self-authenticating nature. In the author's view, the philosophical skepticism of Agrippa and the Pyrrhonists (and many philosophers of our day) is indeed correct that human knowledge would be hopeless without a "beginning;" and moreover, that that beginning must be external to humanity in order to be ultimately trustworthy. By virtue of the doctrines of historic Christian belief, the nature of God is sufficient for such a beginning.

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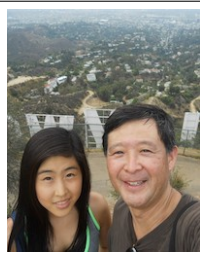
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Mourning of a Mathematician

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Saburo (“Sab”) Matsumoto came to the U.S. at age 15. After earning his Ph.D. at Michigan, Sab worked as a researcher in geometric topology in Australia and Japan before returning to the U.S. Having taught at two Christian universities, he now teaches at a community college, recently compiling a free OER textbook on Liberal Arts Mathematics. He has been married to his artist wife Sonah for 34 years. Sab and his daughter Erika are pictured to the left.

Abstract

In 2018, my youngest child Erika passed away at age 21. As a Christian, as a father, and as a mathematician, I struggled with many faith-related questions concerning her death while mourning and grieving with the rest of my family and friends. Where was God when Erika took her last breath? He could have stopped her from dying, but why didn’t He? Where was He when she prayed? Did He not hear her (and our) cries and prayers? Although the sudden loss was devastating and emotional pain has been excruciating, God has taught me many lessons and even given me a sense of “peace that passes through all understanding.” In particular, some mathematical concepts played a major role in healing the wounds. In this paper I will share some of these lessons and thoughts that have helped me stay close to Him.

1 Introduction

It was Saturday, January 13, 2018, when I got that unexpected call. The caller ID said, “Los Angeles County Coroner’s Office.” I was told that my daughter was dead.

Erika was the youngest of my three healthy, happy children. An active member of the campus ministry Cru, she was a college senior, majoring in culinary arts and hospitality management at Cal Poly Pomona, and she had just finished her internship at a Michelin 3-star restaurant in San Francisco, working for the famous French chef Dominique Crenn. But in the fall of 2017, her best friend, a UCLA senior majoring in pure mathematics, committed suicide following years of depression and eating disorder. Then, just a few months later, Erika also took her own life.

The loss of a family member is always difficult, but losing a child is perhaps most painful. Where is God when we suffer? What was God doing when Erika prayed, earnestly, for her best friend, and for herself? If God is able, why did He not stop her and preserve her life? What was God thinking—is it a good idea to take away a 21-year-old precious child ready to begin her career? My family and I struggled to find answers while experiencing unspeakable grief and sadness. My surviving son and daughter, my wife, and I each mourn and grieve in our own, unique ways.

Perhaps many are familiar with five, seven, or even twelve stages of grief.¹ But the reality is that those stages do not just end or terminate. The process of mourning is, in our mathematical language, more like an infinite sequence (or an infinite series if accumulated grief is considered). But it does not have to be a divergent one. While our mourning may continue, God can still give us a “peace that passes through all understanding.”

¹For example, Elisabeth Kubler-Ross’s “five stages of grief” in her well-known book *On Death and Dying*, published in 1969.

And He has.

In this paper, I will share some comforting and glorious ideas, thoughts, and lessons I have learned through this truly excruciating trial. In particular, some mathematical concepts and models have given me great insight to further appreciate the goodness and the greatness of God, the same God whom Erika loved and worshipped on earth, with whom she now lives in eternity.

2 Lessons from Mathematics

2.1 The Incompleteness Theorems

In 1931, Kurt Gödel proved his celebrated Incompleteness Theorems, which have had far-reaching implications not only in mathematics and logic but also in philosophy. Roughly speaking, the first result states that there are statements in arithmetic that are true but can never be proved. In other words, some truths must be taken “by faith” and not “by sight.” Furthermore, his second result states that no axiomatic system can prove its own consistency. Said differently, no axiomatic system can, within itself, guarantee that the system contains no contradiction; consistency, therefore, must be taken “by faith” as well. The French mathematician Rene Thom is quoted as saying that we mathematicians “have only an incomplete and fragmentary vision of this world of Ideas” [4, p. 299]. The English mathematician and cosmologist John D. Barrow famously said that “Gödel has taught us that not only is mathematics a religion but it is the only religion able to prove itself to be one” [2, p. 142].

The Incompleteness Theorems painfully remind us of our limitations—not only of our knowledge itself but also of the “knowability” of truths. We are clearly aware of this fact experientially. When we ask why certain things happen in life, when we want to know what God’s plans are, we may or may not find the answer. Many events simply do not make sense. Then we wonder if God is really watching—or worse, if God even exists. But of course our Sovereign and Omniscient Creator is orchestrating everything in this universe, all the time, as a master conductor. We are told that no sparrows fall to the ground without His knowledge (Matthew 10:29) and that He even decides what appears to us as random outcomes of casting lots (Proverbs 16:33). The author of Hebrews reminds us how important it is to believe this fact as all of us are to “believe that He exists, and that He proves to be One who rewards those who seek Him” (Hebrews 11:6, NASB).²

As mathematicians we are trained to prove propositions or give counterexamples. You may be inclined, especially during tough times, to ask, “God, why is this happening? Prove to me that this tragic event is a good thing in your overall plan. Explain how ‘all things work together for good’ as Romans 8:28 says.” But often, God remains silent.

Sure, for certain events, there may be a simple explanation or proof. For some other events, however, while an explanation or a proof may exist, it could be beyond our comprehension. Here is an example. As a 7th grader, I learned the formula for the volume of a circular cylinder, $\frac{1}{3}\pi r^2 h$. For many years I wondered where the $\frac{1}{3}$ comes from. Then, in my calculus class, years later, I saw the following question: “Find the volume of the solid obtained by rotating the function $f(x) = \frac{r}{h}x$ on $[0, h]$ about the x -axis.” Clearly, the solid of revolution in question here is the circular cone whose radius and height are r and h , respectively.

²Unless otherwise stated, all Scripture quotations in this paper are from the New American Standard Bible (NASB, 2020).

The answer, using the disc method, looks like this:

$$V = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx = \pi \frac{r^2}{h^2} \frac{1}{3} (h^3 - 0) = \frac{1}{3} \pi r^2 h.$$

Now, calculus students could easily see that the “1/3” came from integrating the square function; however, it is very unlikely that a 7th grader can understand this fact. In the same way, some of God’s truths may look mysterious and incomprehensible to us humans, with our very limited knowledge and understanding. But it does not mean they are false. Perhaps some day we will understand why certain things happened in our brief life on earth just as clearly as we understand why there is a “1/3” in this volume formula. We must then stay patient and admit that there are things we do not understand. “For now we see in a mirror dimly, but then face to face; now I know in part, but then I will know fully, just as I also have been fully known” (I Corinthians 13:12).

Furthermore, yet for some other events, it is quite possible that no proof may exist, comprehensive to us or otherwise. The Incompleteness Theorems suggest that possibility. After all, God is not obligated to give us explanations. In fact, He does not even require that we understand; rather, He requires that we trust. Kurt Gödel probably never imagined this far-reaching, Christian implication of his results—the fact that some truths require faith and not proof.

2.2 Dimensions

In 1884, the English writer Edwin Abbott Abbot published *Flatland: A Romance of Many Dimensions*, a satire on the British society. In this well-known story, all the residents are 1- and 2-dimensional figures (line segments and polygons) in their planar universe, the “Flatland.” Then, one day, the “Sphere” comes down into their 2-dimensional world and passes through the plane, an event totally incomprehensible to the inhabitants of Flatland. This story has been made into a feature film, along with its sequels including *Flatland 2: Sphereland*.

When the authors of the Bible describe heaven, or eternity in general, it is possible that the heavenly realities are so incomprehensible to us the inhabitants of a 3-dimensional universe that no sufficiently accurate description can be written in human language. Ezekiel marveled at God’s revelation frequently, and Daniel even confesses that he “could not understand” what he had seen (Daniel 12:8). Perhaps our understanding of heaven is similar to what the residents of Flatland experienced when they saw the Sphere coming down to their land as a point first, then as a circle growing and shrinking, until it became a point again before disappearing. Of course, as believers we hold that the Bible teaches us exactly what we need to know, albeit with our severely limited understanding.

This “human disability to understand God’s reality” is not limited to heaven and eternity. Perhaps other mysteries like election versus freewill, the Trinity, the infinity of God, and creation *ex nihilo* would all make sense with additional “dimensions” of rationality. In particular, God’s omnipresence and proximity to each of us could be thought of as humans living on the plane while He occupies another entire plane—a separate one—infinitesimally close to ours. Here is a more specific way to look at this: imagine we live on the xy -plane $\{(x, y, z) | z = 0\}$ while His world (or at least a part thereof) is like the horizontal plane $\{(x, y, z) | z = \epsilon\}$ for some $\epsilon > 0$. We would not be able to see Him physically from any point of our human universe (the first plane), but He is there, always very close to each of us (the second plane).

This idea could also illustrate how Satan can travel between God's presence and the human universe, where Job lived, almost instantaneously (Job 1, 2). If that heavenly realm is simply ϵ away but outside of our dimensions, we would never see or know where it is while it is always ever so close to us.

I also wonder if the "realm of the dead" could be similarly illustrated. We often describe the spirit of the dead as "going up" to heaven or someone "ascending the stairway to heaven," but up where? Note that what one refers to as the "up" direction on earth depends on where the person is located on the sphere, and it could point to any direction in the 3-dimensional universe. After the resurrection, Jesus seemed to come in and out of the realm of the living, where we are, as He appeared and disappeared on many occasions. Locked doors were not a problem, either. These can be easily explained by a model like the ϵ -plane described above, so long as there is a way to move from $z = 0$ to $z = \epsilon$. Sometimes I even wonder if this can illustrate the appearance of Erika (and other loved ones who have deceased) in my dreams. Erika often appears in a crystal-clear way in my dreams, where I can see every detail.

Now, I am fully aware that these thoughts (or just my imagination) cannot be "proved," but illustrative models do not require proof. Higher dimensions simply point to the possibility that the beings in the "spiritual realm" can be very "close" (depending on the definition of the metric used, of course) to us in the "physical realm." It is, therefore, possible that Jesus, the angels, and other heavenly hosts, and our deceased loved ones are all ever so close to us while we may not be fully or visually aware of their proximity.

2.3 Non-Euclidean Geometry

What is heaven like? This question often came to my mind when remembering Erika. While we do have *some* information based on the Bible (especially in the book of Revelation), perhaps for us to try to understand the heavenly realm is like for a Flatlander to try to understand 3-dimensional beings. Remember, simple words like "up" and "down" make absolutely no sense to a Flatlander.

Here is another example of a world so different from ours that it is hard to comprehend: non-Euclidean geometry, a hyperbolic space in particular. As many mathematicians struggled in the 18th century to prove or disprove Euclid's Parallel Axiom, Saccheri got so frustrated with the possible outcomes when Euclid's Fifth Axiom is negated that he just called them "repugnant to the nature of the straight line" (to his credit, without claiming them to be contradictions). The very idea that a rectangle (a quadrilateral with four right angles) cannot even exist in this strange universe was just ridiculous to him and others.

One should remember that non-Euclidean geometry is just as consistent as Euclidean geometry. In other words, one could theoretically live in a non-Euclidean world and experience no logical contradictions. As a matter of fact, we do, in a sense, because we live on a giant sphere. Immanuel Kant could not possibly imagine a non-Euclidean universe and dismissed the whole idea, but it is truly ironic that he denied its possibility while living on a positively curved space called the earth. After all, if one considers great circles (geodesics) as "lines" on this sphere, no triangle has the property that its angle sum is 180 degrees, and many other Euclidean theorems would not hold.

In hyperbolic geometry, where the curvature is negative, it is well-known that the angle sum of any triangle is less than 180 degrees, the area of a triangle is actually determined by the angle sum, and no triangles are similar unless they are congruent. Additionally, there are no rectangles while

there are regular pentagons, hexagons, etc., where each angle is a right angle.

When we try to comprehend heaven, maybe it is like those of us living in a Euclidean world trying to comprehend a hyperbolic world, where a “curve” may look “straight” (and vice-versa) and where the residents may dismiss crazy ideas such as rectangles and similar triangles as “absurd,” “repugnant,” and “illogical.” I have a feeling that real heaven is much more glorious than we can ever think, beyond all imagination. On this side of eternity, we are severely limited by our own faculties and senses, biased with our own earthly experience, unable to have heavenly visions accurately.

After reading Erika’s journals, filled with spiritual insights, earnest prayers, and honest observations, there is no doubt that she is in heaven, in the presence of the Lord. An everlasting world? I cannot comprehend it, but I look forward to experiencing it myself.

2.4 The Number Line

The concept of “continuum” can give another illustration of how a “transcendent” God can live or remain among us, close to us, and in us. The set of real numbers, as represented by the number line, is such an intuitive idea yet an extremely difficult one to define. In high school mathematics we often avoid the technicality of a precise definition and give an explanation like this:

Rational numbers are fractions of integers. But there are a lot of real numbers that are not rational, like π and e . Those are called irrational numbers. Rational and irrational numbers together make up the set of real numbers, or the number line.

Many of us are guilty of using circular reasoning when we “define” irrational numbers as those real numbers that are not rational and real numbers as rational or irrational numbers. Of course, one can use Dedekind’s cuts or equivalence classes of Cauchy sequences (both are ways to define real numbers), but as the history of mathematics suggests, it took many centuries for humans to figure out exactly how to define real numbers.

The important point here (no pun intended) is that the structure known today as \mathbb{R} , the set of real numbers, is extremely complex (again, no pun intended). If one begins, as in set theory, with the empty set \emptyset as the definition of 0, followed by $\{0\}$ as the definition of 1, and so on, it is simple to construct the set of natural numbers; it is also easy to extend this notion to define integers and then the set of rational numbers \mathbb{Q} . At this point, because \mathbb{Q} is dense on the number line, one would think the task of filling the line is almost complete. In fact, that is what Pythagoras may have believed, at least until $\sqrt{2}$ was proved to be irrational. Remember, though, that this was some two and a half millennia before Dedekind and Cauchy!

Now, to the dense set \mathbb{Q} , we can even add $\sqrt{2}$ and all other irrational numbers obtained as solutions of polynomials with integer coefficients. These are known as *algebraic* numbers. There are a lot of them (infinitely many, of course). We have just made the set \mathbb{Q} “denser” (maybe). But then one hits a brick wall. There is no algebraic way to get numbers like π , e , or any rational multiple or sum of these numbers, which are clearly real, but not algebraic. We know these numbers as *transcendental* numbers, and it turns out that there are more of these than algebraic numbers.³

³Measure theory suggests that a countable set, like \mathbb{Q} , is so much smaller than the continuum (like \mathbb{R}) that the probability that one picks a rational number when randomly throwing a dart at the number line is 0. This shocks most undergraduate mathematics students.

Now, with these two “levels” of irrational numbers included, the set \mathbb{R} is finally complete.

What is the lesson from this historical development of numbers? Just for the sake of this metaphor, enter the imaginary (again, no pun intended) world of the real line, where we humans are rational numbers, infinitely many and dense, i.e., within any “neighborhood,” there are as many of them as you want to find. We may appear “complete” and filling the entire line. We may be led to think that we are the whole world. But the fact is that there are a bunch of holes. In fact, there are “more” holes than points filled because there are uncountably many irrational numbers, none of which is even “knowable” from the world of rational numbers. They are beyond reach, thus the term “transcendental.”

Theologians speak of the “transcendence of God” as well as the “immanence of God.” The former has to do with God being outside of human understanding and experience whereas the latter indicates His proximity and intimacy with us. The real line then can be thought of as a simple yet stunningly elegant illustration of these two seemingly contradictory ideas.

This transcendence and immanence of God could explain, in part, how Jesus appeared and disappeared, sometimes recognizable and sometimes not, after His resurrection. The two disciples on their way to Emmaus were met by Jesus and walked side by side with Him, not recognizing who He is (Luke 24). But then when He broke bread, they “saw” Him among us, much like finding irrational numbers between two rational ones. But then He simply disappeared, as if to return to the realm of the transcendental. He is, however, forever with us, just as He promised, “Behold, I am with you always, to the end of the age” (Matthew 28: 20). He is, after all, our “Emmanuel.”

During difficult times, many believers ask, “Where is God?” as I did when my daughter passed away. But He is here, with us and in us, “dense” in His presence, far more “present” than we could imagine, although at times He seems so “transcendent” and unreachable from our standpoint. It is by faith we know that “what is seen has not been made out of things that are visible” (Hebrews 11:3).

3 Lessons from the Bible

Needless to say, the Bible has a lot to say about suffering, tribulation, trials, pain, death, and faith. Hundreds of verses can be mentioned, explained, and quoted here. However, I will focus on just a few passages that I, from a mathematical perspective, have found especially helpful during my healing period.

3.1 Philippians 4:7

This verse states, “And the peace of God, which surpasses all comprehension, will guard your hearts and minds in Christ Jesus.” In the New International Version, this peace “transcends” all understanding. Just as those “transcendental” numbers cannot be obtained algebraically from integers, the way to attain this divine peace “transcends” any human logic. Even in prison, Paul thanks the Philippian believers for their gifts, affirms his contentment, tells them to “rejoice (and again rejoice),” and urges them not to be “anxious” but to “let their requests be made known to God.” Then, he adds, the peace of God would guard/keep their hearts and minds in Christ Jesus. What kind of peace is it? It is the type of peace that “surpasses (transcends) all understanding.”

Mathematicians may not like such an expression because we highly value (and some may even worship) understanding. What could possibly surpass *all* of our understanding?

Again, one may recall that underlying message of the Incompleteness Theorems. Some true propositions are beyond proof; some truths are beyond any of our understanding. The type of peace God promises here, once we stop being anxious, goes far beyond anyone's understanding.⁴

3.2 The Book of Job

The book of Job is probably the most relevant book of the Bible when considering human suffering and God's sovereignty. Job, described as "blameless, upright, fearing God, and turning away from evil" (Job 1:1), was blessed with ten children and had become extremely wealthy. Then, suddenly, he lost all ten children, his possessions, and his health.

Job seems to have known quite well that God gives us both blessings and sufferings: "The LORD gave and the LORD has taken away. Blessed be the name of the LORD" (Job 1:21).⁵ Yet, Job 3:11 indicates that he wished to have never been born. He writes, "Why did I not die at birth, come out of the womb and pass away?" In fact, Job eventually shares his frustration and sorrow, with his heart wide-open, in talking to God, stating "I cry out to You for help, but You do not answer me; I stand up, and You turn Your attention against me. You have become cruel to me; with the might of Your hand You persecute me" (30:20, 21).

The last few chapters, however, give us a rich and timeless lesson. When God broke His silence and finally spoke, Job realized that he did not need the answers he thought he wanted. Job asked "Why?" (as so many of us do), but God never directly answered that question. Instead, God began with His own question in Job 38:4, "Where were you when I laid the foundation of the earth?" This is followed by a series of powerful questions underscoring His sovereignty, eternality, immeasurability, and omnipotence. Job then answers, "I am insignificant" (40:4). Someone paraphrased this as "I am not in the same league with God." But God was not done yet, asking Job to "tighten the belt... like a man" (40:7). After many more divine questions, Job clearly understood his role. He said, "Therefore I retract, and I repent, sitting on dust and ashes" (42:6). His role was not to ask God why, but simply to trust.⁶

I once imagined that, were Job a mathematician living today, God may have showered him with such questions as these:

- Were you there when I established the foundation of the world of mathematics, merely with the empty set?

⁴Sometimes people wonder how I can be laughing, joking around, or enjoying vacations after losing my child. True, some people never recover from such a devastating loss. My only answer is this "peace of God." Yes, I do have that "incomprehensible tranquility." No, I cannot explain or understand it. But I thank the Lord for allowing me to have it.

⁵Erika left many journals behind as well as a Bible filled with notes. After her death, we discovered what she had noted, in tiny writing, right next to Job 1:21. Her note reads, "*Dang, that's faith.*" This verse is now engraved on her tombstone.

⁶On a personal note, a few months after Erika's death, I was in Niagara Falls, N.Y., to attend a professional conference. There, strolling right along the world-famous waterfalls, I listened to the last few chapters of Job on my Bible app. Question after question, God makes it clear to Job that God is not man and Job is not God. The powerful thundering sounds of the waterfalls helped me realize the magnificence of a God who created all things. Yes, "I am insignificant" in contrast to the Almighty.

- Who decided that trigonometric and hyperbolic functions are intricately related and there is a hidden transcendental number that can define all of these functions?
- Did you know that from the empty set you can create the infinite set of natural numbers, and then integers, rational numbers, and all reals, complex numbers, and all of \mathbb{R}^n ? Can you create everything out of nothing, as I can?
- Why did algebraists call some groups simple when they are notoriously complicated?
- Why did it take mankind so long—centuries—to figure out the limits of logic and the Incompleteness Theorems?
- Did you know, as I did, that Cantor was doomed when he tried to “solve” the Continuum Hypothesis?

The critical lesson here is this: God does not necessarily want us to understand; He wants us to trust Him. This is similar to an implication of the Incompleteness Theorems: some truths cannot be proved. We must admit our limitations—we are finite, bounded, and incurably depraved beings.

We may think “not knowing why” is the problem. It turns out that “knowing God” solves that problem.

3.3 Book of Ecclesiastes

Like the book of Job, Ecclesiastes is part of the “Wisdom Literature” section of the Old Testament. This remarkable book is arguably the most revealing, poignant, and vivid description of life, its transitory nature, its meaningfulness, and God’s sovereignty.

It is no surprise that this word “vanity” (*havel* in Hebrew) is one of the key words of the book, and one can easily find dozens of explanations or definitions. David Gibson summarizes various aspects of this word with terms such as “short, temporary, vanishes quickly, fleeting, and elusive”[3, pp. 20—21]. William Barrick accurately points out that, while this term can mean (particularly when associated with idols) worthless, useless, or mere breath, none of the texts in Ecclesiastes claims that life is “totally empty and meaningless”[1, p. 2].

How long is this life under the sun, compared to eternity placed in our hearts? In the language of mathematics, if one is to live x years under the sun, for each non-negative real number $x \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{x}{n} = 0.$$

However, this is not to say that life is meaningless, doomed, or empty. God gives us happiness and sadness. There is a time to laugh and a time to mourn. Job was clearly aware of this fact when he faced his challenges. Life is short and often hard. But the whole message of Ecclesiastes, written by the wisest man in history, is to enjoy God in healthy fear of God. The chief end of man is, indeed, to glorify God and to enjoy Him forever.

Another aspect of the word “vanity” involves that which cannot be understood: puzzling, incomprehensible, unfathomable events that do not make sense. In fact, Kaiser even writes, “puzzle of puzzles, all is puzzling”[5, p. 26]. Other synonyms include unreasonable, repugnant (as famously

used by Saccheri in regard to non-Euclidean geometry), ludicrous, bizarre, crazy, irrational, repulsive, hideous, outrageous, unfair, preposterous, and idiotic. Perhaps a simpler description would do: “This world is messed up!”

This frustrating sentiment is echoed by the repeated phrase “striving after wind” (or “vexation of the spirit”). This sense of the word “vanity” then reminds those of us in mathematics of “unknowable” propositions implied by the Incompleteness Theorems. It was indeed quite frustrating for people like Wolfgang Bolyai, who had spent years of his life trying to prove the Parallel Postulate, something we know today as an impossible task.

I have traversed this bottomless night, which extinguished all light and joy of my life. . . Leave the science of parallels alone. . . . I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. . . I turned back when I saw that no man can reach the bottom of the night. I turned back unconsoled, pitying myself and all mankind. . . . I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. . . I thoughtlessly risked my life and happiness (H. Meschkowski, *Noneuclidean Geometry*, 1964, quoted in [4, pp. 161–162])

It was also frustrating for Georg Cantor to try proving the Continuum Hypothesis. After all, he was not aware that the answer is ultimately neither true nor false but rather independent of the other axioms. So why are we no longer frustrated today? It is precisely because we now know that neither the Parallel Postulate nor the Continuum Hypothesis can be proved or disproved from the other generally accepted axioms.

Solomon asserts that life under the sun is “puzzle of puzzles,” filled with incomprehensible things and events. Job also expressed his frustration, along with physical pain, because he could not comprehend many things and events—until God showed up, that is. Job then learned, as mentioned above, that he needed not understand everything. He did not know the Incompleteness Theorems, but he certainly understood that certain truths are beyond our reason and logic. Job was okay with that; we should be as well.

4 Conclusion

Contrary to what may be assumed by some, factors like poverty, lack of education, and being from a broken family are neither necessary nor sufficient conditions for suicidal thoughts.⁷ Mental health issues are not limited to certain groups of people; they do not discriminate. They affect men and women of all ages, every culture and language, regardless of race, ethnicity, talent, wealth, intelligence, occupation, and level of education; and these include mathematicians past and present.

In the mathematical world, suicide victims include Alan Turing, Felix Hausdorff, and Paul Epstein.

Many other mathematicians, while not victims of suicide, suffered tremendously from depression and mental disorder. Georg Cantor is perhaps the most famous example. This revolutionary, the

⁷A few days after Erika’s death, one of our other children found a journal entry where Erika had written about a conversation she had with me that day. Erika added that she wanted to talk to me because (in her words), “Talking to Dad is the closest thing (on earth) to talking to God.” It was with great comfort that I read those precious words. In the end, neither my words nor my presence could help her survive her pain and depression, but at least I was there, trying my best to help her.

father of a theory that forever changed the landscape of all mathematics, had already achieved fame and prestige at the University of Halle before his first episode of mental disorder in his late 30s. Then he too lost his youngest child Rudolph unexpectedly. Cantor never recovered fully from this tragic loss, and all the controversies on his theory of transfinite numbers did not help, either. After suffering from chronic depression and repeating hospitalizations, Cantor died in 1918 in a sanatorium.

As for the man who proved one of the most monumental results, the Incompleteness Theorems, yes, Kurt Gödel too suffered mental illness, refusing to eat, and ultimately died of starvation in 1978.

Do we live in a broken world? Yes. Absolutely.

Yet, at the same time, we have unwavering hope in a world to come. Is it based on understanding? Not necessary. Is it based on faith?

Yes. Absolutely.

For the last twenty or so years, I have been thinking a lot about integration of faith and mathematics [6]. Then God took away my child. But then He taught me truly invaluable, transforming, and eternal lessons, through mathematics as well as His Word, giving me a “peace that passes through all understanding.”

This is integration.

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Exploiting Our Limits: Finitude in Information Security

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Abstract

Human finitude—the set of limitations we have as created beings—is a surprisingly positive aspect of our existence. It is easy to interpret our limits in a negative light, but viewing them instead as a gift from God can reveal blessings that might otherwise be taken for granted. In this paper, we examine the field of information security, discovering that many of its fundamental principles and techniques are designed to account for our created finitude and exploit it toward productive ends. Through this lens, we can appreciate more fully the God-given gift of creaturely finitude.

1 Introduction

Finitude is so fundamental to the human experience that, aside from the occasional philosopher, theologian, and perhaps mathematician, most of us simply take it for granted. Each person has a finite amount of knowledge: as anyone who has embarked on a quest for learning has discovered, the number of things one does not know seems to increase the more one learns. There are also limits to what we can do. Outside of science fiction and fantasy, no one questions the restrictions of existing in one and only one place and moving continuously through three spatial dimensions. Each of us has command of a finite amount of resources: the most powerful people in the world still have bounds to their possessions and influence. We have come to accept as inevitable that our existence in this world has a beginning and an end, and we are bound to traverse that span in a single direction, as the quip goes, “at a constant rate of one second per second.” Despite the limiting nature of these facts, Christian thought embraces the idea that our finitude is not an inevitable handicap but an intentional and good gift from our Creator.

Another (at least seemingly) inevitable aspect of modern life is the pervasiveness of technology. As society’s adoption of technology affects more and more areas of our lives, the importance of protecting our digital security continues to rise. Driven by endless opportunities for profit and power, malicious actors lurk around every corner in cyberspace. The purpose of information security is to develop principles and techniques to combat these adversaries, protecting our information and technological assets from misuse. In this paper, we note an intriguing philosophical theme that runs through this highly technical field: recognition of, and even reliance on, human limitation. In fact, the God-given gift of human finitude can be found underlying both principles and techniques throughout information security.

2 Technology and Finitude

The formal theory underlying computer science has some of its earliest foundations in a recognition of our mathematical and computational limitations. Alonso Church and Alan Turing independently formalized the concept of an “algorithm” in 1936. Following the lead of Kurt Gödel’s then-recent work in mathematics, each proved (via his own definition) that there are inherent, unavoidable limits in what an algorithm can accomplish. The field of complexity theory emerged from attempts to delineate clearly what can and cannot be computed and, even more importantly, what can and cannot be computed *efficiently*. This is the essence of the famous open problem of the relationship between complexity classes P and NP, which has been called “the most fundamental and important mathematical question of our time” [14]. Even the futuristic-seeming field of quantum computing is already understood well enough to define quantum complexity classes and begin placing them in relation to those of classical (non-quantum) computing.

Of particular interest to us is the distinction between problems that are not computable at all and those that are “intractable,” i.e., not computable efficiently. It is specifically because of our finitude that, despite being fundamentally different in theory, these two classes of problems can be considered equivalent in practice. As we will see in Section 6.2, this is especially important to the assumptions that underlie public-key cryptography, arguably one of the most impactful inventions in information security.

Any system, network, or technique of information security is only shown to be secure against a specific, well-defined attacker. An indispensable portion of any research paper in information security or privacy is the “attacker model.” The attacker model provides a clear definition of the assumed capabilities and, importantly, assumed limitations of the entity who is trying to defeat the system. Every security technique, from authentication protocols to key exchanges to cryptographic constructs, is predicated on a set of assumptions about the finitude of the adversary. If any one of these assumptions fails or can be invalidated by a clever attacker, the security properties that the system aims to defend can be violated. For example, some algorithms for distributed consensus (Section 5.3) rely on the assumption that an attacker controls fewer than 50% of the participating systems. If any participant can generate a large number of independent-looking votes, then the result may not represent the actual consensus of legitimate participants.

The field of information security involves guiding principles as well as concrete techniques, and examples of the importance of finitude can be found in both categories. For a representative sample of the former, we will refer to the oft-cited set of information protection principles proposed by Jerome Saltzer and Michael Schroeder in 1975. Though nearly fifty years old, these keen observations continue to inform the design of security-conscious systems to this day. We will also observe assumptions of finitude found in techniques for confidentiality, integrity, and authentication.

3 Theology and Finitude

As Christians, the idea that finitude is not merely a limitation but a property that can be harnessed and put to good use should not come as a surprise to us. The creation narrative in Genesis makes it clear that humankind was already limited when God declared his work “very good” (the Hebrew *tov* here carrying the meaning of “functioning as intended”). Even the finite nature of the rest of creation is a boon to scientific thought, since our human understanding would be insufficient to explore a universe of infinite complexity [5]. Our creatureliness was given to us intentionally by our Creator. It can be tempting to scorn or reject it, but we are instead called to

recognize and affirm that our knowledge, physicality, and time are limited. In this section we draw from several neo-orthodox and existential theologians as well as from Genesis and Ecclesiastes to establish a theological context for finitude as a gift. In subsequent sections, we examine principles and techniques of information security that are made possible because of this gift.

3.1 Finitude Intentionally Created

Christian theology has, over the years, come to recognize finiteness as a good and intentional aspect of humanity's created form. While a consistent examination of finitude in a positive light is a relatively recent theological development, the idea is not altogether new to Christian thought. Emmanuel Falque notes positive appraisal of our limits in theology as early as Thomas Aquinas, who writes that our creaturely bodies are not hindrances for unlimited intellects but fitting homes for limited ones [13]. Theological movements of the nineteenth and twentieth centuries, notably neo-orthodoxy and Christian existentialism, began to give more thought to the significance of our limits, often in connection with the doctrine of creation. Reinhold Niebuhr, for one example, observes that the Bible views our finiteness, dependence, and insufficiency as "facts which belong to God's plan of creation and must be accepted with reverence and humility" [26].

Although there are many things that our limitations prevent us from accomplishing, it is worth noting that God created humankind finite from the very beginning and saw fit to declare it "very good." Adam and Eve were intellectually limited. They lacked the knowledge of good and evil (Gen. 3:22) and were subject to deception by the serpent (Gen. 3:1). They were physically bounded, distinct from one another and placed in a particular garden, which was itself delineated by four rivers (Gen. 2:10–15). Adam was not all-sufficient in resources or ability, needing a partner to work alongside him—something that was just as true in Genesis 2:18 as it is in Ecclesiastes 4:9–11. Finally, both were bound by time, coming into existence at a defined point in the creation narrative (Gen. 2:7,22) and experiencing the flow of time along with the rest of the created world, as illustrated in the sequential days of creation and heard in the refrain of evening and morning throughout Genesis 1. God even placed a moral restriction regarding the tree at the center of the garden. Dietrich Bonhoeffer points out that even this was a gift to Adam and Eve: the gift of true freedom to choose obedience to their Creator [6].

Our finitude, therefore, was imparted by the giver of every good and perfect gift. Rather than focus on the restrictions it imposes, we should seek out and appreciate the new possibilities that it creates. Karl Barth takes this approach in his *Church Dogmatics*, in which he devotes an entire section to "Freedom in Limitation". Throughout the section, he highlights important consequences of our limitations, ideas that we might have taken for granted were they not pointed out. His unifying thesis is that our limits give us specificity. "Limitation as decreed by God," Barth observes, "means circumscription, definition, and therefore determination" [3]. Each of us has a defined place precisely because we are not omnipresent, an allotted time precisely because we are not eternal, and a unique identity precisely because we are not omniscient.

Moreover, our place, time, and identity are not accidental, nor are they self-determined; rather, our Creator chooses them for us. It is an exact place we have been given in all of creation and in all of history. The very act of drawing a boundary is also the act of defining a shape, and in drawing out our limits, God defines for each person's life a specific calling [3]. Colin Gunton, in his study *The One, the Three, and the Many*, criticizes the modern era for losing sight of the "particularity" defined by the created boundaries distinguishing one creature or creation from another. He suggests that *haecceitas*—the "this-ness" of any created thing—is derived from "the way it is held in being

not only by God but also by other things in the particular configurations in space and time in which its being is constituted” [16]. That is to say, we have meaning because we are placed, and we are placed because we are finite.

Scripture reminds us that “we are God’s handiwork, created in Christ Jesus to do good works, which God prepared in advance for us to do” (Eph. 2:10). If God has prepared our work in advance of creating us, then He knows exactly where to locate us and how to gift us to make possible that very work. To use Barth’s distinction, our limitations are not given so that we can do *only* certain things, but so that we can do *precisely* that which has been prepared for us [3].

3.2 Rejecting our Finitude

If each person’s calling is presented to them by God through created limitations, then we should celebrate rather than scorn them. Contemporary theologian Daniel Treier writes at length about the “gift of finitude” as celebrated in the book of Ecclesiastes [35]. A clear theme throughout Ecclesiastes is the folly of attempting to build for ourselves a life or legacy that does not succumb to the eventualities of “time and chance” (9:11). Niebuhr makes a stronger indictment, writing that “sin has its source, not in temporality, but in man’s wilful refusal to acknowledge the finite and determinate character of his existence” [26]. Although there can be nothing evil in a gift from God, the occasion for sin arises when we choose to refuse this gift, so we must resist the temptation to reject or disregard our limitations. Jacques Ellul summarizes the wisdom message of the book of Ecclesiastes thus: “[w]isdom involves first of all recognizing our finiteness” [12].

We may even identify the sin of rejecting our finitude as a form of idolatry. Paul Tillich defines “idolatrous faith” as mischaracterizing that which is finite [33], and Ellul, citing Aarre Lauha, suggests that Ecclesiastes may lend indirect support to this idea. Each of the Teacher’s attempts at finding lasting meaning in temporal things—pleasure (2:11), work (2:21), wealth (2:26), status (4:16)—is given the same label: vanity (*hebel*). Lauha, in examining this refrain, notes that the same Hebrew root is used in the plural as an important synonym for “idols” in Old Testament books of history and prophecy [20]. Ellul posits that the book’s famous incipit might then have carried the connotation “vanity of idols!” in its original cultural-linguistic context [12].

3.3 Embracing our Finitude

Finitude and our response to it are central themes in the theology of Paul Tillich, featured in his treatise *The Courage to Be* and found throughout his *Systematic Theology*. One anthologist even christens him “theologian of the boundaries” [31]. According to Tillich, the hallmark of finitude is the presence of a boundary between a state of “being” and a state of “nonbeing.” All of creation lives on this boundary of finitude, and it is necessary and beneficial for us to work through the collision of being and nonbeing. Wrestling with this tension produces anxiety, and the critical decision we must make is how we will respond. Ideally, we should allow the anxiety of nonbeing to point us to God, who makes possible the courage needed to conquer it and transform it into a recognition of the goodness of our limits [32]. Niebuhr agrees that the temptation to sin arises not from our “finiteness, dependence and weakness” but rather from our “anxiety about it,” identifying a right mindset as the willingness to acknowledge that we are dependent, accept that we are finite, and admit that we are insecure [26].

In order to respond appropriately to our limits as given by God, we must be intentional in recog-

nizing what those limits are. Finitude characterizes various dimensions of our existence. Tillich specifically identifies several of these in his *Systematic Theology*. To begin with, knowledge and reason are themselves finite. Our finite reason then encounters four categories of finitude in being: time, space, causality, and substance [32]. Inspired by Tillich's taxonomy, we will organize our exploration around three major dimensions of human finitude: limits of knowledge and reason, limits of physicality and resources, and limits of temporality and time. The remaining category, finitude of substance, is the idea that our identity, our "being something in particular," is also destined to come to an end. In keeping with Barth's belief that our intellectual, physical, and temporal limitations are precisely what God uses to give us identity and meaning, it seems fit to fold this category into the others.

Tillich first discusses the inherent finitude of human reason. Citing Nicolaus Cusanus and Immanuel Kant, he asserts that reason is subject to the same limitations as the rest of our existence; yet, "[i]n spite of its finitude, reason is aware of its infinite depth" [32]. It does not allow us to circumvent our temporal or physical limits, but, paradoxically, it does afford us an awareness of the infinite absolutes behind the finite categories of being. We see a parallel in Niebuhr's explanation of the "problem of finiteness and freedom," which credits the *imago Dei* impressed upon humanity for the ground on which our finite reason can recognize its own finitude [26]. Ecclesiastes puts it poetically: God has "set eternity in the human heart; yet no one can fathom what God has done from beginning to end" (3:11b). Despite the inability of our finite reason to comprehend eternity in its fullness, we can understand enough to appreciate our own limitedness, and this "capacity for self-transcendence" [26] is what ultimately enables us to identify and embrace our limits.

Our spatial finitude delimits our position not only in physical space, but also in what Tillich calls a "social space" including our spheres of influence and sociocultural context [32]. We face the anxiety of the realization that we occupy "borrowed" space: we do not own in any meaningful or permanent way either the earth we tread or the earth we are made of. "[T]he dust returns to the ground it came from, and the spirit returns to God who gave it" (Eccl. 12:7). Courage in this category affirms our current space while recognizing its impermanence.

Tillich considers temporal finitude to be the central category of the four he identifies. It reminds us of the ephemerality of all creation, revealing the "present" we live in as merely the infinitesimal boundary between a bygone past which knew us not and an unknown future which will one day forget us [32]. This is a major theme in Ecclesiastes, and with good reason. It is easy for us to become anxious over the limits of our time on earth. Not even our legacy escapes the inevitability of oblivion: "No one remembers the former generations, and even those yet to come will not be remembered by those who follow them" (Eccl. 1:11). Biblical scholar Daniel C. Fredericks suggests that it may be consistent with other Old Testament usage to read the book's repeated exclamation "vanity!" as "transience!", finding comfort in the knowledge that even the problems and evil of life are fleeting [15]. The courage identified by Tillich in this category is that which "affirms temporality" [32], and Barth encourages each of us to rejoice in our being "this transient creature at this particular hour" [3].

It is easy to recognize that our time is finite in duration, in its beginning and end, but Barth and Tillich both observe limitations in our *experience* of time as well. We see time not from the point of view of God's eternal simultaneity, but from inside as we progress sequentially from past through present to future, from cause to effect: "For us the past is the time which we leave and are in no longer... It has remained behind, never to be restored" [3]. Yet, Barth concludes, there is no dimension on which to write the story of a human life, no canvas on which to paint the grand

narrative of covenantal promise and redemptive fulfillment between God and his creation without the progression of time. Tillich pushes the “finitude of causality” to its philosophical terminus. In contrast with God’s divine attribute of *aseity*, or self-causedness, we must accept that we are ultimately derivative, admitting in humility that we owe our existence to a cause that is outside of ourselves. Courage from God helps us to find meaning in the affirmation that our contingency gives us a place in reality [32].

4 Exploiting Intellectual Finitude

Accounting for the limits of human comprehension is a theme that is found in principles of computer science generally and information security in particular. Renowned computer scientist Edsger Dijkstra, in his essay *The Humble Programmer*, argues for “intellectually manageable programs,” concluding that to be successful in creating software, one must “respect the intrinsic limitations of the human mind” [10]. As three of Saltzer and Schroeder’s secure design principles will attest, the importance of intellectual manageability is as important in information security as it is elsewhere in computing.

4.1 Secure System Design Principles

The principle of *economy of mechanism* (popularly known as the “KISS” principle) states that the design of secure systems should be “as simple and small as possible” [28]. Due to our limited comprehension, it is more likely that someone will understand a simple design fully and correctly than a complex one. The developers implementing a simple design are therefore less likely to make implementation mistakes or overlook flaws. A small design affords fewer opportunities for such oversights, and fewer avenues for malicious misuse (a “smaller attack surface”). Minimal design has further benefits in that it can result in systems that are simple enough to be formally verified, one of the requirements of another important model of secure design, the *reference monitor* [2]. In this way, we can also account for cases in which our limited understanding still fails to catch a mistake.

We must acknowledge our limited comprehension when configuring a system as well. Saltzer and Schroeder codify the principle of *fail-safe defaults*, asserting that secure systems should begin with a default behavior of preventing access or actions and be configured to allow precisely those actions that it should permit. (This resonates with our discussion of specificity and vocation from Section 3.1.) In theory, the default-deny and default-allow approaches are equivalent: one could allow only the set of permitted accesses or deny only the complement of that set. In practice, however, our limited reason makes default-deny far preferable. Firewall configuration rulesets are a notable example of this principle: a 2004 study found a quantitative correlation between the complexity of rulesets from corporate firewalls and their likelihood of containing specific errors [36]. If omissions are possible, it is safer to enumerate what is permissible than what is impermissible.

The people who use a system are every bit as human as the ones who design and configure it, and Saltzer and Schroeder identify an applicable principle here as well. The principle of *psychological acceptability* underscores the importance of understandability in the interface presented to end-users. If the people who interact with a security mechanism fail to understand how it is meant to be used, they are likely to mis-apply it. If they fail to grasp its importance, they are likely to find a way to circumvent it. Both failures can lead to security issues. Once again, our finite human reason must be acknowledged and accounted for.

4.2 Secrets

Barth recognizes that identity is one aspect of the gift given to us in our intellectual finitude, and the study of *authentication*, or how to prove one's identity, is a significant field in information security. One of the most widely used means of authentication is requiring a user to demonstrate knowledge of a specific datum: a password, passphrase, or PIN. Implicit in this use is the assumption that it is something that no one else knows. We may take it for granted, but this is only possible because of intellectual finitude, the boundary separating one person's knowledge from another's.

Confidentiality, generally provided by encryption, is another important pillar of information security. In a world without intellectual boundaries, the concept of confidentiality would, of course, be nonsense. Even if we were to grant the existence of a single piece of secret knowledge, the knower would still be unable to share it, write it down, or use it in any way without relinquishing its confidentiality. The distinction of mind that results from our finitude provides for the existence of secrets, but more interestingly, it allows them to be used, shared, and kept. One of the fundamental concepts in cryptography is that of a *key*. Cipher algorithms are assumed to be publicly known; it is knowledge of the correct key that enables a person to encrypt or decrypt data. As in password-based authentication, the assumption that an adversary does not know and cannot learn the key is implicit in any cryptosystem.

5 Exploiting Physical Finitude

The second aspect of created finitude we explore is our physical, or spatial, finitude. We are embodied souls, and as such, we occupy a particular space and location. We cannot be everywhere, or in two places at once, and to get from one place to another, being bound to move continuously through space, we must actually travel there bit by bit. The resources and influence at our disposal are similarly limited, both in specificity and extent. No person has the authority or ability to control everything, or anything they wish, but rather each has a finite set of resources and a defined sphere of influence.

5.1 Secure System Design Principles

Saltzer and Schroeder's secure design principles acknowledge our spatial limits as well as the intellectual limits discussed earlier. In their elaboration of the principle of *separation of privilege* they explain how to exploit the bounds of physical access and ownership to improve the security of a system: "Where feasible, a protection mechanism that requires two keys to unlock it is more robust and flexible than one that allows access to the presenter of only a single key. . . . [T]he two keys can be physically separated and distinct programs, organizations, or individuals made responsible for them" [28]. The two-person concept (formerly "two-man rule") mandated for nuclear safety in the United States is an example of applying this principle in the most direct way.

Another consequence of physical finitude, as Tillich observed, is that each person has influence over a limited amount of resources. This idea is implied in the applications of separation of privilege, in that requiring an adversary to circumvent two separate security measures makes it more difficult to carry out a successful attack. The *work factor* principle makes this explicit, urging system designers to "compare the cost of circumventing the mechanism with the resources of a potential attacker" [28]. Security mechanisms are always evaluated with respect to a well-defined attacker model specifying the bounds of the adversary's assumed capabilities. Conventional wisdom in

information security says that any nontrivial system can be compromised given sufficient resources. Only the finitude of the attacker enables us to declare it “secure.”

If we truly wish to maximize the security provided by the finitude of resources, then—inasmuch as it is within our ability—we should attempt to minimize the resources available to each person. From this observation comes what may be the most widely known of Saltzer and Schroeder’s recommendations: the principle of *least privilege*. Entities within a computer system should be granted “the least set of privileges necessary to complete the job” [28]. If an adversary can compromise an account or program, this reduces the options available for parlaying the compromise into a more significant attack. This principle recognizes the good we can derive from our boundedness, and encourages us to maximize that good by setting the tightest bounds possible.

5.2 Multi-Factor Authentication

As we have previously noted, password authentication is a technique that depends on intellectual finitude. On the other hand, the limits of our ability to memorize [21] leave this form of authentication with a Goldilocks dilemma. Some passwords are too easy to guess, others are too hard to remember, and it is difficult to find and enforce a balance that is “just right.” Recognizing our limitations here means supplementing—if not supplanting—password authentication schemes, and this is precisely the purpose of multi-factor authentication.

Multi-factor authentication, in which users prove their identity using more than one “factor,” is another manifestation of the principle of separation of privilege discussed in the previous section. In its most common configuration, the user authenticates by simultaneously demonstrating knowledge of a specific datum and possession of a specific physical item. In terms of separation of privilege, the two individuals are the person who knows the former and the person who holds the latter, and the purpose technically remains to ensure that both individuals are present. When authenticating a single user, this implies that the “individuals” are one and the same person.

Factors used in a multi-factor authentication scheme must be sufficiently different from each other in order to be effective. Two user-chosen passwords are not much better than one, since an attacker with the resources to guess or compromise one password is likely able to do the same thing for the second. By analogy, putting a second lock on a door makes no difference if both of the keys are kept on the same keychain. For authentication, different factors are often expressed as the maxim “something you know, something you have, something you are.” A memorized passphrase differs from a physical security key, which differs from a biometric factor like a fingerprint or retinal scan. The likelihood that the same attacker has both guessed a user’s credentials and stolen a physical device from them is much lower than the likelihood of either event in isolation. These factors work because they exploit different aspects of the attacker’s finitude, some intellectual, others physical.

5.3 Distributed Consensus Algorithms

In recent years, a new means of exploiting our own limitedness of resources to achieve productive ends has come into the public eye in the form of cryptocurrencies and related blockchain technology. A blockchain is a distributed public ledger: a permanent, ordered list of transactions that can be viewed by everyone but is not overseen or controlled by anyone in particular. Blockchains are constructed atop *consensus* algorithms, which allow a number of independent, distributed peers to come to agreement on a computed result—in this case, the content of the ledger. However,

any distributed system that is designed to allow unknown peers to join the computation process must deal with the possibility of Sybil attacks [11]. In a Sybil attack, an adversary creates a large number of malicious identities in order to influence the final result of a distributed computation. Each of these previously-unknown participants can claim to be an independent entity, swaying the final vote with the digital equivalent of ballot-stuffing.

Decentralized cryptocurrencies, then, need a way to prevent attackers from artificially inflating their influence to a point where they can override the existing ledger with a craftily altered one. Bitcoin [25] addresses this by requiring *proof-of-work*: honest participants will only accept a list of transactions (or “block”) if it is accompanied by a solution to a computational puzzle based on its contents. The puzzle is designed so that “mining” a new block by finding a solution is computationally expensive, but verifying a provided solution is easy. This prevents the classic Sybil attack by weighting participants’ influence by the number of CPU cycles they invest in generating solutions to the puzzle. An attacker cannot have an unbounded amount of influence without an unbounded amount of processing power. In essence, Bitcoin is made possible by tying a distributed consensus protocol to finite real-world resources.

6 Exploiting Temporal Finitude

Perhaps the most familiar and oft-contemplated dimension of our created limitation under the sun is that of time. We are placed on the earth one day, we disappear from it another, and this interval is all the time we have for the work set before us. We are limited in time and duration, both that of our existence and that of our influence, and restricted to traveling it in one direction at an unchangeable rate. From this arises the concept of intractability, one of the central ideas in computational complexity theory. A problem is said to be intractable if finding a solution, though theoretically possible, requires an inordinately large amount of time. Computer science has identified a number of such problems, which may be as esoteric as finding the minimum feedback vertex set of a graph [18] or as accessible as finding the fastest way to solve a Rubik’s Cube of given dimension [9]. Based on our observations, we might define intractability from a different angle: an intractable problem is one that is, specifically as a result of our finitude, impossible in practice. Once again, we can identify information security techniques that convert this limitation into an important building block.

6.1 Cryptographic Hashes

Along with encryption, one of the most fundamental primitives of applied cryptography is cryptographic hashing. A secure hash function must satisfy certain additional properties beyond what is required of an ordinary hash [8]. Given a possible output of the hash function, it must be computationally infeasible to find a corresponding input (pre-image resistance). It must also be infeasible to identify two inputs that generate the same output, regardless of what the output is (collision resistance). It is obviously possible to find a preimage or collision by exhaustive search *if* given an infinite amount of time, but the idea of infeasibility here is naturally predicated on an attacker with finite time.

Cryptographic hashes are an example of “one-way functions,” and it is easy to see how their properties result in unidirectional behavior. The hash output cannot exist before the input that generates it, a relationship that is only meaningful because the time in which we are created flows in a single direction, from cause to effect and not the other way around. It is unsurprising, then,

that many uses of hashing in applied cryptography establish or enforce a causal (and therefore temporal) ordering between events. It allows us to apply the created gift of our causal finitude in the context of cryptographic protocols.

Hashing has several idiomatic uses in applied cryptography. One is to create a representative “fingerprint” from arbitrary data that can be used when working with the original would be computationally costly, unwieldy, or vulnerable to attack. This can be seen in the ubiquitous task of password authentication. When a system authenticates a user by password, it must verify that the password provided at login is the same as the one entered when the password was established. However, if the system were to store its users’ passwords directly, an attacker who obtains read access to the database of account information would learn the passwords directly and could then impersonate any user on the system. Standard secure practice is to store instead a value derived from the password using a cryptographic hash function [22]. Recovering the original password from this value is tantamount to finding a preimage for the hash function, which, under the assumption of preimage resistance, is a practical impossibility for finite attackers.

In the presence of a stronger attacker who can eavesdrop on authentication exchanges, a new problem arises. Passwords may be protected at rest, but this is meaningless if they can still be observed in flight. No matter how we attempt to obscure them when sending, if the data expected by the authenticator is always the same, then an eavesdropper can break the scheme simply by replaying data that was previously observed. The most straightforward way to solve this problem is by introducing a random value known as a nonce (so named because it is used “for the nonce,” now and never again). The system sends a randomly chosen nonce to the user, and the user must produce a response based on both their password and the nonce. A natural choice is to hash the two values together in some fashion, ensuring that an eavesdropper cannot learn the password by observing the transmitted response. The server computes the same hash from the user’s password and the current nonce and compares the response to determine whether authentication succeeds.

Nonces are, we observe, another tool for harnessing the temporal finitude of an attacker. They are the cryptographic analogue of Barth’s observation that each person’s life is unique, offered “once and not twice; once and never again” [3]. The problem solved by adding a nonce to the protocol is that an eavesdropper learns the only correct response as soon as it is used. In the challenge/response version, the eavesdropper learns one correct response, but its usefulness is past as soon as it has been observed, and it will be a very long time until, by a coincidence of random number generation, the same nonce is reused. If the attacker were not constrained by the monotonic flow of time or bounded to live a certain span of time, the addition of a nonce would make no difference. Our temporal limitations make the protocol work.

A similar approach to harnessing temporality is found in one-time password schemes. One-time passwords, in a sense, automate the idea behind challenge/response authentication. Like challenge/response, the user must provide different information at each login, but here the system and user both derive the next password independently, eliminating the need for a back-and-forth exchange. The user simply sends the next one-time password and the system verifies it. Implementations of one-time passwords make use of hashes and nonces in much the same way as challenge/response protocols. The earliest suggestion, due to Lamport [19] and used in the S/KEY authentication system [17], is based on hash chains. Beginning with a seed value, a cryptographic hash is applied repeatedly until a set number of values have been generated. These values are then used *in reverse order* as one-time passwords. The user retains the seed, in order to be able to produce the next password in sequence, and an eavesdropper cannot determine what comes next,

only what has gone before. Current one-time password schemes, often used in hardware tokens for two-factor authentication, are essentially challenge/response with a simple counter used as the nonce. This counter is incremented with each successful authentication, as in HOTP [23], or at regular time intervals, as in TOTP [24].

Another application that relies on the one-way property of hashes is known as a commitment scheme. Suppose a medical researcher has discovered the cure for some common ailment. She knows that her discovery will bring her wealth and fame and wishes to publish it in a medical journal. However, she also knows that there is a danger of her work—and, more importantly, the credit—being stolen by an anonymous reviewer during the lengthy publication process. To prevent this, she runs her name and paper text through a cryptographic hash function and publishes the output as an advertisement in a widely-circulated newspaper. Once this “commitment” goes to press, she submits the paper for review. If a dishonest reviewer attempts to publish and claim credit for her work, she can prove to anyone that she knew the details of the cure first by showing them her submission and the newspaper. The unidirectionality of the hash function proves that her knowledge of the paper’s contents had to exist prior to the advertisement running. This scheme also keeps her honest, as the hash corresponds only to her paper exactly as it initially stood; there is no way for her to make an improvement later and claim that it was part of her original submission. Privacy-oriented cryptocurrencies such as Incognito [34] make use of this latter application.

6.2 Public-Key Cryptography

The field of cryptography saw a revolution in 1976 with the introduction of asymmetric cryptosystems, first and famously published in Whitfield Diffie and Martin Hellman’s paper “New Directions in Cryptography” (though they now appear to have been discovered in secret by government security organizations several years prior). Diffie and Hellman’s main contribution was an algorithm for key establishment, allowing two parties communicating over an insecure medium to securely agree on a new, random cryptographic key. Reconstructing the key solely from the information sent between the two parties requires solving the intractable discrete logarithm problem, which means that even an adversary who has eavesdropped on the entire exchange cannot, in practice, derive the same key.

The same idea has been applied to encryption to create public-key encryption and signature schemes such as RSA [27]. As with symmetric cryptography, the security of the cryptosystem is vested in the assumption of intellectual finitude: that an attacker does not know a specific private key. Unlike symmetric encryption, however, these algorithms involve a second key that functions as the inverse of the first, so that data encrypted with one key of a pair can be decrypted with the other. The asymmetry which gives this class of algorithms its name is in the treatment of the two keys: the private key is kept secret, but the other key is made public.

Publishing part of a keypair does come with an important caveat. The precise mathematical relationship between the two keys means that one may contain sufficient information to derive the other. In theory, it is not impossible to break a cipher by calculating the private key, but it can still be considered secure in practice if doing so requires solving an intractable problem. Simon Singh puts this in perspective in his extensive history of cryptography, *The Code Book*: “It is now routine to encrypt a message... so that all the computers on the planet would need longer than the age of the universe to break the cipher” [30]. In other words, it is our temporal finitude that allows us to assume that a private key remains secret, even though its mathematical counterpart is known to an attacker. The RSA cryptosystem, for example, relies on the intractability of determining the

prime factorization of large integers. An attacker with an unlimited amount of time could decipher any encrypted message, but for one with a finite amount of time, it is a practical impossibility.

We can therefore credit our finitude for the multitude of new cryptographic techniques that have emerged from the invention of public-key cryptography. In addition to encrypting data for confidentiality, the keys can be used in reverse so that the owner of the keypair encrypts data using the private key. The result can then be decrypted using the corresponding public key, allowing anyone to verify that the encrypted data was created by the holder of the private key. This forms the basis of what we call “digital signatures,” in which a person can securely “sign” a document, cryptographically binding their identity to the signed data in a way that can subsequently be verified by anyone. Here we even see echoes of Barth’s discussion on how our God-given limits define an identity and calling for each person.

It is important here to dispel some popular ideas about the impact of quantum computing on the idea of intractability. Perhaps the most famous result to date in the quantum arena is Peter Shor’s algorithm for efficient calculation of discrete logarithms and integer factorizations [29], which demonstrates that a sufficiently powerful quantum computer could break both Diffie–Hellman key agreement and RSA encryption. Unless one is familiar with further results in quantum complexity theory, it is reasonable to question whether this implies that quantum computers could one day conquer algorithmic intractability altogether. Popular entertainment has reinforced this misconception (the Netflix miniseries *Devs* is one recent example), but in reality even quantum systems have bounds to be explored. Many problems are known to be “QMA-complete,” or believed to be difficult even for quantum computers [7]. Using this idea, research in post-quantum cryptography has been producing quantum-resistant algorithms since the 2000s [4], and enough variety has emerged that NIST has launched a competition, like those which produced DES, AES, and SHA-3, to standardize a quantum-resistant cryptosystem [1]. We can push back our boundaries with innovation, but we cannot escape our limitations entirely.

7 Conclusion

If we recognize our finitude as a gift, we have a basis on which to confront the prideful thinking and problematic behaviors that arise from rejecting it. Refusal to accept that our knowledge is limited fails to acknowledge that omniscience belongs to God. This temptation drives actions such as information theft and violations of digital privacy. Refusal to accept that our space and resources are limited fails to acknowledge that omnipresence and omnipotence belong to God. Here we see a motivation for botnets, in which attackers seek to increase their own power by compromising and utilizing computing resources that belong to others. Refusal to accept that our time is limited fails to acknowledge that eternity belongs to God, which explains why some self-propagating malware seems to have no purpose beyond outliving its author. In contrast, a right response to finitude is naturally modeled by Jesus Christ, who did not spurn human form and limitation but rather accepted them in humble obedience (Phil. 2:7–8).

Finitude is such an ingrained part of the human experience that we can easily take it for granted. Careful reading of Scripture reveals that our created limitations are part of God’s “very good” creation, and the book of Ecclesiastes provides an affirmation of life in a finite world. If we are intentional about viewing human finitude as a gift from God, we can begin to see its pervasive goodness. Design principles and cryptographic techniques of information security make for a surprisingly apt example, but there are undoubtedly many other places in which the recognition and embracing of our finitude can illuminate God-given gifts that we may otherwise never have seen.

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Identities, Rank of Appearance, and Period of Second Order Linear Recurrences

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Abstract

We consider the second order linear recurrence $U_{n+2} = PU_{n+1} - QU_n$, with P and Q in \mathbb{Z} and initial conditions $U_0 = 0$ and $U_1 = 1$. We show that for all integers r, s, k, l such that $r + s = k + l$, and G_n, H_n satisfying the recurrence relation and initial conditions G_0, G_1 and H_0, H_1 respectively, we have $G_r H_s - G_k H_l = Q^t (G_{r-t} H_{s-t} - G_{k-t} H_{l-t})$ for all integers t . We also give a relationship between the period and the rank of appearance when the recurrence is considered over \mathbb{Z}_p . We obtain as a corollary that the period of the Fibonacci sequence is always even.

1 Introduction

The well known Fibonacci sequence, 0, 1, 1, 2, 3, 5, 8, 13, 21, ... is defined by the second order recurrence relation $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The terms of the Fibonacci sequence satisfy many beautiful identities. Vajda's book [1] has a good compilation of them. In [4], E. Lucas studied the general linear recurrence of order 2. He stated several identities and divisibility properties satisfied by the terms of the sequence. In this paper we prove some of the identities stated by Lucas. These identities generalize those satisfied by the Fibonacci sequence. Following Lucas notation we let P and $Q \neq 0$ be two relatively prime integers and consider the second order linear homogeneous recurrence relation

$$U_{n+2} = PU_{n+1} - QU_n \text{ for } n \geq 0, \text{ and } U_0 = 0, U_1 = 1. \quad (1)$$

over the field of rational numbers \mathbb{Q} or the finite field \mathbb{Z}_p . We denote by $f(x) = x^2 - Px + Q$ the characteristic polynomial of the recurrence and by E be the splitting field of $f(x)$. We assume that the discriminant of $f(x)$, $\Delta = P^2 - 4Q \neq 0$ and denote by α and β the roots of $f(x)$ in E . We set $\delta = \alpha - \beta$. If $f(x)$ is irreducible over \mathbb{Q} or \mathbb{Z}_p then E is a vector space of dimension two over \mathbb{Q} or \mathbb{Z}_p and every element e of E can be written uniquely as $e = a + b\alpha$ with a and b in \mathbb{Q} or \mathbb{Z}_p . The elements α, β and δ of E satisfy, $\alpha \neq \beta$ and

$$\alpha + \beta = P, \quad \alpha\beta = Q, \quad \alpha^2 = P\alpha - Q, \quad \beta^2 = P\beta - Q, \quad \delta^2 = \Delta. \quad (2)$$

If $f(x)$ is not irreducible we may treat α and β as formal symbols over \mathbb{Q} or \mathbb{Z}_p satisfying (2), and all the computations carry over as in the case when $f(x)$ is irreducible. Multiplication in E is given by

$$(a + b\alpha)(c + d\alpha) = (ac - Qbd) + (ad + b(Pd + c))\alpha. \quad (3)$$

The Lucas sequence V_n associated to the recurrence is defined as $V_n = \alpha^n + \beta^n$.

Since $\alpha^{n+2} = P\alpha^{n+1} - Q\alpha^n$ and $\beta^{n+2} = P\beta^{n+1} - Q\beta^n$ we have

$$V_{n+2} = \alpha^{n+2} + \beta^{n+2} = P(\alpha^{n+1} + \beta^{n+1}) - Q(\alpha^n + \beta^n) = PV_{n+1} - QV_n.$$

This shows that V_n satisfies the recurrence relation with initial conditions $V_0 = 2$ and $V_1 = P$.

2 Identities

As mentioned in the introduction, Vajda's book [1], contains more than 100 identities satisfied by the Fibonacci and the Lucas sequence. In this section we will give a generalization, displayed below, of identities (18), (19a), (19b), (20a), and (20b) found on page 28 of [1].

$$G_{n+i}H_{n+j} - G_nH_{n+i+j} = Q^n (G_iH_j - G_0H_{i+j}) \quad (\text{V18})$$

$$V_{n+i}U_{n+j} - V_nU_{n+i+j} = Q^n (V_iU_j - V_0U_{i+j}) = -Q^n U_iV_j \quad (\text{V19a})$$

$$G_{n+i}G_{n+j} - G_nG_{n+i+j} = Q^n (G_iG_j - G_0G_{i+j}) \quad (\text{V19b})$$

$$U_{n+i}U_{n+j} - U_nU_{n+i+j} = Q^n U_iU_j \quad (\text{V20a})$$

$$V_{n+i}V_{n+j} - V_nV_{n+i+j} = Q^n (V_iV_j - V_0V_{i+j}) = -Q^n \Delta U_iU_j \quad (\text{V20b})$$

We start by extending the sequence U_n to negative indices by defining

$$U_{-n} = (PU_{-n+1} - U_{-n+2}) / Q \text{ for } n \geq 1.$$

We have the following proposition relating negative terms to positive terms.

Proposition 1. For every integer n we have

$$U_{-n} = -U_n/Q^n \text{ and} \quad (4)$$

$$V_{-n} = V_n/Q^n \quad (5)$$

Proof. The identity $U_{-n} = -U_n/Q^n$ follows by induction on n and the definitions of U_n , and U_{-n} . For V_{-n} , we have

$$V_{-n} = \alpha^{-n} + \beta^{-n} = (\alpha^n + \beta^n) / Q^n = V_n / Q^n.$$

□

The following Lemma shows that the powers of the roots are good book keeping devices for the elements of the recurrence sequence.

Lemma 1. For every integer r

$$\alpha^r = -QU_{r-1} + U_r\alpha \text{ and } \beta^r = -QU_{r-1} + U_r\beta. \quad (6)$$

Proof. Induction on r . First we will assume that $r \geq 0$.

$$1. \alpha^0 = 1 = 1 + 0 \cdot \alpha = -Q(-(1/Q) + U_0\alpha) = -QU_{-1} + U_0\alpha.$$

$$2. \text{ We have } \alpha^1 = 0 + 1 \cdot \alpha = U_0 + U_1\alpha.$$

3. Assume $\alpha^r = -QU_{r-1} + U_r\alpha$. Then

$$\begin{aligned}\alpha^{r+1} &= \alpha^r \cdot \alpha = (-QU_{r-1} + U_r\alpha)\alpha \\ &= -QU_{r-1}\alpha + U_r\alpha^2 = -QU_{r-1}\alpha + U_r(P\alpha - Q) \\ &= -QU_r + (PU_r - QU_{r-1})\alpha = -QU_r + U_{r+1}\alpha.\end{aligned}$$

For $r < 0$ we observe that

$$\alpha^{-1} = \frac{P}{Q} - \frac{1}{Q}\alpha = -QU_{-2} + U_{-1}\alpha.$$

Using this last identity and induction on r we can show that $\alpha^r = -QU_{r-1} + U_r\alpha$ in this case also. \square

Corollary 2. For every integer r we have

$$U_r = (\alpha^r - \beta^r) / (\alpha - \beta) \quad (7)$$

$$V_r = PU_r - 2QU_{r-1} = U_{r+1} - QU_{r-1} \quad (8)$$

Proof. Since $\alpha^r = -QU_{r-1} + U_r\alpha$ and $\beta^r = -QU_{r-1} + U_r\beta$ we have

$$\alpha^r - \beta^r = U_r\alpha - U_r\beta = U_r(\alpha - \beta).$$

Since $\alpha - \beta \neq 0$ we can solve for U_r to get identity (7). Since

$$\begin{aligned}V_r &= \alpha^r + \beta^r = (-QU_{r-1} + U_r\alpha) + (-QU_{r-1} + U_r\beta) \\ &= U_r(\alpha + \beta) - 2QU_{r-1} = PU_r - 2QU_{r-1}\end{aligned}$$

we get identity (8). \square

Lemma 2. Let r and s be two integers. Then

$$\alpha^r\alpha^s = -Q(U_rU_s - QU_{r-1}U_{s-1}) + (-QU_{r-1}U_s + PU_rU_s - QU_rU_{s-1})\alpha \quad (9)$$

$$U_{r+s} = -QU_{r-1}U_s + PU_rU_s - QU_rU_{s-1} \quad (10)$$

$$\Delta U_rU_s - V_rV_s = -2(\alpha^r\beta^s + \alpha^s\beta^r) \quad (11)$$

$$\Delta U_rU_s - V_rV_s = Q(\Delta U_{r-1}U_{s-1} - V_{r-1}V_{s-1}) \quad (12)$$

$$V_rV_s - 2V_{r+s} = -\Delta U_rU_s \quad (13)$$

$$V_rU_s - V_sU_r = Q(V_{r-1}U_{s-1} - V_{s-1}U_{r-1}) \quad (14)$$

$$V_rU_s + V_sU_r = 2U_{r+s} = V_0U_{r+s} \text{ or } \quad (15)$$

$$V_rU_s - 2U_{r+s} = -U_rV_s. \quad (16)$$

Proof. By identity (6), we have $\alpha^r = -QU_{r-1} + U_r\alpha$, $\alpha^s = -QU_{s-1} + U_s\alpha$. Therefore,

$$\begin{aligned}\alpha^r\alpha^s &= (-QU_{r-1} + U_r\alpha)(-QU_{s-1} + U_s\alpha) \\ &= (Q^2U_{r-1}U_{s-1} - QU_rU_s) + (-QU_{r-1}U_s + U_r(PU_s - QU_{s-1}))\alpha \\ &= (Q^2U_{r-1}U_{s-1} - QU_rU_s) + (-QU_{r-1}U_s + PU_rU_s - QU_rU_{s-1})\alpha\end{aligned}$$

This proves identity (9). Also,

$$\alpha^r \alpha^s = \alpha^{r+s} = QU_{r+s-1} + U_{r+s} \alpha.$$

This implies that

$$QU_{r+s-1} + U_{r+s} \alpha = \alpha^{r+s} = (Q^2 U_{r-1} U_{s-1} - QU_r U_s) + (-QU_{r-1} U_s + PU_r U_s - QU_r U_{s-1}) \alpha.$$

Equating coefficients of the last two expressions in the above equation we get (10). We also have,

$$\begin{aligned} V_r V_s &= (\alpha^r + \beta^r)(\alpha^s + \beta^s) = (\alpha^r - \beta^r)(\alpha^s - \beta^s) + 2(\alpha^r \beta^s + \alpha^s \beta^r) \\ &= \Delta \left[\frac{\alpha^r - \beta^r}{\alpha - \beta} \right] \left[\frac{\alpha^s - \beta^s}{\alpha - \beta} \right] + 2(\alpha^r \beta^s + \alpha^s \beta^r) \\ &= \Delta U_r U_s + 2(\alpha^r \beta^s + \alpha^s \beta^r). \end{aligned}$$

This proves (11). Next, we have

$$\begin{aligned} V_r V_s - \Delta U_r U_s &= 2(\alpha^r \beta^s + \alpha^s \beta^r) = 2Q(\alpha^{r-1} \beta^{s-1} + \alpha^{s-1} \beta^{r-1}) \\ &= Q(V_{r-1} V_{s-1} - \Delta U_{r-1} U_{s-1}). \end{aligned}$$

This proves identity (12). By expanding we have

$$\begin{aligned} 2V_r V_s &= 2(\alpha^r + \beta^r)(\alpha^s + \beta^s) = 2(\alpha^{r+s} + \beta^{r+s}) + 2(\alpha^r \beta^s + \alpha^s \beta^r) \\ &= 2V_{r+s} + V_r V_s - \Delta U_r U_s. \end{aligned}$$

Therefore, $V_r V_s - 2V_{r+s} = -\Delta U_r U_s$ which is identity (13). Identities (14) and (15) follow from

$$\begin{aligned} (\alpha^r + \beta^r)(\alpha^s - \beta^s)/(\alpha - \beta) &= (\alpha^{r+s} + \beta^r \alpha^s - \alpha^r \beta^s - \beta^{r+s})/(\alpha - \beta) \\ (\alpha^s + \beta^s)(\alpha^r - \beta^r)/(\alpha - \beta) &= (\alpha^{r+s} - \beta^r \alpha^s + \alpha^r \beta^s - \beta^{r+s})/(\alpha - \beta) \end{aligned}$$

by adding and subtracting and observing that

$$2(\beta^r \alpha^s - \alpha^r \beta^s)/(\alpha - \beta) = Q(V_{r-1} U_{s-1} - V_{s-1} U_{r-1}).$$

□

Theorem 3. Let G_n and H_n be two sequences with initial conditions G_0, G_1 and H_0, H_1 respectively, satisfying the linear recurrence. If $r + s = k + l$ then

$$U_r U_s - U_k U_l = Q(U_{r-1} U_{s-1} - U_{k-1} U_{l-1}) \quad (17)$$

$$G_r H_s - G_k H_l = Q(G_{r-1} H_{s-1} - G_{k-1} H_{l-1}) \quad (18)$$

$$V_r V_s - V_k V_l = Q(V_{r-1} V_{s-1} - V_{k-1} V_{l-1}) \quad (19)$$

$$\Delta U_r U_s - V_k V_l = Q(\Delta U_{r-1} U_{s-1} - V_{k-1} V_{l-1}) \quad (20)$$

Proof. From identity (9) we get

$$\begin{aligned} \alpha^r \alpha^s &= -Q(U_r U_s - QU_{r-1} U_{s-1}) + (U_r U_{s+1} - QU_{r-1} U_s) \alpha \text{ and} \\ \alpha^k \alpha^l &= -Q(U_k U_l - QU_{k-1} U_{l-1}) + (U_k U_{l+1} - QU_{k-1} U_l) \alpha \end{aligned}$$

By assumption $r + s = k + l$, so $\alpha^r \alpha^s = \alpha^k \alpha^l$. Equating the terms of the left hand side of the last two identities we have $U_r U_s - Q U_{r-1} U_{s-1} = U_k U_l - Q U_{k-1} U_{l-1}$. Rearranging this last identity we get identity (17). To prove identity (18) we observe that $G_1 U_n - G_0 Q U_{n-1}$ and $H_1 U_n - H_0 Q U_{n-1}$ satisfy the recurrence relation and have initial conditions G_0, G_1 and H_0, H_1 respectively. Therefore, $G_n = G_1 U_n - G_0 Q U_{n-1}$ and $H_n = H_1 U_n - H_0 Q U_{n-1}$. So,

$$\begin{aligned}
G_r H_s - G_k H_l &= (G_1 U_r - G_0 Q U_{r-1}) \cdot (H_1 U_s - H_0 Q U_{s-1}) \\
&\quad - (G_1 U_k - G_0 Q U_{k-1}) \cdot (H_1 U_l - H_0 Q U_{l-1}) \\
&= G_1 H_1 U_r U_s - G_0 H_1 Q U_{r-1} U_s - H_0 G_1 Q U_r U_{s-1} \\
&\quad + G_0 H_0 Q^2 U_{r-1} U_{s-1} - G_1 H_1 U_k U_l + G_0 H_1 Q U_{k-1} U_l \\
&\quad + H_0 G_1 Q U_k U_{l-1} - G_0 H_0 Q^2 U_{k-1} U_{l-1} \\
&= G_1 H_1 (U_r U_s - U_k U_l) - G_0 H_1 Q (U_{r-1} U_s - U_{k-1} U_l) \\
&\quad - H_0 G_1 Q (U_r U_{s-1} - U_k U_{l-1}) \\
&\quad + G_0 H_0 Q^2 (U_{r-1} U_{s-1} - U_{k-1} U_{l-1}) \\
&= G_1 H_1 Q (U_{r-1} U_{s-1} - U_{k-1} U_{l-1}) \\
&\quad - G_0 H_1 Q^2 (U_{r-2} U_{s-1} - U_{k-2} U_{l-1}) \\
&\quad - H_0 G_1 Q^2 (U_{r-1} U_{s-2} - U_{k-1} U_{l-2}) \\
&\quad + G_0 H_0 Q^3 (U_{r-2} U_{s-2} - U_{k-2} U_{l-2}) \\
&= Q [G_1 H_1 (U_{r-1} U_{s-1} - U_{k-1} U_{l-1}) \\
&\quad - G_0 H_1 Q (U_{r-2} U_{s-1} - U_{k-2} U_{l-1}) \\
&\quad - H_0 G_1 Q (U_{r-1} U_{s-2} - U_{k-1} U_{l-2}) \\
&\quad + G_0 H_0 Q^2 (U_{r-2} U_{s-2} - U_{k-2} U_{l-2})] \\
&= Q (G_{r-1} H_{s-1} - G_{k-1} H_{l-1}).
\end{aligned}$$

This proves identity (18). From identity (12) we get

$$V_r V_s - Q V_{r-1} V_{s-1} = \Delta (U_r U_s - Q U_{r-1} U_{s-1}) \quad (21)$$

$$= \Delta (U_k U_l - Q U_{k-1} U_{l-1}) \text{ by rearranging (17)} \quad (22)$$

$$= V_k V_l - Q V_{k-1} V_{l-1} \text{ by using (12) again.} \quad (23)$$

Therefore, setting the left hand side of (21) equal to (23) and the right hand side of (21) equal to (22) we get

$$\begin{aligned}
V_r V_s - Q V_{r-1} V_{s-1} &= V_k V_l - Q V_{k-1} V_{l-1} \text{ and} \\
\Delta (U_r U_s - Q U_{r-1} U_{s-1}) &= V_k V_l - Q V_{k-1} V_{l-1}.
\end{aligned}$$

Rearranging these two last identities we get

$$\begin{aligned}
V_r V_s - V_k V_l &= Q (V_{r-1} V_{s-1} - V_{k-1} V_{l-1}) \text{ and} \\
\Delta U_r U_s - V_k V_l &= Q (\Delta U_{r-1} U_{s-1} - V_{k-1} V_{l-1}).
\end{aligned}$$

□

The following theorem is the main result of this section.

Theorem 4. For all integers k, l, r, s , and t with $k + l = r + s$ we have

$$U_r U_s - U_k U_l = Q^t (U_{r-t} U_{s-t} - U_{k-t} U_{l-t}) \quad (24)$$

$$G_r H_s - G_k H_l = Q^t (G_{r-t} H_{s-t} - G_{k-t} H_{l-t}) \quad (25)$$

$$V_r V_s - V_k V_l = Q^t (V_{r-t} V_{s-t} - V_{k-t} V_{l-t}) \quad (26)$$

$$\Delta U_r U_s - V_k V_l = Q^t (\Delta U_{r-t} U_{s-t} - V_{k-t} V_{l-t}) \quad (27)$$

Proof. This follows by iterating identities (17), (18), (19), and (20). \square

We will refer to identities (24), (25), (26) and (27) as $U(r, s, k, l, t)$, $GH(r, s, k, l, t)$, $V(r, s, k, l, t)$, and $\Delta UV(r, s, k, l, t)$ respectively.

Now we can see that $GH(n + i, n + j, n, n + i + j, n)$ produces

$$G_{n+i} H_{n+j} - G_n H_{n+i+j} = Q^n (G_i H_j - G_0 H_{i+j}). \quad (V18)$$

Replacing G by V and H by U into (V18) and using (15), since $V_0 = 2$, produces

$$V_{n+i} U_{n+j} - V_n U_{n+i+j} = Q^n (V_i U_j - V_0 U_{i+j}) = -Q^n U_i V_j. \quad (V19a)$$

Replacing G by U and H by V into (V18) and recalling that $U_0 = 0$ produces

$$U_{n+i} V_{n+j} - U_n V_{n+i+j} = Q^n (U_i V_j - U_0 V_{i+j}) = Q^n U_i V_j. \quad (V19b)$$

Replacing G by U and H by U into (V18) and recalling that $U_0 = 0$ produces

$$U_{n+i} U_{n+j} - U_n U_{n+i+j} = Q^n (U_i U_j - U_0 U_{i+j}) = Q^n U_i U_j. \quad (V20a)$$

Replacing G by V and H by V into (V18) and using (13), since $V_0 = 2$, produces

$$V_{n+i} V_{n+j} - V_n V_{n+i+j} = Q^n (V_i V_j - V_0 V_{i+j}) = -Q^n \Delta U_i U_j. \quad (V20b)$$

Corollary 5. For every integer n, i , and j

$$U_n^2 - U_{n+i} U_{n-i} = Q^{n-i} U_i^2 \quad \text{Catalan's identity} \quad (28)$$

$$U_n^2 - U_{n+1} U_{n-1} = Q^{n-1} \quad \text{Cassini's identity} \quad (29)$$

$$U_m U_{n+1} - U_n U_{m+1} = Q^n U_{m-n} \quad \text{D'Ocagne's Identity} \quad (30)$$

Proof. Catalan's identity is obtained from $U(n, n, n + i, n - i, n - i)$. Cassini's identity is given by $U(n, n, n + 1, n - 1, n - 1)$. D'Ocagne's identity results from $U(m, n + 1, n, m + 1, n)$ and recalling that $U_0 = 0$. \square

3 Divisibility Properties

The terms of the recurrence sequence U_n and the Lucas sequence V_n satisfy several divisibility properties. Below, we will prove some of them.

Proposition 6. Let n and d be positive integers, then

- (i) U_d divides U_{dn} .
- (ii) If n is even V_d divides U_{dn} .
- (iii) If n is odd V_d divides V_{dn} .

Proof. (i) By identity (6) we have $\alpha^d = U_d\alpha - QU_{d-1}$ and $\alpha^{dn} = U_{dn}\alpha - QU_{dn-1}$. Therefore, $\alpha^d \equiv -QU_{d-1} \pmod{U_d}$ and it follows that $\alpha^{dn} \equiv (-QU_{d-1})^n \pmod{U_d}$. We also have, $\alpha^{dn} \equiv U_{dn}\alpha - QU_{dn-1} \pmod{U_d}$. Together, these two identities imply $U_{dn}\alpha + (QU_{dn-1} + (-QU_{d-1})^n) \equiv 0 \pmod{U_d}$. Therefore, $U_{dn} \equiv 0 \pmod{U_d}$ and $QU_{dn-1} \equiv (-QU_{d-1})^n \pmod{U_d}$.

(ii) By definition $\alpha^d + \beta^d = V_d$, so we get $\alpha^d \equiv -\beta^d \pmod{V_d}$. Since n is even we have $\alpha^{dn} \equiv \beta^{dn} \pmod{V_d}$. This implies that $U_{dn} \equiv \alpha^{dn} - \beta^{dn} \equiv 0 \pmod{V_d}$.

(iii) As in part (ii), we have $\alpha^d \equiv -\beta^d \pmod{V_d}$. Since n is odd we have $\alpha^{dn} \equiv -\beta^{dn} \pmod{V_d}$. This implies that $V_{dn} \equiv \alpha^{dn} + \beta^{dn} \equiv 0 \pmod{V_d}$. \square

Proposition 7. Let n and d be positive integers, then

- (i) $U_{dn} = U_d \left[\sum_{i=0}^{n-1} \binom{n}{i} (-Q)^i U_d^{n-i-1} U_{d-1}^i U_{n-i} \right]$.
- (ii) If $n = 2k$ then $U_{dn} = U_{dk} V_{dk}$.
- (iii) If $n = 2k + 1$ then $V_{dn} = V_d \left[(-1)^k Q^{kd} + \sum_{i=0}^{k-1} (-1)^i Q^{id} V_{2(k-i)d} \right]$.

Proof. (i) By Lemma 1, we have $\alpha^d = U_d\alpha + h_d$ and $\alpha^{nd} = U_{nd}\alpha + h_{nd}$ where $h_d = -QU_{d-1}$ and $h_{nd} = -QU_{nd-1}$. Therefore,

$$\begin{aligned}
U_{nd}\alpha + h_{nd} &= \alpha^{nd} = (\alpha^d)^n = (U_d\alpha + h_d)^n = \sum_{i=0}^n \binom{n}{i} (U_d\alpha)^{n-i} h_d^i \\
&= \sum_{i=0}^n \binom{n}{i} (U_d)^{n-i} \alpha^{n-i} h_d^i \\
&= \sum_{i=0}^n \binom{n}{i} U_d^{n-i} h_d^i \alpha^{n-i} = \sum_{i=0}^n \binom{n}{i} U_d^{n-i} h_d^i (U_{n-i}\alpha + h_{n-i}) \\
&= \sum_{i=0}^n \binom{n}{i} (U_d^{n-i} h_d^i U_{n-i}\alpha + U_d^{n-i} h_d^i h_{n-i}) \\
&= \sum_{i=0}^n \binom{n}{i} U_d^{n-i} h_d^i h_{n-i} + \alpha \sum_{i=0}^n \binom{n}{i} U_d^{n-i} h_d^i U_{n-i} \\
&= \sum_{i=0}^n \binom{n}{i} U_d^{n-i} (-QU_{d-1})^i (-QU_{n-i-1}) \\
&\quad + \alpha \sum_{i=0}^{n-1} \binom{n}{i} U_d^{n-i} (-QU_{d-1})^i U_{n-i} \\
&= \sum_{i=0}^n \binom{n}{i} (-Q)^{i+1} U_d^{n-i} U_{d-1}^i U_{n-i-1} + \alpha \sum_{i=0}^n \binom{n}{i} (-Q)^i U_d^{n-i} U_{d-1}^i U_{n-i}.
\end{aligned}$$

Since $U_0 = 0$, we have

$$\begin{aligned} U_{dn} &= \sum_{i=0}^n \binom{n}{i} (-Q)^i U_d^{n-i} U_{d-1}^i U_{n-i} \\ &= \sum_{i=0}^{n-1} \binom{n}{i} (-Q)^i U_d^{n-i} U_{d-1}^i U_{n-i} \\ &= U_d \sum_{i=0}^{n-1} \binom{n}{i} (-Q)^i U_d^{n-i-1} U_{d-1}^i U_{n-i}. \end{aligned}$$

$$(ii) \quad U_{dn} = \frac{\alpha^{dn} - \beta^{dn}}{\alpha - \beta} = \frac{\alpha^{2dk} - \beta^{2dk}}{\alpha - \beta} = \frac{\alpha^{dk} - \beta^{dk}}{\alpha - \beta} \cdot (\alpha^{dk} + \beta^{dk}) = U_{dk} V_{dk}.$$

(iii) Let $x = \alpha^d$ and $y = \beta^d$. We have $x^i + y^i = V_{id}$ and $(xy)^i = Q^{id}$. Also,

$$\begin{aligned} V_{dn} &= \alpha^{d(2k+1)} + \beta^{d(2k+1)} \\ &= x^{2k+1} + y^{2k+1} \\ &= (x + y) (x^{2k} - x^{2k-1}y + \dots + (-1)^i x^{2k-i}y^i + \dots - xy^{2k-1} + y^{2k}) \\ &= V_d ((x^{2k} + y^{2k}) - (x^{2k-1}y + xy^{2k-1}) + \dots + (-1)^k x^k y^k) \\ &= V_d ((x^{2k} + y^{2k}) - xy(x^{2k-2} + y^{2k-2}) + \dots + (-1)^k x^k y^k) \\ &= V_d ((x^{2k} + y^{2k}) - xy(x^{2(k-1)} + y^{2(k-1)}) + \dots + (-1)^k x^k y^k) \\ &= V_d (V_{2kd} - (xy)V_{2(k-1)d} + \dots + (-1)^i (xy)^i V_{2(k-i)d} + \dots + (-1)^k (xy)^k) \\ &= V_d (V_{2kd} - Q^d V_{2(k-1)d} + Q^{2d} V_{2(k-2)d} + \dots + (-1)^i Q^{id} V_{2(k-i)d} + \dots + (-1)^k Q^{kd}) \\ &= V_d \left((-1)^k Q^{kd} + \sum_{i=0}^{k-1} (-1)^i Q^{id} V_{2(k-i)d} \right). \end{aligned}$$

□

Proposition 8. If $\gcd(P, Q) = 1$ and $n \geq 1$, then $\gcd(U_n, Q) = 1$ and $\gcd(V_n, Q) = 1$.

Proof. Induction on n . Since $\gcd(U_1, Q) = \gcd(1, Q) = 1$ and $\gcd(U_2, Q) = \gcd(P, Q) = 1$ the result is true for $n = 1$ and $n = 2$. Assume that

$$\gcd(U_1, Q) = \gcd(U_2, Q) = \dots = \gcd(U_{n-1}, Q) = \gcd(U_n, Q) = 1.$$

We will show that $\gcd(U_{n+1}, Q) = 1$. Since $U_{n+1} = PU_n - QU_{n-1}$ we have $\gcd(U_{n+1}, Q) = \gcd(PU_n - QU_{n-1}, Q)$. If $\gcd(PU_n - QU_{n-1}, Q) = d$, then since d divides Q and $\gcd(P, Q) = 1$ we have $\gcd(P, d) = 1$. Since $\gcd(P, d) = 1$, d divides $PU_n - QU_{n-1}$, and Q , so d divides U_n . By induction hypothesis $\gcd(U_n, Q) = 1$. We conclude that $d = 1$. Let $d = \gcd(V_n, Q)$. By equation (8), $V_n = PU_n - 2QU_{n-1}$. So $PU_n \equiv 0 \pmod{d}$. Since $\gcd(U_n, Q) = \gcd(P, Q) = 1$ we have $d = 1$. □

Theorem 9. Suppose that $\gcd(P, Q) = 1$ and $m \geq n \geq 1$, then

$$(i) \quad \gcd(U_m, U_n) = U_{\gcd(m, n)}.$$

$$(ii) \quad \text{If } m/d \text{ and } n/d \text{ are odd } \gcd(V_m, V_n) = V_{\gcd(m, n)}.$$

Proof.

(i) Let $d = \gcd(m, n)$ and $d_1 = \gcd(U_m, U_n)$. By Proposition 6, U_d divides U_m and U_n . So, $U_d \leq d_1$. Since $d = \gcd(m, n)$ there exist positive integers r and s such that $d = rm - sn$. If we replace m by rm and n by sn into identity (30) we have

$$Q^{sn}U_{rm-sn} = U_{rm}U_{sn+1} + QU_{sn}U_{rm-1} - PU_{rm}U_{sn} \quad (31)$$

By Proposition 6, U_m divides U_{rm} and U_n divides U_{sn} . Since d_1 divides U_m and U_n , identity (31) implies that d_1 divides $Q^{sn}U_{rm-sn}$. Since d_1 divides U_n , by Proposition 8 we must have $\gcd(d_1, Q) = 1$. Therefore d_1 divides $U_{rm-sn} = U_d$. This shows that $d_1 \leq U_d$. We have shown that $U_{\gcd(m,n)} \leq d_1 \leq U_{\gcd(m,n)}$, so $\gcd(U_m, U_n) = U_{\gcd(m,n)}$.

(ii) Let $d_2 = \gcd(V_m, V_n)$. By Proposition 6 (iii), V_d divides V_m and V_n , so $V_d \leq d_2$. There exists integers r and s such that $rm + sn = d$. Since $r(m/d) + s(n/d) = 1$, and $m/d, n/d$ are both odd we have either r is odd and s is even or r is even and s is odd. Suppose r is odd and s is even. By assumption $V_m \equiv 0 \pmod{d_2}$ and $V_n \equiv 0 \pmod{d_2}$. This implies that $\alpha^m \equiv -\beta^m \pmod{d_2}$ and $\alpha^n \equiv -\beta^n \pmod{d_2}$.

$$\begin{aligned} \alpha^d &\equiv \alpha^{rm+sn} \equiv \alpha^{rm}\alpha^{sn} \equiv (\alpha^m)^r (\alpha^n)^s \equiv (-\beta^m)^r (-\beta^n)^s \\ &\equiv -\beta^{rm}\beta^{sn} \text{ since } r \text{ is odd and } s \text{ is even} \\ &\equiv -\beta^{rm+sn} \equiv -\beta^d \pmod{d_2}. \end{aligned}$$

Therefore, $V_d \equiv \alpha^d + \beta^d \equiv 0 \pmod{d_2}$. That is, d_2 divides V_d . We conclude that $V_d = d_2$. □

4 Rank of Appearance and Period Modulo a Prime

In this section we assume that $Q \neq 0$ in \mathbb{Z}_p , since otherwise, the recurrence becomes a recurrence of order 1.

Definition 10. The rank of appearance of p in U_n is the smallest positive integer ρ such that $U_\rho = 0$. The period of the recurrence is the smallest positive integer π such that $U_\pi = 0$ and $U_{\pi+1} = 1$.

Since in this case, the splitting field E_p of the characteristic equation is finite, so α and β have finite order in E_p^\times , the multiplicative group of nonzero elements of E_p . We denote the order of α and β by $\text{ord}(\alpha)$ and $\text{ord}(\beta)$ respectively. First, we consider the case when the characteristic polynomial of the general recurrence relation

$$x_{n+k} = a_{k-1}x_{n+(k-1)} + a_{k-2}x_{n+(k-2)} + \cdots + a_2x_{n+2} + a_1x_{n+1} + a_0x_n \quad (32)$$

with initial conditions $W_0 = W_1 = \cdots = W_{k-3} = W_{k-2} = 0$ and $W_{k-1} = 1$ and $a_0 \neq 0$ has a root of multiplicity k . We have the following theorem.

Theorem 11. Let $f(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_2x^2 + a_1x + a_0$ be the characteristic polynomial of (32). Assume that $f(x) = (x - \alpha)^k$, that is, $\alpha \neq 0$ is a root of multiplicity k . If $p \geq k$ then $W_p = W_{p+1} = \cdots = W_{p+(k-3)} = W_{p+(k-2)} = 0$ and $W_{p+(k-1)} = \alpha$.

Proof. It is well known, see Remark 6.23 in [3], that in this case $W_n = g(n)\alpha^n$ with

$$g(n) = g_0 + g_1n + g_2n^2 + \cdots + g_{k-2}n^{k-2} + g_{k-1}n^{k-1},$$

g_i in E_p and $g(0)\alpha^0 = g(1)\alpha^1 = g(2)\alpha^2 = \cdots = g(k-3)\alpha^{k-3} = g(k-2)\alpha^{k-2} = 0$ and $g(k-1)\alpha^{k-1} =$

1. We observe that in our case $E_p = \mathbb{Z}_p$. Since $p \geq k$ the coefficients g_i are uniquely determined by the initial conditions. Also, since $\alpha \neq 0$, we have $g(0) = g(1) = \dots = g(k-3) = g(k-2) = 0$. It follows that $W_p = g(p)\alpha^p = g(0)\alpha^p = 0$, $W_{p+1} = g(p+1)\alpha^{p+1} = g(1)\alpha^{p+1} = 0$, \dots , $W_{p+(k-2)} = g(p+(k-2))\alpha^{p+(k-2)} = g(k-2)\alpha^{p+(k-2)} = 0$ and $W_{p+(k-1)} = g(p+(k-1))\alpha^{p+(k-1)} = g(k-1)\alpha^{k-1}\alpha^p = \alpha$ since $g(k-1)\alpha^{k-1} = 1$ and $\alpha^p = \alpha$. \square

Corollary 12. If the characteristic polynomial of the second order linear recurrence $x_{n+2} = Px_{n+1} - Qx_n$ has a multiple root then $U_p = 0$ and $U_{p+1} = \alpha$ in \mathbb{Z}_p for all primes. That is, the rank is p and the period is $p \text{ ord}(\alpha)$.

Proof. That $U_p = 0$ and $U_{p+1} = \alpha$ in \mathbb{Z}_p follows immediately from the theorem above. Since p is prime the rank is p . The statement about the period is a consequence of Lemma 5 below. \square

For the rest of this section we assume that $\Delta \neq 0$ in \mathbb{Z}_p .

Lemma 3. Let $m = \text{lcm}(\text{ord}(\alpha), \text{ord}(\beta))$. Then $U_m = 0$ and $U_{m+1} = 1$. That is, the rank of appearance and the period exist and satisfy $\rho \leq \pi = m$.

Proof. Since $1 = \alpha^m = U_m\alpha - QU_{m-1}$ and $1 = \beta^m = U_m\beta - QU_{m-1}$ we have $0 = U_m(\alpha - \beta)$. Since $\alpha - \beta \neq 0$ we have $U_m = 0$. Also, $\alpha^{m+1} = U_{m+1}\alpha - QU_m = U_{m+1}\alpha$. Therefore, $1 = \alpha^m = U_{m+1}$. By definition of the rank of appearance and period $\rho \leq \pi$. By the minimality of the period $\pi \leq m$. Also, $\alpha^{\pi+1} = U_{\pi+1}\alpha - QU_\pi = \alpha$. So, $\alpha^\pi = 1$ and $\beta^\pi = 1$. These identities imply π is a multiple of $\text{ord}(\alpha)$ and $\text{ord}(\beta)$. Therefore, $m \leq \pi$. \square

Lemma 4. $U_n = 0$ modulo p if and only if ρ divides n .

Proof. By Proposition 6, part (i) if ρ divides n then U_ρ divides U_n and it follows that $U_n = 0$. Conversely if $U_n = 0$, let $n = q\rho + r$ with $r < \rho$. Then by identity (10) and $U_{q\rho} = U_q U_\rho = 0$ we have

$$\begin{aligned} 0 = U_{q\rho+r} &= -QU_{q\rho-1}U_r + PU_{q\rho}U_r - QU_{q\rho}U_{r-1} \\ &= -QU_{q\rho-1}U_r \end{aligned}$$

By Theorem 9, $\gcd(U_{q\rho-1}, U_{q\rho}) = 1$. So $U_{q\rho-1} \neq 0$ modulo p . Also, $Q \neq 0$ modulo p . Therefore, $U_r = 0$ modulo p . This is a contradiction since $r < \rho$. \square

Lemma 5. $\pi = \rho \text{ ord}(U_{\rho+1})$.

Proof. Let $u = \text{ord}(U_{\rho+1})$ let T_n^2 satisfies the second order linear recurrence relation with initial conditions $T_0^2 = 0$ and $T_1^2 = U_{\rho+1}$. As mentioned in the proof of Theorem 33 $T_n^2 = T_1^2 U_n - T_0^2 Q U_{n-1} = U_{\rho+1} U_n$. By the definition of T_n^2 it follows that $U_{\rho+n} = T_n^2$. Therefore, $U_{2\rho} = U_{\rho+\rho} = T_\rho^2 = U_{\rho+1} U_\rho = 0$ and $U_{2\rho+1} = T_{\rho+1}^2 = U_{\rho+1} U_{\rho+1} = U_{\rho+1}^2$. Similarly, if T_n^3 satisfies the second order linear recurrence relation with initial conditions $T_0^3 = 0$ and $T_1^3 = U_{\rho+1}^2$. Then $T_n^3 = T_1^3 U_n - T_0^3 Q U_{n-1} = U_{\rho+1}^2 U_n$. By the definition of T_n^3 it follows that $U_{2\rho+n} = T_n^3$. Therefore, $U_{3\rho} = U_{2\rho+\rho} = T_\rho^3 = U_{\rho+1}^2 U_\rho = 0$ and $U_{3\rho+1} = T_{\rho+1}^3 = U_{\rho+1}^2 U_{\rho+1} = U_{\rho+1}^3$. Continuing in this manner one can see that $U_{u\rho} = 0$ and $U_{u\rho+1} = U_{\rho+1}^u = 1$. \square

Theorem 13. Let $d = \gcd(\rho, \text{ord}(Q))$ and $u = \text{ord}(U_{\rho+1})$. Then

$$(i) \quad Q^\rho = U_{\rho+1}^2.$$

- (ii) $\text{ord}(Q^\rho)$ divides $\text{ord}(U_{\rho+1}^2)$.
- (iii) $\text{ord}(U_{\rho+1})$ divides $2 \text{ord}(Q)/d$.
- (iv) d divides π and $\text{ord}(Q)/d$ divides π .

Proof. (i) Since $U_\rho = 0$ we have $\alpha^{\rho+1} = U_{\rho+1}\alpha$ and $\beta^{\rho+1} = U_{\rho+1}\beta$. So, $\alpha^\rho = U_{\rho+1}$ and $\beta^\rho = U_{\rho+1}$. This implies that $Q^\rho = (\alpha\beta)^\rho = U_{\rho+1}^2$.

(ii) This follows immediately from (i).

(iii) We have

$$1 = (Q^{\text{ord}(Q)})^{\rho/d} = (Q^\rho)^{\text{ord}(Q)/d} = U_{\rho+1}^{2 \text{ord}(Q)/d}.$$

So, u divides $2 \text{ord}(Q)/d$.

(iv) Since d divides ρ and $\pi = \rho u$, d divides π . Also, $Q^{\rho u} = U_{\rho+1}^{2u} = 1$. So, $\text{ord}(Q)$ divides ρu . Therefore, $\text{ord}(Q)/d$ divides $\rho u/d$. Since $\gcd(\rho/d, \text{ord}(Q)/d) = 1$, $\text{ord}(Q)/d$ divides u . This implies that $\text{ord}(Q)/d$ divides π . \square

Corollary 14. The period π of the Fibonacci sequence $F_{n+2} = F_{n+1} + F_n$ is even. Moreover, if the rank of appearance ρ is odd then 4 divides π .

Proof. For the Fibonacci sequence, $Q = -1$. So, $\gcd(\rho, \text{ord}(Q)) = \gcd(\rho, 2)$. There are two cases to consider. If $\gcd(\rho, 2) = 2$ then ρ is even and it follows that π is even. If $\gcd(\rho, 2) = 1$ then ρ is odd, so $(-1)^\rho = -1$. This implies that 2 divides $\text{ord}(U_{\rho+1}^2)$. That is, 4 divides $\text{ord}(U_{\rho+1})$ and we conclude that 4 divides π . \square

Remark 2. If $\alpha \neq \beta$ and α and β are in \mathbb{Z}_p then $\text{ord}(\alpha)$ and $\text{ord}(\beta)$ divide $p-1$ so $U_{p-1} = 0$. If $\alpha \neq \beta$ and α and β are not in \mathbb{Z}_p then $\beta = \alpha^p$. Since $Q = \alpha\beta = \alpha\alpha^p = \alpha^{p+1} = -QU_p + U_{p+1}\alpha$ we have $U_p = -1$ and $U_{p+1} = 0$. If $U_{p-1} = U_{p+1} = 0$ then the identity $0 = U_{p+1} = PU_p - QU_{p-1} = PU_p$ implies that $PU_p = 0$. If $U_p = 0$ then by working backward we get $U_{p-2} = U_{p-3} = \dots = U_1 = 0$. This gives a contradiction since $U_1 = 1$. Therefore, $P = 0$. In this case, $U_n = 0$ for n even, $U_n = (-Q)^n$ for n odd, $n > 1$, the rank of appearance is 2 and the period π satisfies $\pi = 2 \text{ord}(Q)$.

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God and Mathematical Objects: Can Both Exist Simultaneously?

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Abstract

Is theism compatible with a Platonist view of mathematical objects? Asserting that both are true leads to both philosophical and theological problems. In this paper we introduce several of these problems and discuss three philosophical models which attempt to affirm the existence of both God and mathematical objects.

1 Introduction

Does God exist? Less provocatively, do numbers exist? Any definite (i.e., not agnostic) position on these two questions will fit somewhere under the umbrella of one of Einar Bøhn's four categories: Atheistic Nominalism, Atheistic Platonism, Theistic Nominalism, and Theistic Platonism.¹ Atheism and Theism are the views that God does not or does exist (respectively), and Nominalism and Platonism are the views that abstract objects do not or do exist,² (respectively) [2]. In this paper, I will not attempt to survey these four positions or to support any one of them; rather, I have outlined these four broad positions in order to help frame our discussion here as exploring the coherence of a particular position that fits under the umbrella of Theistic Platonism: the belief in both God and a branch of Platonism called Mathematical Platonism. Can both God and mathematical objects exist simultaneously?

What are numbers, and what are they like? What about geometric objects such as points, lines, and shapes? How about algebraic objects like groups and isomorphisms, or topological objects such

¹This term of Bøhn's, Theistic Platonism, is very general and encompasses all views that affirm theism and Platonism. The reader should be careful to distinguish this from "Platonic Theism" in section 8 which is, confusingly, one specific type of Theistic Platonism. Hopefully the reader will find it helpful that I will not use the term "Theistic Platonism" any more in this paper after this paragraph.

²One could argue that this is not truly a binary choice. In [1], Balaguer outlines a spectrum of four positions: Platonism, which claims that abstract objects exist outside of space-time and independently of our minds; immanent realism, which affirms that abstract objects exist, but which locates them in the physical world; conceptualism, which holds that abstract objects do exist but only as ideas in our minds; and nominalism, which denies that abstract objects exist. For example, under this schema, Bridges holds to conceptualism, staking out a position in which abstract objects are "objects abstracted by the intellect and [which] have the properties they do as a result of the way the intellect grasps its objects" [3, p. 278].

On the other hand, Craig argues that Bridges's perspective is not a middle ground but just another type of nominalism [4]. Indeed, depending on how one answers the thorny question of how abstract objects should be defined (which we will briefly address in Section 3) immanent realism and conceptualism can both be construed as actually denying the existence of abstract objects. From this perspective, immanent realism and conceptualism are under the umbrella of nominalism, and so the choice between Platonism and nominalism would be binary after all. In this paper we will not solve this problem of whether Platonism and nominalism are points on a spectrum or a binary choice.

as simplicial complexes and metric spaces? By these questions, I do not mean to ask what it is that the various fields of mathematics teach. We will not be exploring the *content* of mathematics. Rather, my questions have an ontological flavor to them.

Following our intuition, it seems that we did not make these mathematical objects up; it *feels* like when we are learning our numbers as children or learning algebra and geometry later on, we are not creating it all in our heads but rather we are learning about something that is already there. This mirrors the feelings that a mathematician has when she discovers some brand new math—it feels like discovery, not like creation. Still following our intuition, it seems like it does not matter what I think about the Pythagorean Theorem; it just is whatever it is, independent of my thoughts, the various symbols used to represent it, my culture, or the first culture that discovered it. Are numbers, shapes, and equations such as $2 + 2 = 4$ never brought into being by anybody? At this point, a Christian might pause to wonder whether God himself, the creator of all things, brought (or even *could* bring) numbers into existence.

What else can we say while we are following our nose about the nature of mathematical objects? It seems that mathematical objects are not located anywhere in particular.³ Further, it is hard to see how they could ever change or how they could be affected by something concrete or affect something else. Stewart Shapiro comically writes, “The scientific literature contains no reference to the location of numbers or to their causal efficacy in natural phenomenon or to how one could go about creating or destroying a number. There is no mention of experiments to detect the presence of numbers or determine their mathematical properties. Such talk would be patently absurd” [20, p. 27]. Again, a Christian might pause. I can see how I myself have no power to affect mathematical objects, but is not God all powerful and sovereign over everything? If so, how can it be that God is all powerful but that at the same time, math is unchangeable?

It seems that for most people who have interacted much with mathematics⁴ but not thought deeply about related philosophical problems, our default unexamined intuition would seem to support the belief that numbers and other mathematical objects are real. It is what feels most natural. Kelsey Houston-Edwards says the idea that mathematical objects are *real* in some deep sense “aligns with the way people *feel* while doing math. It positions mathematicians as investigators of sorts. They’re like explorers in a sea of numbers and shapes and ideas. It’s how actual mathematicians talk” [11]. That is, our intuition seems to support the idea that numbers, shapes, and so on, are real things. Obviously we must be careful; our unexamined default intuition would also seem to support a flat earth and a geocentric solar system, which is a good reminder that intuition in and of itself is not sufficient justification to hold any position.

We will call a belief in the reality of numbers and other mathematical objects Mathematical Platonism. To be more specific, Mathematical Platonism is the belief that mathematical objects are eternal and necessary objects which cannot affect or be affected by anything. They have no location in space-time, and their existence is independent of minds. (We will go into more detail about Mathematical Platonism in Section 3.) Mathematical Platonism is not so much a distinct philosophy from Platonism as it is a way Platonism is applied to mathematical objects. Mathematical Platonism is what Platonists might believe about mathematics. Therefore Mathematical Platonism can fit within the Theistic Platonism frame (although to be clear, Mathematical Platonism is silent

³I am talking about for example, the number 2 itself (if such a thing exists), not the ink on this page which denotes it.

⁴I am not only referring to people who have earned a degree in mathematics. This intuition would be present in most children in an elementary school math class.

about theism).

Even though for many, Mathematical Platonism is a default unexamined perspective, many philosophers have taken issue with this view for various reasons. There are many philosophical problems that arise alongside Mathematical Platonism, and many other ontologies are held up as a way to resolve some of these tensions. As alluded to above, some, but not all, of these problems arise from tensions between Christianity and Mathematical Platonism. Other problems arise from within Mathematical Platonism itself. In this paper we will explore questions about the compatibility of Christianity and Mathematical Platonism, focusing on this overall question: *Are the inherent tensions between Christianity and Mathematical Platonism reconcilable, or does logic dictate that we must reject one to hold the other?*

In sections 2 and 3, we will discuss several relevant propositions about the God of Christianity and about Mathematical Platonism, respectively. Section 5 is a discussion of the contradictions that seem to arise as a result of holding both Christianity and Mathematical Platonism. Following that, we discuss three ways Christian philosophers have attempted to resolve these difficulties, ways which we will categorize based on two binary variables: 1) created or uncreated and 2) dependent on God or independent from God. The first variable is straightforward: does a particular model construe mathematical objects as created by God or not? As for the second variable, regardless of whether mathematical objects are created by God, are they considered independent from him or not? The first model we consider is Theistic Activism, in section 6. Under Theistic Activism, mathematical objects are held to be both created by and dependent on God. Then in section 7 we consider Divine Conceptualism, in which mathematical objects are taken to be uncreated but still dependent on God. Finally, a model is presented in section 8 called Theistic Platonism in which mathematical objects are uncreated and independent from God.⁵

2 Propositions About God

As a belief system, Christianity encompasses a plethora of propositions about God, humanity, the world, ethics, history, the purpose and meaning of life, and many other things. However, in this paper, when I talk about Christianity or Christian theism, I mean something much narrower: some very specific propositions about the God of classical Christianity that are relevant to the discussion of God and mathematical objects. We will be particularly interested in the Christian vision of God as a sovereign and self-existing creator of all things.⁶ I do not mean to imply that these few ideas compose the whole of Christian Theism, nor its core or essence. The three statements below are all specifically about God, but I do not in any way argue that these are the definition of God or the core of his being, nor even the most important points about him. They are simply three statements implied by Christianity which have particular relevance to the discussion of God and mathematical objects.

What did God create, and how did he create it?

- “In the beginning, God created the heavens and the earth. . . . And God said, ‘Let there be light,’ and there was light” (Genesis 1:1, 3).

⁵It is contradictory to suppose that mathematical objects could be simultaneously created by God and independent of him, for “being created by X ” is a type of “dependence on X ,” so there is no created-and-independent model for us to explore.

⁶In fact, Islam, Judaism, and some other monotheistic belief systems would hold similar views about God as a sovereign, self-existent creator.

- “In the beginning was the Word, and the Word was with God, and the Word was God. He was in the beginning with God. All things were made through him, and without him was not any thing made that was made” (John 1:1-3).
- “[God’s son] is the image of the invisible God, the firstborn of all creation. For by him all things were created, in heaven and on earth, visible and invisible, whether thrones or dominions or rulers or authorities—all things were created through him and for him” (Colossians 1:15-16).

When we think about the idea of creating or making something, images of a construction site or an art studio come to mind in which the beauty, order, and structure of a building or painting are made to come together by its creator out of the chaos of raw materials: beams, bricks, wood, metal, ceramic, paint, and canvas. Christians believe that in creating, we are acting as image bearers, for God is the ultimate creator of the universe. God’s original creation of the universe though is substantially different from any act of creation we might perform, for God created the world *ex nihilo*, out of nothing. The builder gathers materials (concrete, wood, etc.) and arranges them together in just the right way to form a house, but God created the world *out of nothing*. “By creation we mean the work of God in bringing into being, without the use of any preexisting materials, everything that is” [7, p. 392].

Regarding *what* God created, consider the fifth question of the New City Catechism: “What else did God create? God created all things by his powerful Word, and all his creation was very good. Everything flourished under his loving rule” [10]. How should Christians interpret the words “all things” and “everything” in this catechism and in the passages above? The Genesis passage refers to God’s creation of the “heavens and the earth,” and the two New Testament passages refer to “all things.” Wayne Grudem writes that we should take these phrases to mean “the entire universe” and “all the parts of the universe, both visible and invisible things” [9, p. 124-5]. Certainly Grudem is arguing that God created a huge host of things, but it is not clear whether he means to include abstract objects. Millard Erickson is a little more all-encompassing in his assessment, saying that God created “everything that is” [7, p. 392]. He writes, “God did not create merely a certain part of reality, with the remainder attributable to some other origin; he has made all of reality,” [7, p. 397] which is a very strong statement and would most naturally be interpreted as including abstract objects. However, he later writes that “creation in the proper sense refers to bringing into existence all of physical reality as well as all spiritual beings other than God himself” [7, p. 399], which brings into question whether he has abstract objects in view at all, for it is generally agreed that such objects are not part of physical reality, nor are they considered spiritual beings. In view of this, we will encapsulate the biblical doctrine of God as creator in the statement that follows, while for the present leaving ambiguous the meaning of the term “everything.” We will highlight several statements in this way throughout the paper. Note that for convenience, the complete list of these statements is found in the appendix on page 268.

Statement (GC). *Everything (besides God himself) is created by God.*

Next, Christians hold that God is existentially independent of anything and everything else; he is self-existent. This is what we will mean when we say that God exists *a se*, or that he has the property of *aseity*. In describing this doctrine, William Lane Craig points out that it is closely tied to the doctrine of creation, for if God is the creator, the source of everything whatsoever besides himself, then everything depends on him; there is nothing left for him to depend on [5]. The argument is that since everything depends on God, he cannot depend on anything. This gives us the following statement.

Statement (GA). *God exists a se.*

Christianity also teaches that God is the omnipotent, sovereign king over all creation. Erickson writes that God “is free to do whatever he wills” [7, p. 928], and Grudem writes, “there are no limits on God’s power to do what he decides to do” [9, p. 98]. This understanding of God’s power seems to imply that God has the power to change or destroy a thing if he chooses. He could make the world spin the other direction or make an apple tree bear grapes. Indeed, we have seen him calm the storm and heal the blind. Our question here is whether God’s power extends over mathematical objects, as we will explore in later sections.

Statement (GO). *God is omnipotent.*

I have intentionally left these statements somewhat ambiguous, for as we shall see, how these doctrines are interpreted with respect to mathematical objects varies by philosopher.

3 Mathematical Platonism

Before we dive into describing Mathematical Platonism, I should point out that while the philosophical stances of (regular) Platonism and Mathematical Platonism are quite related (as one would expect from their names), they are not identical. The term “Platonism” itself can refer to Plato’s philosophy of a realm of ideal forms serving as models used by the demiurge as he created the world [12]. Modern metaphysical Platonism is a bit different: it is the belief that there exist abstract objects such as properties, propositions, universals, relations, and the like [1]. This latter view is the version of Platonism to which we are referring. As we discussed in section 1, Mathematical Platonism is to Platonism as economic conservatism is to conservatism. Economic conservatism is a conservative outlook on economics, but an economic conservative need not be conservative about everything. For example, Bill Clinton was considered to be economically conservative and socially liberal. Similarly, Mathematical Platonism is a Platonistic outlook on mathematical objects. Theoretically, one could be a Mathematical Platonist but not hold a Platonistic view on, say, universals. The focus of this paper is Mathematical Platonism, and the focus of Mathematical Platonism is on mathematical objects. Mathematical objects include such objects as numbers, sets, shapes, groups, functions, operations, fields, topological spaces, vectors, matrices, manifolds, etc.⁷

In *The Stanford Encyclopedia of Philosophy*, Øystein Linnebo defines Mathematical Platonism as the view that mathematical objects exist, that they are abstract, and that they are independent of intelligent minds [13]. Shapiro refers to this view as *realism in ontology* and observes that “among ontological realists, the most common view is that mathematical objects are acausal, eternal, indestructible, and not part of space-time” [20, p. 27]. As we did in section 2 about God, we are going to formalize these ideas into discrete statements about mathematical objects, which we will be able to focus on and refer to throughout the rest of the paper.⁸ As a reminder, all of these statements are summarized for convenience in the appendix on page 268.

⁷A reader need not be a mathematician to understand this paper, nor need she have an understanding of this list of mathematical objects. The basic concepts of number and shape will suffice.

⁸It is worth reflecting on the fact that while [GC](#), [GA](#), and [GO](#) are derived from theology which is derived from scripture (and tradition and reason), the statements we will formalize about mathematical objects will be derived largely from our intuition. That is to say, Mathematical Platonism seems to be the natural perspective to hold about mathematical objects. Again, this is only an observation about Mathematical Platonism, not an argument for its truth. Our goal here is not to determine whether Mathematical Platonism itself is true or false anyway, but rather to decide whether it is true or false that Mathematical Platonism is compatible with [GC](#), [GA](#), and [GO](#).

The idea of *necessity* is the opposite of *contingency*. A *contingent* object may have existed or not existed, depending on the circumstances. For example, the beet and radish plants in the garden in front of me would not exist had my wife chosen to plant flowers instead. In contrast, a *necessary* object could not but exist, no matter what. As a relatively non-controversial example (among Christians of course), God is a necessary object; there is no possible world in which he does not exist. Nothing can happen to alter his existence. Less clear is whether a mathematical object such as the number 2 is necessary. Starting with our intuition again, it does appear to be the case that the number 2 could not fail to exist. No matter what my wife planted in the garden, how could it possibly *not* be the case that a pair of them would correspond to the number 2? How could there be a world in which one discrete unit and another, together, would not be accurately described by the number 2? Numbers strike us as things that *have* to exist, and further, they have to exist *just as they are*. We cannot conceive of, say, a different number system. Of course, we could use different symbols or a different radix⁹ to denote numbers, but the underlying numbers themselves seem to just *be there*, existing, no matter how we refer to them. This apparent necessity of numbers and other mathematical objects gives us the following statement.

Statement (MN). *Mathematical objects are necessary.*

In Shapiro’s quote above, we see that one commonly (but not universally) accepted description of abstract objects (and thus also of mathematical objects) is that they are acausal or causally inert. An object’s being *causally inert* means that it does “not have the potential to enter into an ordinary causal chain” [2, p. 5]. As Peter van Inwagen colorfully writes, abstract objects’ existence “has nothing to do with causation. One can no more cause a purely qualitative property or proposition to exist than one can extract a cube root with a forceps. Causation is simply irrelevant to the being (and the intrinsic properties) of abstract objects. And, if abstract objects and their intrinsic features cannot be effects, neither can they be causes. An abstract object can be neither of the terms in any causal relation” [22]. Van Inwagen is saying that a mathematical object such as a circle cannot possibly be caused to exist or otherwise affected by something else, nor can the circle cause anything else; it cannot cause something to exist, to change, or to stop existing. Mathematical objects simply have no effects. We will formalize this idea with the following statement.

Statement (MCI). *Mathematical objects are causally inert: they cannot participate in causal chains as causes or effects.*

As noted previously, Mathematical Platonism asserts that mathematical objects are independent of intelligent minds. For one thing, this means that a mathematical object would exist regardless of whether any mind thinks of it. It is not brought into being by or in any way affected by the will, decision, desire, or action of any mind. As an example, my car is *not* independent of intelligent minds because although there are times when nobody (at least no human) is thinking of it, engineers decided to design it and factory managers decided to build it. It would not exist without the actions of some (very intelligent) people’s minds. Similarly, Harry Potter is *not* independent of intelligent minds because he was thought up by J. K. Rowling. On the other hand, something like the number 2 *is* independent of intelligent minds, at least according to Mathematical Platonism (and our intuition would seem to agree). Since it seems that the number 2 existed before humanity, it is hard to see how 2 could be a product of a mind. It certainly seems like neither you nor I could

⁹The radix is the number of unique digits (including 0) used to name the natural numbers. We generally use a decimal (or base 10) system, for we use ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Another somewhat familiar system is binary, the two-digit system ubiquitous in computer science which uses only 0’s and 1’s. The point, though, is that even though the number we call “three” in English can be denoted by the decimal number 3 or the binary number 10 or the Roman numeral III, each of these representations has the same referent.

alter or affect 2 in any way.

Statement (MIIM). *The existence of Mathematical objects is independent of intelligent minds.*

Mathematical Platonism also claims that mathematical objects are eternal. Now the word *eternal* can be interpreted in at least three ways. Imagining a timeline charting an object's existence, the object may have a certain beginning point but continue existing into the future without end,¹⁰ or it may have existed indefinitely into the past with no beginning point but have a definite endpoint,¹¹ or it may exist indefinitely into the past and into the future, as in the Christian conception of God. This last option is what I will mean when I talk about objects being eternal. There is no point in time at which an eternal object does not exist.

Statement (ME). *Mathematical objects are eternal.*

Another commonly held (but not universally held, cf. [5]) view is that abstract objects, and thus also mathematical objects, are non-spatiotemporal; they do not occupy space-time. The time element of this idea we have already begun to discuss above by characterizing mathematical objects as eternal, but this idea goes a bit further by implying that not only do mathematical objects exist at every point in time, but they cannot change, for if they were to change, there would be two points in time at which a mathematical object would not be identical to itself, as a child is distinct from his adult self. However, being non-spatiotemporal means that besides merely *existing* at every point of time, mathematical objects have no other relation to time. The empty set and the hexagon and the number 2 do not and cannot, say, evolve over the centuries.¹²

Next there is the *space* aspect of space-time. To say that mathematical objects are non-spatiotemporal means that one cannot find geometric lines lurking in Libya. There is no telescope or microscope that will reveal the location of the number 2. An Aristotelian perspective would hold abstract objects to exist *in rebus*, in things. So the number 2 would be located in the space where a bipedal animal exists and in a pair of chopsticks. However Platonist views generally reject this perspective.

Statement (MNST). *Mathematical objects are non-spatiotemporal.*

4 Abstract Objects

In the next section, we will discuss apparent contradictions between Mathematical Platonism (specifically MN, ME, MCI, MNST, and MIIM) and the existence of God (specifically GC, GA, and GO), but first, since mathematical objects are usually considered to be examples of abstract objects, it is worth thinking briefly about abstract objects in general. Notoriously difficult to define, abstract objects have no single agreed-upon definition. Amusingly, Welty describes the abstract/concrete distinction to be “in disarray, ontologically speaking” [8, 94]. Gideon Rosen claims that the most widely accepted definition of an abstract object is that it is non-spatiotemporal and causally inefficacious [19]. However, none of the other sources I read used that precise definition, some refuted it, and others did not even try to define abstract objects. For example, Shapiro defines a concrete object as one that is “‘physical’ or ‘spatio-temporal,’” so an abstract object would be defined as

¹⁰For example, Christians believe that the human soul comes into existence at conception but that if one is a Christian, the soul has no end. (Whether or not the soul of a nonbeliever has an endpoint in time is the subject of debate between those who hold to “Eternal Conscious Torment” and those who hold to “Annihilationism” which are completely outside the scope of this paper.)

¹¹One might imagine a deity who eventually dies.

¹²Of course, this is not to say that humanity's *understanding* of mathematics does not change and evolve over time.

a non-spatiotemporal object, but he says nothing about causal efficacy [20, p. 162]. In contrast, Craig uses causal impotence exclusively as the dividing line between concrete and abstract objects, and he explicitly rejects the idea that abstract objects are non-spatiotemporal, proffering examples such as the equator, the center of mass of the universe, and God himself [5]. The causal impotence of an object X affirms that X cannot affect anything, but not that X cannot itself be affected by anything. (This is why Craig uses the term causal “impotence” rather than something like causal “intertness” or causal “inaptness”.) Then there is Bøhn, who defines abstract objects by two criteria, first that they are causally inapt (they cannot be causes or effects, unlike Craig’s concept), and second, that they do not obey the law of the excluded middle with respect to properties,¹³ which was a unique idea among my sources [2]. To muddy the waters further, Bøhn suggests that perhaps the abstract/concrete divide is not even a partition of the set of all objects but rather a spectrum along which each object can be placed. For example, he might suggest the following objects are listed in order of decreasing abstractness: the empty set, the set containing the number 1, the set containing George Washington, George Washington himself.

In light of the fuzzy, ill-defined line between concrete and abstract objects, many philosophers do not even attempt to carefully define the distinction. “Usually, discussion of the reality of abstract objects is able to proceed on the basis of shared examples without a clear delineation between concrete and abstract” [6, p. 8] (*cf.* [8, 14, 21]). The idea is that if you give a philosopher a list of objects, she could fairly easily sort them into two categories: abstract and concrete. Van Inwagen writes that “this procedure will yield pretty consistent results,” though he acknowledges that some objects will be difficult to categorize [21, p. 109]. From van Inwagen’s (and others’) perspective, we all more or less know the difference between concrete and abstract objects even though we cannot precisely define it. While this may be largely true, the lack of agreement about the concrete/abstract distinction still leads to confusion. For example, Chris Menzel ([14]) (and most philosophers) would categorize numbers as abstract objects, but Bøhn and Craig, using different definitions, would say Menzel’s conception of numbers as ideas in the mind of God (see section 7) implies that numbers are concrete objects instead. I find this kind of confusion unhelpful.

As our interest here is not in abstract objects generally but in mathematical objects, I will attempt to avoid the difficulties posed by the ambiguity of the nature and definition of abstract objects by focusing solely on mathematical objects. This will allow us not to get hung up on the definition of abstract objects, whether abstract objects exist, or whether God’s thoughts are abstract or concrete. Most people believe that if abstract objects exist, then mathematical objects are a prototypical subcategory of abstract objects. It follows that whatever my sources write about abstract objects will usually apply to mathematical objects, so whenever a quote in this paper refers to abstract objects, the reader can assume, unless otherwise specified, the quote applies to mathematical objects as well.

¹³This is a concept I did not find in any of my other sources. The idea is that for any given property (say, the property of redness) a concrete object either has the property or it does not; it obeys the Law of the Excluded Middle with respect to properties. On the other hand, for an abstract object, there is a third or “middle” option: something like “does not apply.” The abstract object 2 cannot be said to have the property of redness or to not have it; rather, redness just does not apply to 2. Thus, 2 does not obey the law of the excluded middle with respect to properties. I do not find an appeal to the law of the excluded middle with respect for properties helpful (or perhaps I misunderstand Bøhn’s meaning) because for example, I think we can positively say that the number 2 is *not* red. Further, it does not seem that concrete objects necessarily follow the law of the excluded middle with respect to properties either because they are able to *partially* or *unevenly* have properties. Does my house have the property of being green? Well, the window shudders and a few of the interior walls are painted various shades of green, but most of the house is not green. It is not clear that my house should be considered to have the property of greenness. Does it have the property of being large? Well, yes and no; it is larger than a cat but smaller than a planet.

I am going to take a page out of van Inwagen’s book and decline to define mathematical objects, instead relying on a common understanding of what they are. Paradigmatic examples of objects we could classify as mathematical objects are sets, numbers, shapes, and propositions about these objects, such as “ $2 + 2 = 4$ ” or “*There are infinitely many prime numbers.*”

5 The Apparent Contradictions

In this section we will outline several philosophical problems that arise in simultaneously positing the existence of the Christian God and the existence of mathematical objects. With a few exceptions, we will not attempt to resolve these problems in this section, but in the following sections, we will see how specific philosophical models address them. It is worth pointing out for the sake of completeness that while there are philosophical quandaries at the intersection of Christianity and Mathematical Platonism, other problems arise in holding both naturalism and Mathematical Platonism together. For example there is the access problem: “How can humans, as physical organisms inhabiting a physical universe, have intuitive knowledge of a causally inert realm of abstract objects?” [20, p. 221]. It is arguably easier for Christians to answer this question with God as an explanatory item in our worldview. For example, one might argue if mathematical objects exist in the mind of God as Theistic Activism (section 6) and Divine Conceptualism (section 7) posit, one can appeal to the *Imago Dei*. Menzel writes, “we are *like* God in certain important respect such as consciousness and rationality” [14]. One could then take a position that in God’s creating us in his image, he intentionally instilled mathematical perception in our minds. Much more would need to be said in order to construct a proper argument in this direction. The point is simply that while this paper focuses on philosophical problems between Christianity and Mathematical Platonism, there are also serious problems between naturalism and Mathematical Platonism.

5.1 The Eternality Problem

Statement [ME](#) implies that mathematical objects do not have a beginning point. How then could there be any moment at which they were created? It seems that [ME](#) implies there are objects which are uncreated. But recall that [GC](#) claims that all things are created by God. Thus some *objects* are uncreated, and all *things* are created. As van Inwagen points out, the words *thing* and *object* are generally synonymous in these discussions¹⁴ [21]. Therefore while [GC](#) states that God created every *thing*, [ME](#) seems to imply that the *things* which are mathematical objects are uncreated since they have no beginning point. This is the eternity problem, and we will write it out explicitly as a contradiction.

Contradiction (ETERNALITY). *Mathematical objects have no beginning point, and they began when God created them.*

5.2 The Causal Problem

Next, it seems clear that to create something is one way of *causing* that thing. As van Inwagen writes, “Creation is, in the broadest sense of the word, a *causal* relation” [22, p. 5]. It follows that if [GC](#) is true then God *caused* mathematical objects to exist. Explicitly, God is the cause, and

¹⁴In fact, he states that the words *thing*, *item*, *entity*, and *object* are synonymous [21]. Elsewhere, he more specifically defines an ‘object’ to be “anything that can be the value of a variable, anything that can be referred to by the use of a pronoun” [22], and we are treating the term the same way in this paper. Thus persons—including God—are objects, though in these cases, it feels more respectful to refer to them by the synonym *entity* or even to just stick with *person*, understanding that all persons are objects (though the converse is of course false).

mathematical objects are the effect. This of course contradicts [MCI](#), which claims that mathematical objects cannot participate in causal chains. Thus we have the causal problem.

Contradiction (CAUSAL). *Mathematical objects are caused, and mathematical objects are uncaused.*

5.3 The Mind-Independence Problem

The next problem is similar to the last: If the existence of mathematical objects is independent of intelligent minds ([MIIM](#)), then does not this imply that the existence of mathematical objects is independent of God's mind? But how could this be so if God created them ([GC](#))? After all, the Christian vision of God is not of some impersonal, mindless force (like the Force in *Star Wars*), but of a thinking *person*. Yet the statement [MIIM](#) would seem to imply that no intelligent mind—including God's—is involved in the creation or existence of mathematical objects. We will call this the Mind-Independence Problem.

Contradiction (MIND-INDEPENDENCE). *The existence of mathematical objects is due to God's mind, and the existence of mathematical objects is due to nobody's mind.*

5.4 The Freedom and Sovereignty Problem

Given [MN](#), mathematical objects *must* exist. They could not have *not* existed. Mathematical objects' necessity seems to imply that if they were created by God ([GC](#)), God had no choice but to create them. Their necessity seems also to imply that God has no power to change or destroy mathematical objects (e.g., to make the set \mathbb{N} of natural numbers a finite set, to abolish circles, or to change the proposition " $2 + 2 = 4$ " from true to false). Yet if God was *constrained* to create mathematical objects, and he is unable to edit them, so to speak, does not this compromise both his freedom and his sovereignty? This is the Freedom and Sovereignty Problem.

Contradiction (FREEDOM). *God has creative power over everything, and God does not have creative power over mathematical objects.*

5.5 The Uniqueness Problem

If [MIIM](#) and [MN](#) are true, it seems to follow that mathematical objects are self-existent, not relying on anyone or anything else for their existence. In other words, it seems that mathematical objects have the property of *aseity*. Furthermore, Craig brings up a perspective held by the early church Fathers, which is that eternality implies deity [5]. Thus if [ME](#) is true, this would imply that mathematical objects are divine, so mathematical objects are divine, *a se* objects alongside God. Christian theology generally considers God to be the *only* entity which is the ultimate transcendent reality, the source of all things. This gives us a picture of mathematical objects as a sort of competitor for God's primacy. (This is Craig's deepest concern about Mathematical Platonism, leading him to reject it.)

Contradiction (UNIQUENESS). *God is the unique transcendent reality existing a se, but also, mathematical objects have a real, transcendent, a se existence.*

5.6 The Dependency Problem

According to [GA](#), God cannot rely or depend upon anything else, including any mathematical object. Yet consider the doctrine of the Trinity. Though the doctrine is mysterious, it is clear at least that it relies on the numbers 1 and 3 and on a notion of perhaps not *equality* but at least some kind of equivalence relation.¹⁵ Since God is understood to be a necessary being, he could not have been otherwise, implying that he does not have control over how he is, over his properties. For example, he cannot choose not to be a Trinity of three persons. This being the case, one can argue that there is a sense in which God's existence fundamentally relies on at least some mathematical objects.¹⁶ Thus we reach another contradiction.

Contradiction (DEPENDENCY). *God depends on nothing, and he depends on (at least some) mathematical objects.*

5.7 Two Quick Solutions

The eternity problem and the causal problem are actually not difficult to solve. The solution to the eternity problem is to adjust our notion of creation. Up to now, whenever we have used the word *creation*, we have essentially been implying *creation at a point in time*, but this is not the only way to understand the term. According to Hebrews 1:3, “[God’s Son] is the radiance of the glory of God and the exact imprint of his nature, and *he upholds the universe* by the word of his power . . .” About the same Son, Colossians 1:17 says “And he is before all things, and *in him all things hold together*.” (The emphasis is mine in both verses.) Christians are not deists; we do not believe that when God initially created the world “in the beginning,” he infused it with ontological momentum,¹⁷ so that it could go on existing on its own, as a clockmaker might create a pocket watch, wind, start, and leave it alone to tick away independently. Rather, as the verses suggest, Christians hold that everything is existentially dependent from moment to moment on the creative power of God. If God were to withdraw his sustaining hand, the universe would cease to exist. Stephen Williams writes, “The notion of providence encapsulates the conviction that God sustains the world that he has created and directs it to its appointed destiny” [24, p. 711]. Thus while Christians do hold that there is a beginning point to our concrete universe when God initially brought everything into existence, we do not view his creative work in general as a single finished event but as a continuous, ongoing act. Menzel refers to this as *continuous creation* [14].

With this enhanced understanding of creation in mind, we begin to see how the eternity problem may be solved. To say that mathematical objects are created is only to say that at every point in time, God creates or sustains or upholds mathematical objects, not necessarily that they have a beginning. Therefore the eternity problem is easily dissolved without compromising either theology or Mathematical Platonism.

The causal problem is not too difficult to solve either, provided we allow a slight adjustment of [MCI](#). As noted earlier, there is not general agreement about what abstract objects and mathematical

¹⁵An *equivalence relation* is a relation \approx which is reflexive (for all objects X , it is always true that $X \approx X$), symmetric (for all objects X and Y , if $X \approx Y$ then $Y \approx X$), and transitive (refer to footnote 21 for the definition of transitive). We evoke equivalence relations here because there is some sort of equivalence relation between the members of the Trinity. Again, the three are not *equal* to each other, but all of them are in a mysterious way, one essence, one God.

¹⁶This case is often made in the more general context of abstract objects, where it is argued that God depends on his properties (which are abstract objects), such as omniscience, wisdom, aseity, and existence. For example, see [17] and Gould and Davis’s chapter in [8].

¹⁷I picked up the helpful phrase “ontological momentum” from [14].

objects are. In particular, not everyone considers causality to be fundamentally related to their ontology. Of those who do, some, like Craig, take issue with [MCI](#). Craig writes, abstract objects are “not related to other objects as causes to effects” [5, p. 6]. Notice that this concept is subtly different than [MCI](#): A mathematical object *can* participate in a causal chain, but if it does, it must be the last link in the chain, so to speak. That is, mathematical objects may *be* caused, but they themselves are unable to cause anything else. Overall it would not be overly controversial to replace [MCI](#) with the following statement.

Statement (MCI-2). *Mathematical objects are causally impotent; they cannot cause or affect anything.*

Replacing [MCI](#) with [MCI-2](#) in our Mathematical Platonism schema quickly and cleanly dissolves the causal problem.

6 Created and Dependent: Theistic Activism

The first philosophical model we consider which attempts to iron out the apparent discrepancies between Christianity and Mathematical Platonism is Theistic Activism. According to Thomas V. Morris, this model demonstrates the “extraordinary compatibility of core elements of these two profound metaphysical visions” [15]. The first point is the assertion that abstract objects—and therefore also mathematical objects—are created.¹⁸ Morris and Menzel write, “God is absolute creator of necessary as well as contingent reality, and... literally all things do depend on him” [16]. God is not only the creator of objects that may or may not exist, such as cats, trees, stars, and you and me; he is further the creator of everything that exists necessarily, including mathematical objects.¹⁹ How is God responsible for this entire “framework of reality,” as they call it? It is a product of God’s creative intellectual activity. God’s mind is continually, eternally creating and sustaining the whole host of mathematical objects, and they are in a sense located in his mind.

It is not clear whether mathematical objects are considered to be merely *products* of God’s thinking or God’s very thoughts themselves; perhaps different theistic activists would take different positions. For example in describing this model, Gould directly identifies abstract objects with divine thoughts and concepts [8, p. 8], while in contrast, Craig describes this model as holding abstract objects to be “caused by God’s intellectual activity” [5, p. 55]. Either way, whether mathematical objects *are* God’s thoughts and concepts or only *caused by* God’s thoughts and concepts, Theistic Activism affirms that mathematical objects are in some sense located in God’s mind and dependent on him for their existence.

The Theistic Activism model wholeheartedly affirms [GC](#), although we will need to take another look at [GA](#) and [GO](#) below. As for statements about mathematical objects, Theistic Activism would affirm [MN](#), [ME](#), and [MCI-2](#). [MNST](#) is accepted as well, for mathematical objects are viewed as existing in the mind of God, and God is generally understood to be non-spatiotemporal.

Theistic Activism would fundamentally reject [MIIM](#), as mathematical objects are here seen to *be* elements within the mind of God, so this model would opt instead for the following statement.

Statement (MIIM-2). *The existence of mathematical objects is independent of all intelligent minds except God’s.*

¹⁸Thus Theistic Activism is also referred to as “Absolute Creation.”

¹⁹Morris and Menzel assert that God is not only responsible for necessarily existing objects, but also for necessary truths [16].

MIIM-2 still allows us to affirm our intuition that mathematical objects are independent of *our* human minds. My thoughts have no bearing on a number or set, for example.

In my view, the slight modifications which the Theistic Activism model makes to Mathematical Platonism are not so dramatic as to prevent us from viewing this model as affirming Mathematical Platonism. However, a crucial question remains: does Theistic Activism succeed in resolving the apparent contradictions between Christianity and Mathematical Platonism? The Eternality Problem is easily solved as described in section 5. Likewise, the Mind-Independence Problem is solved since MIIM has been replaced with MIIM-2. As for the causal problem, Theistic Activism simply denies MCI and adopts MCI-2 instead, for as Gould and Davis explain, “the relation between thought and thinker is most naturally understood as a *productive* relation: the thinking produces his thoughts. Thus, it is natural to think abstracta are created” [8, p. 100].

What about the Freedom and Sovereignty Problem? Morris and Menzel affirm MN, writing that “God’s creation of the framework of reality is both eternal and necessary—it never was, never will be, and could not have been, other than it is” [16]. Does this imply that God is not free to *not* create mathematical objects? In other words, is God constrained to create them? Their answer is ‘yes,’ but they point out that according to Theistic Activism, “the necessity of [God’s] creating the framework is not imposed on him from without, but rather is a feature and result of the nature of his own activity itself, which is a function of what he is.” Since mathematical objects are (or are products of) God’s own thoughts and concepts, his necessary creation of them is an activity flowing from his nature.

Theologians already agree that God is in a sense constrained to be himself, to act consistently with his own nature. For instance, he cannot sin or fail to be omnipotent or omniscient. Erickson writes that God “is not bound by anything external to himself. His only limitations are those of his own nature and the choices he has made” [7, p. 401]. Let us formalize this idea, as it will be convenient to refer to later too.

Statement (The Nature Principle). *God always and exclusively acts in ways consistent with his own nature.*

The point here is that by the Nature Principle, God necessarily acts consistently with his nature to create the framework of reality (including mathematical objects) in exactly the form it exists; he could not make it otherwise without negating himself, without denying his own nature. Therefore, no, God is not free to not create mathematical objects, but this is only because of the already acceptable proposition that he is not free to go against his nature.

This has addressed the freedom portion of the Freedom and Sovereignty Problem, but what about the sovereignty concern? Morris writes, “theists can acknowledge the standard Platonist view that God is not in control of abstract objects or necessary truths, in the sense that he cannot annihilate or alter them intrinsically, while at the [same] time maintaining that these things depend on God for their existence and intrinsic characteristics” [15, p. 171]. Thus the theistic activist must alter GO as follows.

Statement (GO-2). *God is omnipotent in the sense that while all things concrete and abstract depend on God for their being, nevertheless God does not have the power to alter some types of things (including mathematical objects), whether by changing them or destroying them, or by not having created them in the first place.*

An appeal to the Nature Principle would explain why this modification of GO does not diminish God's sovereignty or power. Exercising his power to alter mathematical objects would go against his logical, ordered character; moreover, it is against the framework of reality that he himself creates and maintains.

Turning to the Uniqueness Problem, Morris and Menzel do not consider God to be the only transcendent reality, but they describe him as "the source of absolutely everything there is;... he is in the deepest sense possible the ground of all being" [16]. The Theistic Activism model holds that God is not the only eternal and necessary being, yet it does not follow that God is one god among many. God is still supreme over all things, including mathematical objects. After all, mathematical objects only exist *in* God's mind. He is the source and ground of every one of them. He is the ultimate author of the framework of reality. The fact that mathematical objects are necessary and eternal like God does not threaten God's unique place as the king of the universe, supreme over everything.

How does Theistic Activism address the Dependency Problem? First of all, since mathematical objects are located in the mind of God, we can say that God does not depend for his existence on anything *external to himself*. Saying that he depends only on his own thoughts or on his own nature seems to take the teeth out of the Dependency Problem; however new problems arise from this solution. Notably, there is what is often called the Bootstrapping Problem. If God creates mathematical objects, and if God's existence depends on the existence of at least some mathematical objects (consider the number 3), then is he not creating what he depends on to exist? Another way of stating the question is this: if God depends on mathematical objects, and mathematical objects depend on God, then doesn't it follow that God depends on himself? If so, this initially seems incoherently circular, but Morris and Menzel argue that "there is no absurdity or unacceptable circularity about this when properly understood," and their justification hinges on an asymmetry of dependence; they distinguish between how God depends on his nature and how his nature depends on him. The dependence of God's nature on God is a *causal* dependence relation; God causes his nature by creating it. In contrast, God's dependence on his nature is a *logical* dependence relation. What does this mean? We will say that "*A* logically depends on *B*" if the statement "if *B* exists, then *A* exists" is true. Consider the statement,

$$\text{If God does not exist, then mathematical objects do not exist.} \quad (1)$$

Recall that the statement "if *P* then *Q*" is always true whenever *P* is false. As God does exist, it follows that (1) is true. Recall also that any if-then statement is logically equivalent to its contrapositive, and thus it is true that

$$\text{If mathematical objects exist, then God exists.} \quad (2)$$

Therefore God logically depends on mathematical objects.²⁰ While the relations of causal dependence and logical dependence relations are individually transitive,²¹ they are not necessarily

²⁰Interestingly, we can generalize this by replacing 'God' in this formulation by any necessarily existing object, and replacing 'mathematical objects' with any object whatsoever, and we would conclude that the necessary object logically depends on the other arbitrary object. I do not think this type of dependence is a very deep, existential kind of dependence. In particular, it is not a causal or explanatory kind of dependence. It does not seem to me that it diminishes a necessary object in any way to logically depend on something else. In particular I do not think this kind of dependence compromises God's aseity.

²¹To say that a relation \sim is transitive is to say that if $A \sim B$ and $B \sim C$, then $A \sim C$. Familiar transitive relations from mathematics include equality ($=$) and inequality ($<$, \leq , $>$, and \geq). Familiar transitive relations from

transitive together. To illustrate this, consider a simpler example using the commonplace transitive relations *older than* and *taller than* in statements (3) and (4) below.

“Abby is *older* than Brooke” (3)

“Brooke is *taller* than Carlos.” (4)

The conjunction of statements (3) and (4) does not give us any information about Abby’s age or height relative to Carlos. Abby may be older than, younger than, or the same age as Carlos, and likewise, she may be taller than, shorter than, or the same height as him. Next let us forget about Carlos and suppose that

“Brooke is *taller* than Abby.” (5)

It would be silly to conclude that the conjunction of statements (3) and (5) implies that “Abby is older than herself” or that “Abby is taller than herself.” The point of this illustration is that it is illogical to appeal to transitivity to claim that that since God *logically* depends on his nature and his nature *causally* depends on God, it therefore follows that God depends (logically or causally) on himself. Thus Theistic Activism does not imply that God incoherently depends on himself or creates himself, which would be philosophically problematic.

To summarize, the driving motivation behind this model seems to be to preserve the widest and most natural understanding of the statement GC by locating mathematical objects in God’s mind. Theistic Activism’s biggest problems lie with GO, which is replaced by GO-2. The model would basically affirm GA, but since God is held to logically depend on some mathematical objects (which are products of his own mental activity), proponents would clarify the statement as follows.

Statement (GA-2). *God exists a se; he does not depend on anything external to himself for his existence.*

7 Uncreated and Dependent: Divine Conceptualism

Although this model differs from Theistic Activism on the question of whether mathematical objects are created, the differences are subtle. While Theistic Activism is wishy washy about whether mathematical objects are God’s thoughts or are merely *products* of his thoughts, Divine Conceptualism takes a definite stand: mathematical objects (and abstract objects in general) *are* God’s thoughts and beliefs.²²²³ Menzel argues that properties and relations are God’s concepts, and propositions are his thoughts, and he specifically identifies numbers as *properties* of mathematical sets [14]. Of sets, he argues, “they can be regarded as part of a collecting activity on God’s part” [14, p. 81]. Similarly, Alvin Plantinga describes Divine Conceptualism as “a view popular among medieval philosophers...: the view that abstract objects are really divine thoughts. More exactly,

everyday experience include “being the ancestor of” and “being inside of.” To illustrate, if Abby is Brooke’s ancestor and Brooke is Carlos’s ancestor, then Abby is Carlos’s ancestor, and if an apple is inside of a backpack and that backpack is inside of a car, then the apple must be inside of the car too. (Although, on the other hand, if Lucy is inside of a wardrobe, and the wardrobe is inside of a house, it does not necessarily follow that Lucy is in the house...)

²²In [22], van Inwagen argues that while it is clear enough how propositions (at least true ones) can be God’s thoughts, it does not make sense to say that other abstract objects, such as numbers, can be God’s thoughts.

²³This is another point at which it is advantageous that we are focusing on mathematical objects rather than abstract objects: there is disagreement about whether God’s concepts and thoughts should be considered concrete or abstract. Thus, Divine Conceptualism (and possibly Theistic Activism) may be advocating for the existence of abstract objects in God’s mind, or from another perspective, as the rejection of the existence of abstract objects since all of the objects usually considered abstract objects (e.g., properties, propositions, and mathematical objects) are instead put in the category of God’s thoughts which are *concrete* objects (cf. Craig in [5] and Craig and Yandell in [8, pp. 97, 100]).

propositions are divine thoughts, properties divine concepts, and sets divine collections” [18].

Greg Welty writes that Divine Conceptualism (which he calls “Theistic Conceptual Realism”) “holds that [abstract objects] are necessarily existing, uncreated divine ideas that are distinct from God and dependent on God” [8, p. 81]. From this statement we see that first of all, this model rejects **GC**, replacing it with the following statement.

Statement (GC-2). *Everything concrete (besides God himself) is created by God.*

While there are philosophical reasons for adopting **GC-2** in place of **GC**, it brings up a theological question: why is it valid for a Christian to reject **GC**? We will put off answering this question until section 8 in which we discuss Platonic Theism, which replaces **GC** with **GC-2** in the same way.

Welty’s quote above also shows that the Divine Conceptualism model upholds **MN**. That mathematical objects are necessary entails that they are also eternal, so **ME** is also upheld.

If Divine Conceptualism claims that mathematical objects depend on God but are not created by him, does Divine Conceptualism affirm **MCI**? Suppose I set a clock on a high shelf. The clock is then depending on the shelf for its elevation and even for its continued existence, for if the shelf stops supporting the clock, it will fall to the floor and break. We could argue that the shelf is *causing* the clock to not fall. It seems that one could argue that if object x depends on object y , then x and y are participating in a causal chain in which y is affecting x , and so x is not causally inert. Thus since Divine Conceptualism holds mathematical objects to be dependent on God, mathematical objects are not causally inert; they are caused by God. There is nothing about Divine Conceptualism that would cause one to doubt that mathematical objects are causally impotent though, so therefore, like the Theistic Activism model, the Divine Conceptualism model affirms **MCI-2** in place of **MCI**.

The similarity between Theistic Activism and Divine Conceptualism allow us to piggyback on many of the arguments made in section 6. For the same reasons described in that section, Divine Conceptualism affirms **GA-2**, **GO-2**, **MNST**, and **MIIM-2**, and the Divine Conceptualism model can provide solutions to the problems outlined in section 5 in the same way as Theistic Activism.

8 Uncreated and Independent: Platonic Theism

I will call the next model Platonic Theism, following Gould [8]. Platonic theists such as van Inwagen attempt to resolve the problems raised in section 5 by viewing mathematical objects as uncreated and independent of God [22, 23]. This model preserves all five statements of Mathematical Platonism (**MN**, **ME**, **MCI**, **MNST**, and **MIIM**), but it requires some amount of curtailing of **GC**, **GA**, and **GO**.

Adherents of Platonic Theism believe **MIIM** is true and that it rules out **GC**; they hold mathematical objects to be uncreated and consider the idea of creating or causing necessary, eternal objects to be incoherent. Thus Platonic Theism, like Divine Conceptualism, replaces **GC** with **GC-2**. With this concept of mathematical objects, there is conveniently no Causal Problem (mathematical objects are simply not caused) nor a Mind-Independence Problem (the mind-independence of mathematical objects does not conflict with **GC-2**), but Christian adherents of this view must now explain why it is theologically valid to reject **GC**.

The Platonic theist’s basic premise is that when Biblical authors talk about God creating ‘everything’ or ‘all things’ in passages such as John 1 and Colossians 1, they were not actually talking

about mathematical objects (nor other abstract objects); rather, the Biblical authors meant that everything *concrete* was created by God. The argument is that although the Biblical authors probably did not have the abstract/concrete distinction in mind as they wrote, nevertheless what they intended to communicate was that God created absolutely all concrete objects. Platonic theists do not deny the truth of scripture. They only disagree with the absolute creationist about how one should interpret these ‘everything’ passages.²⁴

To illustrate how a Platonic theist interprets these ‘everything’ passages, imagine a child coming to school late, walking into her classroom and finding it empty. The janitor in the hallway tells her, “everyone is outside on the playground.” Even though the janitor used the word “everyone,” presumably he is not attempting to mislead the student and convince her that every single person in the world is on the playground. Rather, in the context of their exchange, the girl and the janitor would understand that the janitor’s use of the word “everyone” has a restricted domain, namely, *all the other students in that particular class*. The janitor’s response was not false or deceitful. In describing this view, Craig gives a similar example in [5]: when you say “there’s nothing in the refrigerator,” you don’t mean to imply that the refrigerator doesn’t even contain air molecules.

Back to metaphysics, the argument is that in verses like John 1:3, “All things were made through him, and without him was not any thing made that was made,” the domain of the phrase *all things* is the set of concrete objects, not the entire unrestricted collection of all objects of every type. (See [5, 6, 15] for counterarguments.) In this way, adherents of Platonic Theism can stand on philosophically solid ground in affirming verses like John 1:3 and Colossians 1:15-16 while at the same time claiming that God is not the creator of mathematical objects.

What about the Freedom and Sovereignty Problem? Under Platonic Theism, God’s freedom is restricted; he does *not* have the power to create or change or abolish mathematical objects. The proposition that the sum of two odd numbers is even is a necessary truth that God cannot change. This does compromise GO, or at least it requires a reinterpretation of it. Platonic Theism would opt for the following modification.

Statement (GO-3). *God’s omnipotence extends absolutely over the realm of concrete objects.*

Under Platonic Theism, God in some sense depends on mathematical objects (like the number 3 and equivalence relations, as described in section 7), which means the Platonic theist needs to modify or reject GA. GA-2 is not an option, for adherents of Platonic Theism do not consider abstract objects as being in the mind of God or depending on him in any fundamental way. Platonic Theism will at least affirm the following weaker version of GA.

Statement (GA-3). *God exists a se in the sense that he depends on no concrete object for his existence.*

This allows proponents of Platonic Theism to quickly answer the Dependency Problem: it is simply not the case that God depends on nothing. He does depend in some way on mathematical objects.²⁵ They would further argue along with proponents of Divine Conceptualism that whatever kind of dependence God has on mathematical objects, it is not a causal or explanatory dependence. There

²⁴This is so often the case when there are debates about theology among Christians. Whether you look at complementarian/egalitarian theology, Calvinism/Arminianism, differing perspectives about the current work of the Spirit, or a hundred other questions, it is rarely the case that one side claims scripture is true and the other claims it is false. Instead, both sides believe in the truth and authority of scripture, but they disagree about how certain key passages and broad Biblical narratives should be understood and interpreted.

²⁵and on other abstract objects too, such as *wisdom* and *omniscience*

is no sense in which God’s dependence on mathematical objects in this trivial way diminishes his glory, majesty, greatness, and supremacy in the slightest. While any dependence on concrete objects would be theologically problematic, that is not the case for abstract objects.

The Uniqueness Problem is perhaps the hardest problem for Platonic Theism, since this model is the one that holds Mathematical Platonism most intact. Under this model, it is the case that there are uncountably many²⁶ necessary, eternal, uncaused, self-existent entities in existence alongside God. As J. Thomas Bridges phrases it, this model “commits [the Christian philosopher] to a realm of entities existing eternally alongside God without being caused to exist by God. Such a position does seem to violate a straightforward understanding of the doctrines of aseity and sovereignty” [3, p. 279]. First of all, proponents would probably start by denying certain church Fathers’ assertion that eternity and necessity imply deity. Just because the number 2 is uncreated, eternal, and necessary, it does not follow that this number is a god or at an equal level of greatness to God in any sense. Even if God has the properties of eternality and necessity in common with mathematical objects, it does not follow that they are equal to him in supremacy or deity, any more than the fact that God shares the property of personhood with humans makes humans equal with God. The proposition that God is not the only *a se* being does not in any way diminish his transcendence, his majesty, his deity, or his supremacy. Platonic Theism does not imply that anything like a Greek pantheon of *actual gods* is real, where Zeus, Athena, Poseidon, etc., are on some kind of equal level with God; Platonic Theism only holds that things like the number 2 and points and squares are real. There is simply no threat to God’s supremacy from that quarter.

9 Conclusion

Christian philosophers like Craig believe that Mathematical Platonism raises insurmountable obstacles because Mathematical Platonism diminishes God. His solution is to reject Mathematical Platonism. The three models we have presented above can be viewed on a spectrum, categorized by how much they agree with Craig. Farthest from Craig’s view is Platonic Theism, which on the whole simply disagrees that Mathematical Platonism is any threat to God’s supremacy. Closest to Craig’s view (but still far enough away to continue to affirm a version of Mathematical Platonism) is Theistic Activism, which preserves a very strong vision of God as creator and a fairly robust understanding of God’s aseity and omnipotence. Divine Conceptualism is somewhere in the middle. All three models modify, nuance, or clarify at least some of the initial statements we made in sections 2 and 3 about God and mathematical objects.

Overall, it seems that a reasonable person could take the position that Mathematical Platonism and Christianity are fully compatible, with Theistic Activism, Divine Conceptualism, and Platonic Theism all offering viable paths toward reconciling the problems at their intersection. Of course, it does not follow that Christianity implies Mathematical Platonism or vice versa. Nor have we considered whether Christianity is compatible with the proposition that Mathematical Platonism is false. If Christianity and the proposition that Mathematical Platonism is false are philosophically or theologically incompatible, then clearly Christians must embrace Mathematical Platonism. However (at the risk of the fallacy of appeal to authority), since there are learned Christian philosophers (such as Craig) who deny Mathematical Platonism, at the very least we have reason to *suspect* that Christianity could be philosophically compatible with the proposition that Mathematical Platon-

²⁶To say that a set S is *uncountable* is to say that unlike the infinite sets \mathbb{Z} (the set of integers) and even \mathbb{Q} (essentially the set of fractions of integers), the set S is so large that even an infinite list of elements of S could not contain every one of them. Another way to say this is that an uncountable set is a bigger size of infinity than the infinity which is the size of the integers.

ism is false. If that were to be the case, Christians would be free to affirm or deny Mathematical Platonism without compromising their faith either way. However, all of that is outside of what we have had the time and space to consider in this paper. Our humbler conclusion here is simply that Christianity and Mathematical Platonism seem to be compatible; both can simultaneously be true.

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11 Appendix

| | Theistic Activism | Divine Conceptualism | Platonic Theism |
|--|----------------------|-------------------------|--------------------|
| [GC] Everything (besides God himself) is created by God. | ✓ | | |
| [GC-2] Everything <i>concrete</i> (besides God himself) is created by God. | | ✓ | ✓ |
| [GA] God exists <i>a se</i> . | | | |
| [GA-2] God exists <i>a se</i> ; he does not depend on anything external to himself for his existence. | ✓ | ✓ | |
| [GA-3] God exists <i>a se</i> in the sense that he depends on no <i>concrete</i> object for his existence. | | | ✓ |
| [GO] God is omnipotent. | | | |
| [GO-2] God is omnipotent in the sense that while all things concrete and abstract depend on God for their being, nevertheless God does not have the power to alter some types of things (including mathematical objects), whether by changing them or destroying them, or by not having created them in the first place. | ✓ | ✓ | |
| [GO-3] God's omnipotence extends absolutely over the realm of concrete objects. | | | ✓ |
| [The Nature Principle] God always and exclusively acts in ways consistent with his own nature. | ✓ | ✓ | ✓ |
| [MN] Mathematical objects are necessary. | ✓ | ✓ | ✓ |
| [ME] Mathematical objects are eternal. | ✓ | ✓ | ✓ |
| [MCI] Mathematical objects are causally inert: they cannot participate in causal chains as causes or effects. | | | ✓ |
| [MCI-2] Mathematical objects are causally impotent; they cannot cause or affect anything. | ✓ | ✓ | |
| [MNST] Mathematical objects are non-spatiotemporal. | ✓ | ✓ | ✓ |
| [MIIM] The existence of Mathematical objects is independent of intelligent minds. | | | ✓ |
| [MIIM-2] The existence of mathematical objects is independent of all intelligent minds except God's. | ✓ | ✓ | |

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An Interdisciplinary Approach to Integrating Faith in the Calculus Classroom

Diana Schepens (Whitworth University)



Diana Schepens (B.A., Houghton College, M.S., Ph.D., Montana State University) enjoys applying mathematics to other STEM disciplines. Her research includes nonlinear dynamical systems in application to the study of microbial consortia, and geometric approaches to molecular shape space in crystallography. 2022 was her first ACMS conference and she is excited to find a community interested in history, philosophy, and theological connections with mathematics.

Abstract

Calculus curriculum is often used as prerequisite material for upper division math and science courses. As such, it can be difficult to find space in the curriculum to accomplish intentional faith integration activities or assignments. I propose that rather than trying to find space for explicit faith integration content, instructors adopt an interdisciplinary approach to the presentation of the calculus curriculum so that faith integration can be woven together with calculus throughout an entire semester.

1 Introduction

The majority of students majoring in science, technology, engineering, and math (STEM) are required to take classes from the calculus sequence. These classes teach them the mathematical skills they will need for their discipline, as well as help develop their critical and logical thinking skills. At Christian liberal arts universities there is often an expectation that instructors integrate faith with learning, but the calculus sequence leaves little room for additional faith-based assignments or class discussions. Universities without a Christian mission typically do not permit overt integration of faith in the classroom. Both settings can benefit from a more subtle and interdisciplinary integration of faith.

Consider an analogy to illustrate the motivations for this paper. Imagine a plain colored, standard reading chair. This object is a staple in many homes, and a lovely complement to most furniture sets. The reading chair serves a very practical purpose in providing comfortable seating to those who wish to read, or sit and visit. This is analogous to the calculus sequence within the STEM fields—practical and foundational to the ability of students to participate comfortably within their discipline. The explicit model of faith integration would be analogous to throwing a colorful blanket over the chair. Instructors superimpose an assignment about faith or worldview, or initiate contrived class discussions about God, on top of the already busy calculus curriculum. To those who see how the structure of the chair supports the blanket it can be a great addition, adding to the depth of comfort and experience of using the chair. Unfortunately, our students often do not see the benefits, but instead experience the blanket as a superficial imposition, in the way of fully experiencing the practical uses of the chair. Students would view the integration of faith and mathematics in a more approachable way if instructors were to highlight the natural color variations of the fibers woven throughout the fabric, and a depth of appreciation for the complexities of the construction, while not impeding the practical uses of the chair. I would like to suggest that rather than contrived assignments or discussions of faith, calculus instructors use an interdisciplinary ap-

proach to faith integration, tapping into the very nature and history of calculus which is rich with foundations of faith and the Divine.

In this paper, I will give some practical examples of ways instructors can begin practicing an interdisciplinary integration of faith throughout the calculus curriculum. Instructors are encouraged to begin by choosing one area or topic to approach in this way, while also pursuing readings and study to broaden their own knowledge in the history, philosophy, and the Divine that is part of the rich background to the study of calculus.

2 Practical Examples

This section contains some examples of ways instructors can implicitly and naturally weave discussions of faith and God throughout the calculus curriculum. Many examples are connected to concepts of zero and infinity, which are inherently linked to both faith and the historical development of calculus. These examples are not intended to be exhaustive, but rather a sample of possibilities and a starting point for those who wish to develop this approach to faith integration.

2.1 Removable Discontinuities

Consider an exercise in finding limits such as

$$\lim_{x \rightarrow 1} \frac{(x^3 - x^2 - x + 3)(x - 2)}{(x - 2)}.$$

Instructors use an exercise like this to introduce removable discontinuities and may even present a sketch of the graph as in Fig (1). Focus is given to the algebraic methods for evaluating this limit (factor and cancel), and illustrating that a hole exists at the point (1, 2).

Instructors who wish to integrate faith in calculus should not miss this opportunity, as exercises such as this one demonstrate the concept of faith to students. Rather than initially approaching this exercise from an algebraic perspective we could allow students to use technology to investigate

$$\lim_{x \rightarrow 1} \frac{x^4 - 3x^3 + x^2 + 5x - 6}{x - 2},$$

the same exercise as above presented in its non-factored form. Students will discover that direct substitution does not work, but that the graph of the function does not indicate any visible issues. A point or hole, being without length or width, should more accurately be invisible in a sketch of the graph. Before presenting the algebraic methods and sketch with the hole drawn visibly, instructors could take this opportunity to encourage student discussion about the accuracy of our perceptions, and the faith needed to believe in what we cannot see when we are given “evidence of things unseen.” (Hebrews 11:1) Instructors could then ask students to brainstorm a list of other moments in life where the accuracy of initial perceptions should be reevaluated. In my own classes this approach has led to discussions of various topics such as the existence of God, the struggles with hidden disabilities, mental health, racism, and fears, among others. With even this short five minute discussion about perception and faith, students can glimpse the connections between their own humanity, God, and the mathematical concepts underlying limits.

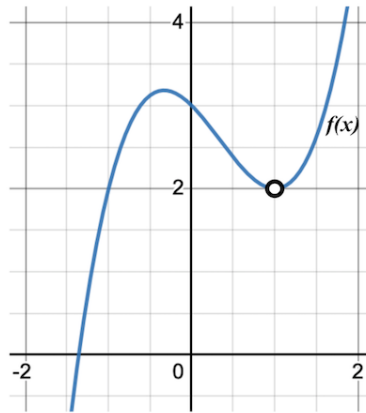


Figure 1: Sketch of $f(x) = \frac{(x^3 - x^2 - x + 3)(x - 2)}{(x - 2)}$ with the removable discontinuity at $(1, 2)$ clearly visible.

2.2 Limits at Infinity

Limits give instructors another opportunity to integrate faith while exploring the topic of evaluating limits at infinity,

$$\lim_{x \rightarrow \infty} f(x).$$

What does it mean for x to “go to infinity”? When we write $x \rightarrow 3$ we can easily see the value of 3 on the x -axis, but when we write $x \rightarrow \infty$ we are indicating a point that exists only in our imagination - when x reaches a magnitude larger than the largest value imaginable. As Amir D. Aczel writes, “God as infinity cannot be described or comprehended. The Ein Sof¹ is far beyond what a human mind can hope to glimpse.” [2]

The discomfort of thinking about the infinite nature of x can be avoided by focusing on the methods used to evaluate these limits. But, as Francis Su states, “Truth is a basic human desire. We crave it, even if it might bring us uncomfortable information.” [7] Consider how students might be challenged if instead instructors allowed them to sit with the discomfort of the infinite. What does it mean when students evaluate

$$\lim_{x \rightarrow \infty} \frac{3x}{e^x} = 0,$$

and conclude that the denominator grows faster than the numerator, though both are going to same point of infinity? Or that

$$\lim_{x \rightarrow \infty} \sqrt{x^2 - 2} - x = -\frac{1}{2},$$

and “the infinite value of the first term is only $\frac{1}{2}$ smaller than the infinite value of the second term” - as students often interpret the indeterminate form $\infty - \infty$?

Exercises such as these provide an opportunity to briefly discuss Cantor’s thoughts, “The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.” [5] Giving our students the opportunity to reflect on the

¹Hebrew, translated as *infinity*

meaning behind infinity and the implications of its acceptance allows them to tap into the beauty that is described in Ecclesiastes 3:11, “He has made everything beautiful in its time. Also He has put eternity in their hearts, except that no one can find out the work that God does from beginning to end.”

In the spirit of weaving the concepts of faith throughout the curriculum I do not suggest contriving a class discussion on the philosophies of infinity. Rather I suggest that instructors practice seamlessly incorporating quotes and moments of brief reflection when introducing topics about infinity, and encouraging students to engage with the concepts at a level beyond the algebraic methods needed to evaluate the exercises. “Mathematical concepts of the infinite can do much to engage and propel our thinking about God.” [3]

2.3 Zero and Derivatives

The concept of zero holds a special place in the development of calculus, and as a companion to infinity it can be just as uncomfortable for students to reflect on. Charles Seife writes, “Cultures girded themselves against zero, and philosophies crumbled under its influences, for zero is different from other numbers. It provides a glimpse of the ineffable and the infinite. This is why it has been feared and hated—and outlawed.” [6]

Foundational to the definition of a derivative,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

is the idea that the slope of the tangent line to a curve can be computed by finding the slope of a secant line between two points and allowing those points to approach each other until their distance apart, h , reaches zero. Instructors draw curves, secant lines, and tangent lines to help students understand how this could represent the slope of the tangent line, but many students miss the troubling nature of the fact that both the numerator and the denominator of this fraction are headed towards zero. By emphasizing this fact we can allow students to see a “glimpse of the ineffable and the infinite” when we introduce derivatives. In this context, we have the opportunity to tell our students about infinitesimals and the story of how Newton and Leibniz manipulated their equations and notations to either avoid zero and infinitesimals, as in Newton’s case, or to embrace them, as in Leibniz’s case. Within the development of calculus it was those who embraced Leibniz’s notation and accepted a method that required faith as much as logic who were able to further the development of calculus and its implications. [6]

2.4 Integration

Instructors can begin the topic of definite integration by posing the question “What is it that makes you who you are?” Students may answer personality, physical body, or experiences, to which one could further ask “Is it only the big and meaningful experiences that contribute to who you are, or is it the *unified whole* of all life experiences combined?” As calculus instructors we can easily see that this discussion is moving towards the idea that we are the integration (unified whole) of all properties and experiences in life, down to the smallest infinitesimal detail.

Through this introduction, the notation of an integral becomes not just arbitrary symbols communicating how to compute a value, but it is also telling the story of integrating parts into a unified

whole. Students are encouraged to think about what makes a person who they are and how every area of their life is integrated into a cohesive unit.

I had a student recently who interpreted his life experiences as

$$\int_{\text{Birth}}^{\text{Death}} \text{Happiness \& Sorrow } d(\text{time}) = \text{Life}.$$

This is a great example of both the philosophical context of our personal development, and the connection between the differential, $d(\text{time})$, and the limits of integration, Birth to Death. As we teach areas under a curve, or volumes of rotated solids, we can refer back to this analogy to remind our students that an object or region is fundamentally the unified whole of its infinitesimally small parts.

2.5 Series

Zeno's paradox is a fun way to introduce the power of infinite series. When telling the story of Zeno's paradox instructors can pose the idea as an open-ended scenario: What would happen if I was in a race with someone much slower than me and I let that person have a head start 2 meters ahead of me? We will each be running the race at constant speed, but my speed will be much faster than theirs. The buzzer goes off and by the time I get to where they started, 2 meters from my own starting position, they have moved ahead 1 meter. By the time I move ahead another meter they have also moved ahead, this time by 0.5 meters. If we continue in this way, will I ever be able to pass them?

It is challenging to not let our students sidestep the discomfort of the infinite and the infinitesimal when they say that the example is silly and "steps don't get that small". We should enlighten our students to the fact that this Aristotelian-based philosophy and the rejection of the infinite and the void may give those who wish for logical certainty in the universe comfort, but does not allow for the establishment of calculus or the mathematical language that describes the natural world. [4] This philosophy is often so engrained in students' thinking that they are unaware of the far reaching implications within mathematics. Students who are willing to reflect on these concepts become more at peace with the abstractness of infinite series.

2.6 Dimensions

In this last example, students in a multivariable calculus course are presented with the concept of dimensions larger than three. The response of the main character in Edward Abbott's book *Flatland* illustrates the power of multidimensional thinking, as illustrated by the question A. Square asks of the Sphere about higher dimensions:

"My Lord, your own wisdom has taught me to aspire to One even more great, more beautiful, and more closely approximate to Perfection than yourself. As you yourself, superior to all Flatland forms, combine many Circles in One, so doubtless there is One above you who combines many Spheres in One Supreme Existence, surpassing even the Solids of Spaceland. And even as we, who are now in Space, look down on Flatland and see the insides of all things, so of a certainty there is yet above us some higher, purer region, wither thou dost surely purpose to lead me—O Thou Whom I shall always call,

everywhere and in all Dimensions, my Priest, Philosopher, and Friend—some yet more spacious Space, some more dimension-able Dimensionality, ...” [1]

Multivariable calculus instructors introduce the idea of level curves, visualizing two dimensional surfaces which exist in three dimensions, and level surfaces, visualizing three dimensional objects which exist unseen in four dimensions. The initial encounter by students with dimensions higher than three often results in incredulity and even distrust. They ask how such a thing can exist when the only language they have to describe it is mathematical and not easily accessible by sight or touch. This is an opportunity to demonstrate to students a practice of faith and belief in something abstract and beyond imagination. Instructors can encourage students to utilize these abstract constructions as an opportunity to expand their ability for belief and faith. It is a reminder of Job 11:7, “Can you fathom the mysteries of God? Can you probe the limits of the Almighty?”

3 Conclusion

The aim of these examples of implicitly integrating faith throughout calculus is not to have one main moment or assignment to discuss faith or worldview, but rather to incorporate history, short stories, questions about faith, and questions about life often during class time. To do this instructors need to have an interdisciplinary mindset about mathematics: at each moment and for each topic we are not just bringing calculus to our students, but also the interest in the history, philosophy, theology, or psychology. The books cited in this paper are a good starting point for those interested in studying more about the interdisciplinary nature of calculus. In particular, Charles Seife’s book, *Zero: The Biography of a Dangerous Idea*, is a quick and approachable read with many fun and engaging stories around the idea of zero.

Conveying our own interest in weaving together various disciplines translates to our students the theological foundations of calculus and the complexities inherent in the discovery of calculus and its methods. This does not take away time from the development of mathematical skills, or hinder the practical applications of the curriculum. Rather it enhances these with a depth of appreciation for the colorful variations woven throughout the fabric of calculus.

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A Homily on Hardy's *Apology*

Andrew Simoson (King University)



Andrew Simoson (B.S., Oral Roberts University; Ph.D., University of Wyoming) has taught at King University since 1980. He has published extensively with the MAA, and received two Fulbright awards: to Botswana, and Tanzania. His first ACMS conference was in 1987. The many mathematicians he has met through Bob Brabenac's amazing legacy in this assembly has been a very real blessing, both in friendship and in being mentored as a professor.

Abstract

Imagine responding to G.H. Hardy's well-known 1940 *A Mathematician's Apology* wherein he claimed that (i) "Engineering is not a useful study for ordinary men." (ii.) "There is the real mathematics of the real mathematicians, and there is what I call the trivial mathematics—which includes its practical application, the bridges and steam-engines and dynamos." (iii.) "Most people can do nothing at all well." (iv.) "Exposition . . . is work for second-rate minds." This meditation consists of various open-ended rebuttal thoughts to these claims—which may prompt further ACMS discussion.

1 Introduction

Allow me to begin this homily on G.H. Hardy's classic 1940 *A Mathematician's Apology* with an apology of my own. Of all the topics upon which to give an ACMS talk, I never would have had the temerity to choose this one unless cornered to do so. For who am I to say to a mathematical giant, *Excuse me, did you really mean to say what you did?* Yet one day in January 2022, I was cornered by a note from a friend: "It seems to me that the *book* simply demands an introduction by a mathematician, and I think you're the perfect person to do that."

The friend was Paul Nahin, an award-winning mathematical expositor and a long-time electrical engineer, and the *book* was a few manuscript chapters of his forthcoming *The Mathematical Radio* whose subtitle is *An Engineer's Reply to G.H. Hardy*. As a clever literary device to win an audience among the mathematical community, Paul couches his narrative on how the radio works—whose starting point is Maxwell's equations of electricity and magnetism—as if he were counter-arguing Hardy's bold *Apology* assertion that *engineering mathematics is for second-rate minds*, and, by extension, that the same stigma brands applied mathematics. To amplify this analogy, imagine that Paul Nahin is an Apostle Paul figure and Hardy is a Festus figure, a Roman procurator in the Province of Judea as described in Acts 26:34. Therein, in Festus's courtroom from so long ago, Paul defends himself against a sedition charge. Will Paul Nahin have any more success with Hardy than did St. Paul with Festus, who, after hearing the plaintive's passionate argument, responded to him by saying, "Your great learning has made you mad!"?

2 Hardy's Claims

With that introduction, we itemize Hardy's *Apology* claims. The first two pertain directly to the claim of a gulf between pure and applied mathematics and the remaining ones give insight into the first two items.

1. Engineering is not a useful study for ordinary men [9, p. 117].
2. There is the real mathematics of the real mathematicians, and there is what I call the trivial mathematics ... which includes its practical application, the bridges and steam-engines and dynamos [9, p. 64, 139].
3. Mathematics is a young man's game [9, p. 70].
4. A man's first duty is to be ambitious [9, p. 77].
5. Exposition is for second-rate minds [9, p. 61].
6. Most people can do nothing at all well [9, p. 67].
7. It is one of the first duties of a professor to exaggerate a little both the importance of his subject and his own importance in it [9, p. 66].

3 A Little History

To set a foundation upon which to critique this list, we go back in time and then leap forward in varying incremental steps to specific episodes until the present date. We ask, *When did we first have the luxury of having both pure and applied mathematics?* Even as far back as the Pleistocene epoch, people have always used a little math so as to pay tribute or taxes, settle property disputes, and build safe structures. But beyond that, when did mathematics bifurcate into quasi-distinct knowledge fields of pure and applied? Somewhat arbitrarily, we submit that a meaningful answer is when we began explaining natural phenomena without reference to mumbo-jumbo.

In this context, one of the first philosophers to do so is Thales (620–546 BC). Among Euclid's collected propositions attributed to Thales is Proposition 26: *If two triangles have angle-side-angle or have angle-angle-side in common then the triangles are congruent.* In fact, this proposition, as the legend goes, is how Thales determined distances of ships from shore [8, vol. I, pp. 35–36, 304–305]. In Figure 1 on the following page, imagine that Thales T atop a coastal tower—whose base point A is directly below T —sights ship S (using a larger-than-life compass-like instrument, one arm being a vertical pole and the other free to swivel and lock into position so as to align with his line of sight); whereupon Thales rotates the vertical pole so that the swivel arm now points to marker B (among many markers) along the visible coastline. Since $\triangle TAS$ and $\triangle TAB$ are congruent, then sea distance AS is coastal distance AB .

Thales' trick is both pure and applied geometry; it's a general abstract theorem and untold specific instances of that truism. Two hundred years later, purportedly emblazoned above the entrance to one of the world's first great schools, Plato's Athens Academy, was the motto,¹ “Let no man ignorant of geometry enter here.” Plato (c. 429–347 BC) taught that “all the useful arts [such as making shoes] are reckoned mean,” and proposed that an ideal education consisted in studying (i) arithmetic, (ii) plane geometry, (iii) solid geometry, (iv) astronomy, (v) music, and (vi) philosophy² [15, book VII, pp. 264–278]. From out of this pedagogical tradition we have Archimedes (287–212 BC). Most mathematicians—among the few opinions about which we all might share—believe that Archimedes was among the greatest mathematicians of all time—perhaps on everyone's top-ten

¹Scholars suspect that this inscription was figurative rather than one cut in stone. A more literal rendition is, “Let no one ignorant of geometry come under my roof” [22].

²Plato's last subject includes natural philosophy, or what is now called science.

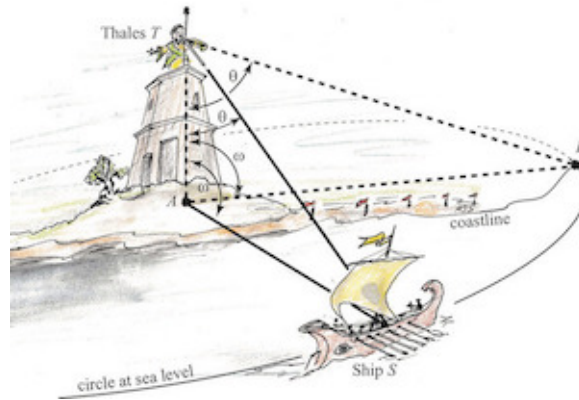


Figure 1: Thales measuring a ship's distance at sea, author sketch.

list.³ Yet he was also an incredible engineer. Witness Plutarch's description of how Archimedes' boast, that given a place to stand, he could move planet Earth itself, was put to the test by Syracusan King Hiero, who challenged Archimedes to launch "a ship of burden which could not be drawn out of the dock without great labour and many men," whereupon,

[Archimedes, loading the ship] with many passengers and a full freight, sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.

Yet Plutarch (46–119) goes on to say that Archimedes, as Plato had believed before him and Hardy would believe long after him, "repudiated as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit." Instead, Plutarch continues, Archimedes "placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life."⁴ What are these purer speculations? My favorite is Archimedes' Diophantine riddle of the Sun God's Cattle:

If thou are diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colors white, black, yellow, and dappled. In each herd were bulls, mighty in number according to these proportions: the white bulls were equal to half and a third of the black together with the whole of the yellow, . . .

and so on, ultimately to give us a total of eight equations. Although Archimedes probably never solved his own riddle, the least positive possible answer is approximately 7.7603×10^{4658} cattle [2, pp. 232–238]. But just posing this puzzler is the very nature of pure mathematics.

Now we leap forward to shortly after the fall of Rome. During house arrest prior to execution, Boethius (c. 477–524) wrote *The Consolation of Philosophy* popularizing Plato's educational slate, now partitioned into the quadrivium and the trivium: arithmetic, geometry, astronomy, music, grammar, rhetoric, and logic. All of this material focused on pure rather than applied knowledge, at least in theory.

³Hardy said, "Archimedes will be remembered when Aeschylus is forgotten," [9, p. 81].

⁴As evidence that Archimedes held such a view of engineering, Plutarch simply says that Archimedes did "not deign to leave behind him any commentary or writing on such subjects [as engineering]." Nevertheless, Plutarch's explanation suggests that Plutarch himself—and most likely a host of others—held this view [16, pp. 376–378].

To illustrate a study item from, say, the pure field of arithmetic, Figure 2 is my adaptation of a puzzle from Boethius' *De Arithmetica*,⁵ which was his rough translation of a Greek manuscript of Nicomachus of Gerasa (60–120). Imagine the gray square is a 4×4 matrix A , with cell entries a_{ij} in row i and column j , where i and j go from 1 to 4. Imagine also that the integers are written in Roman numerals.⁶

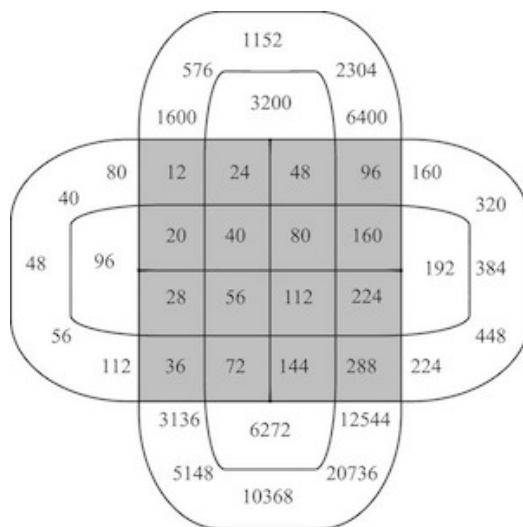


Figure 2: An arithmetic puzzle from Boethius' *De Arithmetica*.

Back to the puzzle: the integers outside the gray square can each be obtained in two distinct ways by arithmetic operations on the integers inside the gray square. In particular, we allow equality between the (i) sum of two integers, (ii) the product of two integers, (iii) doubling an integer, and (iv) squaring an integer. *Can you, the student, find them? Moreover, how are the integers themselves within the gray box related to each other?* For example, from A 's row-3, we have $a_{31} \times a_{34} = 6272 = a_{32} \times a_{33}$; and $a_{31} \times a_{33} = 3136 = (a_{32})^2$. *Unimpressed?* Remember that these integers are in Roman numerals,⁷ a system never designed for computational purposes. Instead, those computations were performed using an abacus of sliding beads along rods or a counting board with movable tokens. To solve our puzzle, a student would translate, say, two desired numerals from Roman (or Greek) into proper arrangement onto one of these instruments; manipulate the tokens thereon, performing the desired operation; and translate the result into Roman numerals. Ironically, such puzzles as this one develop students' facilities with the abacus—useful skills which a potential employer might prize—but the pure aim of the puzzle was to develop insight into the nature of number so as to discover more (interesting yet arguably useless) relations within number theory.

Next, we leap forward in time to shortly after the bombshell technological innovation of the printing press. My favorite artist of all time is Albrecht Dürer (1471–1528). His 1525 four-volume *The Art of Measurement with Compass and Straightedge* was one of the first printed mathematical texts in German. Therein, he introduces a construction technique for a general family of curves known as the *trochoid*.

⁵Boethius' book was "the source of all arithmetic taught in schools and universities for over a thousand years," [17].

⁶To represent the integer ten thousand, linguistic anthropologist Stephen Chrisomalis says that the Romans used the symbol $\textcircled{\Phi}$, explaining that the Roman symbol for one thousand in the time of the Punic wars was initially written as $\textcircled{\text{D}}$, and that over time it "opened up from the bottom to look like an M," [5, p. 39].

⁷At least up until after the days of Fibonacci's 1202 *Liber Abbaci* introducing the fantastic computational capabilities of the Indo-arabic number system.

Figure 3c is a snippet from Dürer’s woodcut “The Circumcision” showing what is now called a *limaçon*—which Dürer constructed using a modified compass.⁸ To use the instrument, set the compass down so that O remains fixed on the page; arrange arms A and B as desired, so determining angle θ and ϕ ; define step-size increments for angles θ and ϕ at the joints O and Q ; advance arms A and B by those angle step-sizes and mark your page at point P , and repeat. One hundred years later, geometers such as René Descartes (1596–1650) transformed Dürer’s mechanical procedure into algebraic terminology and onto coordinate systems, so much so that the realm of mathematics, which had been classical geometry with a bit of algebra, has become algebra with a bit of classical geometry.

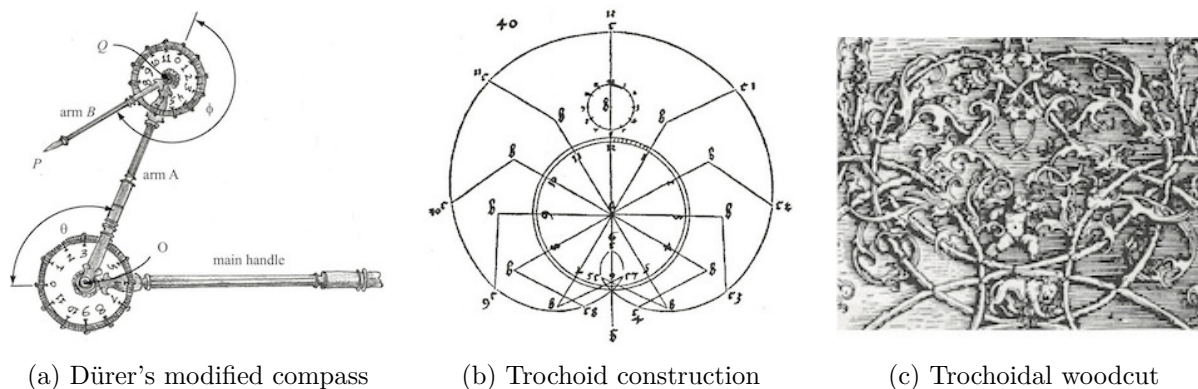


Figure 3: Albrecht Dürer’s trochoid family of curves.

The point of this story is that Dürer applied a mathematical idea to help create and market art. Once this primitive idea percolated into what we might call the mainstream of mathematics, *aha*, the mathematical community delighted in making ever more fanciful curves.⁹ This dynamic between pure and applied repeats itself time after blessed time. Upon which shall we focus? If you are someone like Stephen Wolfram of *Mathematica* fame, why not fine-tune all the world’s great mathematical algorithms for universal usage into a convenient computer algebra system (CAS) and make a fortune along the way. However, if you are someone like Grigori Yakovlevich Perelman who resolved the 1904 Poincaré conjecture and nevertheless turned down the million dollar Clay Millennium Prize, then you might be like Archimedes who, in the sack of Syracuse,

... was intent upon working out some problem by a diagram. In [the midst of Archimedes’ calculations], a soldier commanded him to [move]. Declining to do so before he had worked out his problem, the soldier, enraged, drew his sword and ran him through [16, p. 380].

Just one more leap, this time to the dawn of the twentieth century. At the end of his life, Alfred Nobel (1833–1896), who had made a fortune marketing explosives, was faced with distributing that wealth, ultimately choosing to endow monetary prizes celebrating extraordinary practical discoveries. In 1901, the first prizes¹⁰ were awarded in physics, chemistry, medicine, literature, and peace, but not mathematics. Why? The bottom line answer: *Everyone knows*, whether we like it or not, *that pure mathematics is impractical*. Tracing mathematical development through time we see that it has continually bifurcated, spawning the disciplines of engineering, accounting,

⁸Figure 3a is my sketch of Dürer’s modified compass. Figure 3b is [7, book 1, figure 40]. Figure 3c is [6, cut 184]. For further details on how Dürer manipulated his compass to construct the trochoid in the “Circumcision” see [19, chapter VII, pp. 163–187].

⁹To explore one mathematician’s top-ten curves of all time, see [11].

¹⁰In 1968, prizes were also extended to economics.

physics, economics, statistics, operations research, computer science, communications technology, and artificial intelligence. Mathematics jettisons, as it were, all the practical aspects of its success, so as not to thwart its grand quixotic quest at understanding in full the elusive abstractions of number and space. To personify mathematics, imagine that mathematics is Diogenes and the world view of what is important is Alexander the Great; one afternoon, approaching from the west, Alexander asked Diogenes, who¹¹ was drawing figures in the sand, what he could do for him; and Diogenes replied, “Just move out of the sunlight.”

4 The Great Gulf Claim

Now, I’ve exaggerated a bit, more or less in the spirit of Hardy’s claim (7). Yet, hopefully, these last few paragraphs have set a foundation upon which to discuss his other claims.

Let’s consider points (1) and (2) about engineering. Hardy co-authored about 100 papers with John Littlewood (1885–1977) whose comments often enlighten Hardy’s observations, such as this recollection: One day at Trinity College of Cambridge University,

in 1912, it was proposed that the Engineering students should be taught some *real* mathematics by the *mathematical staff*. I asked F. J. Dykes, the Lecturer in Engineering, what he would like me to select; all he said was: *Give the buggers plenty of slide rule* [12, p. 142].

I used the slide rule for all my college physics courses in the early 1970s, but teaching the slide rule back in the early twentieth century is like teaching the abacus and the counting board back in the days of Boethius. Yet Professor Dykes had a valid point, namely, *His students needed help at that fundamental level!* When my older son, with a master’s in mechanical engineering, landed his first position at General Electric, he was placed into a multi-year training program where all the new recruits were taught afresh the ins and outs of the trade. Part of this trade is to realize that the company must be commercially competitive—and while experimentation with new ideas is encouraged—*make sure you all follow the company line*. This same spirit was much in vogue, say, in the days of Johannes Gutenberg (c. 1400–1468) and the printing press; as his team was trying to mass produce copies of the Bible, Gutenberg continued tinkering with engineering improvements to the press, so much so, that his services were terminated shortly after the first few of about 200 complete holy writ copies rolled out the door. This *it’s-good-enough* production-line approach to exploring knowledge, I believe, forms much of the target for Hardy’s denigrating remarks about engineering.

When Hardy uses the word *useful* in a positive sense, he classifies as useful much of college mathematics as well as electricity and magnetism and fluid dynamics [9, p. 133–135]. Hardy’s main point: “What is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics.” When Hardy uses the word pejoratively, he concludes

If useful knowledge is . . . knowledge which . . . contributes to the material comfort of mankind, then the great bulk of higher mathematics is useless [9, p. 135].

Blunt and harsh words from a master mathematician who is heartily convinced of both the innate value and ridiculousness of a liberal arts education! When reading Hardy’s *Apology*, dynamical

¹¹Paintings and sculptures of this classic interaction often depict Diogenes in a drunken stupor, but I like to think he was doodling.

systems professor Daniel Silver reminds us “that Hardy enjoyed teasing his audience,” citing George Pólya (1887–1985) who recalled, “Hardy liked to shock people mildly by stating unconventional views . . . because he liked arguing” [18]. For example, listen to Hardy as he closes his 1928 Josiah Gibbs lecture in New York City,

A month’s intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful and at least ten times as entertaining as the same amount of “calculus for engineers” [10].

Such strident language may charm some, yet it fails to amuse everyone.¹² In a 1941 review, Frederick Soddy (1877–1956), a 1921 Nobel laureate in chemistry, called Hardy’s *Apology* “cloistral clowning” [21]. Despite this caustic view, consider the robust nature of Soddy and Hardy’s relationship: at the end of his review, Soddy mentioned that during a dull faculty meeting in 1931, he passed a note to Hardy asking for the sum on the left of equation (1).

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (2n+1)^{-2m} = \sum_{n=1}^{\infty} \frac{(2n+1)^{-2}}{1 - (2n+1)^{-2}} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{4n(n+1)} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{4}. \quad (1)$$

“In an incredibly short space of time”—and while the meeting was yet in progress—Hardy passed Soddy the rest of equation (1). As a more thoughtful commentary, here’s Hardy’s friend, Norbert Wiener (1894–1964),

When I returned to Cambridge after working with engineers for many years, Hardy used to claim that the engineering phraseology of much of my mathematical work was a humbug, and that I had employed it to curry favor with my engineering friends at MIT. He thought I was really a pure mathematician in disguise, and that these other aspects of my work were superficial. This, in fact, has not been the case [1, p. 158].

The physicist turned novelist C.P. Snow (1905–1980), who had read Hardy’s original 1940 *Apology* manuscript prior to publication, and who wrote the *Apology*’s foreword appearing in editions since 1967, echoed these thoughts in his 1959 Rede lecture, saying

Pure scientists have by and large been dim-witted about engineers and applied science. Their instinct was to take it for granted that applied science was an occupation for second rate minds. I say this sharply because thirty years ago I took precisely that line myself [20].

As a last expert witness on Hardy’s points (1) and (2), we call on Mary Cartwright (1900–1998), a Ph.D. student of Hardy’s and a co-author with Littlewood on a number of papers. Here’s her insight into how they thought.

Hardy, who taught me, was very much against applied mathematics, but [he once] wrote that a certain problem is most easily grasped in terms of cricket¹³ averages. [And] Littlewood and I did a lot of work on the theory of ordinary differential equations

¹²See [3] for an exhaustive compendium and commentary on reviews of Hardy’s *Apology* and for Hardy’s allusions therein.

¹³On sabbatical at Princeton in 1928, G.H. Hardy—an ardent cricketer from early childhood—wrote, “I read reports in the papers by the hour and worship Babe Ruth and Lou Gehrig [with] batting averages ranging from 0.387” [1, pp. 132–133].

arising from problems of radio engineering. Littlewood [had] worked on antiaircraft gun fire in the First World War, and he translated our problems, which were suggested by radio values and oscillations, capacitance and inductance, etc., into dynamical problems and called all the solutions of our equations ‘trajectories’ as if they were the paths of missiles shot from a gun [4].

That is, no matter how much we might profess to be a pure mathematician, we often transgress into applied mathematical thinking while doing mathematics. Moreover, Cartwright and Littlewood’s collaborative pioneer work on Van der Pol’s nonlinear differential equation showed “the existence of two stable [solution] sets of subharmonics of different orders” and also of strange “fine structures [now] recognized as typical manifestations of the *butterfly* effect” and of chaos theory in general [13]—a domain that most mathematicians might agree is a rich inextricable blend of pure and applied mathematics.

5 The Remaining Claims: Items 3 through 6

Figure 4 is my sketch of Mary Cartwright, a cat, and a radio: while Mary explains modeling the radio, equations stream from the radio, somewhat alarming the befuddled cat. In particular, Van der Pol’s equation streams to the top right of the radio, and Maxwell’s third equation streams to the left. Incidentally, this image may form the cover art for Paul Nahin’s book.



Figure 4: Mary Cartwright and cat with radio.

Now to comment on the remaining enumerated Hardy points: *Is mathematics only a game for young men and women?* point (3). Hardy says, “I do not know of an instance of a major mathematical advance initiated by a man past fifty” [9, p. 72]. Yet,

Littlewood remained active in mathematics even at an advanced age: his last paper was published in 1972, when he was 87. One of his most intricate papers, concerning Van der Pol’s equation and its generalizations, was written when he was over seventy. 110 pages of hard analysis [12, p. 15–16].

Paul Erdős (1913–1996) toured the world, visiting one university after another like a Johnny Appleseed planting trees of joint work around the globe, until nearly the end. In San Diego for a Joint Mathematics Meetings conference, 11 January 2013, Richard Guy (1916–2020) gave a talk on continued fractions. To a packed room of about 500, he introduced himself, “You didn’t, most of you, come to hear math, but to see a dinosaur.” But really, most of us were thinking, *No, we came to see a dinosaur and to hear some math.*

With respect to point (4), *is ambition truly necessary?* Hardy reminds us that he came to mathematics because, “I wanted to beat other boys [at exams]” [9, p. 144]. Littlewood, too, had this competitive element. As Belá Bolobás describes,

In 1971, a paper was submitted claiming a proof of RH [the Riemann Hypothesis]. [Both Littlewood and I examined the proof and] after a few hours of painstaking work he [Littlewood] was relieved to find a mistake [12, p. 16].

That is, Littlewood was hoping that he himself would be the one to prove RH! Littlewood, however, acknowledges that not everyone is similarly wired. Of Edmund Landau (1877–1938), Littlewood recalls, “It was said round 1912 that it gave him [Landau] the same pleasure when someone else proved a good theorem as if he had done it himself” [12, p. 125]. If doing mathematics were like playing baseball or cricket, some of us would be all-stars and others, like me, would perhaps, at best, be umpires. The former require a killer instinct and the latter just plain obstinacy.

What about point (5)? Hardy’s 1928 Josiah Gibbs lecture won the 1932 Chauvenet Prize for best expository article in mathematics as published in North America. With respect to writing, Hardy confessed in a letter to Bertrand Russell,

I wish you could find some tactful way of stirring up Littlewood to do a little writing. Heaven knows I am conscious of my huge debt to him. But, in our collaboration, he will contribute ideas and ideas only: and that *all* the tedious part of the work has to be done by me. If I don’t, nothing would ever get published [1, p. 155].

It is well-known that Hardy wrote well, and was proud of his ability to put words together. His saying that “exposition is for second-rate minds” is like Babe Ruth¹⁴ saying, “Anyone can swing a bat.” In other words, stellar writing is a rare gift.

For point (6), *is it true that few, if any of us, can do anything well?* My two-year old grandson can walk upright and utter complete sentences! Wow, I’m impressed. That’s my standard for excellence. But suppose you are a Leonardo da Vinci. His deathbed last query [14] to his assistant was, “*Dimmi, dimmi se mai fu fatta cosa alcuna,*” that is, “Tell me, tell me if anything ever got done.” *Oh my, God bless us all.*

6 A Parting Challenge

Finally, has this homily tarnished Hardy’s *Apology*? Indeed no! His presentation is a rare soul-searching commentary on the art of being a mathematician by a world-class mathematician. To highlight this dynamic, Hardy deliberately opened his *Apology* not with a conversation between himself and a mathematician but between himself and the poet Alfred Hausman (1859–1936). In particular, Hardy recalls debating Hausman on point (5) at length at high table in the Hall at Cambridge, saying, “We argued [the point] all through dinner” [9, p. 62–63]. In that same spirit, and especially in the ACMS context, let’s continue the conversation, asking as does Hardy, “Why is it really worthwhile to make a serious study of mathematics? [And] what is the proper justification of a mathematician’s life?” [9, p. 65].

¹⁴On sabbatical at Princeton in 1928, Hardy—an ardent cricketer from early childhood—wrote, “I read reports in the papers by the hour and worship Babe Ruth and Lou Gehrig [with] batting averages ranging from 0.387” [1, pp. 132–133].

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When to go Backwards in *Sorry!*

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Abstract

In the classic game *Sorry!*[®], players try to move tokens around the board from Start to Home in as few turns as possible. Moves are determined by cards, some of which provide alternative options—in particular one card specifies “forward 10 spaces or backwards one space.” Because backward moves can lead to a shortcut Home, we ask when using that option is optimal. To make this analysis we model the movement of a single token around the board as a Markov process, from which we can determine the expected number of turns for different strategies. We also look at the effect on the length of the game for other card actions and speculate how this might play out with multiple tokens and players.

1 Introduction

It could be argued that mathematicians either have more fun or less fun when playing games, depending on whether one views an interest in the underlying mathematics as adding to or distracting from the natural fun of the game. Virtually every game has mathematical aspects and it is natural to use mathematical analysis to better understand the games we play. The goal of the analysis may be to determine the best play at any stage of the game, such as for Yahtzee [8, 11]. It might be to determine the comparative value of positions, such as Monopoly [1, 2]; or the chance of success of an attack, as in RISK [5, 10]. Or it might determine the expected length of a game, such as Chutes and Ladders [3, 4] or Hi Ho! Cherry-O [12]. For games where the probability of moving between “states” can be determined precisely, Markov chains provide convenient models which lead to the determination of the expected number of turns to achieve a certain game objective (such as getting a token to the end space). Johnson [7] gives a nice intro with examples using several games. When a game includes some choices for the player (a “strategy”), optimality can be determined by comparing Markov model results for each strategy. This approach can also be used to explore the effects of possible changes to the game rules. Of course many games include interaction between the players, which can be adverse or helpful; to reduce complexity analyses are often carried out on a “single-player” version of the game.

In this report we look at the game *Sorry!*[®]¹, seeking to determine strategies that minimize the expected number of turns to move a single token from the Start space to the Home space. Moves in *Sorry!* are determined by a card selected from a deck (rather than a die or spinner) and some of the cards have options for what the move can do. Hence the player may use a particular strategy for choosing between the options and we can compare the expected number of turns. In particular, we wish to determine where on the board opting for a small backwards move instead of a large forward

¹*Sorry!* is produced by Hasbro

move improves the player’s chances of taking advantage of a shortcut. The regular *Sorry!* game has up to four players, competing to move their four tokens from Start to Home and some of the cards allow moves that can affect their opponents’ token locations. While we won’t attempt to model the interaction between players, we will discuss its possible impact on the expected number of turns to move each token around the board.

Because moves in the game are determined by selecting cards from a deck, which changes after each draw, the probabilities of moving between “states” (spaces on the board) depend on what cards have previously been drawn. This is different from games that have independent turns, such as those determined by a roll of dice. So the *Sorry!* scenario is technically not Markovian. However, in Section 3 we will make an argument for using probabilities that do not depend on the state of the deck of cards and that the Markov model provides a reasonable approximation, at least to the extent of being able to compare strategies.

2 The Game

Sorry! was introduced by Parker Brothers in 1934, and is similar to earlier games Parchessi (USA) and Ludo (England), all of which are in the vein of “race” style games like Parcheesi (India) and a variety of Amerindian games [9]. Over the years there have been changes to the rules for *Sorry!*; we will focus on the 1992 version². The *Sorry!* board, shown in Figure 1, has a continuous track around its perimeter, a Start space for each player, and a Safety Zone (with matching color) branching off the outer track leading to the player’s Home space. Tokens move from a player’s Start space around the board clockwise toward their Safety Zone, to reach their Home space. There is a “shortcut” from the Start to the Safety Zone, but it can only be traversed through backwards moves. The track also has “slides” which advance a token that lands on the beginning of the slide by exact count.

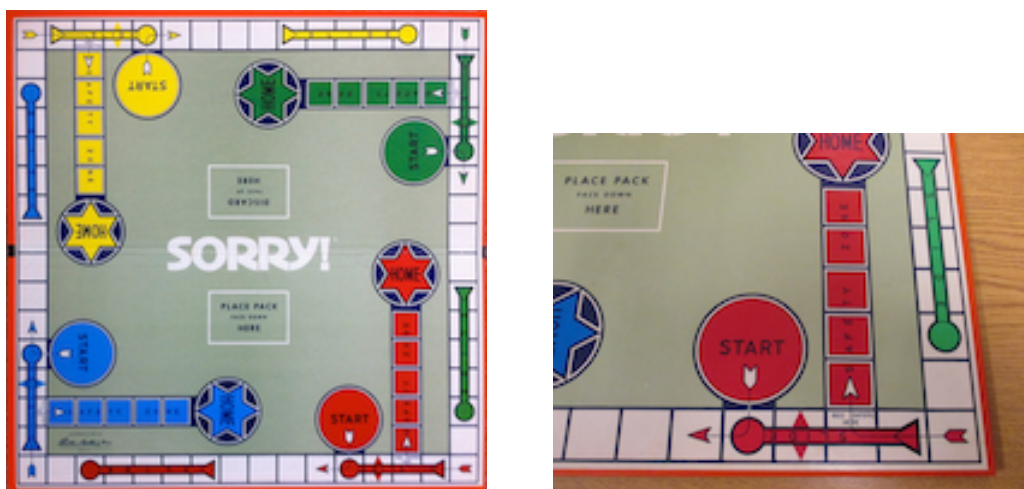


Figure 1: The *Sorry!* board

Movement is determined by selecting the top card on the deck, which is reshuffled whenever it is completely depleted. The deck has 45 cards: five 1s; four each of 2, 3, 4, 5, 7, 8, 10, 11, 12; and Sorry.

²rules for other versions can be found online but apparently not in one place

They have these actions:

- a 1 card either moves a token from their Start space to the closest space on the track or moves a token one space forward;
- a 2 card either moves a token from Start or moves a token two spaces forward; then the turn continues by drawing again;
- a 3,5,8, or 12 card moves a token forward that many spaces;
- a 4 card moves a token backwards four spaces;
- a 10 card gives the option to move a token forward ten spaces or backwards one space;
- a 7 card gives the option to move one token seven spaces forward or to split the seven space moves between two tokens;
- an 11 card gives the option to move a token forward eleven spaces or to swap positions with an opponent's token;
- a Sorry card allows the player to substitute a token in their Start space for any opponent token, which is sent back to its corresponding Start space.

These are the key rules of play:

- tokens must reach Home by exact count (if splitting a 7 both parts must be used);
- all card actions must be carried out unless not allowed, even if the result is disadvantageous; if the move cannot be made it still counts as a turn; the 11 card swap is never required;
- tokens are not affected by slides of their same color;
- tokens in their Safe Zone cannot be moved by another player's Sorry or 11 action; tokens cannot move into another player's Safe Zone;
- if a token ends its move on a space already occupied by a token of a different color, then the other token is sent to its Start space; if the tokens are of the same color, then that move is not allowed.

For our analysis we will play one-person *Sorry!*, which uses only one token and no opponents. In this case the 7 and 11 cards only have the option to move the token that many spaces forward and the Sorry card will have no effect on the token while still counting as a turn. So the 10 is the only card that gives the player an option. We are interested in determining **when it is advantageous to move backwards one space rather than forward ten spaces**. Obviously this is only practical near the beginning of the path around the board as it might enable use of the shortcut. In order to determine precisely on which spaces to make the choice to go backwards, we will compare the expected number of turns to move the token from Start to Home. For future reference we number the spaces near Start, referring to the blow-up in Figure 1: 0 for the space where tokens move from Start; 1,2,3,... for the spaces clockwise around the outer track from there; -1 for the space counterclockwise—the “shortcut space”.

3 The Markov Set-Up

For the sake of analysis we consider each space that the single token can occupy to be a *state*. Then to model the movement as a Markov process we must determine the probabilities for moving between each pair of states and these probabilities must only depend on the states. As mentioned before, because moves are determined by cards drawn from a deck, the state of the deck of cards would not be the same every time a token is on a specific space on the board so the probability can change. In order to use the Markov model analysis we will assign what might be considered “long-term probabilities” based on the proportion of each type of card in the deck. Below we give rationale for why this should be a reasonable and effective accommodation.

In this model there are 61 allowable states: the Start and Home spaces, the 54 spaces on the outer track (including the “shortcut” space and excluding the start spaces of the six opponent-colored slides), and the five Safe Zone spaces. Once we have determined the probabilities of moving between all of the states, we create the 61×61 transition matrix A (partially shown in Figure 2) where each entry $a_{i,j}$ is the probability of moving from state i to state j on a single turn. For examples, the

| | Start | space -1 | space 0 | space 1 | space 2 | ... |
|----------|----------|----------|----------|----------|----------|-----|
| Start | 0.800000 | 0 | 0.119216 | 0.010096 | 0.000567 | |
| space -1 | 0 | 0.089456 | 0.111111 | 0.008105 | 0.098985 | |
| space 0 | 0 | 0 | 0.089456 | 0.111111 | 0.008105 | |
| space 1 | 0 | 0.008077 | 0 | 0.089456 | 0.111111 | |
| space 2 | 0 | 0 | 0.008077 | 0 | 0.089456 | |
| ⋮ | | | | | | ⋱ |

Figure 2: Transition matrix A

1,1 entry is the probability of the token not leaving the Start space, the 2,4 entry is the probability of a token going from the -1 space to the 1 space, and the 0 in entry 5,2 reflects that it is not possible for a token to go from space 2 to space -1 in a single turn. The full transition matrix can be requested from the author³. The i, j entries of the matrix products A^2 , A^3 , A^n give the probabilities of moving from state i to state j in exactly 2, 3, or n turns, respectively. In particular, the entries along the first row (corresponding to the Start space) of each A^n give the probabilities of moving from Start to each state in exactly n turns. This is also the probability that turn $n + 1$ starts on that space. The sum of the 1, j entries of all of the powers of A give the expected number of turns that start in state j . Grinstead and Snell [6, Chapter 11] give a thorough presentation of how to find the expected game length: we trim the row and column from A corresponding to the *absorbing state* (Home space) to form matrix B . Then the *fundamental matrix* $N = (I - B)^{-1}$ is found. Each entry along the first row of N is the number of turns that a token is expected to start in that state. So the sum of the entries on the first row of N gives the expected number of total turns for the token to get from Start to Home. Briefly, this can be seen since N is equal to the sum $I + B + B^2 + B^3 + \dots$ whose i, j entry adds the probabilities of getting from state i to state j in 0, 1, 2, 3, etc., turns. This entry then equals the expected number of turns that a pawn that was once in state i will start a turn in state j . Figure 3 shows the beginning of the 60×60 matrix N . The sum of all the entries of any row of N gives the expected number of turns for a token starting in that state to reach Home, thus the row 1 sum is the expected length from Start to Home, but also the row 2 sum is the expected number of turns for a token already on space -1 to get Home, etc.

³email gswain@ashland.edu

| | Start | space -1 | space 0 | space 1 | space 2 | ... |
|----------|----------|----------|----------|----------|----------|-----|
| Start | 5.000000 | 0.015329 | 0.662139 | 0.151296 | 0.036626 | |
| space -1 | 0 | 1.103852 | 0.149431 | 0.034275 | 0.141975 | |
| space 0 | 0 | 0.015321 | 1.105736 | 0.149994 | 0.037285 | |
| space 1 | 0 | 0.013535 | 0.016935 | 1.106279 | 0.151772 | |
| space 2 | 0 | 0.015060 | 0.015361 | 0.017510 | 1.108245 | |
| ⋮ | | | | | | ⋱ |

Figure 3: Fundamental matrix $N = (I - B)^{-1}$

Next we focus on determining each entry of the transition matrix A . This will depend on the chance of drawing a card from the deck that determines this move, and this is where using the deck of 45 cards rather than a spinner or die causes some difficulty. We expect that as the deck is exhausted, the probability of getting a specific card changes; however this is not the case. For any one specific card it has an equal chance of being in any of the 45 places in a shuffled deck. So for example, the chance of drawing an 8 card from a fresh deck is clearly $\frac{4}{45}$, which is also the probability that the last card in the deck is an 8, but also the same chance of the n th card in the deck being an 8. Of course this assumes we don't know anything specific about which cards were already drawn from the deck; as soon as we have any such information, then the corresponding conditional probability is not the same. In reality the probabilities, for example $P(\text{draw } 8 | \text{token on space } j)$, depend on what cards moved the token to the space. In order to be able to use the Markov process, we will assume that, at the start of each turn, we are at a random spot in a shuffled deck with no prior information about which cards may have already been drawn.

So the same $\frac{4}{45}$ probability ($\frac{5}{45}$ for the 1 card), holds for any card on any turn, except that whenever a 2 card is drawn the same turn continues with another card drawn. This card could also be a 2 and so a single turn could be quite complicated, especially if the deck is exhausted in the middle of a turn. The most 2s that could be drawn consecutively is eight: the last four cards in the deck are 2s and the first four cards in the reshuffled deck are also 2s. As an example of calculating probabilities for turns that involve repeated 2s we consider *two 2s followed by an 8* (a 2, 2, 8 turn). If the first card drawn is in any of the top 43 positions in the deck then this probability is $\frac{4}{45} \frac{3}{44} \frac{4}{43}$; if the first card drawn is in position 44 then the probability is $\frac{4}{45} \frac{3}{44} \frac{4}{45}$ (the 8 is drawn from the reshuffled deck); and if the first draw is the last card in the deck then the probability is $\frac{4}{45} \frac{4}{45} \frac{4}{44}$. Weighting these three possibilities assuming our place in the deck at the beginning of any turn is uniformly random, we see that the probability of drawing two 2s and an 8 is

$$\frac{43}{45} \left(\frac{4}{45} \frac{3}{44} \frac{4}{43} \right) + \frac{1}{45} \left(\frac{4}{45} \frac{3}{44} \frac{4}{45} \right) + \frac{1}{45} \left(\frac{4}{45} \frac{4}{45} \frac{4}{44} \right) \approx 0.000566654.$$

Similar calculations show that the probability of drawing eight 2s and an 8 is about 0.000000000000098. The transition probabilities in matrix A need to be accurate enough to reflect even these small probabilities.

We can now explain some entries in matrix A (Figure 2): the 1,1 entry is $\frac{36}{45} = 0.8$ reflecting all cards except 1s and 2s, and the 2,3 entry is $\frac{5}{45} = 0.\bar{1}$ since only a 1 card will move the token from space -1 to space 0. Since several different turns can have the same end result—for example a 7, a 2,5, a 2,2,3, and a 2,2,2,1 all move the token 7 spaces—many entries in matrix A involve adding several probabilities. For example, the 3,3 entry, which corresponds to staying on space 1, reflects a Sorry card draw (probability $\frac{4}{45} = 0.0\bar{8}$) or a 2,2,4 draw (probability 0.000566654 as above). The

role of slides, at the end of a move or in the middle of a move that involves drawing a 2, means that probabilities for moving between states that are the same number of spaces apart may not be the same depending on where they are on the board. Also, the requirement to reach Home by exact count restricts the action of some cards as the token gets closer to the end. All of this makes creating matrix A a technically demanding task.

4 When to go Backwards

The strategic question is, “When do I take the *move back one space* option when a 10 card is drawn?” When the token is fewer than 10 spaces from Home this move is required, but in all other circumstances it is the player’s choice. In general one wants to maximize forward progress on a move, so even moving backwards to take advantage of a slide will do worse than moving 10 spaces forward. The one exception is when a backwards move enables the use of the shortcut. The obvious scenario is if we are on space -1 , since the space we might designate “ -2 ” is really space 58 in the normal clockwise direction.

Opting to move backwards from space -1 will be our base case strategy, and this will be reflected in the transition probabilities. The “strategies” we consider are whether we will always move backwards from space 0, from space 1, etc., specifically where we should switch from opting to move backwards to moving forward to optimize the expected number of turns. While space -1 is the only one for which this automatically leads to the shortcut, it is plausible that a subsequent 4 card (“move back four spaces”) or additional 10 cards could combine to achieve the shortcut from other spaces. While the chances may be small, the huge net forward progress of the shortcut will affect the expected number of turns to get the token Home.

| when to move backwards | expected # of turns |
|--------------------------|---------------------|
| only on space -1 | 25.41097 |
| only on spaces -1 or 0 | 25.42297 |
| up to space 1 | 25.43240 |
| up to space 2 | 25.45093 |
| up to space 3 | 25.46879 |
| \vdots | \vdots |
| never | 25.41984 |

Figure 4: Expected number of turns for different “10 card strategies”

These strategies are easily compared by modifying appropriate rows in the transition matrix. Figure 4 lists the calculated expected lengths. There is no need to extend the table as the expected lengths will increase as we extend the “backwards” strategy further clockwise from the beginning space. These results show that the optimal strategy is to **only move backwards with a 10 card when on space -1** . This result should be surprising to players of the game, as most players would move backwards from space 0 as well. But of course that makes sense if they have other tokens in play and can “wait” for a fortuitous draw. The wait for a 4 card or one of the three remaining 10 cards is expected to be about 6 turns (using the geometric distribution where each turn is an independent Bernoulli trial with $p = \frac{7}{44}$). Realistically a player would not want to “wait” on space 0 since it blocks other tokens from leaving start, and on space 1 or 2 they would have an expected wait of 11 turns for a 4 card ($p = \frac{4}{44}$).

The last entry in the table shows that it makes almost no difference if the backwards option is never

(unless required) used. This is surprising but is explained by the very small probability of a token ever getting to space -1 —that token must have come from space 0 using a 10 card or from space 3 by means of a 4 card action! Clearly in the single token version the shortcut is rarely achieved. On the other hand, in the full version a player has a greater opportunity of success, but also can be thwarted by opponents.

5 Other Questions Answered, Somewhat

We can use the Markov model to answer other questions about the effects on the expected number of turns of other aspects of the *Sorry!* rules.

- How much does the opportunity to draw again on 2 speed up the game? The adjusted Markov model, where a 2 does not draw again, yields an expected game length of 27.6218 turns as opposed to 25.41097. The increase of about two turns is reasonable as we can expect a 2 card to appear about twice in a 25 turn game (around half of the deck), hence that many additional turns.
- In the regular game being able to split 7 moves between two tokens allows a player to take advantage of more slides and makes getting the exact count to go Home easier. This efficiency is not available when only one token is in play. We can simulate this effect by allowing the single token to move any number of spaces up to 7 (as though they had another token to use the rest of the moves on). This modification yields an expected length of just 19.83479 turns.
- It is difficult to assess the effects of the 11 card swap action and the Sorry card, which depend on having multiple players. It seems fair to assume no net effect of the swap option for the 11 card as on average you would get moved backwards by someone else as often as you get to move forward through swapping. The effect of the Sorry card is not only to move a token forward, and the defensive player back, but it also saves or adds the turns spent in trying to get a 1 or 2 card to leave the Start space. To model a worst case scenario we stipulate that the single token moves back to Start any time a Sorry card is drawn or it moves to the 0 space if it is at Start. The expected length under these assumptions is 36.08460 turns!
- How many turns are used just to move from Start? The 1,1 entry of the fundamental matrix is 5, which means we don't expect to get out until the fifth turn, on average. This agrees with the geometric distribution with $p = \frac{9}{45}$. On the other hand, draws at the beginning of the game start with a fresh deck and we can calculate the precise expected value for where the first 1 or 2 card appears in the deck by

$$1 \frac{9}{45} + 2 \frac{36}{45} \frac{9}{44} + 3 \frac{36}{45} \frac{35}{44} \frac{9}{43} + \cdots + 37 \frac{36}{45} \cdots \frac{1}{10} \frac{9}{9} = 4.6.$$

So, even with multiple players and at any point in a game, we can expect 4 to 5 turns to be used just to move a token from Start. Of course when a player has other tokens in play these turns aren't wasted. Regardless, it takes about 20 turns for a token to get Home once it is out of Start.

- What effect does having to reach Home by exact count have on the length? This turns out to have a much greater effect than even the Start restrictions. Adjusting the transition matrix to allow for overshoots of Home, the expected game length is 15.03224, a savings of more than 10 turns! Some of this efficiency can be achieved in the full game if you can rush a token to the Safe Zone, then wait for a fortuitous card while spending turns on other tokens.

Clearly the significant difference between our single token analysis and the multi-token game is the opportunity to wait for a fortuitous card; at least we have a clearer idea now of what the best case scenario is.

6 Conclusions

By modeling the movement of a token around the *Sorry!* board we are able to see the effect, in terms of how many turns it takes to get the token Home, of applying different strategies for using the 10 card actions. We also explored the effects of “splitting a 7” and the Sorry card. Additionally, we discussed whether and how the one-token game may relate to the regular multi-player, four-token game. By no means have we “figured out” the game, but we have shown a process that can help analyze strategies for efficient play. One significant aspect of what we’ve done is that we have applied the Markov model to a situation that is not actually Markovian. We’ve argued that this gives a reasonable approximation to reality and should still provide a useful and effective way to *compare* strategies.

One might wonder how good the approximation really is? To explore this, Rupesh Maharjan, a computer science student at Ashland University, wrote code to play the one-token game, with the deck shuffled only at the start and whenever it was depleted. His program counted the number of turns it took to get the token Home and averaged this over a million games. The result, 24.2627 turns, is very close to our Markov model estimate of 25.41097. It would be interesting to explore how to adjust the transition probabilities to reflect in some way the likely state of the card deck when the token is on each space throughout the game, seeking to make the Markov estimates even closer to the simulated average.

Of course the same analysis we’ve done can be done for two tokens (61^2 states!) or even four tokens, assuming the creation of the transition matrix can be automated. Then priority strategies, such as “get all tokens out of Start” versus “get a token Home”, can be compared. It may also be fun to model a pair of opposing tokens to explore how adversarial or cooperative strategies compare. Or you can just get out your *Sorry!* game and play for the fun of it and try to ignore any mathematical questions that come to mind!

7 Acknowledgements

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Biblical Principles with Benedictine Hallmarks in the Mathematics Curriculum

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Abstract

Many of the hallmarks of Christian education identified in the Bible by Benedict of Nursia fit naturally in mathematics curriculum. Successes and challenges to incorporating Benedictine Hallmarks were considered from a newly revised Core course on Mathematics in Modern Society. Multiple methods were adopted to integrate these values into the course. Short readings from scripture and mathematics literature that related to the campus values were posted on the course webpage for discussion each week. In-class activities included reflective exercises on applying these values in mathematics. Community was built through these activities as students listened to each other's ideas and techniques. Hospitality was modeled to the instructor's classes through office hours held in the tutoring center. A term paper requiring students to describe mathematics involved in their hobby or area of study encouraged many to practice stewardship of resources using their mathematical knowledge or to express mathematics relevant to their cultures.

Benefits of learning and obeying God's Word are described throughout the Bible, with chapters such as Psalms 1 and 119 dedicated to the importance of God's Word. In Matthew 4, Jesus also provided examples of how to rebuke temptation using God's laws in Deuteronomy. Benedict of Nursia, a sixth century Catholic monk, recognized the necessity of God's Word as he quoted and paraphrased passages from God's Word in writing his monastic rule for seeking God with others [9]. *The Rule of Saint Benedict* still directs the life of Benedictine monasteries as they carry out their work.

While schools sponsored by Benedictines do not function with the same structured schedules as the monasteries, they share many of the Benedictine values as hallmarks of Christian education. At colleges such as Saint Martin's University, these common values are Awareness of God, Community Living, Dignity of Work, Hospitality, Justice, Listening, Moderation, Peace, Respect for Persons, Stability, and Stewardship [7, 9]. Saint Martin's University also arranges these values more concisely into the campus core themes of Faith, Reason, Community, and Service [7]. Saint Martin's University is a small, private liberal arts university of about 1500 students in Lacey, WA. Saint Martin's also recruits diverse students and employees, and is categorized as a Minority Serving Institution [5].

The Benedictine hallmarks provide a bridge between teaching with a Biblical worldview and the campus culture of Saint Martin's University. While the Benedictine values apply to the entire Saint Martin's University community, the university Core courses are structured to develop individuals in the Benedictine values. The Core mathematics course, Mathematics in Modern Society, specifically focuses on the theme of reason through examining problem solving in data, statistics, voting, and finance. This course serves students in majors without a specific mathematics course requirement,

such as Communications, English, Fine Arts, History, Political Science, Social Work, Sociology, and starting in 2022, Elementary Education. When the author first taught the Core mathematics course in Fall 2019, Saint Martin's University celebrated a different Benedictine hallmark each week, which he explicitly connected to our course study of mathematics and the Bible through short class discussion. The author continues to share many of the activities and Bible verses while encouraging discussion with other mathematics and statistics courses as well.

1 Searching for Knowledge

One of the most remembered activities of students the author reconnects with or who join him for multiple classes in sequence is a scavenger hunt on campus. While most activities assigned in class are intended to be completed that session, for this assignment students are instructed that they will need to find the required information to be discussed and submitted for grading at the next class meeting. The exercise asks students to complete the statement:

“The reflected image is:”

Students are instructed in writing on the course learning management system and verbally in class to find the statement on the dedication plaque next to the You Mirror, which was gifted to Saint Martin's University from Academy of International Education, Saint Martin's University's sister school in Tokyo, Japan. Directions are also provided for finding the You Mirror and plaque. Most students find the mirror, plaque, and quote successfully before the next class, sometimes with additional help from the instructor or classmates. The complete statement reads:

“The reflected image is the person responsible for success in academic pursuits.”

After checking that the class has recorded the statement, the instructor asks them which Benedictine value(s) this represents? While there is often more than one answer, the most common response has been Dignity of Work. Students usually discuss and comment on the importance of their responsibility in their academic work. Each semester, the instructor reminds the class of Paul's exhortation in Ephesians 4:1, “As a prisoner for the Lord, then, I urge you to live a life worthy of the calling you have received.” The instructor's encouragement to the class is that their current calling involves active participation and responsibility in their mathematics course and studies to learn skills necessary for later courses, work, and life. The action of physically finding this quote seems to help many students remember this lesson, the associated Benedictine value, and Biblical teaching.

2 Hospitality in Service

Each semester, the author holds a few of his weekly office hours away from his office to make himself available where the students are instead of requiring them to find him. In addition to posting office hours and locations on the office door and course learning management system, the author also regularly invites classes to visit him with questions or for group work. Some semesters, the author reserves an unused classroom space before or after a class to take student questions or help them review recently covered material. However, most semesters he schedules a few office hours in the university Center for Student Success. These options provide more space than the office for multiple students to work together and with the instructor.

Working in the Center for Student Success also familiarizes students with a designated study area on campus where they can work together and get help from peer tutors even when the instructor is not available. Moving away from the office limits the range of course materials the instructor can bring with him, but recent notes and a laptop or campus computer are usually enough to address most student questions. Completing course preparations or scholarship also do not usually happen when working with students in public work spaces, but the benefit of welcoming and showing care for students outweighs these limitations. Students have shared that the instructor's accessibility helps them build a habit of working in the same place each week and provides the reassurance of having an instructor or tutor to clarify questions with.

When Saint Martin's University celebrated Hospitality Week during Fall 2019, the author announced and posted on the office door and learning management system for all of his courses that his office hours would be held in the Center for Student Success. In sharing his observation of hospitality that week and through the semester with the classes, the instructor reminded them of Benedict's encouragement that, "Anyone who knocks at the door of the community should be welcomed as Christ, for he himself will say: I was a stranger and you welcomed me" [9, p. 81]. The instructor also shared a few verses later where in Matthew 25:40, Jesus teaches, "Truly I tell you, whatever you did for one of the least of these brothers and sisters of mine, you did for me." May the Holy Spirit continue to guide the author and Saint Martin's University in better showing students and each other the love of Jesus.

3 Reason and Peace

As the first exam approaches each semester, the author posts in the learning management system of each course an article from *The Atlantic* entitled *The Myth of 'I'm Bad at Math'* [2], and asks students to read and reflect on it. The first exam for the Core mathematics course also happened to occur during the campus Reason Week of Fall 2019, putting students' reasoning skills to the test. While there are many sources on mindset, this article is more accessible for introductory students. Both in class and individual meetings throughout the semester, the instructor encourages students:

Do not be anxious about anything, but in every situation, by prayer and petition, with thanksgiving, present your requests to God. And the peace of God, which transcends all understanding, will guard your hearts and your minds in Christ Jesus. (Phil. 4:6-7)

The author also shares that he finds these two of the easiest verses to memorize and two of the most difficult to apply. Many students have responded that they found the article helpful in seeking peace and reassurance to adopt a growth mindset for overcoming difficulties.

4 Stewardship and Calling

The final project for the Core mathematics course was a short term paper describing the mathematics involved in a topic of each student's choosing related to their hobby or interest. Topics the author had not anticipated included the role of mathematics in the Benedictine value of stewardship. Many students wrote about financial literacy, including factors and statistics on debt, credit, and retirement savings. One sociology student wrote a paper on the costs and difficulties of home ownership, describing taxes, homeowner's insurance, and the amortization of mortgage payments. While she was passionate about the topic, the author was surprised when this sociology student's academic advisor shared that the Core mathematics course was her favorite class that semester.

The year after this student graduated, the university wrote a story about her work to educate and help individuals find work and living accommodations.

A native Hawaiian student also wrote about the importance of stewarding nature well. She cited and interpreted statistics from reports correlating tourism and the decline in coral reefs near Oahu before the pandemic, as well as improvements in the coral reef during the pandemic when tourism slowed. Another student studying history education described statistics on high school dropout rates for local school districts and factors related to improving graduation rates. She concluded with excitement to make a difference for her future students. These projects also encouraged the instructor in hearing about the passions and callings God has given these students.

5 Outcomes and Conclusions

Instead of searching for results, Jesus taught in Matthew 6:33 to, “seek first the kingdom of God and his righteousness, and all these things will be added to you.” As the author does so, God has provided opportunities to explicitly build bridges between teaching with a Biblical worldview and the Benedictine hallmarks practiced by the Saint Martin’s University community. A few of these outcomes are notable, while the author may not have heard of others yet.

Working at a faith-affiliated school that grants faculty freedom to share about faith in and out of class and encourage students and coworkers in theirs has allowed the author opportunities to sow seeds of the good news of Jesus. Students were encouraged to find and value peace from anxiety, a gift directly from God that he has also allowed people to discover through research. Many students now have a lifelong lesson to remember the dignity of work and to walk worthy of the calling we have received, which includes projects applying mathematics. Throughout this journey, these classes have met and worked in community to actively practice Listening to shared learning and growing experiences. These lessons and activities generally provide beneficial discussion for other mathematics courses as well.

The work of incorporating Biblical principles with Benedictine hallmarks continues with prayer for Saint Martin’s University, the community, and local congregations. A possible next step includes developing a Core ethics course on dignity of work in STEM pedagogy. Readings from the vast literature on Culturally Relevant Pedagogy are being organized to responsibly structure such a course, with flexibility for careful listening to the interests of Saint Martin’s University’s students of diverse experiences and backgrounds. Much like a Core mathematics course, this Core ethics course will also be fun to teach from a Biblical worldview.

Benedictine values are practiced by many other communities outside of monasteries and institutions they have founded. People of Catholic and other Christian traditions who resonate with *The Rule of Saint Benedict* and seek to live by it as monks working in the world can grow and serve as laypeople known as Oblates of Saint Benedict [6]. Benedictine values may also be explored at other institutions through study and courses on church history. At Whitworth University, Jerry Sittser has lead students in one month January-term courses on the history of Christian spirituality at a Presbyterian camp in the Cascade Mountains [8]. While Sittser noted that students found living *The Rule of Saint Benedict* a challenge, they also learned to value play as part of Sabbath rest, how to pray in work, and that the church is more than what is often experienced here and now in modern, western society.

In addition to working at a Benedictine university, the author learned more about Benedict of

Nursia and the values from his *Rule* by reading an English paraphrase and commentary by Jonathan Wilson-Hartgrove, a Baptist minister [9]. A four month daily reading following the schedule of the monasteries is available from The Order of Saint Benedict [4]. Through reading Wilson-Hartgrove's paraphrase with this schedule, the author prayerfully formed the described cultural bridges to the Biblical worldview for teaching his classes. Similar exploration of recorded Christian practices may be helpful to classes at other institutions as well.

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Functions and Vectors for Data Science

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Simon Tse earned his Ph.D. in Mathematics from the University of British Columbia, and M.Phil and B.S. from the Chinese University of Hong Kong. His research interest includes analysis and simulation of pattern formation with PDEs, and, more recently, mathematics education. He is married to Jenny Wu, whom he met at an evangelical student fellowship in Hong Kong. They have been building a family together with three children Stephanie, Paul and Theodore.

Abstract

The past decade has witnessed “learning from data” catapulting into great public attention. In such a context, I describe an undergraduate course, *Mathematics of Data Science*, that immediately follows courses for *Linear Algebra* and *Multivariable Calculus*, aiming to fill the educational need of a new generation of math majors and minors. I also discuss how the organization of the math content led to synergy with a few other upper-level courses and how the teaching links to new modes of scientific reasoning and Christian apologetics.

1 Introduction

1.1 Beginning with Calculus

Calculus is still the main starting point of North American university education in mathematics. Although there have been decades of effort to rethink the calculus sequence: differential and integral calculus of a single variable, followed by multivariable or vector calculus, still hold the role of “ushers” of mathematics to almost every university student with a need or an interest to major or minor in mathematics. In a society that increasingly values “learning from data” over “learning from formulas,” how can we use these standard course prerequisites to help students understand “data”?

Observing that more and more mathematics programs are offering a freshman-oriented course on some aspects of linear algebra, I have developed a course titled “Mathematics of Data Science” (hereafter abbreviated as MDS) that is designed to be taken immediately after finishing a course in linear algebra and a course in multivariable calculus. This paper begins with the background of the course’s conception, followed by the mathematical ideas behind the chosen learning outcomes, along with some pedagogical and philosophical reflections, and ends with discussions on how MDS fits into a math program and the role of Christian witness at a personal and an institutional level in the midst of a data-driven paradigm shift in the society.

1.2 The rise of data science education

Why would a course like MDS be of interest to the next generation? How is it relevant to a mathematics department in a liberal arts university, often with a modest variety of mathematics courses? First and foremost, it is clear that the *content* of data science is of interest to the next generation. Seeing how nowadays people are “hyped up” by promising prototypes of “self-driving” cars, “computer-generated” artwork, and business “intelligence,” to name a few, it is unsurprising to

see more and more mathematics programs re-branding themselves accordingly. Indeed, according to Davenport and Patil, the authors of highly read article back in 2012: *Data Scientist: The Sexiest Job of the 21st Century* [10], the majority of data scientists surveyed just a decade ago, “had PhDs in some scientific field, were exceptional at math and knew how to code,” but “[n]ow there are hundreds of degree programs in data science,” says the same authors ten years afterward [11]. It does seem that various departments and institutions have been pivoting to adapt to the societal expectation of teaching “some understanding of data” in a quantitatively oriented degree - and a mathematics department will inevitably need to give a good answer to that.

Why would MDS, at the undergraduate level, be a viable way of responding to this demand? If you have browsed through new educational resources, especially from engineering, business, and statistics programs, you may agree that current offerings look like a bimodal distribution. They either promote low-barrier courses requiring minimal math and coding background (e.g. the “Data Science Specialization” by Coursera) or specialist courses with more rigorous prerequisites (e.g. courses that are part of a master-level program that examine the concepts underpinning modern algorithms). It seems to me the former caters to “data science users” and the latter caters to “data science specialists.” Davenport and Patil’s findings concur that “most of [these new data science programs] are master’s degree programs,” even though initiatives at the doctoral, undergraduate, or even high-school levels also exist [11].

Is there really a gap and such bimodality in mathematics education in the context of data science? As far as I know, a systematic analysis may not exist yet. However, increasingly, some leading scholars in their fields of expertise observe some gaps of knowledge of this sort. For instance, machine learning researchers Deisenroth et al. opined in the foreword of their new book [4] that “Current machine learning textbooks primarily focus on machine learning algorithms and methodologies and assume that the reader is competent in mathematics and statistics.” On the other hand, outside of academia, a quantitative finance specialist, Wilmott, in his book [12], observes that “... there are quite a lot of books that describe the *use* of machine-learning techniques ... not much code, and very elementary mathematics ... then there are the specialist books. Each devoted to a narrow field, but extremely deep.”

It leads me to wonder, “Is it true that a comprehensive mathematical foundation course for data science demands the academic maturity of a postgraduate student?” After a survey of the admission requirements and course structures of some of the more mathematically oriented master-level programs, I concluded that it isn’t. Moreover, a motivated math major or even minor could do well in the foundational mathematics concepts. The good old calculus and linear algebra combination is still an excellent background knowledge for a mathematics student pursuing data science and machine learning studies. In addition, I found something even more surprising - I realized that classic calculus texts such as the two-volume by Apostol [1, 2], written more than 60 years ago with an explicit goal of blending multivariable calculus with linear algebra still fits these technical requirements quite well! The main caveat is that we must re-imagine how the idea of “function” and “vector” should be interpreted in the modern data science context. Some simplification and organization of materials are also necessary to make MDS feasible within a semester.

In the following, I shall begin by defining what the word “data” means to mathematical sciences and use it to describe how knowledge in calculus and linear algebra can be used to support a mathematical foundation for data science. I will provide brief descriptions of the topics covered in MDS that has been offered at Trinity Western University in British Columbia, Canada, since 2021.

1.3 A mathematical interpretation of “data”

The English word “data” is, in fact, Latin. It is the plural form of “datum,” which means “the givens.” This shade of meaning is faithfully translated in French as “les données” and “נתונים” (natonim) in Hebrew, which both mean “that which are given.” While this meaning does not directly entail a numerical representation of data, the modern meaning does. From the Merriam-Webster dictionary, we find these three entries for how the word “data” is used nowadays:

1. factual information (such as measurements or statistics) used as a basis for reasoning, discussion, or calculation;
2. information in digital form that can be transmitted or processed;
3. information output by a sensing device or organ that includes both useful and irrelevant or redundant information and must be processed to be meaningful.

I interpret these loosely to correspond to (1) scientific, (2) technological, and (3) engineering aspects of the meaning of “data” to society. In consideration of these aspects of applications, I extract from the above discussion two mathematically useful senses of “data” as follows:

The empirical sense: that which is given empirically as a means to confirm or falsify a preconceived hypothesis or to inform or generate a new hypothesis;

The inferential sense: that consists of a possibly huge but finite set of numbers by which truths for a certain domain of knowledge can be inferred

Various milestone events in science easily corroborate “the empirical sense of data.” For example, Nightingale collected data about injured soldiers and conducted exploratory data analysis to put forward a novel hypothesis that the main reason for mortality was preventable diseases but not combat-related trauma. Another example is the Eddington experiment, when observational data from a solar eclipse helped establish Einstein’s theory of relativity - a preconceived hypothesis - over Newtonian mechanics as the mathematical framework of choice for cosmology.

However, when you think about the kind of mathematics or statistics and the form of scientific reasoning involved, these two examples are rather different. In the latter, the fact that the *a posteriori* data differed *less* from the *a priori* numerical values computable from theories is the reason why Einstein’s theory was better. This relationship between data and mathematical theory has a striking property that the numbers computed from a scientifically confirmed mathematical theory would render numbers measured from empirical observation *dispensable*. In fact, this is a reason why computer simulations of physical laws can be used to replace frequent empirical measurements. This is the legacy of hypothesis-driven science.

In contrast, I opine that, for the purpose of educating math undergraduates, we should focus on the “inferential sense of data” that relates better to novel modes of reasoning in data science, especially for applications in artificial intelligence. To further elaborate on this second sense of “data,” let us consider a thought experiment in a hypothetical world that upends “the empirical sense of data.” Suppose creatures lived in a universe constructed by fractals, such as the like of Mandelbrot set, and suppose these creatures are given the engineering prowess to both measure a lot of data points in this world and also computationally infer, *based on the data*, that $z^2 + c = z$ is

the law by which this world was formed. Suppose also that this theory computed from data matches perfectly with subsequent observations. What, then, would “empirical data” mean? Wouldn’t such a data-driven theory production render “data measured” and “data computed” one and the same? This thought experiment touches the ongoing debate of the value of data-driven science vs. the traditional hypothesis-driven science.

Not dwelling on philosophical debates, my proposition is that a course like MDS that concerns the “inferential sense of data” and develops mathematical concepts for obtaining an understanding of numerical data is valuable to both math majors and minors. More specifically, I propose that the following three categories of mathematical ideas would enrich a mathematics student’s understanding of “learning from data”:

- representations by finite sets of numbers, as vectors, matrices, and tensors
- the concept of abstract functions as mappings between high-dimensional spaces
- the interconnection of approximation, optimization, and convergence

These categories are chosen to broadly give a sample of how mathematical thinking can help clarify the “whats” and “hows” of foundational algorithms in data science. In MDS, we treat the “inferential sense of data” by formalizing “data” as vectors, and “inference” as learning how to approximate functions, followed by a broadening of differential calculus as the way to solve optimization problems to incrementally improve approximations. A notable omission in this list of categories is the “why” – interpretations of the model functions constructed and analyzed.

We found it more feasible to help students who have just finished the Calculus sequence to transition from associating “functions” as a collection of “algebraic manipulation techniques”, to “functions” as maps lacking a determinate form before data comes in. These categories represent the kind of mathematics that are more readily stated in a precise way and amenable to manual computations that build up conceptual understanding. Examples of content that can be found in other introductions to data science, but omitted in MDS, include data engineering best practices, statistical reasoning and data visualization techniques.

With these categories in mind, I now summarize the mathematics that I found helpful to cover in MDS. They are selected for this article based mainly on two criteria:

- Is the concept a logical extension of what are usually covered in calculus and linear algebra?
- Does a profitable reformulation exist to reduce conceptual complexity to help provide a big-picture view of the theoretical ideas in data science?

2 Some Useful Mathematics for Data Science

The following content contains highlights of key ideas for each of the seven core topics covered, interspersed with pedagogical notes as appropriate.

2.1 Modelling Phenomena by Mathematical Functions.

The overarching purpose of the kind of mathematics presented in MDS is predicated on the premise that we are ultimately trying to understand how to *learn* a multi-input multi-output function f that maps from a domain X to a codomain Y , but we lack precise knowledge of how to define X , Y and f and thus seek ways to learn about these sets and the mapping between them from data.

The abstractness of the standard definition of function was emphasized by encouraging students to first come up with non-numerical examples of inputs and outputs that they are interested in, to conceptualize what I called the “phenomenological aspect” of a function. This stems from the observation that in data science, the term “function” is more often replaced by “model,” which connotes some procedure of “model training” using data. Students will subsequently see examples of functions in applications of data science, such as

- a linear combination of basis functions with undetermined coefficients (for linear regression)
- nested compositions of multi-input multi-output linear functions with single variable nonlinear activator functions (for artificial neural networks)
- probability distributions with parameters inferred from data (for Gaussian mixture models)

A key learning goal of this topic is to appreciate various ways a function can be defined with a potentially high-dimensional domain or codomain.

2.2 Introducing Differentiability by Linear Approximations.

The protagonist of calculus—derivative—appears immediately after broadening the concept of functions and is motivated by the need to understand how multiple outputs of a function vary with multiple inputs. Having prior exposure to the concept of differentiability, students now learn a more advanced formulation due to Fréchet for normed vector spaces. Beginning from a single variable function $y = f(x)$, and an anchor point (x_0, y_0) with $y_0 = f(x_0)$, the mapping f of x to y values is said to be differentiable at the point (x_0, y_0) if and only if it can be approximated by a linear function in a neighborhood, i.e., the approximation $y - y_0 \approx m(x - x_0)$ commits an error that diminishes “faster than linearly” as $x \rightarrow x_0$. This leads to the idea that the first derivative is the scalar multiplier on differentials, commonly understood as slope: $m = \frac{dy}{dx}$.

This formulation leads naturally to the multiple-input multiple-output version $\vec{y} - \vec{y}_0 \approx J(\vec{x} - \vec{x}_0)$ that achieves a similar goal. Even though the concept of “norm” is not to be introduced at this stage, students would still be able to appreciate the fact that the Jacobian matrix is a natural extension of the first derivative concept:

$$J = \frac{d\vec{y}}{d\vec{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

That such a matrix comes about from a rectangular arrangement of individual first partial derivatives $\frac{\partial y_i}{\partial x_j}$ —each of them describing how a single output varies with a single input—illuminates the

need to define matrix-vector multiplication properly. Matrix multiplication is now seen as a fundamental step to approximate differentiable functions. Likewise, the chain rule can be motivated naturally by further allowing each of y_i and x_i to vary in time so that

$$\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{d\vec{x}} \frac{d\vec{x}}{dt}.$$

Students mostly have no trouble relating this to the notations used in single variable calculus, and the formal resemblance of notations motivates students to appreciate that a proper way to define the multiplication on the right-hand side above is needed, and they would discover that it agrees precisely with how matrix multiplication was taught in an earlier course of linear algebra.

2.3 Vector-Based View of Matrix Algebra for Non-Square Matrices.

Except for orthogonal matrices that describe coordinate change, most advanced topics in linear algebra that primarily concern square matrices (e.g., properties of determinants, characteristic polynomials, Jordan normal form, and matrix exponential) are omitted in MDS. Rather, we discuss matrices that are either very wide ($n \gg m$) or very tall ($m \gg n$), using examples of models in data science that takes high-dimensional inputs (multiple camera video footages of a self-driving car) to low-dimensional outputs (power and steering angles at each of the four wheels), vs. those that take low-dimensional input (short textual prompts that describe a picture) to high-dimensional outputs (high-resolution AI-generated artworks).

This way of presentation has two kinds of pedagogical effects. Firstly, it paints a different intuitive picture of various fundamental theorems in linear algebra. For instance, students are excited to see that the rank theorem: $r = \text{rank } A \leq \min\{m, n\}$ implies how very high-dimensional models could become feasible to construct as the “principal” dimension depends on the lower dimension (e.g., power and steering at the four wheels, short textual prompts). Secondly, by purposely distancing from manual calculations about square matrices, like factoring characteristic polynomials (which are only tractable enough to be assigned as homework for very small n), students are guided to think more like a data scientist, where exact formulas are rare (e.g., that the probability of $A\vec{x} = \vec{b}$ holds is almost zero when \vec{b} comes from actual noisy data). They would be more motivated to learn about the normal equation $A^T A \vec{x} = A^T \vec{b}$ and the concept of orthogonal projection.

One other important highlight in this topic is the singular value decomposition (SVD) of an arbitrary matrix that represents a linear map $\Phi : X \rightarrow Y$ with $\dim X = n$ and $\dim Y = m$, presented in a compact outer-product expansion form (which Strang called “the third part of the fundamental theorem of linear algebra” in [9]):

$$A = \sigma_1 \vec{w}_1 \vec{v}_1^T + \sigma_2 \vec{w}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{w}_r \vec{v}_r^T,$$

where the right singular vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r \in \mathbb{R}^n$ are coordinate vectors representing elements of the domain X and the left singular vectors $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_r \in \mathbb{R}^m$ are coordinate vectors representing elements of the codomain Y , and where the descending list of positive singular values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

naturally induces ordered bases by telling us how important each pair of coordinate vectors (\vec{v}_k, \vec{w}_k) is, and that we need exactly $r = \text{rank } A$ such pairs of coordinate vectors to fully represent the original matrix A . The pairs with higher singular value multipliers are called the “principal components.”

2.4 Norms as the Key Instrument in Optimization

Instead of a discussion of the venerable spectral theorem for square matrices based on the fundamental theorem of algebra, we opt to characterize singular values through an optimization procedure due to Rayleigh’s energy-based definition of eigenvalues that relates to induced matrix norms:

$$\sigma_1 = \max_{\|\vec{x}\|=1} \|A\vec{x}\| = \max_{\|\vec{y}\|=1} \|A^T\vec{y}\| = \|A\| = \|A^T\|,$$

$$\vec{v}_1 = \operatorname{argmax}_{\|\vec{x}\|=1} \|A\vec{x}\|, \vec{w}_1 = \operatorname{argmax}_{\|\vec{y}\|=1} \|A^T\vec{y}\|.$$

(The subsequent singular values are then searched in orthogonal complements of the span of singular vectors already found, e.g., $\vec{v}_{k+1} \in (\operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\})^\perp$ etc., the fact that these orthogonal complements have falling dimensions entail the ordering of singular values $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$ is appreciated by some brighter students. This approach has various advantages in conceptual simplicity (see [9]).

On the one hand, this paves the way for the application of the method of Lagrange multipliers to connect singular values to the multipliers that occur at the critical points of a constrained optimization problem—the central kind of mathematical problem in the implementation of machine learning algorithms. On the other hand, a demonstration of the Monte Carlo method to solve the constrained optimization problem—i.e., trying random unit vectors to maximize the norm, highlights the role of loss functions defined through norms, and the usefulness of approximate numerical representations in data science.

2.5 Multi-linear Derivatives and Chain Rule by Propagating Values on Graphs

The idea of a positive/negative definite Hessian matrix is a difficult concept in multivariable calculus. Usually, calculus textbooks opt to introduce the “second derivative test” for multivariable scalar-valued functions omitting vector and matrix notations. In contrast, at this point of the course, students are more ready to see that the order of derivative in a Taylor expansion corresponds to the order of tensor that approximates a multi-linear operator, as shown below:

$$f(\vec{x}) \approx f(\vec{x}_0) + \nabla f(\vec{x}_0)(\vec{x} - \vec{x}_0) + (\vec{x} - \vec{x}_0)^T H(\vec{x}_0)(\vec{x} - \vec{x}_0) + \cdots,$$

where $f(\vec{x}_0)$ is zeroth order, a scalar; $\nabla f(\vec{x}_0)$ is first order, a vector; $H(\vec{x}_0)$ is second order, a matrix. When a minimum occurs at $x = x_0$ so that $\nabla f(x_0) = 0$, then it becomes apparent why the quadratic form in the next term defined by the Hessian matrix $H(x_0)$ deserves an extra adjective “positive definite” when it is strictly positive regardless of the direction of the vector differential $\vec{x} - \vec{x}_0$, and this then clearly describes a minimum.

The previous introduction of the outer product expansion form of SVD would also help students visualize tensors as the result of stacking: scalar components can be stacked to form vectors; likewise, vectors can be stacked to form matrices, and matrices can be stacked to form 3rd order tensors, etc.

A comprehensive discussion of neural networks would be beyond the scope of MDS. Still, it would be instructive to show the “flow diagram” of how the chain rule is computed for a few nested compositions of multi-input multi-output linear maps with scalar nonlinear activator functions using directed graphs - one of the hardest parts of the course for students to grasp. This is

because drawing directed graphs representing the value propagation mechanism of the chain rule is technically very different from the mathematical formalism used elsewhere in MDS. But for those who would like to specialize in artificial neural networks later, this is a prerequisite knowledge to the engineering aspects of the model training algorithms through calculating gradients.

2.6 Connecting Iterative Algorithms of Optimization back to the Nature of Real Numbers

At this point, we take some time to review with students the concept of approximation (through error control) and the concept of optimization (through searching for critical points) that we encountered earlier. We will then introduce the third foundational concept—convergence—the main goal of iterative algorithms that solve optimization problems. The idea of convergence brings back students’ memory of the limit operator and presents an excellent opportunity to introduce them to the grander scheme of the nature of real numbers. The logical basis for “number-crunching” in data science can now be traced back to the classical axioms of the real number set:

The Nested Interval Property —which I call the “axiom of approximation.” Given an order relation inherited from rational numbers, a nested sequence of intervals with shrinking widths is supposed to contain a unique element contained by all intervals. The understanding of this axiom connects several ideas in numerical analysis.

Dedekind Completeness Property —which I call the “axiom of optimization.” Every bounded subset of real numbers admits a least upper bound (supremum) and a greatest lower bound (infimum), and continuity of a function concerns that these bounds in the range of function values be attainable as the maximum and minimum values of the function. This axiom is an essential part of the logical foundation of optimization.

Cauchy Criterion of Convergence —which I call the “axiom of convergence.” It is pointed out that this is applicable whenever a norm is present to give a measure of distance between numbers, between vectors, or between functions. This axiom asserts that a unique limit exists for any sequence (e.g., from an iterative procedure) whose tail eventually does not differ with respect to the given norm. This axiom is a core idea in various branches of analysis.

The fact that these three axioms are logically equivalent as the foundation of analysis brings about a deep sense of awe and presents an opportunity for faith integration. We Christians who believe that the Word of God created the world can be encouraged in our faith that He fits all things perfectly together in His creation and be grateful that we are privileged as creatures made in God’s image to understand this amazing connection between the material world and the world of our souls and thoughts.

2.7 Visualizing Probability Distributions and Major Theorems with Sequences

The previous topic on the nature of sequences of numbers affords us an analytic foundation to understand random variables, trials, the law of large numbers, and the central limit theorem. Students are reminded that data science obtained its success because of the sheer amount of data we can process nowadays. The value of “Big Data” is thus connected mathematically to how a large amount of data allows us to sequentially define coarsening processes necessary for humans to interpret models’ predictions. After outlining the basic ideas and interpretations of probability

space, partial Riemann sums were computed to demonstrate the similarities of the definitions of expectation, cumulative density functions, and marginalization of joint distributions between discrete and continuous random variables.

Other key coarsening processes demonstrated numerically are that of the additive laws of large numbers and the central limit theorem. In the context of MDS, which does not treat theory of probability comprehensively due to the lack of time and student preparation, I found it more productive to resort to simulations and numerical demonstration of the validity of these key theorems by computing and visualizing partial sums of series, followed by the expectation maximum algorithm used to recover parameters in a Gaussian Mixture Model, and thus connecting back to the previous theme of using optimization to learn functions. It is only at this stage that students will see a formal treatment of noise in data as probability distributions, because all the mathematics that precedes this topic do not involve stochasticity. The sources of noise due to round-off errors, instability of floating-point computations, or aleatory and epistemic uncertainty should be discussed in other courses, even though they are quite frequently mentioned in online resources about numerical algorithms and statistical learning.

3 Discussions

We conclude this paper by discussing a few aspects of how MDS fits into the learning experience of university mathematics and the role of the Christian witness.

3.1 Student Learning Experience and Perception of Mathematics

One of the course's philosophies is partially borrowed from Wolfram's idea that we ought to learn mathematics in a way, "assuming computers exist" [13]. This course encourages students to learn mathematics with both a computational and theoretical mindset, as manual algebraic manipulations give way to exploratory computations and abstract conceptual reasoning. Students experience a collaborative, cloud-based knowledge acquisition process utilizing Google Colaboratory's (<https://colab.research.google.com/>) Jupyter notebook interface [6] in class and in solving their homework. We used it both to demonstrate mathematical facts by computations and visualizations and used uploaded images and big simulation datasets to demonstrate the practical meaning of relevant theorems. Even students without a background in Python could appreciate how these Jupyter notebooks function as "customized online calculators" that help them with visual intuition and imbue a sense of connection to real-life data, thanks to a literate programming style [7]. Students in MDS would more likely see mathematical work as a creative discovery process using a set of abstract but powerful concepts.

3.2 Connections to Other Upper-Level Courses

As hinted in Section 2, MDS has several natural follow-up courses. More application and computation-oriented students would benefit from taking a course in scientific computing or numerical analysis. In contrast, more philosophically-oriented students mystified by the power of numbers to approximate reality are now in a better position to understand the real line continuum, definition of sequential convergence, and to perceive the conceptual clarity offered by the analysis of real-valued functions. The exploration of probability and distribution in MDS would likely engender some interest in a more rigorous coverage by a course in mathematical statistics or probability based on

calculus. Finally, students interested in modeling natural phenomena will find a course in differential equations interesting. They are especially well-positioned to appraise the role of linearity and linearization in a course of differential equations after MDS.

3.3 The Role of Christian Liberal Arts in Data Science

In my opinion, if a smaller institution wants to take the initiative in developing courses for data science, fine-tuning current program offerings may be a viable alternative to creating tag-on mini-programs dedicated to data science. It is true that more and more people are interested in data science. Still, I observe that many good students who want some exposure to data science are not exclusively interested in it, and MDS is a course for such students if they major or minor in mathematics. Moreover, while large comprehensive universities have more resource to create courses and hire dedicated instructors, there is likely inertia in the course structuring that resist comprehensive changes. In such a time of change, smaller flexible departments can act like nimble “British war-ships” if the faculty can reach a consensus faster to prevail against the “Spanish Armadas” of much larger programs offered by public universities. Also, a Christian liberal arts institution arguably offers better humanities training to all its undergraduates to ingrain an ethical sense and a concern for society. A graduate that possesses a good command of mathematical and coding skills from a liberal arts university may turn out to be a strong value proposition for prospective employers as the integration of data-driven skills with ethical judgment is increasingly called for. We could even imagine how a Christian data scientist could have made a difference in the debacles reported in recent years due to the unethical abuse of data and algorithms.

3.4 Abductive Mode of Reasoning and Christian Apologetic

As an instructor and learner, the process of developing MDS shook the foundations of how I understand scientific knowledge as a Christian educated in the modern western tradition. When I examined the subject more closely, the tricky business I found was that data science’s dominant mode of reasoning was different from that of pure mathematics. Some might think it is a natural extension of statistical inference, but some philosophers of science hold a different opinion: “By contrast, Bayesian confirmation theory makes no reference at all to the concept of explanation.” [5]

The overarching mode of reasoning identified by subject experts ([4], one of the major reference texts used in MDS) is neither deduction nor induction, but abduction (sometimes informally called “inference to the best explanation.”) Some philosophers of science argue that important discoveries like Kepler’s discovery of the elliptic orbit of Mars were based on abductive reasoning [5].

As an applied mathematician trained mostly in deductive reasoning, I see this unfamiliar mode of scientific reasoning as both a challenge and an opportunity for Christian testimony. My current opinion is to hold an open mind that an abductive framework of reasoning could give rise to an alternative form of Christian apologetics different from a deductive point of view [8] or an inductive framework [3], and could be perceived in this postmodern age as a means to harmonize a rational concept of faith and science derived from data.

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Ellipses in Taxicab Geometry

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Abstract

The taxicab metric measures distance in the way that one does by walking north/south and east/west on city blocks. Taxicab geometry is typically presented as an example of a non-Euclidean geometry with an interesting metric in introductory courses in geometry and topology. Much is known about taxicab geometry, beginning with the astonishing fact that taxicab circles are squares. In this talk, we define and classify taxicab ellipses. Furthermore, we give formulas for both the circumference and area of taxicab ellipses. The derivations of these formulas do not require mathematical knowledge beyond Precalculus, so this research project illustrates the accessibility of taxicab geometry to beginning researchers.

1 Introduction

1.1 What is Taxicab Geometry?

Taxicab geometry results from replacing the usual distance function with the taxicab distance function. For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the taxicab distance from P to Q is calculated using the taxicab distance function

$$d_T(P, Q) = |x_2 - x_1| + |y_2 - y_1|.$$

As one can see, this distance function models the distance that a taxicab must travel when it moves around on an urban street grid. For that reason, it is sometimes called the Manhattan distance; for example, see [1].

The most popular introduction to the study of taxicab geometry appears to be the book by Krause [5]. An exhibit at the Chicago Museum of Science and Industry curated by Karl Menger in the 1950s appears to be the first time that the geometry was called “taxicab geometry.” [3]. Martin Gardner provided an introduction to it in one of his columns in *Scientific American* [2].

1.2 Previous Study of Conic Sections

Past research about taxicab geometry has largely followed some of the lines of research established in the last chapter of Krause’s book [5]. Among the topics of study have been the conic sections. In a series of articles in the *Pi Mu Epsilon Journal*, several authors studied conic sections. After Reynolds [7] wrote an article that closely follows the development of taxicab conic sections in [5], Moser and Kramer [6] established how to find the distance from a point to a line and determined the general form of a parabola with a given focus and directrix.

Iny [4] responded to Moser and Kramer by pointing out that neither Reynolds nor Moser and Kramer provided a definition of a line. Then Iny defined a line as the set of points that are equidistant from two given points. This definition captures all Euclidean lines but also captures sets of points in \mathbb{R}^2 that are lines in taxicab geometry but not in Euclidean geometry. Subsequently, Iny showed how to produce a taxicab ellipse that cannot be given by the definition given by Krause, i.e. $\{P \mid d_T(P, F_1) + d_T(P, F_2) = n\}$ where F_1 and F_2 are foci and n is a positive real number.

Although there are compelling reasons to use Iny’s definition of a taxicab line, in what follows, we restrict our attention to taxicab lines that are also Euclidean lines.

2 Taxicab Circles

We begin by studying taxicab circles. We find formulas for their circumference and area, and we compare those formulas with the corresponding formulas for Euclidean circles.

Definition 1. Let Q be a point in \mathbb{R}^2 and let r be a positive real number. A **taxicab circle with radius r** is the set of points that are a distance r from Q , i.e. a circle is $\{P \mid d_T(P, Q) = r\}$.

In what follows, we often refer to taxicab circles as “circles.” We revert to “taxicab circle” when the context demands it.

In Euclidean geometry, one defines π to be the ratio of the circumference to the diameter of a circle. We do the same for taxicab circles.

Definition 2. Suppose a taxicab circle has circumference of length C and diameter of length d . Then

$$\pi_T = \frac{C}{d}.$$

It is an exercise in Krause’s book [5] to show that $\pi_T = 4$.

2.1 Circumference

Theorem 3. *The circumference of a taxicab circle of radius r is $2\pi_T r$.*

To motivate the proof of Theorem 3, we find the circumference of a taxicab circle with radius 3 centered at the origin.

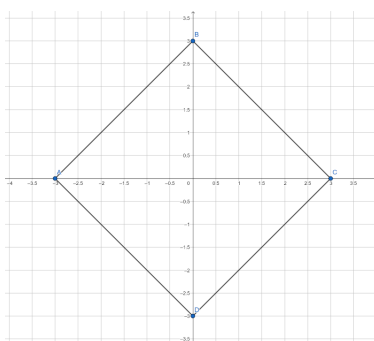


Figure 1: A taxicab circle of radius 3

As noted above, a taxicab circle is actually square. In this case, each side of the square has length $3 + 3 = 6$ since each side of the square covers three horizontal units and three vertical units. Thus, the length of one side is 6. Since there are four sides, the circumference of this taxicab circle is $4 \cdot 6 = 24$.

Proof. We start by considering a taxicab circle with radius r . We know that each taxicab circle has four vertices that are each some distance r away from the center. These vertices are the endpoints of four line segments, two with slope 1 and two with slope -1 . Using the definition of distance in taxicab geometry, we know that the length of one of these line segments is equal to the distance between the two vertices that act as endpoints for that line segment. By definition of distance in taxicab geometry, the distance between these two endpoints is the difference in their horizontal coordinate distance plus the difference in their vertical coordinate distance. Notice that no matter which line segment you choose, this distance will always be $2r$. This is the length of one line segment. Using the same reasoning, we see that each line segment has length $2r$. Thus, the circumference of the taxicab circle is $4(2r)$.

Now $\pi_T = 4$, so we may replace 4 with π_T . Thus, the circumference formula for a circle in taxicab geometry is $2\pi_T r$. \square

We note that the circumference formula for a taxicab circle is the same as the formula for a Euclidean circle, except that the value of π differs in the two geometries.

2.2 Area

Next, we turn our attention to the area of a taxicab circle.

Theorem 4. *The area of a taxicab circle with radius r is $\frac{\pi_T r^2}{2}$.*

To motivate the proof of Theorem 4, we find the area of a taxicab circle with radius 3 centered at the origin.

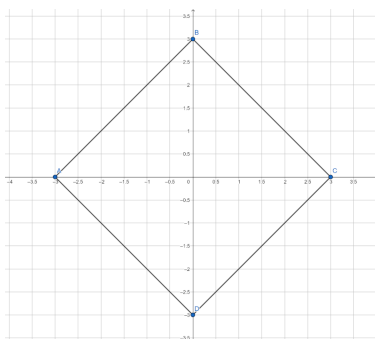


Figure 2: A taxicab circle of radius 3

Construct a square on the segment extending from the origin to $(3, 0)$ as in Figure 2.

Then do the same for each of the other segments extending from the origin to a vertex of the taxicab circle as in Figure 3.

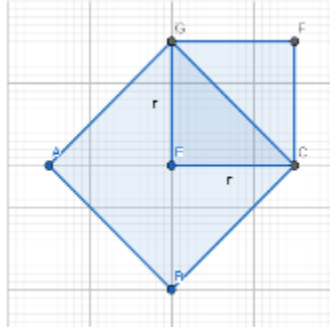


Figure 3: A taxicab circle with radius 3 along with a squared radius

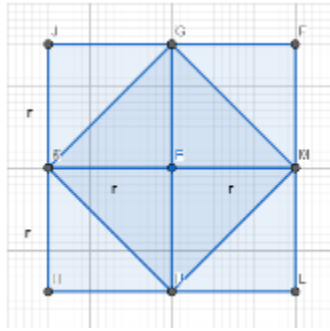


Figure 4: A taxicab circle inscribed in a square with side length $s = 6$

We see that the area of the taxicab circle with radius 3 is half of the area in each of the four squares that were constructed. Since each square has area $3 \cdot 3 = 9$, the total area of the taxicab circle is $\frac{1}{2}(4)(3)(3) = \frac{\pi_T \cdot 3^2}{2} = 18$. Note the use of $\pi_T = 4$. It is used again in the proof of Theorem 4.

Proof. Consider a taxicab circle with radius r centered at the origin. Construct a square on each of the 4 segments that extend from the origin to a vertex of the taxicab circle. This construction produces 4 squares of area r^2 and a large square with side length $2r$. Moreover, each of the smaller squares is divided into two congruent right triangles by a side of the taxicab circle. In each pair of congruent right triangles, one triangle contributes area to the taxicab circle. Thus, the area of the taxicab circle is the area of the larger square minus the area of four of the congruent right triangles:

$$(2r)^2 - 4 \left(\frac{1}{2} \right) (r)(r) = 4r^2 - 2r^2 = 2r^2.$$

Since $\pi_T = 4$, we now have the area formula for a circle in taxicab geometry:

$$A = \frac{\pi_T r^2}{2}.$$

□

Thus, we see that the expression for the area of a taxicab circle is $\frac{1}{2}$ times the expression for the area of a Euclidean circle.

3 Taxicab Ellipses

Taxicab ellipses are not as well-studied as taxicab parabolas. They are unlike Euclidean ellipses in several different ways. For example, they may have different shapes depending on the relative location of the foci. More surprisingly, although there is no known closed-form formula for the perimeter of a Euclidean ellipse, we provide an explicit formula for the perimeter of a taxicab ellipse below.

Following the lead of [5], we make the following definition.

Definition 3. A taxicab ellipse E is given by

$$E = \{P \mid d_T(P, F_1) + d_T(P, F_2) = n\},$$

i.e., a taxicab ellipse is the locus of points P for which the sum of the distance from P to a focus F_1 and the distance from P to a focus F_2 is constant. We note that in order to have a non-empty set of points, one must choose n so that $n > d_T(F_1, F_2)$. We call n the **distance sum** of E , and we call $d_T(F_1, F_2)$ the **focal distance** of E .

3.1 Possible Shapes

We begin our study of taxicab ellipses by noting that they may have two distinct Euclidean shapes.

Theorem 5. *Let F_1, F_2 be points in \mathbb{R}^2 , and let $n > d_T(F_1, F_2)$. Suppose $E = \{P \mid d_T(P, F_1) + d_T(P, F_2) = n\}$ is a taxicab ellipse. If F_1 and F_2 lie on a horizontal or vertical line, then E is a hexagon. Otherwise, E is an octagon.*

The proof of Theorem 5 is omitted. However, one may convince oneself of its truth by drawing several examples using graph paper.

Figures 5a and 5b illustrate the two possible shapes of a taxicab ellipse.

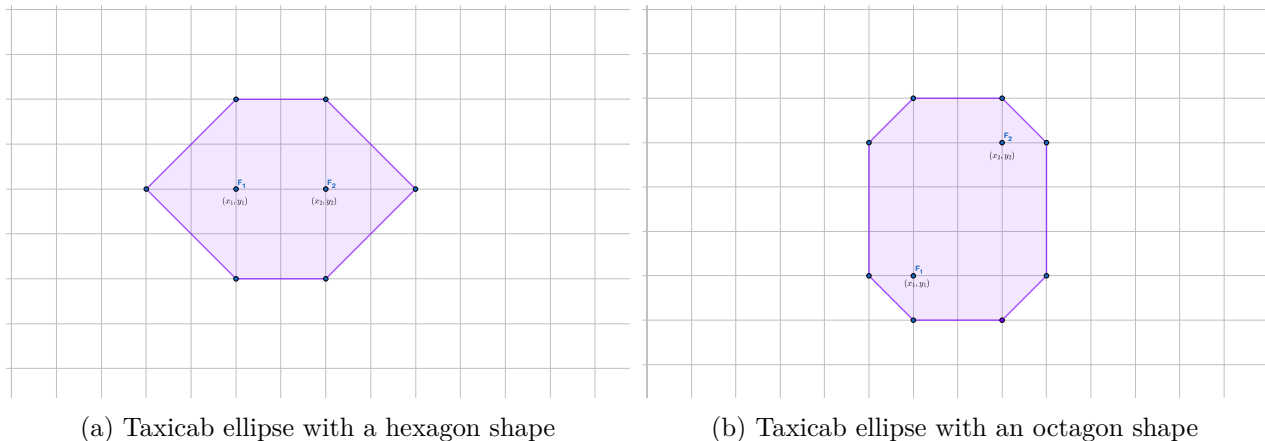


Figure 5: Taxicab ellipses with different shapes

In the case that a taxicab ellipse is an octagon, one may choose an alternative set of foci that produce the same ellipse.

Lemma 1. Suppose E is a taxicab ellipse with distance sum n and with foci F_1 and F_2 that do not lie on the same horizontal or vertical line. Then there exist points G_1 and G_2 such that the ellipse produced by using G_1 and G_2 as the foci and n as the distance sum is the same as E .

Proof. Suppose E is a taxicab ellipse with foci F_1 and F_2 that do not lie on the same horizontal or vertical line. Let $F_1 = (x_1, y_1)$ and let $F_2 = (x_2, y_2)$. Let G_1 be the point that lies at the intersection of the vertical line passing through F_1 and the horizontal line passing through F_2 . Similarly, let G_2 be the point that lies at the intersection of the horizontal line passing through F_1 and the vertical line passing through F_2 . Then $G_1 = (x_1, y_2)$ and $G_2 = (x_2, y_1)$.

Let n be the distance sum of E . Consider the ellipse E^* formed by using G_1 and G_2 as foci with distance sum n . We claim that $E = E^*$. First, note that we may write $d_T(F_1, F_2) = |x_1 - x_2| + |y_1 - y_2|$. Now

$$d_T(G_1, G_2) = |x_1 - x_2| + |y_2 - y_1| \quad (1)$$

$$= |x_1 - x_2| + |y_1 - y_2| \quad (2)$$

$$= d_T(F_1, F_2), \quad (3)$$

so the distance between G_1 and G_2 is the focal distance of E .

Let $P = (p, q)$ be a point on E^* . We next show that $d_T(P, F_1) + d_T(P, F_2) = n$. This is accomplished by using the coordinates of all of the points involved:

$$d_T(P, F_1) + d_T(P, F_2) = |p - x_1| + |q - y_1| + |p - x_2| + |q - y_2| \quad (4)$$

$$= |p - x_1| + |q - y_2| + |p - x_2| + |q - y_1| \quad (5)$$

$$= d_T(P, G_1) + d_T(P, G_2) \quad (6)$$

$$= n. \quad (7)$$

Thus, every point on E^* is also a point on E . A similar argument shows that if Q is a point on E , then Q is also a point on E^* . Hence, $E = E^*$. Given the distance sum n , we may use either F_1 and F_2 or G_1 and G_2 as the foci of E . \square

3.2 Perimeter

We are now ready to give the formula for the perimeter of a taxicab ellipse.

Theorem 6. *Suppose F_1 and F_2 are points in \mathbb{R}^2 . Let $n > d_T(F_1, F_2)$. Suppose $E = \{P \mid d_T(P, F_1) + d_T(P, F_2) = n\}$ is a taxicab ellipse. Then the perimeter of E is*

$$4n - 2d_T(F_1, F_2) = \pi_T n - 2d_T(F_1, F_2).$$

Proof. Suppose $E = \{P \mid d_T(P, F_1) + d_T(P, F_2) = n\}$ is a taxicab ellipse. There are two cases to consider with respect to the relative locations of F_1 and F_2 .

Case 1: Suppose F_1 and F_2 lie on the same horizontal or vertical line. Let l be the line that contains F_1 and F_2 . Now E is a hexagon, so E consists of six line segments. We may divide these six line segments into two sets; call them S_1 and S_2 .

S_1 contains two line segments that are parallel to l and have length $d_T(F_1, F_2)$. Thus, the line segments in S_1 contribute $2d_T(F_1, F_2)$ to the perimeter of E .

S_2 contains four line segments. They may be grouped into pairs accordingly to their slopes. In one pair, each line segment has slope 1, while in the other pair, each line segment has slope -1 . Notice

that the line segments in S_2 may be rearranged to form a taxicab circle. Let r be the radius of this taxicab circle. By Theorem 3, the line segments in S_2 contribute $2\pi_T r$ to the perimeter of E .

It remains to show that $r = \frac{n - d_T(F_1, F_2)}{2}$. Let P be an endpoint of one of the line segments in S_2 . Now $d_T(P, F_1) \leq d_T(P, F_2)$ or $d_T(P, F_2) \leq d_T(P, F_1)$. Without loss of generality, suppose that $d_T(P, F_1) \leq d_T(P, F_2)$. Let $r = d_T(P, F_1)$. Then $d_T(P, F_2) = r + d_T(F_1, F_2)$. From the definition of E , we have

$$d_T(P, F_1) + d_T(P, F_2) = n.$$

Substituting in the left-hand side of this equation, we obtain

$$\begin{aligned} d_T(P, F_1) + d_T(P, F_2) &= n \\ r + (r + d_T(F_1, F_2)) &= n \\ 2r + d_T(F_1, F_2) &= n \\ 2r &= n - d_T(F_1, F_2) \\ r &= \frac{n - d_T(F_1, F_2)}{2}. \end{aligned}$$

We have established that the two line segments in S_1 contribute $2d_T(F_1, F_2)$ to the perimeter of E , while the four line segments in S_2 contribute $2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right)$ to the perimeter of E . It follows that the perimeter of E is given by

$$2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right) + 2d_T(F_1, F_2).$$

Case 2: Suppose that F_1 and F_2 lie on neither the same horizontal line nor the same vertical line. Then E is an octagon, and E consists of eight line segments. We may divide these line segments into two sets as we did in case 1; call the two sets S_3 and S_4 .

S_3 contains four line segments that are parallel to the coordinate axes. In particular, two segments are parallel to the x -axis, while two segments are parallel to the y -axis. Form two pairs of segments so that in each pair, there is a segment that is parallel to the x -axis and a segment that is parallel to the y -axis. In each pair, the total length of the two segments is $d_T(F_1, F_2)$ by the definition of taxicab distance. Since there are two such pairs of segments in S_3 , the segments in S_3 contribute $2d_T(F_1, F_2)$ to the perimeter of E .

Next, we consider the line segments in S_4 . They have the same properties as the segments in S_2 in the proof of Case 1. Thus, we may use a similar argument to show that they contribute $2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right)$ to the perimeter of E . One important difference in the argument presented below is that one must account for two additional points that are each closest to one of the segments in S_4 .

To begin, we add two additional points to our discussion. Let G_1 be the point that lies at the intersection of the vertical line passing through F_1 and the horizontal line passing through F_2 . Similarly, let G_2 be the point that lies at the intersection of the horizontal line passing through F_1 and the vertical line passing through F_2 . Then by Lemma 1, the distance from F_1 to F_2 is the same as the distance from G_1 to G_2 .

We claim that it is possible to rearrange the segments in S_4 into a (Euclidean) square. Toward that end, we first note that two of the segments in S_4 have (Euclidean) slope 1, while the other two segments in S_4 have (Euclidean) slope -1 . Each of the segments is closest to exactly one of the points in $\{F_1, F_2, G_1, G_2\}$. Suppose S is the segment in S_4 that is the closest to F_1 . Let P be one of the endpoints of S , and let $r = d_T(P, F_1)$. Then $d_T(P, F_2) = r + d_T(F_1, F_2)$. By using a chain of reasoning similar to that used in Case 1, we obtain

$$r = \frac{n - d_T(F_1, F_2)}{2}.$$

One may use a similar argument to establish this value of r for points on each of the remaining three segments. For the segment closest to G_1 and the segment closest to G_2 , one must use the identity $d_T(F_1, F_2) = d_T(G_2, G_1)$ to get the same result. Thus, one can construct a taxicab circle of radius $r = \frac{n - d_T(F_1, F_2)}{2}$ from the segments in S_4 . Then Theorem 3 tells us that the circumference of that circle is $2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right)$.

In conclusion, we have established that the four line segments in S_3 contribute $2d_T(F_1, F_2)$ to the perimeter of E , while the four line segments in S_4 contribute $2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right)$ to the perimeter of E . It follows that the perimeter of E is given by

$$2\pi_T \left(\frac{n - d_T(F_1, F_2)}{2} \right) + 2d_T(F_1, F_2).$$

The resulting expression simplifies to

$$4n - 2d_T(F_1, F_2) = \pi_T n - 2d_T(F_1, F_2)$$

as claimed. □

3.3 Area

Next, we present the area formula for a taxicab ellipse.

Theorem 7. *The area of a taxicab ellipse with foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$, distance sum n , and focal distance $d_T(F_1, F_2)$ is*

$$\frac{n^2 - d_T(F_1, F_2)^2}{2} + |x_1 - x_2||y_1 - y_2|.$$

In the proof of this theorem, we first calculate the area of a rectangle that contains the ellipse and then subtract the area of a taxicab circle created by the excess. Figures 6a and 6b illustrate the situation.

Proof. Suppose E is a taxicab ellipse with foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$, distance sum n , and focal distance $d_T(F_1, F_2)$. Without loss of generality (by Lemma 1), suppose $x_1 \leq x_2$ and $y_1 \leq y_2$. Let $r = \frac{n - d_T(F_1, F_2)}{2}$ as in the proof of Theorem 6.

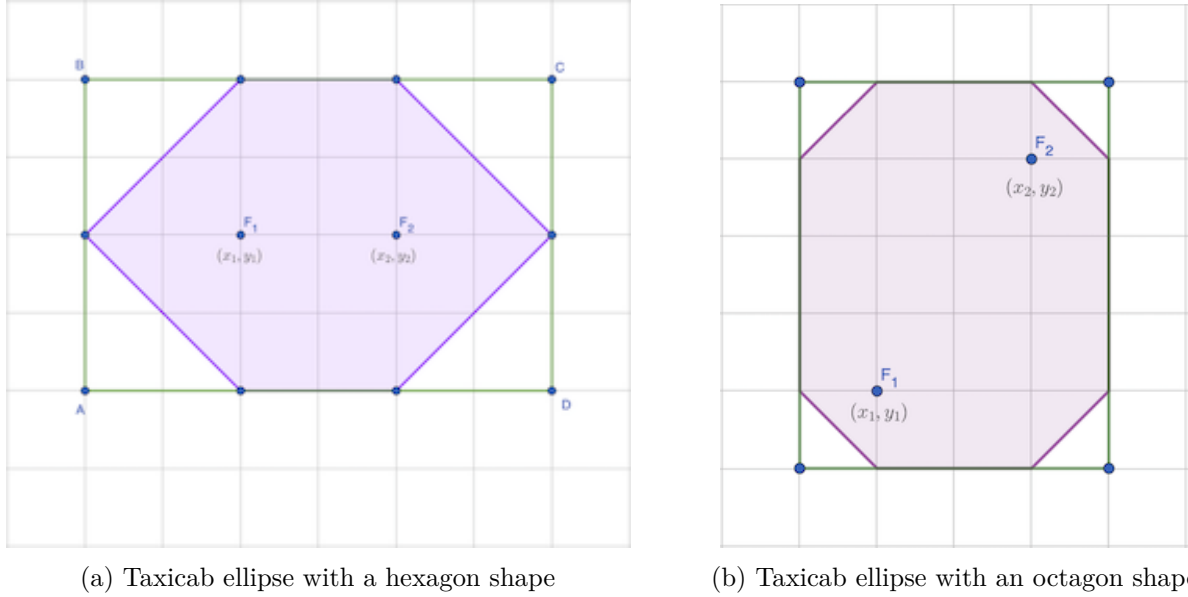


Figure 6: Taxicab ellipses with bounding rectangles

To calculate the area A_E of E , we begin with an overestimate. Consider the rectangle $BCDF$ with vertices at

$$B = (x_1 - r, y_1 - r), C = (x_1 - r, y_2 + r), D = (x_2 + r, y_2 + r), \text{ and } F = (x_2 + r, y_1 - r).$$

Notice that rectangle $BCDF$ contains E and that its area is given by

$$\begin{aligned} A_{BCDF} &= (2r + |x_1 - x_2|)(2r + |y_1 - y_2|) \\ &= 4r^2 + 2r(|x_1 - x_2| + |y_1 - y_2|) + |x_1 - x_2||y_1 - y_2|. \end{aligned}$$

Recall that $d_T(F_1, F_2) = |x_1 - x_2| + |y_1 - y_2|$. It follows that

$$\begin{aligned} A_{BCDF} &= 4 \left(\frac{n - d_T(F_1, F_2)}{2} \right)^2 + 2 \left(\frac{n - d_T(F_1, F_2)}{2} \right) d_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| \\ &= 4 \left(\frac{n^2 - 2nd_T(F_1, F_2) + d_T(F_1, F_2)^2}{4} \right) + (n - d_T(F_1, F_2))d_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| \\ &= n^2 - 2nd_T(F_1, F_2) + d_T(F_1, F_2)^2 + nd_T(F_1, F_2) - d_T(F_1, F_2)^2 + |x_1 - x_2||y_1 - y_2| \\ &= n^2 - nd_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2|. \end{aligned}$$

Now to correct the overestimate to the correct value of the area of E , we notice that the area of E is the area of rectangle $BCDF$ minus the area of a taxicab circle of radius r . One can form this taxicab circle of radius r by joining four congruent right triangles from the corners of rectangle $BCDF$. Each right triangle has legs of taxicab length r and a hypotenuse of taxicab length $2r$ that lies on a line with slope 1 or -1 .

By Theorem 4, the area of the taxicab circle formed by the four congruent right triangles is $\frac{\pi_T r^2}{2}$.

Since $r = \frac{n - d_T(F_1, F_2)}{2}$, the area of E is:

$$\begin{aligned}
A_E &= n^2 - nd_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| - 2 \left(\frac{n - d_T(F_1, F_2)}{2} \right)^2 \\
&= n^2 - nd_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| - 2 \left(\frac{n^2 - 2nd_T(F_1, F_2) + d_T(F_1, F_2)^2}{4} \right) \\
&= n^2 - nd_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| - \frac{n^2 - 2nd_T(F_1, F_2) + d_T(F_1, F_2)^2}{2} \\
&= n^2 - nd_T(F_1, F_2) + |x_1 - x_2||y_1 - y_2| - \frac{n^2}{2} + nd_T(F_1, F_2) - \frac{d_T(F_1, F_2)^2}{2} \\
&= \frac{n^2}{2} - \frac{d_T(F_1, F_2)^2}{2} + |x_1 - x_2||y_1 - y_2| \\
&= \frac{n^2 - d_T(F_1, F_2)^2}{2} + |x_1 - x_2||y_1 - y_2|.
\end{aligned}$$

□

4 Future Work

We note two intriguing directions for future work. First, as mentioned in Section 1.2, the results produced above assume that all lines are Euclidean lines. However, if one takes Iny's definition of a line in [4], then there are taxicab lines that are not Euclidean lines. One might extend the present investigation by determining whether and how to modify the theorems presented here using Iny's definition of a line.

Second, we believe that it may be possible to find equation(s) for a taxicab ellipse. As is well-known, a Euclidean ellipse centered at the origin with horizontal semi-major axis of length a and vertical semi-minor axis of length b has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Based on our investigations, it appears that it may be possible to express the linear segments that make up a taxicab ellipse using its focal distance $d_T(F_1, F_2)$ and distance sum n . Rosen gives an equation for a taxicab ellipse with foci located at $(-a, 0)$ and $(a, 0)$, where a is a positive integer [8]. Perhaps a more general equation may be found.

5 Acknowledgements

The author carried out this research project with the assistance of undergraduate students Jaime Acre and Caleb Cornelius during the summer of 2021 on the campus of Southern Nazarene University. Our work was supported financially by the Southern Nazarene University Catalysts (a science and math alumni group) and the Oklahoma Space Grant Consortium.

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Banquet Tributes and the Brabenec Lectureship

For the past several conferences ACMS has honored long-standing members at the conclusion of its Friday evening banquet. This year two tributes were given to those who, though “absent from the body,” we trust are now “present with the Lord,” namely, Alice Iverson and Gene Chase. In addition, a special tribute was prepared for Robert Brabenec, founder of ACMS, who was present at the conference, but who had recently stepped down from his role of Executive Secretary. A lecture series was also created in Bob’s honor. This section contains the remarks that were prepared for [Alice](#), [Gene](#), and [Bob](#), followed by a brief description of the [Brabenec Lectureship](#).

Tribute to Alice Iverson (Prepared by Dave Klanderman)

Alice Iverson (January 26, 1924–June 22, 2020) served as a mathematics professor at North Park College (later renamed North Park University) in Chicago, Illinois for 48 years. Her academic leadership extended for over three decades. She was the first woman to chair the Division of Science and Mathematics, and the chairperson of the Committee to Introduce Computer Science Major to North Park College. Having earned her master’s degree in mathematics from Northwestern University, she never wavered in support for women in science, technology, engineering, and the mathematical science. A lifelong learner, she spent many years exploring the United States with her husband (Marvin) and children (Carol and Keith). She loved religious history, and was always open to new ideas, approaches, and insights. She cared for all living things—especially plants, of which she had many!



Alice was a longtime member of ACMS, and served on the ACMS Board from 1995 to 1999. She was Vice President from 1995 to 1997, and President from 1997 to 1999. In fact, she was the first of four women who served as ACMS President. I first met her at the 1991 ACMS Biennial Meeting at Wheaton College. I later saw Alice at annual meetings of the mathematics division of the Associated Colleges in the Chicago Area (ACCA) as well as additional ACMS conferences. Each time I met her, she greeted me warmly and made me feel welcome in these academic settings.

In addition to supporting women in the fields of science, technology, engineering, and (in particular) the mathematical sciences, Alice was a tremendous role model for them. One of my longtime colleagues at Trinity Christian College—and an alumna of North Park College, Dr. Mary Webster Moore—considers Alice a mentor who influenced her decision to pursue a career in mathematics education. Mary noted that she contacted Alice when she completed her doctoral degree, and Alice offered her words of congratulations and encouragement. The timeline of Alice Iverson’s long career at North Park closely paralleled that of Rich Kooy, my departmental colleague who retired shortly after I arrived at Trinity. He mentioned speaking and working with Alice at some of these same conferences and meetings during several decades.

Alice was an active member of Ravenswood Presbyterian Church in Chicago, Illinois.

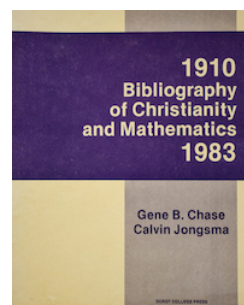
Tribute to Gene Chase (Prepared by Calvin Jongsma)

I don't recall how Gene Chase and I initially met, but it was likely at the first ACMS conference, which we both attended in 1977. The brainchild of Bob Brabenec, the goal of that conference was to explore a Christian perspective on the foundations of mathematics. As background reading, Bob had compiled a list of nine books and articles dealing with the integration of faith and mathematics, and he offered to send copies of these out at cost to whoever was interested. Reflecting on this, I proposed to Bob that setting up a clearinghouse for collecting and disseminating these sorts of materials could be a good way to facilitate an ongoing dialogue on the topic, and that an annotated bibliography of Christianity and mathematics might be a good first step.



Gene and I were both interested in such a venture. Gene had developed a reading list for his senior-seminar students at Messiah, and I was keen on analyzing the different approaches to relating Christianity and mathematics, which then became the basis of a talk I gave at the Pittsburgh Coalition's 1980 Jubilee Conference. Bob initially hoped that a bibliography might materialize at the 1978 Working Conference convened to plan the following year's ACMS conference, but Gene was unable to attend. The two of us were instead commissioned by participants at the 1979 ACMS conference to compile such a bibliography. I wrote up a brief prospectus, which was appended to the conference proceedings, and we asked for participants' assistance in suggesting materials for us to include.

With naïve optimism, we projected that this could probably be done in relatively short order. Gene and I were aware of more works than were on Bob's list, but we had no idea just how many items we might find on the topic. We scoured library resources and wrote to academics across the world for information to assist us—this, incidentally, was before the internet or search engines existed, and before email became an accessible means of communication! Gene and I were surprised to discover far more materials addressing this topic than we had expected, certainly more than would fit on a simple xeroxed handout. Gene sent a pre-publication, computer-generated draft of our work to Bob, which he included as an appendix to the proceedings of the 1981 ACMS Conference, but our ultimate goal was to produce an inexpensive (\$2/copy), nicely-formatted, stand-alone pamphlet. This was finally ready in June 1983. Our *Bibliography of Christianity and Mathematics: 1910—1983* catalogued and annotated around 300 items, including a number of things that had been generated in the meantime by speakers at the first four ACMS conferences.



I began my career at Dordt in 1982, so I was soon consumed by the demands of teaching. Like Gene, I required students to develop their ideas on the relationship between Christian faith and mathematics. I did so in a capstone history of mathematics course I taught on an alternate year basis. But the bibliography project per se was no longer a front-burner concern for me, due largely to institutional assignments and writing projects. Gene and I continued to see one another at ACMS conferences, however, and we occasionally shared information by mail about new items we came across that should be included in a second edition of our *Bibliography*.

Gene's interest in the *Bibliography* never flagged. He was actively at work revising the *Bibliography* in 1990 in response to an ACMS Board request. He devoted sabbatical time around then working on a second edition; my contribution was mainly giving feedback to his work and forwarding some

newly annotated items. Gene hoped to expand the *Bibliography* backward in time to include writings prior to the 20th century, and he undertook a study of philosophers and mathematicians whose theological ideas influenced their view of mathematics. He gave an ACMS talk on this topic in 1991, projecting that a second edition might be ready for publication by 1992 or 1993, although that did not materialize.

The next time I heard from him about the bibliography project was some ten years later. Gene's proposal for another sabbatical in 2004/2005 was to include revising and expanding the *Bibliography*. Not having a matching sabbatical, I offered to give him input and feedback as time permitted, but I was unable to provide any substantial assistance. Gene's report on his work at the 2005 ACMS Conference, which I was unable to attend, reads like a draft introduction to a second edition of the *Bibliography*, but I did not learn what the final outcome of his work was. Once again, I was heavily committed to other projects and responsibilities that demanded my time and energy.

Gene retired from Messiah in the early 2010s; I followed suit a few years later. In 2019 I belatedly reached out to Gene to see what state the *Bibliography* was in and whether he would be interested in collaborating on a new edition. Gene's response was rather tentative: "Yes. But not in a rush, just as I have time." I was busy working on a major writing project, so I too was not in a real rush to start. Then in September, 2020, I received the sad news that Gene had died after a three-week battle with COVID-19.

That seemed to put an end to cooperating on a second edition of the *Bibliography*, but in email exchanges over subsequent months with Gene's wife Emily and his son John, the three of us managed to decipher which of Gene's boxes of materials and computer files might be pertinent to a future edition. Consequently, I now have a raft of digital material related to this: about 1500 files, 2 GB of data. I have not had time to delve very far into this material to know everything it contains, so I can't predict what might result from it. But if a second edition finally does see the light of day, it will be based on the work Gene put into it over the years, and it will still be a truly collaborative effort.

Since the first edition of our *Bibliography*, the nature and availability of documents treating the topic of Christianity and mathematics have changed dramatically. Many materials are now more readily available online, but they are also more ephemeral. What should a new edition of the *Bibliography* include? Gene wished to include materials from earlier eras, and ACMS has become more inclusive over the years by focusing on computer science and statistics as well as mathematics. Is such an expansion of the *Bibliography* wise or feasible? And what format should be used at a time that is far more digital than when the *Bibliography* first came out? Moving to a second edition will require substantial logistical and editorial rethinking. This task may be more than I can commit to and accomplish on my own, so—assuming there's sufficient interest in a second edition—I'll welcome serious input from other ACMS members.

Gene's academic work was always one of service to God, his students, and his colleagues, and his intellectual passion was to promote the development of Christian perspectives on various aspects of the field. His contributions to the ACMS have been much appreciated and, God willing, may still bear fruit into the future.

Tribute to Bob Brabenec (Prepared by Russell Howell)

As you all know, Robert L. Brabenec has a long-standing passion for the history and context of mathematics, and the great Swiss mathematician Leonhard Euler is someone he admires—not only for his mathematics, but especially for his character. In that regard, I think that one of the finest tributes to Euler came in 1783 from the Marquis of Condorcet, a French mathematician and philosopher who worked not only with Euler, but also with Benjamin Franklin: “He preferred instructing his pupils to the little satisfaction of amazing them.” Of course, anyone who knows Bob would say something similar: “He prefers instructing his pupils to the little satisfaction of amazing them.” Note the switch to the present tense. Even though retired, Bob continues to instruct, and in many ways we are all his pupils.



But *how* does Bob instruct us? Ironically, the *way* Bob instructs his pupils has the *effect* of their being amazed by him. I could list his many accomplishments: receiving teaching awards from Wheaton, and the Illinois Section of the MAA; author of a book in real analysis, and a subsequent MAA publication on resources for that subject; founder of the ACMS . . . The list goes on and on, but I would like to suggest that we can see how Bob instructs us by modifying the original Euler tribute just a bit: He prefers *serving* his pupils to the little satisfaction of serving himself.

We all have to write letters of recommendation, but Bob went out of his way to serve me by writing several important letters completely unrelated to recommendations. He penned the first during my senior year at Wheaton. The Vietnam War was at its peak. I had gone through the ROTC program, and my application to delay entry into the Army until completing graduate school was denied. Bob took it upon himself to write a letter to the National Association of Evangelicals to see if they could sway some influence in getting the decision reversed. One can never be sure what levers of government resulted in such a reversal, but it came just two days before commencement. For sure my career path would have been completely different otherwise, and may very well not have involved any graduate education.

Then, during my first year at The Ohio State University, I got a letter out of the blue from Bob. I still have it. In the second paragraph he said, “Graduate school can be a very narrowing experience. I’d recommend that you consider the following books for reading that can help provide a wider, richer context for mathematics.” And in that letter Bob went well beyond just giving advice. A sentence later on illustrates that history sometimes repeats. He wrote, “I felt bad when I watched Michigan beat Ohio State in football.” I wonder, how many times have I taken the initiative to write to alumni without their first writing me?

Bob’s service extends well beyond his own students. He is the founder of the “Wheaton Cares” program, which involves students from a variety of majors meeting in his home once per week to write cards of encouragement to fellow students who are going through difficult times, and to pray for them. Time limits my talking about other kinds of involvement with students, but I must share the judgment of Mary Vanderschoot, chair of Wheaton’s mathematics department. I was in an email exchange with her in 2013 about a planned gathering during homecoming to honor Bob for completing 50 years of teaching at Wheaton. I was not able to make it, and asked whether there was anything brewing for celebrating his retirement. Here is what she wrote: “We will certainly plan something special for his retirement – I’m not sure when that will be. I think Bob wants to

continue teaching for another year or so.” Okay, Mary, you were only off by about seven years! But listen to what she wrote next: “He is more involved with student ministry than anyone I know.”

The serving in which Bob engages takes on a variety of forms, and crosses many generations. In 2008 I was honored when he invited me to join him, Dick Stout, and Wheaton student Jesse Liebe for a month of research in Cambridge, England. We stayed at Hawthorne House, a facility that is owned by Wheaton College. Bob insisted on paying for its rental. The picture you see is one that a passerby took of us one day. I can’t remember where we were going, but evidently Bob and Dick were headed to a twin primes dress-alike contest! Jesse is now a lawyer, and it didn’t surprise me when I found out that, in 2011, he invited Bob to his wedding in North Carolina. Of course, it was also not surprising to learn that Bob went.



I’ve wondered how I should sum up a thank-you to Bob besides the nerdy way I did at the conclusion of a visit to Wheaton in 2003 in response to an invitation to give some talks. I wrote on a chalk board, “Bob, $\int_0^q \sec^2$, Russ.” Yeah, that equals $\tan q - \tan 0 = \tan q$. Indeed, “ $\tan q$,” Bob, for your investment in my life, for both inspiring and amazing me. I count it a rich blessing to continue to be your pupil, and for many years to have been your colleague. As a final thought, let me share an inscription. Its application to Bob can aptly come from us all.

Bob loves the UK, with its rich traditions, and the stunning architecture in Cambridge. In London, there are the fabulous structures designed by Christopher Wren. One of Wren’s masterpieces is Saint Paul’s Cathedral. Located in the crypt below Saint Paul’s is Wren’s tomb, with a Latin inscription that translates, “He lived beyond the age of ninety, not to himself but for the public good.” Now, this is a tribute to Bob, not a memorial service, and we all hope that he will live beyond the age of ninety, though he certainly has lived “not to himself, but for the public good.”

It is the second half of Wren’s inscription that is most applicable to Bob. To set the stage for it, let me show you a picture that was taken yesterday during lunchtime. On the right is Bob, who, when I was his advisee, chaired Wheaton’s mathematics department. Coincidentally, I now happen to be chair of Westmont’s mathematics department, and pictured to the left of me is Mary Vanderschoot, who is one of my former advisees. She is now chair of Wheaton’s mathematics department. To the left of Mary is one of her former advisees, Anna Aboud, who next term is slated to be chair of Westmont’s mathematics department!



To the elucidate the rest of Wren’s inscription let me ask a favor of everyone here. If you were a student of Bob at Wheaton, would you mind standing up? Now, if you were a student of a student of Bob at whatever institution, would you please stand? Finally, if your life has been enriched by Bob, either directly or indirectly, would you please stand?

The remaining portion of Wren’s inscription reads “Lector, si monumentum requires, circumspice.” That translates to “Reader, if you seek his monument, look around.” When you look around London you can truly see Wren’s monuments in the form of striking architectural wonders. That same inscription applies to Bob, but the monuments are people. Bob, we know you do not nor never will seek monuments, but you actually already have them in abundance—just look around. Thank you for modeling what it means to be a great teacher, colleague, and friend. Please hear and accept from us now the sound that symbolizes our deepest thanks, respect, admiration, and—yes—amazement.

The Brabenec Lectureship

As a final gesture of appreciation for Robert Brabenec's many years of service, the ACMS Board announced the establishment of the *Brabenec Lectureship*, with funds that will allow an ACMS member to serve as the *Brabenec Lecturer* for an academic year. The Board solicits nominations for this position. Institutions will then submit applications to the Board with the request that the lecturer visit their campuses to deliver a talk dealing with faith-integration issues, and possibly additional lectures on mathematical topics. Priority will be given to institutions that do not normally have funding for outside speakers. Russell Howell, pictured to the right of Bob Brabenec, was selected as the first holder of the Brabenec Lectureship position. A full description of this program can be found at <https://acmsonline.org/brabenec-lectureship/>.



Appendix 1: Conference Schedule

| Monday, May 30 | |
|----------------|--|
| 6:00–8:00 pm | Off-campus dinner for pre-conference attendees |

| Tuesday, May 31 | |
|-----------------|--|
| 7:30–8:30 am | Breakfast (1899 Dining Hall) |
| 9:00–12:00 pm | Pre-Conference Workshops 1 (Wilden 102, 103) – <i>Mathematics in Context: Its History, Philosophy, and Connections with the Christian Faith</i> (Organizers: Bob Brabenec, Russell Howell, Dick Stout, Dusty Wilson) – <i>Integrating Ethics into Your Statistics and Computer Science Courses: Integrated Ethics Labs</i> (Organizers: Lori Carter, Catherine Crockett, Stacy DeRuiter) |
| 12:00–1:00 pm | Lunch (1899 Dining Hall) |
| 1:00–4:00 pm | Pre-Conference Workshops 2 (Wilden 102, 103) |
| 5:00–6:00 pm | Dinner (1899 Dining Hall) |
| 5:15–11:30 pm | Optional Dodgers/Pirates game |
| 6:30–9:30 pm | Informal Social Gathering (President's Dining Room) |

| Wednesday, June 1 | |
|-------------------|--|
| 7:30–8:30 am | Breakfast (1899 Dining Hall) |
| 9:00–12:00 pm | Pre-Conference Workshops 3 (Wilden 102, 103) |
| 12:00–1:00 pm | Lunch (1899 Dining Hall) |
| 2:30–4:30 pm | Reception at the home of Francis, Natalie, and Nathanael Su (Pasadena) |
| 5:30–6:30 pm | Dinner (1899 Dining Hall) |
| 7:00–7:45 pm | Opening Session (Wilden Lecture Hall) – Welcome: Dave Klanderman – Azusa Pacific University Provost Greeting: Rukshan Fernando – Introductions, Devotional, and Announcements: Bryant Mathews |
| 7:45–8:45 pm | Plenary 1: Matt DeLong (Wilden Lecture Hall) <i>Saving Beauty: On Myshkin, Misery, Meaning, and Mathematics</i> |
| 8:45–9:15 pm | Refreshments (Wilden Atrium) |
| 9:00–11:00 pm | Informal Social Gathering (President's Dining Room) |

| Thursday, June 2 | |
|------------------|---|
| 7:00–7:30 am | Morning Prayer: Kristin Camenga (Wilden Lecture Hall) |
| 7:30–8:30 am | Breakfast (1899 Dining Hall) |
| 8:40–9:00 am | Preparatory Session (Wilden Lecture Hall) – Devotional: Derek Schuurman – Announcements: Bryant Mathews |
| 9:00–10:00 am | Plenary 2: Jason Thacker (Wilden Lecture Hall) <i>Ethics in the Age of AI: Navigating Emerging Technologies with Biblical Wisdom</i> |
| 10:00–10:30 am | Refreshments (Wilden Atrium) |
| 10:30–11:55 am | Parallel Contributed Presentations 1 (Wilden 102, 103, 104, Lecture Hall) |
| 12:00–1:00 pm | Lunch (1899 Dining Hall) |
| 1:30–3:25 pm | Parallel Contributed Presentations 2 (Wilden 102, 103, 104, Lecture Hall) |
| 3:30–4:00 pm | Refreshments (Wilden Atrium) |
| 4:00–5:55 pm | Parallel Contributed Presentations 3 (Wilden 102, 103, 104, Lecture Hall) |
| 6:00–7:00 pm | Dinner (1899 Dining Hall) |
| 7:15–8:15 pm | Plenary 3: Talithia Williams (Wilden Lecture Hall) <i>Power in Numbers: The Rebel Women of Mathematics</i> |
| 8:15–11:00 pm | Informal Social Gathering (President's Dining Room) |
| 8:30–11:00 pm | Choir Practice: Karl-Dieter Crisman (Wilden Lecture Hall) |

Conference Schedule (Continued)

| Friday, June 3 | |
|------------------|--|
| 7:00–7:30 am | Morning Prayer: Kristin Camenga (Wilden Lecture Hall) |
| 7:30–8:30 am | Breakfast (1899 Dining Hall) |
| 8:40–9:00 am | Preparatory Session (Wilden Lecture Hall) – Devotional: Kristin Camenga – Announcements: Bryant Mathews |
| 9:00–10:00 am | Plenary 4: Talithia Williams (Wilden Lecture Hall) <i>Scam or Scriptural: Could Plant-Based Diets Be God's Healing Mechanism?</i> |
| 10:00–10:30 am | Refreshments (Wilden Atrium) |
| 10:30–11:45 am | Parallel Contributed Presentations 4 (Wilden 102, 103, 104, Lecture Hall) |
| 11:45–12:45 pm | Lunch (1899 Dining Hall) |
| 12:45–4:15 pm | Excursions (Huntington and Norton Simon Museums) |
| 5:00–6:00 pm | Plenary 5: Jason Thacker (Wilden Lecture Hall) <i>Always Known, but Rarely Loved: Facial Recognition Technology and the Nature of Privacy</i> |
| 6:10–6:20 pm | Group Photo |
| 6:30–8:00 pm | ACMS Business Meeting and Banquet (Upper Turner Campus Center) |
| 8:00–8:45 pm | Honorees (Upper Turner Campus Center) |
| 8:45–11:00 pm | Informal Social Gathering (President's Dining Room) |
| 9:00–10:00 pm | Choir Practice: Karl-Dieter Crisman (Wilden Lecture Hall) |
| Saturday, June 4 | |
| 7:30–8:30 am | Breakfast (1899 Dining Hall) |
| 8:50–9:00 am | Announcements: Bryant Mathews (Wilden Lecture Hall) |
| 9:00–9:20 am | ACMS Business Meeting (Wilden Lecture Hall) |
| 9:20–10:20 am | Plenary 6: Matt DeLong (Wilden Lecture Hall) <i>Child's Play: Mathematical Exploration and Exposition at MathPath</i> |
| 10:20–11:00 am | Refreshments (Wilden Atrium) |
| 11:00–12:00 pm | Worship Service: Josh Wilkerson, Karl-Dieter Crisman (Wilden Lecture Hall) |
| 12:00–1:00 pm | ACMS Board Meeting (outside tables during lunch) |

Parallel Session Schedule

| Thursday, June 2 | | | | |
|------------------|---|--|---|---|
| Session 1 | Wilden 102 | Wilden 103 | Wilden 104 | Lecture Hall |
| 10:30–10:55 | Michael Veatch <i>International vaccine allocation models</i> | Christina Osborne <i>The First Step Towards Higher-Order Chain Rules for Abelian Calculus</i> | Sarah A. Nelson <i>Innovative Assessments via Creative Projects</i> | Ricardo Cordero-Soto <i>Restoring Faith: A Philosophy and Model for Faith Integration in Mathematics</i> |
| 11:00–11:15 | Kevin Vander Meulen <i>Geometry as a Playground for Expanding Our Horizons: A review</i> | Samuel Alexander <i>Formal Differential Variables and an Abstract Chain Rule</i> | India White <i>Cultural Competency: Providing Equitable Math Instruction through Cultural Relevance</i> | Bradley McCoy <i>Thematic Approach to Program-Level Faith Integration Curriculum</i> |
| 11:20–11:35 | Adam Goodworth, Russell Howell <i>Complex Analysis, Stability, and Cerebral Palsy</i> | Jesús Jiménez <i>Fibonacci-Type Identities for Second-Order Linear Recurrences</i> | Amanda Harsy, Sarah Klanderman, Marie Meyer, Michael Smith, Brittany Stephenson, Cara Sulyok <i>Surviving a Pandemic Using Alternate Assessments</i> | Cory Krause <i>Reflections on Maddy's Naturalized Epistemology of Mathematics</i> |
| 11:40–11:55 | Esther Lee, Lisa Hernández <i>TDA and the Housing Market</i> | Joseph DiMuro <i>PreTeXt for Accessible Math Documents</i> | Rachel Grotheer <i>To Grade or not to Grade: Testing the Waters of "Ungrading"</i> | David Freeman <i>A Belief Expressionist Explanation of Divine Conceptualist Mathematics</i> |
| Session 2 | Wilden 102 | Wilden 103 | Wilden 104 | Lecture Hall |
| 1:30–1:55 | Ken Arnold <i>AI and writing: Laziness or thoughtfulness?</i> | Francis Su <i>The Game of Cycles</i> | Adam Hammett <i>How to Discover (ALL) Divisibility Tests</i> | Ricardo Cordero-Soto <i>Discovered or Invented: The Ontology of Mathematics</i> |
| 2:00–2:15 | Kari Sandouka <i>Engaging the Imperial Army</i> | Bryant Mathews <i>The Game of Cycles on Spiders and Hammocks</i> | Kristin A. Camenga <i>A First Attempt at a Corequisite Course for Calculus I</i> | Daniel Rodman <i>God and Mathematical Platonism</i> |
| 2:20–2:35 | Derek Schuurman <i>A Christian Field Guide to Technology for Engineers and Designers</i> | Karl-Dieter Crisman <i>Voting on Cyclic Orders: Results and Ideas in Undergraduate Research</i> | Chris Cyr <i>Partnered Problem-Solving as a Gateway to Active Learning</i> | Chad Mangum <i>Circular Reasoning in Theology and its Intersections with Mathematics</i> |
| 2:40–2:55 | Benjamin Mood <i>Reflections on Group Programming Projects in Computer Science Classes</i> | Nicholas Zoller <i>Ellipses in Taxicab Geometry</i> | Shawn Wirts <i>Student Support in Gateway Mathematics Courses: Academic Intercession</i> | Matt Lunsford <i>A Tale of Two Mentors: G.H. Hardy and C.S. Lewis</i> |

Parallel Session Schedule (Continued)

| Thursday, June 2 | | | | |
|------------------|---|--|--|---|
| Session 3 | Wilden 102 | Wilden 103 | Wilden 104 | Lecture Hall |
| 3:00–3:25 | Victor Norman <i>Using POGIL to Enhance Engagement and Improve Understanding in the Classroom</i> | Bryan Dawson <i>The nearly Universal, Easy-to-Use Level Comparison Test for Series</i> | Judith Canner <i>Threshold Concepts in Quantitative Reasoning: Development and Impact</i> | Andrew Simoson <i>A homily on Hardy's Apology</i> |
| 4:00–4:25 | Jeffrey H. Green, Marian Zaki <i>Rethinking the Software Engineering Code of Ethics Using a Biblical Worldview Lens</i> | Troy Riggs <i>Why Save the Best for Last? Introducing Calculus through Asymptotics</i> | Saburo Matsumoto <i>Mourning of a Mathematician</i> | Jason Ho, Dave Klanderman, Sarah Klanderman, James Turner <i>Using Graphic Novels in the Teaching and Learning of Mathematics and Physics</i> |
| 4:30–4:45 | Kari Sandouka, Miranda Vander Berg <i>The Antecedents of Success for Developing Core Skills</i> | Samuel Alexander <i>Big-Oh Notation, Elections, and Hyperreal Numbers</i> | Tanner Auch <i>The Great Integration</i> | Jeremy Case, Patrick Eggleton, Evidence Matangi, Mandi Maxwell <i>Intentional Community: Fostering Faith . . . Developing Mathematicians</i> |
| 4:50–5:05 | Michael Stob <i>What does $P = 0.035$ Mean?</i> | Anika Homan, Jocelyn Zonnefeld <i>Exploring the Min-Plus and Max-Plus Finite Tropical Semirings</i> | Thomas Shifley <i>Incarnational Mathematics: Teaching Math as a Ministry</i> | Sharon Robbert <i>Teaching Math Technology</i> |
| 5:10–5:25 | Stacy DeRuiter <i>Collaborative Tests in Applied Statistics Courses</i> | Daniel Majcherek <i>Multiplicative Lattices: A Brief Recount</i> | Beth McCoy <i>Mentoring Christian Women in STEM</i> | Mike Janssen <i>Liberal Arts Mathematics for Human Flourishing</i> |
| 5:30–5:55 | Alana Unfried <i>Measuring Statistics and Data Science Attitudes: A Modern Approach, and Why You Should Get Involved</i> | Justin Marks <i>Pursuing Discriminative Statistics for Buried Object Detection Using Ground Penetrating Radar</i> | Jane Chunjing Ji <i>How has Mathematics Helped Me Know God Better?</i> | Calvin Jongsma <i>Incorporating Perspectival Elements in a Discrete Mathematics Course</i> |

Parallel Session Schedule (Continued)

| Friday, June 3 | | | | |
|----------------|---|--|--|--|
| Session 4 | Wilden 102 | Wilden 103 | Wilden 104 | Lecture Hall |
| 10:30–10:45 | Kaitlyn Fitzgerald <i>Data Visualization Tips and Tricks: Teaching and Learning How to be Better Communicators of Data</i> | Elliott Best <i>Topology and the Trinity</i> | Thomas J. Clark <i>Wonder Driven Mathematical Modeling</i> | Kim Jongerius <i>Mathematical Minds: a Senior Seminar Experience for the Mathematically-Minded</i> |
| 10:50–11:05 | Pete Kosek <i>Creating a statistical Literacy Course for Liberal Arts</i> | Corban Harwood <i>Mathematics as a Language of Creation</i> | Michael Martinez <i>Virtual Class Activities with Desmos</i> | Sharon McCathern, Elizabeth Rivas <i>Developing a Writing Course on Mathematical Vocations</i> |
| 11:10–11:25 | Matthew Hawks <i>Redeeming Statistics with DEI</i> | Josh Wilkerson <i>Micah 6:8 Mathematics</i> | Valorie Zonnefeld <i>Training Pre-Service Teachers to Foster a Growth Mindset Culture in their Classrooms</i> | Diana Schepens <i>The Infinite and the Void: a Multidisciplinary Approach to Integrating Faith in the Calculus Classroom</i> |
| 11:30–11:45 | Michael Izbicki <i>Teaching Mathematical Sciences in North Korea as a Christian</i> | Stoyan Dimitrov <i>Randomness and Faith Again: Impressive Bijections and Pseudosecret Messages in the Bible</i> | Elizabeth DeWitt <i>Math Intersect Creativity</i> | Cameron Sweet <i>Christian Values in the Mathematics Curriculum: Incorporating Biblical Principles in a Core Mathematics Course with Benedictine Hallmarks for life with God and Other People</i> |

ACMS would like to thank the following parallel session moderators:

Aisha Chen, Assistant Professor of Engineering and Computer Science, Azusa Pacific University
 Thomas Clark, Associate Professor of Mathematics, Dordt University
 Karl-Dieter Crisman, Professor of Mathematics, Gordon College
 Edwin Ding, Associate Professor of Applied Mathematics, Azusa Pacific University
 Kaitlyn Fitzgerald, Assistant Professor of Statistics, Azusa Pacific University
 Daniel Hogue, Physics and Mathematics Faculty, Azusa Pacific University
 Louise Huang, Assistant Dean of the College of Liberal Arts and Sciences, Azusa Pacific University
 Paul Kaak, Executive Director of the Office of Faith Integration, Azusa Pacific University
 Cory Krause, Assistant Professor of Mathematics, LeTourneau University
 Sharon McCathern, Associate Professor of Mathematics, Azusa Pacific University
 Bradley “Peanut” McCoy, Professor of Physics, Azusa Pacific University
 Elizabeth Rivas, Mathematics Lecturer, Azusa Pacific University
 Lu Ruan, Associate Professor of Computer Science, Azusa Pacific University
 Amanda Sinner, Statistics and Mathematics Faculty, Azusa Pacific University
 Kathryn Tickle, Mathematics Student, Azusa Pacific University
 Dusty Wilson, Mathematics Faculty, Highline College

Appendix 2: Abstracts

Abstracts for Invited Speakers

Saving Beauty—On Myshkin, Misery, Meaning, and Mathematics

Matt Delong, Wilden Lecture Hall, Wednesday, 7:45–8:45 p.m.

Over a decade ago, I was the lead author on the “Beauty” chapter of *Mathematics Through the Eyes of Faith*. In that chapter, we made a case for seeing beauty in mathematics. Beginning with some well-known examples, we articulated one definition of beauty, listed some standards of beautiful mathematics, discussed the roles of objectivity and subjectivity, and framed a theological perspective on what beauty is and what it is for. In the years since, my life experiences, particularly periods of unexpected darkness and suffering, have deepened and changed many of my perspectives. In this talk I will reflect on the chapter written and how I would write it differently now. I will also invite us to puzzle together over Dostoevsky’s claim, “I believe the world will be saved by beauty,” from the mouth of Prince Myshkin in *The Idiot*, and what it might mean for our mathematics, teaching, theologizing, and living.

Ethics in the Age of AI: Navigating Emerging Technologies with Biblical Wisdom

Jason Thacker, Wilden Lecture Hall, Thursday, 9:00–10:00 a.m.

Every day we are faced with deep and complex ethical challenges as new technologies are developed and used. As we seek to navigate these issues in a pluralistic society, we must slow down to ask some of the most fundamental questions about life in a digital age in order to understand how technology is shaping everything about our lives including how we view God, ourselves, and the world around us. This session provides a look at three major ethical concerns in light of AI—including privacy, work, and war—before illustrating the Christian ethic of human dignity in a pluralistic society. The Christian ethic of

Power in Numbers: The Rebel Women of Mathematics

Talithia Williams, Wilden Lecture Hall, Thursday, 7:15–8:15 p.m.

The movie “Hidden Figures” brought visibility to the lives of African American women who served as NASA “human computers” in the 1960s, women who dreamed the impossible in a field where their presence was lacking. When it comes to inspiring the future productivity and innovation of our nation, mathematicians are on the front lines. In this talk, I’ll discuss my personal journey as a woman of color in mathematics and share ways we can excite public interest in mathematics, building upon the rich legacy of the Hidden Figures that have come before us. As we shift the fixed mindset around mathematics ability, we can begin conversations that improve public perception of STEM and bring people from all backgrounds into this important work.

Scam or Scriptural: Could Plant-Based Diets be God's Healing Mechanism?

Talithia Williams, Wilden Lecture Hall, Friday, 9:00–10:00 a.m.

I grew up eating a traditional southern diet. Collard greens, macaroni and cheese, candied yams, fried chicken, smoked pork ribs, you name it . . . I ate it. In the south, we greet perfect strangers with a glass of sweet tea in one hand and a bowl of squash casserole in the other. We communicate love with food. But what does God have to say about how we're supposed to eat? Could a raw, plant-based diet give our bodies the fuel it needs to cure itself? In this talk, I'll share our personal family journey towards better nutritional health and the data from scripture that supports a Genesis 1:29 lifestyle.

**Always Known, but Rarely Loved: Facial Recognition Technology
and the Nature of Privacy**

Jason Thacker, Wilden Lecture Hall, Friday, 5:00–6:00 p.m.

Highly sophisticated surveillance systems are being deployed to track, identify, and control people in all parts of the world. These systems employ a number of technologies—including facial recognition, artificial intelligence, and various forms of data tracking—which raise major ethical concerns with centralized power in the hands of authoritarian leaders and personal privacy. Questions abound over the ethical use of these technologies for the public good often without a solid foundation of the nature and role of personal privacy in Christian ethics. Drawing on the concept of the imago Dei and the Christian moral tradition, this session provides a foundation for ethical guidance in the utilization of facial recognition tools that honor God and love our neighbors, upholding personal privacy and protecting the innocent among us.

Child's play: Mathematical Exploration and Exposition at MathPath

Matt DeLong, Wilden Lecture Hall, Saturday, 9:20–11:20 a.m.

MathPath is a four-week residential summer program for students aged 11-14 showing high promise and interest in mathematics. As Academic Director of the camp, I hire faculty and oversee the course offerings. I also help grade the Qualifying Test and run the Writing Plenaries at camp. In this talk I will give an overview of the MathPath program, share some of my favorite Qualifying Test problems from recent years, and discuss our approach to developing the campers as mathematical writers.

Mathematics in Context: Its History, Philosophy, and Connections with the Christian Faith

(Organizers: Bob Brabenec, Russell Howell, Richard Stout, Dusty Wilson)

The ACMS was founded in part to facilitate the discussion of issues related to the integration of faith and learning in mathematics courses. Such a task seems to require some knowledge of the foundational and philosophical issues in mathematics, as well as a sense of the historical background that generated them. This workshop is designed to introduce topics that could be used throughout the curriculum, but are generally not a part of most undergraduate or graduate programs. The first day will focus on historical topics, primarily from the 18th and 19th centuries (e.g., the rise of axiomatization, introduction of new geometries, development of complex numbers, work of Cantor and set theory). The second day will explore philosophical issues, such as the implications of Gödel's theorems, a summary of classical philosophies of mathematics, and a discussion of current issues in the philosophy of mathematics. No prior background in these topics is assumed.

A link to some documents provided for this workshop can be found by clicking [here](#).

Integrating Ethics into your Statistics and Computer Science Courses: Integrated Ethics Labs

(Organizers: Lori Carter, Catherine Crockett, Stacy DeRuiter)

Are you struggling to find ways to naturally integrate your faith into your teaching? Would you like to become part of a project that will impact students well into their careers? The ongoing Integrated Ethics Labs project provides professor- and student- friendly ethics labs that can be integrated into computer science, data science and statistics courses throughout multi-year programs. We believe, as have others before us, that ethics have become an increasingly important part of the undergraduate education of students in these disciplines. We also believe that Christians should be at the forefront of this movement. In this workshop, participants will:

- be introduced to the existing materials available at integratedethicslabs.org;
- experience several of the ethics labs that tie ethics directly to the subject matter;
- get tips on how to include ethics labs in a course or throughout a program;
- adapt and/or create ethics labs to be used in their own courses;
- practice presenting a lab and receive feedback;
- learn how to get involved in the Integrated Ethics Labs project.

Thursday, 10:30-10:55 a.m.

International Vaccine Allocation Models

Michael Veatch (Gordon College), Wilden 102

It's always a plus when you can talk about your research in class—this one fits in a class on optimization. The COVID-19 Vaccine Alliance (COVAX) has called for donations of vaccines to low-income countries largely on the basis of equity. We propose a model to explore another motivation: reduced spread and reduced opportunity for the virus to mutate to more infectious variants. The approach combines an epidemiology model, a simple model of the occurrence of variants, and an optimization that chooses the international vaccine allocations. This nonlinear optimization problem can be solved by iteratively solving linear problems. A similar model, without the emergence of variants, has been applied to COVID-19 data for vaccine allocation in the U.S. In contrast, we seek to show potential benefits of international vaccine allocation, but with a simplified model that does not accurately predict the pandemic.

The First Step Towards Higher-Order Chain Rules for Abelian Calculus

Christina Osborne (Cedarville University), Wilden 103

One of the most fundamental tools in calculus is the chain rule for functions. Huang, Marcantognini, and Young developed the notion of taking higher order directional derivatives, which has a corresponding higher order iterated directional derivative chain rule. When Johnson and McCarthy established abelian functor calculus, they constructed the chain rule for functors which is analogous to the directional derivative when $n = 1$. In joint work with Bauer, Johnson, Riehl, and Tebbe, we defined an analogue of the iterated directional derivative and provided an inductive proof of the analogue to the HMY chain rule. Our initial investigation of this result involved a concrete computation of the case $n=2$, which will be presented in this talk.

Innovative Assessments via Creative Projects

Sarah A. Nelson (Lenoir-Rhyne University), Wilden 104

After taking more traditional (standardized) tests as students, we tend to rely on such forms of assessment as teachers. Over the years, I have been trying to find alternate forms of assessment that afford my students a richer “exam” experience that is also more enjoyable. Ultimately, I want my students to form deeper connections between something they know really well and/or enjoy very much and the material we are covering in class together. So I (as a mathematics professor) started transitioning my traditional final exam to a final project instead.

During this session, we will go through my journey creating and adapting assessments that promote and encourage student creativity. I will share rubric(s) and samples of student work. Attendees will leave with ideas for what types of products students are capable of producing as well as suggestions for creating effective rubrics.

Restoring Faith: A Philosophy and Model for Faith Integration in Mathematics

Ricardo Cordero-Soto (California Baptist University), Wilden Lecture Hall

Accrediting visits have led California Baptist University to ask how each of its programs systematically integrates faith into their respective disciplines. In the Department of Mathematical Sciences, we are replacing the word “integration” with “restoration.” With a philosophy of Faith Restoration, we incorporate various Faith Themes across all of our courses. These Faith Themes are to produce majors that have restored faith into the thought and practice of the mathematical sciences. The philosophy and implementation of faith restoration shall be discussed with specific examples.

Thursday, 11:00-11:15 a.m.

Geometry as a Playground for Expanding our Horizons: A review

Kevin Vander Meulen (Redeemer University), Wilden 102

The book *Shape* is a popular book by Jordan Ellenberg published by Penguin Press in 2021 that explores the “hidden geometry of information, biology, democracy, and everything else.” I will provide a summary of the book, including some commentary on what the book has to offer.

Formal Differential Variables and an Abstract Chain Rule

Samuel Alexander (The U.S. Securities and Exchange Commission), Wilden 103

One shortcoming of the chain rule is that it does not iterate: it gives the derivative of $f(g(x))$, but not (directly) the second or higher-order derivatives. We present iterated differentials and a version of the multivariable chain rule which iterates to any desired level of derivative. We first present this material informally, then we discuss how to make it rigorous (a discussion which touches on formal foundations of calculus).

Cultural Competency: Providing Equitable Math Instruction through Cultural Relevance

India White (Big Ideas Learning), Wilden 104

Scores from the National Assessment of Educational Progress suggest that if current educational practices in math classrooms remain the same, the achievement gap between Caucasians and African descent demographics will be eliminated in approximately 217 years. To close the academic achievement gap for all students, teachers must be equipped with methods to conduct equitable instruction that is culturally relevant. In this session, attendees will learn methods for providing cultural relevance within specific academic settings. Participants will be empowered with strategies that will engage students in a relevant manner as well as acquire pedagogy, perspective, and practices to implement in the math classroom.

Thematic Approach to Program-Level Faith Integration Curriculum

Bradley “Peanut” McCoy (Azusa Pacific University), Wilden Lecture Hall

Though efforts at integrating faith with academic disciplines are often left to the judgment and creativity of individual faculty, there are benefits to curricula that are coordinated throughout an undergraduate program. Notably, program-level curricula have potential for more comprehensive and cohesive development of students’ faith integration experiences. In this talk, I will present a faith integration curriculum that focuses on four themes throughout an undergraduate physics program: characteristics of scientists, nature of science, science in society, and theological implications. I will also explore how a similar approach might work in math or other disciplines with generalized themes of vocation, nature of the discipline, societal impact and responsibility, and theological implications.

Thursday, 11:20-11:35 a.m.

Complex Analysis, Stability, and Cerebral Palsy

Adam Goodworth, Russell Howell (Westmont College), Wilden 102

Many people are not aware that complex analysis has applications in stability and control systems. This presentation will discuss the general complex function theory that is the basis for the “Nyquist Stability Criterion,” and how this construct was used recently in a Kinesiology study that involved children with cerebral palsy.

Fibonacci-Type Identities for Second-Order Linear Recurrences

Jesús Jiménez (Point Loma Nazarene University), Wilden 103

We will present some generalizations of a few well known Fibonacci identities and use them to estimate the period of the corresponding recurrence modulo a prime.

Surviving a Pandemic Using Alternate Assessments

Sarah Klanderman (Marian University), Marie Meyer, Michael Smith,
Brittany Stephenson, Cara Sulok (Lewis University), Wilden 104

The COVID-19 pandemic is one of the most unexpected global challenges of the early 21st century. It has been devastating to healthcare systems and provided unprecedented challenges to educators worldwide as on-ground classes were suddenly transitioned online. In addition to surviving an unknown virus, educators had the additional challenge of determining how to continue to teach and assess students during this new normal. In this presentation, we will share several assessment practices we used in online and hybrid courses. We will include lessons learned from transitioning “traditional” assessments, group projects, alternative/mastery exams, and oral exams to online and hybrid classes. We will also provide a comparison of student surveys from pre- and post-March 2020 which include questions about test anxiety, growth mindset, and perceptions of learning mathematics.

Reflections on Maddy’s Naturalized Epistemology of Mathematics

Cory Krause (LeTourneau University), Wilden Lecture Hall

Penelope Maddy is a respected philosopher of mathematics who has spent a significant portion of her career pursuing questions related to the foundations of mathematics and set theory. In particular, she has been motivated to investigate (from a naturalistic perspective) how it is possible for set theorists to make progress in a field where the apparent objects of study are not empirically accessible. While beginning with a realist perspective similar to Gödel’s, she eventually settled on a position she believes grounds set-theoretic statements yet is free from ontological commitments. In my talk, I will sketch the development of her views and give a (very modest!) assessment of her current position. Finally, I hope to comment on some possible ways someone with a prior commitment to theism might differ methodologically from her idealized “second philosopher” in discerning the objectivity of set-theoretic statements.

Thursday, 11:40-11:55 a.m.

TDA and the Housing Market

Lisa Hernández, Esther Lee (California Baptist University), Wilden 102

Topological Data Analysis (TDA) is a way to use the topological features found in visual representations of data in order to gain insights into possibly complex data. We will give a brief introduction of TDA followed by an example of its application to the US housing market. Features derived from TDA are useful in classifying and clustering time series and detecting breaks in patterns.

PreTeXt for Accessible Math Documents

Joseph DiMuro (Biola University), Wilden 103

A few years ago, my university started getting serious about accessibility. They announced that, eventually, all course materials for all classes would need to be accessible to those with visual or hearing disabilities. I panicked; how does one make math documents accessible to those with visual disabilities? Fortunately, God redeemed my panic and led me to a new document-making system in development: PreTeXt. In this talk, I will show some of what PreTeXt can do, and discuss my plans for using it in the future.

To Grade or not to Grade: Testing the Waters of “Ungrading”

Rachel Grotheer (Wofford College), Wilden 104

The fall of 2020 brought a host of changes for everyone; for me, that included starting a job at a new institution. After a lot of research into mastery and standards-based grading systems, and reflecting further on my own teaching philosophy, I decided to make a fresh start at my new institution to rework all my classes to use these systems for assessment. I have continued to refine these and branch out even further into other types of “ungrading.” This talk outlines what ungrading is, why I decided to try it, the various methods I’ve tried, and the challenges and benefits I discovered along the way.

A Belief Expressionist Explanation of Divine Conceptualist Mathematics

David Freeman (University of Cincinnati Blue Ash College), Wilden Lecture Hall

Many have pointed out that the utility of mathematical objects is somewhat disconnected from their ontological status. For example, one might argue that arithmetic is useful whether or not numbers exist. We explore this phenomenon in the context of Divine Conceptualism (DC), which claims that mathematical objects exist as thoughts in the divine mind. While not arguing against DC claims, we argue that DC claims can lead to epistemological uncertainty regarding the ontological status of mathematical objects. This weakens DC attempts to explain the utility of mathematical objects on the basis of their existence. To address this weakness, we propose an appeal to Liggins’ theory of Belief Expressionism (BE). Indeed, we point out that BE is amenable to the ontological claims of DC while also explaining the utility of mathematical objects apart from reliance upon their existence. We illustrate these themes via a case study of Peano Arithmetic.

Thursday, 1:30-1:55 p.m.

AI and Writing: Laziness or Thoughtfulness?

Ken Arnold (Calvin University), Wilden 102

Phone keyboards and email apps offer us AI-generated suggestions of words and even complete phrases to enter with a single tap or swipe. But these suggestions don't just reduce physical effort; they reduce cognitive effort as well, nudging writers towards quick and lazy platitudes. In contrast, God commands us to use our words to communicate grace and truth, thinking of the needs of our neighbors above our own. Could AI systems possibly encourage that sort of writing? I think the answer is yes, and that we can take a "swords-to-plowshares" approach of re-forming existing AI technologies for this purpose. I will discuss some of my ongoing work in this direction.

The Game of Cycles

Francis Su (Harvey Mudd College), Wilden 103

The Game of Cycles, introduced in my book *Mathematics for Human Flourishing*, is played on a simple connected planar graph together with its bounded cells, and players take turns marking edges with arrows according to a sink-source rule that gives the game a topological flavor. I first started playing this game with Christopher Jackson, an incarcerated man who is a featured contributor to my book. With Chris and several other mathematicians, we wrote a paper together about this game, and I'll share some of the things we discovered.

How to Discover (ALL) Divisibility Tests

Adam Hammett (Cedarville University), Wilden 104

Ever wonder how certain divisibility tests were discovered? Tests for primes—*e.g.*, an integer is divisible by 3 whenever the sum of its digits is—are well-known, but how would we "discover" this, and are there others to be found? It seems an exhaustive analysis of so-called "block-truncation" divisibility algorithms, and ways to construct them, is not present in a singular piece of number-theoretic literature or simply has not been tackled. We present such an approach: Given an odd prime p different from 5, we show how to discover infinitely many divisibility tests for p . Our argument rests only on p being relatively prime to 10, and as such we may replace p by any positive integer m relatively prime to 10. Moreover, our analysis will prove to be exhaustive in the sense that there could not possibly be other block-truncation divisibility tests. Our results also extend to other bases.

Discovered or Invented: The Ontology of Mathematics

Ricardo Cordero-Soto (California Baptist University); Dusty Wilson (Highline College),
Wilden Lecture Hall

Where does mathematics come from? Is it discovered as claimed by realists or invented as per the nominalists? The transcendent mystery seems too far-fetched to believe the latter and the former too dismissive of the hard work of mathematicians. And then there is the problem of God and abstract objections. Into this fray, we propose a new model for understanding mathematics; a logic-loop of Refinement, Invention, Discovery, and Exploration. Leaning into an egalitarian approach to the philosophy of mathematics as well as the philosophy of mathematical practice, this talk outlines a R.I.D.E. forward into discovery and invention.

Thursday, 2:00-2:15 p.m.

Engaging the Imperial Army

Kari Sandouka (Dordt University), Wilden 102

How do you engage your students in the classroom? Does this change when students are not a major in your discipline? This presentation will cover a pedagogical game implemented in my Software Engineering and in my Project Management courses to help engage students from a diverse set of backgrounds. The presentation will be interactive to learn from each other by discussing what went well and what could improve next time.

The Game of Cycles on Spiders and Hammocks

Bryant Mathews (Azusa Pacific University), Wilden 103

The Game of Cycles is a combinatorial game introduced by Francis Su in 2020 in which players take turns marking arrows on the edges of a simple plane graph, avoiding the creation of sinks and sources. Su and his collaborators have found winning strategies on cycle graphs with chords and on graphs with certain types of symmetry. This talk will discuss winning strategies (Mathews, McCathern, and Tickle) on hammock graphs with three strands as well as a proof (Mathews) that player two can win on any 3-legged spider graph with legs of odd length. The latter proof uses a notion of state isomorphism to decompose a game state into states of smaller pieces of a graph, leading to nim-sum calculations with Grundy values.

A First Attempt at a Corequisite Course for Calculus I

Kristin A. Camenga (Juniata College), Wilden 104

Faced with two 40-student Calculus I sections when our typical cap is 30 and gaps in student preparation exacerbated by the COVID pandemic, we implemented an optional 1-credit co-requisite course for Calculus I in Spring 2022. The Calc I Lab course met once a week for two hours and had an extended deadline that allowed students to add the course until after the first exam of the semester. I will share about the structure and activities of the course, successes and challenges, and adaptations for the future.

God and Mathematical Platonism

Daniel Rodman (Taylor University), Wilden Lecture Hall

Are numbers real? According to Mathematical Platonism, numbers and other mathematical objects are real in the sense that they are unchanging, necessary, independent of minds, and outside of space-time. Yet this might seem to be in tension with the Christian understanding of God as the sovereign creator of the universe. We will briefly discuss three different models that Christian philosophers have held which attempt to reconcile the apparent contradictions at the intersection of Mathematical Platonism and theology.

Thursday, 2:20-2:35 p.m.

A Christian Field Guide to Technology for Engineers and Designers

Derek Schuurman (Calvin University), Wilden 102

This presentation will introduce a newly published book about faith and technology from InterVarsity Academic Press (2022) titled *A Christian Field Guide to Technology for Engineers and Designers*. An outline of the book along with the main themes and chapters will be summarized.

Voting on Cyclic Orders: Results and Ideas in Undergraduate Research

Karl-Dieter Crisman (Gordon College), Wilden 103

Sometimes it is hard to find topics for undergraduate research that are accessible to students with very little background, while still providing the opportunity for seniors or graduate students to use (and learn) quite advanced mathematics.

A discipline that does afford this is the mathematics of voting and choice. In this talk, we describe the basic concepts and results of a multi-year project on voting on ways to sit around a dinner table, or “cyclic orders.” Different students tackled this project equally well using simple sets and functions, then using combinatorics and experimental mathematics, and finally even combining linear algebra and groups in representation theory! Most importantly, ideas for other similar topics for listeners to try will be given.

Partnered Problem-Solving as a Gateway to Active Learning

Chris Cyr (Covenant College), Wilden 104

For those teachers (like myself) who have only ever experienced lecture-style classrooms as students, the idea of implementing more active learning strategies might seem daunting or even downright scary. In this talk, I outline the development of a partnered teaching practice I used in my Calculus 2 and 3 classes from 2019-2021, and report on the results of using student group work more heavily in my Calculus 2 class in spring 2022. I hope to highlight the benefits of these approaches while being honest about the drawbacks, and to provide audience members with some useful ideas for employing active learning strategies in their own classrooms.

Circular Reasoning in Theology and its Intersections with Mathematics

Chad Mangum (Clemson University), Wilden Lecture Hall

This talk supports the epistemological claim that all human reasoning is, at its core, circular. In particular, we claim that questions which naturally arise in the fields of theology, philosophy, and related disciplines, to the extent that they make claims of ultimate existence or ultimate reality, are necessarily circular. Examples and analogies from mathematics will be given to enhance this central claim.

Thursday, 2:40-2:55 p.m.

Reflections on Group Programming Projects in Computer Science Classes

Benjamin Mood (Point Loma Nazarene University), Wilden 102

Many computer science classes require students to do group programming projects. These projects are necessary in order to teach students how to function as part of team in preparation for their jobs. Unfortunately, group projects in computer science can often suffer from a variety of problems that can affect both students and professors, including poor/difficult teammates, problems of grading when it is unclear what happened in the group, non-interesting projects, difficulty in planning out a long project, etc. This presentation will give a reflection of what has worked well, what didn't worked well and why not, and ideas for how to improve group computer science projects in your future classes.

Ellipses in Taxicab Geometry

Nicholas Zoller (Southern Nazarene University), Wilden 103

The taxicab metric measures distance in the way that one does by walking north/south and east/west on city blocks. Taxicab geometry is typically presented as an example of a non-Euclidean geometry with an interesting metric in introductory courses in geometry and topology. Much is known about taxicab geometry, beginning with the astonishing fact that taxicab circles are squares. In this talk, we define and classify taxicab ellipses. Furthermore, we give formulas for both the circumference and area of taxicab ellipses. The derivations of these formulas do not require mathematical knowledge beyond Precalculus, so this research project illustrates the accessibility of taxicab geometry to beginning researchers.

Student Support in Gateway Mathematics Courses: Academic Intercession

Shawn Wirts (Fresno Pacific University), Wilden 104

We review multi-year efforts to support Pre-Calculus and Calculus GE courses offered at Fresno Pacific University, a private 4-year HSI in California's San Joaquin Valley. Data includes instructor guided high-support models and peer support models, such as traditional Supplemental Instruction, in-class TAs, and hybrid roles.

A Tale of Two Mentors: G.H. Hardy and C.S. Lewis

Matt Lunsford (Union University), Wilden Lecture Hall

G.H. Hardy was a prominent British mathematician during the first half of the twentieth century. In 1940, Hardy published an essay defending his career choice of becoming a mathematician, *A Mathematician's Apology*, which has become a definitive piece in the history of mathematics. C.S. Lewis, a prominent British author, scholar of the English language, and Christian apologist, published his autobiographical work, *Surprised by Joy*, in 1955. This talk will share the personal journey of how this Christian mathematician has been influenced by both of these intellectual giants of the 20th century.

Thursday, 3:00-3:25 p.m.

Using POGIL to Enhance Engagement and Improve Understanding in the Classroom

Victor Norman (Calvin University), Wilden 102

Process-Oriented Guided Inquiry Learning (POGIL) is teaching and learning technique that has been proven by research studies to improve students' understand of material, enhance their engagement with the material and each other, and improve students' "soft-skills," such as communication and collaboration. In this talk, I will describe how POGIL works in the classroom, describe my experiences with it, and, if time permits, do a POGIL activity with participants. I will conclude with describing what POGIL materials already exist, and how one can get involved with the POGIL "movement" in Computer Science (and Mathematics and Statistics).

The Nearly Universal, Easy-to-Use Level Comparison Test for Series

Bryan Dawson (Union University), Wilden 103

The use of infinitesimal methods in calculus can simplify computations for students, including the determination of convergence or divergence of a series. The level comparison test for series with nonnegative terms is an example. Featuring a computation that is similar in difficulty to the test for divergence, this test hinges on whether the reciprocal of the "omegath" term of the series lies in the "convergence zone" or in the "divergence zone." In this talk the test is described and demonstrated. (The level comparison test is introduced in the textbook *Calculus Set Free: Infinitesimals to the Rescue*, Oxford University Press, 2022.)

Threshold Concepts in Quantitative Reasoning: Development and Impact

Judith Canner (California State University Monterey Bay), Wilden 104

In our teaching practice, we often identify "threshold concepts" that are consistently troublesome for our students, but when they "get it" we see a transformation in their ability to understand the topic. In general, threshold concepts are 1) transformative to learner thinking; 2) bounded by the discipline; 3) integrative with other concepts 4) irreversible once understood; and 5) troublesome for learners to understand. At Cal State Monterey Bay, we recognize that threshold concepts in quantitative reasoning are not exclusive to mathematics and statistics courses and we believe that a focus on threshold concepts in quantitative reasoning reinforces the transfer of knowledge from general education mathematics and statistics to discipline-specific quantitative courses. We will share how we identified five threshold concepts for quantitative reasoning through interdisciplinary faculty workgroups, their definitions and examples, and the ways we use quantitative reasoning threshold concepts within our curriculum.

A Homily on Hardy's *Apology*

Andrew Simoson (King University), Wilden Lecture Hall

An electrical engineer has asked me to write an introduction to his forthcoming book on The Mathematical Radio which is all the mathematics including Maxwell's equations on how the radio works. In that book—as a literary device—he imagines, as it were, responding to G.H. Hardy who, in his 1940 famed *A Mathematician's Apology*, denigrated the realm of applied mathematics. Therein Hardy wrote (i) "Engineering is not a useful study for ordinary men." (ii.) "There is the real mathematics of the real mathematicians, and there is what I call the trivial mathematics—which includes its practical application, the bridges and steam-engines and dynamos." (iii.) "Most people can do nothing at all well." (iv.) "Exposition . . . is work for second-rate minds." In the 82 years since Hardy made such claims, how might we respond to Hardy—imagining that he were alive today and his opinions had remained unchanged? This proposed talk is a meditation on various open-ended rebuttal thoughts, which I hope may prompt some lively ACMS discussion beyond this presentation.

Thursday, 4:00-4:25 p.m.

Rethinking the Software Engineering Code of Ethics Using a Biblical Worldview Lens

Jeffrey H. Green, Marian Zaki (Houston Baptist University), Wilden 102

Many professional guilds have implemented a code of ethics to guide their members. In this presentation we are going to consider one such code of ethics: The Software Engineering Code of Ethics and Professional Practice by ACM/IEEE-CS. This code has some aspects which resonate with Christian ethics broadly and can be rooted in biblical worldview. In addition, there are opportunities to explore places where Christian worldview can amplify and elaborate the code. In this presentation we will highlight some ways that professors can teach the code in the context of Christian higher education so that students are both professionally prepared and wise to the ways in which the secular world reflects God's truth. Of particular interest will be our role as stewards of creation, building a Scriptural understating of privacy, and of the principles of Judgment and Discretion and how they can be applied while developing new software solutions.

Why Save the Best for Last? Introducing Calculus through Asymptotics

Troy Riggs (Union University), Wilden 103

In the traditional calculus textbook, reasoning about relative rates of growth or orders of magnitude is not encouraged until well after the introduction of the exponential function. And a toolbox of rigorous techniques for handling relative rates of growth usually waits for infinite series. Historically however, rates of growth and infinite series were utilized to develop many of the results of the calculus. In this brief paper, I propose that the notion of infinitesimals (now made rigorous) and Dawson's method of approximation in the Hyperreal numbers can be employed to introduce calculus students "on the front end" to asymptotic intuition without compromising a rigorous approach to the material.

Mourning of a Mathematician

Saburo Matsumoto (College of the Canyons), Wilden 104

In 2018, my youngest child Erika passed away at age 21. As a Christian, as a father, and as a mathematician, I struggled with many faith-related questions concerning her death while mourning and grieving with the rest of my family. Where was God when Erika took her last breath? He could have stopped her from dying, but why didn't He? Where was He when she prayed? Did He not hear our cries and prayers? Although the sudden loss was devastating and emotional pain has been excruciating, God taught me many lessons—from mathematics, the society in general, and His Word—and even gave me a "peace that passes through all understanding." In particular, some mathematical concepts played a major role in healing the wounds. In this presentation I will share some of these lessons and thoughts that have helped me stay close to Him during this difficult time

Using Graphic Novels in the Teaching and Learning of Mathematics and Physics

Jason Ho (Dordt University), Dave Klanderman (Calvin University),
Sarah Klanderman (Marian University), James Turner (Calvin University), Wilden Lecture Hall

Are you looking for innovative teaching strategies for geometry or other mathematics and physics courses? If so, then join us for a discussion of several graphic novels and their potential for successful teaching and learning at the high school and university levels. Find out how engaging stories, combined with mathematical and scientific meaning found in both text and image, can help to excite students, enrich learning, and explain mathematical concepts. We report on recent data collected from multiple mathematics and physics classes that extend prior research on the use of graphic novels to teach English Language Arts (Boerman-Cornell and Kim, 2020) and will inform a book focused on the STEM disciplines that is currently in development (Boerman-Cornell, Ho, Klanderman, and Klanderman, in press).

Thursday, 4:30-4:45 p.m.

The Antecedents of Success for Developing Core Skills

Kari Sandouka, Miranda Vander Berg (Dordt University), Wilden 102

The authors examine antecedents of success on the Education Testing Service Proficiency Profile. The authors use the ETS proficiency profile scores for matched pairs of students who took the exam in their freshman and junior years of college. Exploring the pattern of general education courses taken by students that increased their scores for the core skill areas (reading, writing, mathematics, and critical thinking) is a focal point of the research. Other factors such as gender, discipline of study, and standard predictors identified by research literature were included. The practical implications are to adapt to the test-optional environment for higher education.

Big-Oh Notation, Elections, and Hyperreal Numbers

Samuel Alexander (The U.S. Securities and Exchange Commission), Bryan Dawson (Union University)
Wilden 103

When computer scientists use big-oh notation, they implicitly are using a nonstandard number system (as in nonstandard analysis) without realizing it. We present an amusing thought experiment to illustrate what would go wrong if one attempted to use real numbers in place of big-oh notation. Using an election analogy, we introduce the hyperreal numbers and show that if one must replace big-oh notations with numbers, the hyperreals are a better fit.

The Great Integration

Tanner Auch (Bethel University), Wilden 104

When considering how math and faith integrate, it's quite natural to think of the two concepts as fairly independent of each other. What should math look like for a Christian? We can use God's revelations about His broader purposes to help us seek out godly purposes for math. As Christians who view building the kingdom of God as an ultimate concern, we can ask and hope to answer many questions about integrating our subject and our faith. Math can serve to complement our Christian walks as we obey the Great Commandment, the Great Commission, and the Great Requirement.

Intentional Community: Fostering Faith . . . Developing Mathematicians

Jeremy Case, Patrick Eggleton, Evidence Matangi, Mandy Maxwell (Taylor University),
Wilden Lecture Hall

So often students hinder their mathematical development by distancing themselves from the subject. By simply trying to "meet the requirements," students not only lose the benefits of exploring and struggling and developing logical explanations as mathematicians, but they also deter their trust in God's plan and opportunity to serve others. By incorporating rituals and patterns to develop intentional community in the classroom, opportunities to both foster faith and to accept a personal mathematical identity are encouraged. This session shares the observations made by both students and faculty regarding their development in both faith and mathematics as a result of efforts made to establish intentional community in non-major mathematics classrooms. Our goal is to foster virtues that are both mathematical and scriptural as the community works together.

Thursday, 4:50-5:05 p.m.

What does $P = 0.035$ Mean?

Michael Stob (Calvin University), Wilden 102

P values are commonly misunderstood. We only have to ask our students in introductory statistics to explain what $P=0.035$ means to know that. They often tell us that this means that the probability that the null hypothesis is true is 3.5%. This misconception is not confined to students as any cursory review of accounts of research written for the “general public” will reveal. But there is a disguised version of this misconception that is pervasive, even among “experts.” Namely, some experts believe that $P = 0.035$ is strong evidence that the null hypothesis is false.

Exploring the Min-Plus and Max-Plus Finite Tropical Semirings

Jocelyn Zonnefeld, Anika Homan (Dordt University), Wilden 103

Recent work has characterized properties of algebraic structures by exploring their associated directed graphs, which can be thought of as a visual representation of the structure’s Cayley tables. Our work expanded on previous efforts regarding the directed graph of the finite tropical semiring. Building on the previous analysis of the structure of the directed graph, we explore graph-theoretic implications of the minimum and maximum definitions of tropical addition. We give formulas for the in-degree of vertices in the directed graphs and the number of vertices with a given in-degree. We also analyze the connected components of the directed graphs and present formulas for the number of connected components and the greatest common divisor of the vertices in a given component. If time allows, we also discuss the cycles in connected components along with the lengths of paths in a given directed graph.

Incarnational Mathematics: Teaching Math as a Ministry

Thomas Shifley (George Fox University), Wilden 104

The integration of faith in teaching mathematics has for me taken two forms: Using mathematics to point to the order and beauty of God, and using the relationships formed from teaching a notoriously difficult subject to mentor and disciple students. In this talk I will focus on the latter and share how I use my twelve plus years of experience in the outreach ministry Young Life to inform how I approach teaching mathematics. Just as Young Life stresses the incarnational ministry of Jesus, where “The Word became flesh and blood, and moved into the neighborhood” (John 1:14 MSG), I emphasize the role of real and authentic relationships in the classroom. I will share some of my experience, and simple, practical things I do to cultivate relationships with students, where the goal is to share not just my knowledge, but my life with students.

Teaching Math Technology

Sharon Robbert (Trinity Christian College), Wilden Lecture Hall

At Trinity Christian College, we have tried two different ways to help our math majors understand how to use math technology appropriately: infused “just in time” within specific courses and as a stand-alone comprehensive class. In this presentation, I will explain the pros and cons of each approach and share both course design strategies for others to try.

Thursday, 5:10-5:25 p.m.

Collaborative Tests in Applied Statistics Courses

Stacy DeRuiter (Calvin University), Wilden 102

Findings from the science of teaching and learning, statistical education guidelines, and professional requirements all support the practice of integrating collaborative analysis of real-world datasets in applied statistics courses. Group project reports are one obvious way to assess this kind of work, but in courses where collaborative work features prominently and tests are used for assessment, are there creative ways to incorporate it into tests in a way that still measures and rewards individual effort and learning? I will describe a scheme for collaborative take-home statistics tests that I have trialed in the past two academic years, including technical implementation details including using Rmarkdown, RStudio and GitHub Pages to streamline the process; examples and lessons learned; and (pending IRB approval for Spring 2022 surveys) student feedback on the experience.

Multiplicative Lattices: A Brief Recount

Daniel Majcherek (Liberty University), Wilden 103

The study of complete lattice monoids as an abstraction of ideal theory in commutative rings began with Krull's "Axiomatische Begründung der allgemeinen Idealtheorie, Sitzungsbericht der physikalisch-medizinischen Gesellschaft zu Erlangen," 56 (1924), 47-63. This was later further developed by M. Ward and D.P. Dilworth. In this talk, I will give a brief recounting of the history and development of multiplicative lattices, current work being done regarding these structures, and my personal contributions.

Mentoring Christian Women in STEM

Beth McCoy (Azusa Pacific University), Wilden 104

This panel will invite women in Math, Physics, and Engineering, ranging from recent graduates to mid-career women with experience as mentors, to discuss mentoring strategies and needs of Christian women in STEM fields. Based on our experiences, typical guidance from mentors neglects the frequent question of "What does it mean to be a Christian woman in my field?", including the wide variety of options related to academic, family, and career considerations that question entails. The panel will discuss approaches that explore options without prescribing answers in order to help women to thrive in these fields.

Liberal Arts Mathematics for Human Flourishing

Mike Janssen (Dordt University), Wilden Lecture Hall

In his retiring address as president of the MAA in 2017 (and the subsequent book released in 2020), Francis Su laid out a vision for mathematics as integral to a life of flourishing. In this talk, we will describe and share resources for an inquiry-oriented liberal arts math course informed by the vision Su describes.

Thursday, 5:30-5:55 p.m.

**Measuring Statistics and Data Science Attitudes:
A modern Approach, and Why You Should Get Involved**

Alana Unfried (California State University Monterey Bay), Wilden 102

Research and experience tell us that student attitudes matter when learning statistics. Educators have been studying student attitudes toward statistics for decades, but lacking modern instruments for assessing attitudes and a mechanism for studying attitudes at the national scale. These needs are met with the NSF-funded grant “Developing Validated Instruments to Measure Student/Faculty Attitudes in Undergraduate Statistics and Data Science Education” (DUE-2013392). Through the grant, my research team is creating a family of validated instruments to measure student attitudes toward statistics or data science, instructor attitudes toward teaching statistics or data science, and the learning environment. I will describe the goals of the grant, the process for developing the six proposed instruments, and findings from our Spring 2022 national administration of the Student Survey of Motivational Attitudes toward Statistics (S-SOMAS). I encourage anyone who teaches a statistics or data science class to get involved with Fall data collection.

**Pursuing Discriminative Statistics for Buried Object Detection
Using Ground Penetrating Radar**

Justin Marks (Biola University), Wilden 103

Ground penetrating radar arrays are used to detect and classify buried objects, such as utility pipes and squirrel holes. To enhance detection and classification algorithms, we consider a handful of statistics computed from ground image datacubes. These statistics include total variation, singular value metrics, and distance between intensity distributions. This work is in collaboration with Jason Wilson at Biola

How has Mathematics Helped Me Know God Better?

Jane Chunjing Ji (University of Louisiana at Monroe), Wilden 104

As someone who converted from an atheist Chinese to a re-born Christian, I found mathematics never hindered me from knowing God (while many of my non-Christian friends think it should). Surprisingly, it has helped me to digest and then appreciate many “choking points” in the Bible. This may not be a surprise for a mature Christian though. God is omniscient; even though the Bible is surely not a math textbook, it has plenty of math ideas embedded. However, here I am not only referring to the numbers and geometric shapes mentioned in the Bible. In this presentation, I will share several specific examples on how mathematics concepts (such as ratio, rate of change, set, function, etc.) helps me interpret His words and then hear His voice more clearly. I believe these examples can be used to integrate the teaching and learning of God’s Word with mathematics. Furthermore, I hope my attempt could serve as a small springboard and contribute to a pedagogical supplementary to many Christian scholars’ (e.g., Vern Poythress, Russell Howell, etc.) great work of connecting God and mathematics on metalevel (Howell, 2015).

Thursday, 5:50-5:55 p.m. (Continued)

Incorporating Perspectival Elements in a Discrete Mathematics Course

Calvin Jongsma (Dordt University), Wilden Lecture Hall

This talk explores some ways I've incorporated religious, historical, philosophical, and foundational perspectives in an intermediate level discrete mathematics course, drawing from my 2019 textbook in Springer's Undergraduate Texts in Mathematics series, *Introduction to Discrete Mathematics via Logic and Proof*. As a life-long member of ACMS from its inception and as someone with a professional background in the history and philosophy of mathematics as well as in mathematics proper, I've tried to incorporate perspectival elements in every college mathematics course I taught. Teaching a discrete mathematics course to a diverse audience of students over three decades encouraged me to develop my own material and allowed me to incorporate elements that gave voice to my perspectival interests. Fortunately, while Springer put constraints on the size of my text and forced me on several occasions to advocate for my goals and approach, the end result was a mathematics text that still represented my perspectival passions.

Friday, 10:30-10:45 a.m.

Data Visualization Tips and Tricks: Teaching and Learning How to be Better Communicators of Data

Kaitlyn Fitzgerald (Azusa Pacific University), Wilden 102

Data visualization offers an exciting and relevant entry point to get students thinking critically about data and the ways they consume and communicate information in their everyday lives. Data visualization has become one of the most ubiquitous mediums through which we consume information. Yet, there are known cognitive pitfalls and difficulties in statistical reasoning that are often not taken into account when visualizing data or communicating statistical evidence.

Drawing on literatures in cognitive science, information visualization, and uncertainty communication, I discuss these common pitfalls as well as data visualization best practices for more effective communication. I briefly report findings from a randomized experiment where colleagues and I used simple data visualization principles to improve how statistical evidence is communicated to policy-makers in education. I then discuss implications for our own communication practices as researchers and educators and highlight activities that can be used in the classroom to expose students to data visualization best practices.

Friday, 10:30-10:45 a.m. (Continued)

Topology and the Trinity

Elliot Best (Dallas Theological Seminary), Wilden 103

God has revealed himself in the world, and through the lens of faith, we find his glory all throughout. The doctrine of the Trinity has a central and significant role in our profession of faith. Therefore, as we live, exploring and interpreting our world, we eagerly expect to find reflections of Trinitarian glory. Mathematics, the enjoyment of God's patterns, has its place in that great symphony of praise, and, with regard to the Trinity, topology might be its first chair. Whereas most texts first introduce topology as the strange world it envisions, then present a formal definition, and finally explore that definition through examples, a careful construction of the definition out of the geometric perspective more clearly reveals the nature of topological thinking and opens the door for speculating as to how the unity, diversity, and mutual indwelling of the Godhead shine through the topological world.

Wonder Driven Mathematical Modeling

Thomas J. Clark (Dordt University), Wilden 104

Mathematics that begins in wonder is more likely to draw in the attention of students and help them to find value and meaning in the topic. In calculus and differential equations courses, many modeling topics can begin with an appeal to students' curiosity about how some physical phenomena works. Rooting this mathematical curiosity further in a robust doctrine of Creation further reveals how Faith is inherently relevant to mathematics. Modeling then becomes an opportunity to explore the quantitative structures of God's world. In this presentation we will explore how to build lessons on the foundation of wonder with several examples outlined in the contexts of calculus and differential equations.

Mathematical Minds: a Senior Seminar Experience for the Mathematically-Minded

Kim Jongerius (Northwestern College), Wilden Lecture Hall

This talk will present organizational details of a senior seminar with a calculus II prerequisite and a charge to help students synthesize their collegiate careers at a strongly faith-based institution. We'll look at a current syllabus, talk a bit about the evolution of the course, and hear student feedback about their experiences in the course.

Friday, 10:50-11:05 a.m.

Creating a Statistical Literacy Course for Liberal Arts

Pete Kosek (Sterling College), Wilden 102

A few years ago, Sterling College created a new mathematics general education course designed for a liberal arts education called, “Introduction to Statistical Reasoning.” This course is designed for the average college student to become informed consumers of statistics. This talk will go through the creation, implementation, and feedback from this course.

Mathematics as a Language of Creation

Corban Harwood (George Fox University), Wilden 103

Mathematics bridges the gap between known empirical measurements and burgeoning scientific theory, allowing us to learn more about God’s creation and character as we reason through our mind and senses. Believing in a mindful Creator God, Galileo Galilei resisted the Catholic Church’s dogmatic views on science and championed the use of experimentation in understanding the universe. Yearning to glorify God through the earnestness of his scientific studies, James Clerk Maxwell brought about the second great unification of physics by binding together a wide range of experimental results. Drawing insight from Maxwell’s equations for light, Erwin Schrödinger used the language of partial differential equations to develop quantum mechanics as a deeper understanding of atomic measurements and the periodic table of elements.

This talk will highlight mathematicians, scientists, philosophers, and theologians who have contemplated the role of mathematics in understanding God’s Creation. We will focus on the way that mathematical models bridge disciplines and connect fundamental science to public facts—all while being inherently flawed—and discuss what that means to us as human beings and mathematicians.

Virtual Class Activities with Desmos

Michael Martinez (Charleston Southern University), Wilden 104

As many schools have switched to virtual and hybrid classroom environments, it is becoming necessary to offer interactive classroom activities that students can do online. In this talk, we will introduce the Desmos activity building system and show how it can be used to transform your in-person activities into online activities.

Developing a Writing Course on Mathematical Vocations

Sharon McCathern, Elizabeth Rivas (Azusa Pacific University), Wilden Lecture Hall

APU’s general education curriculum includes a sophomore-level writing course intended to be relevant to students’ major field of study, in which students analyze texts and write in several genres in their major and related disciplines. We developed a new course for math and applied math majors which fulfills this requirement and also includes a significant faith component. Course readings and assignments are focused on developing students’ understanding of their callings as Christians and as mathematicians, in addition to building awareness of career options. I will discuss the goals and implementation of the course, as well as student reactions from the first time it was taught.

Friday, 11:10-11:25 a.m.

Redeeming Statistics with DEI

Matthew Hawks (U.S. Naval Academy), Wilden 102

Three major pioneers in the field of statistics promoted eugenics. How can Christian statistics instructors overcome this immoral foundation and attract students “from every nation, from all tribes and peoples and languages?” In this talk, gain three tangible classroom instruction tools you can use next semester! (1) Formulate a course policy that explicitly recognizes past exclusions. Such a statement addresses concerns that a student may not “belong” in the field. (2) Ask a daily question to nurture respect for each individual. Students will open up to their peers and have opportunities for daily validation. (3) Incorporate a simple biographical assignment to expose students to contributions from underrepresented individuals. Help students recognize that inherent, immutable characteristics are not predictors of future success in statistics. Redeem statistics and welcome the next generation to the field!

Micah 6:8 Mathematics

Josh Wilkerson (Regents School of Austin), Wilden 103

This presentation will unpack how the discipline of mathematics, rightly understood in a Christian context, equips students to do justice, love kindness, and walk in humility with God (Micah 6:8). This presentation will expand on the experience of Regents School of Austin in shaping the philosophy of its mathematics program around several key works: Math for Human Flourishing, Citizen Math, Building Thinking Classrooms, and Beauty for Truth’s Sake. Several practical examples from the Regents curriculum will be shared to demonstrate how the ideal of integrating faith and Christian service into the math classroom becomes a reality that is meaningful for students.

Training Pre-Service Teachers to Foster a Growth Mindset Culture in their Classrooms

Valorie Zonnefeld (Dordt University), Wilden 104

Research demonstrates that a student’s mindset plays an important role in achievement and that mindsets are domain specific. Carol Dweck claimed that mathematics needs a mindset makeover and has shown that teachers can foster a growth mindset through their pedagogical choices. This paper shares how one university trains preservice teachers in mathematics pedagogies that are key to fostering a growth mindset. These practices include educating students on brain function, equitable access, metacognition strategies, feedback practices, the importance of productive struggle, and learning from mistakes.

Friday, 11:10-11:25 a.m. (Continued)

**The Infinite and the Void: a Multidisciplinary Approach
to Integrating Faith in the Calculus Classroom**

Diana Schepens (Whitworth University), Wilden Lecture Hall

Calculus curriculum is often used as prerequisite material for upper division math and science courses. As such, it can be difficult to find space in the curriculum to accomplish intentional faith integration activities or assignments. The concepts of zero and infinity are intricately linked to both faith and the development of calculus. We propose that rather than trying to find space for explicit faith integration content, instructors adopt an implicit multidisciplinary approach to the presentation of the calculus curriculum so that faith integration can be woven together with calculus throughout an entire semester. This talk will give concrete ways for instructors to shift to a multidisciplinary perspective and examples of faith integration that can be used throughout a typical calculus curriculum.

Friday, 11:30-11:45 a.m.

Teaching Mathematical Sciences in North Korea as a Christian

Michael Izbicki (Claremont McKenna College / PUST), Wilden 102

The Pyongyang University of Science and Technology (PUST) is the only foreign run university in North Korea. It is founded on explicitly Christian principles, and most of the faculty are Christian. But North Korea has a tumultuous relationship with Christianity. The government blames Christians for hundreds of atrocities, and Christianity is notoriously repressed in the country. In this talk, I will present why Christian faculty like myself feel called to work with North Korea, and how we navigate the spiritual and political tensions on the Korean Peninsula.

**Randomness and Faith Again:
Impressive Bijections and Pseudosecret Messages in the Bible**

Stoyan Dimitrov (University of Illinois at Chicago), Wilden 103

The real world governs statistical laws derived under the assumption of complete randomness. Inspired by the JMM 2021 talk of Randy Prium, we will discuss one more time how the believe in God is compatible with this fact. We will show some impressive correspondences in the Bible, as well as some “secret messages” in it (known as the Bible code) rebutted by Ramsey theory.

Friday, 11:30-11:45 a.m. (Continued)

Math Intersect Creativity

Elizabeth DeWitt (Trinity Christian College), Wilden 104

Many mathematicians would describe ourselves and our work as creative. Helping students at all levels develop their own mathematical creativity can be both one of the most challenging and most rewarding aspects of teaching. And yet, non-mathematicians do not always see the overlap between math and creativity. I plan to explore ways mathematicians see creativity in the hopes that the such discussions lead us to be better communicators of the intersection.

**Christian Values in the Mathematics Curriculum:
Incorporating Biblical Principles in a Core Mathematics Course
with Benedictine Hallmarks for Life with God and Other People**

Cameron Sweet (St. Martin's University), Wilden Lecture Hall

Many of the hallmarks of Christian education identified in the Bible by Benedict of Nursia fit naturally in mathematics curriculum. Successes and challenges to incorporating Benedictine Hallmarks were considered from a newly revised Core course on Mathematics in Modern Society. Multiple methods were adopted to integrate these values into the course. Short readings from scripture and mathematics literature that related to the campus values were posted on the course webpage for discussion each week. In-class activities included reflective exercises on applying these values in mathematics. Community was built through these activities as students listened to each other's ideas and techniques. Hospitality was modeled to the instructor's classes through office hours held in the tutoring center. A term paper requiring students to describe mathematics involved in their hobby or area of study encouraged many to practice stewardship of resources using their mathematical knowledge or to express mathematics relevant to their cultures.

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